Mathematical Theory of Finite Element Methods

Programming assignment #1

FEM for 1-D Boundary Value Problems

Write a program for solving two-point boundary value problems by Ritz-Galerkin method using linear finite elements.

Submit a report with graphs of your numerical results/solutions and tables of the errors $e(x)$ in L^2, H^1 norms and the semi-norm defined as below:

$$\|e\|_{L^\infty(\omega)} := \max_j |e(x_j)|,$$

Give a conclusion or summary or comments of your programming report.

Specifications

1. Use double precision.
2. Use 20, 40, 80,160 linear finite elements, respectively. Plot your solutions. Plot the error for Problems 1&2.

Computational examples – solve the following problems:

1. Consider the following convection-diffusion problem:

$$(-k(x)u' + u)' = 1, \quad x \in (0,1), \quad u(0) = 0, \quad u'(1) = 1.$$

Here the coefficient $k(x)$ and the solution $u(x)$ are piece-wise smooth, namely:

$$k(x) = \begin{cases} 1 & x < 0.5, \\ 0.5 & x \geq 0.5, \end{cases} \qquad u(x) = x + \begin{cases} (1 - e^x)/(2\sqrt{e}) & x < 0.5, \\ (1 - \sqrt{e})/(2\sqrt{e}) & x \geq 0.5. \end{cases}$$

2. The deflection of a uniformly loaded, long rectangular plate under axial tension force and fixed ends, for small deflections, is governed by the second order differential equation. Let F represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by:

$$W''(x) - \frac{F}{D}W(x) = -\frac{qx}{2D}(l - x), \quad 0 < x < l, \quad W(0) = W(l) = 0,$$

where l is the length of the plate, and D is the flexural rigidity of the plate. Let $q = 200\ lb/in^2$, $F = 100\ lb/in$, $D = 8.8\text{x}10^7\ lb\ in$, and $l = 50\ in$.
The exact solution is given by: $a = \frac{Fl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a}\{-t^2 + t - \frac{2}{a} + \frac{2}{a sinh(\sqrt{a})}[sinh(\sqrt{a}t) + sinh(\sqrt{a}(1-t))]\}.$$

3. Repeat problem 2 while using a different boundary condition at the right end, i.e.,

$$W'(l) = 0 \text{ instead of } W(l) = 0$$