Mathematical Theory of Finite Element Methods

Programming assignment #2

FEM for 1-D Boundary Value Problems

Write a program for solving two-point boundary value problems by Ritz-Galerkin method using quadratic finite elements.

Submit a report with graphs of your numerical results/solutions and tables of the errors $e(x)$ in both L2 and H^1 norm.

Give a conclusion or summary or comments of your programming report.

Specifications

1. Use double precision.
2. Use 20, 40, 80 quadratic finite elements, respectively. Plot your solutions. Plot the errors.

   Report your order of convergence for each problem.

Computational examples – solve the following problems:

1. The deflection of a uniformly loaded, long rectangular plate under axial tension force and fixed ends, for small deflections, is governed by the second order differential equation. Let F represent the axial force and q the intensity of the uniform load. The deflection W along the elemental length is given by:

$$W''(x) - \frac{F}{D}W(x) = -\frac{qx}{2D}(l-x), \quad 0 < x < l, \quad W(0) = W(l) = 0,$$

where l is the length of the plate, and D is the flexural rigidity of the plate. Let $q = 200\ lb/in^2$, $F = 100\ lb/in$, $D = 8.8\text{x}10^7\ lb\ in$, and $l = 50\ in$.
The exact solution is given by: $a = \frac{Fl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a}\{-t^2 + t - \frac{2}{a} + \frac{2}{asinh(\sqrt{a})}[sinh(\sqrt{a}t) + sinh(\sqrt{a}(1-t))]\}.$$

2. Repeat problem 1 while using a different boundary condition at the right end, i.e.,

$$W'(l) = 0 \text{ instead of } W(l) = 0$$

The exact solution is given by: $a = \frac{Fl^2}{D}$, $b = \frac{ql^4}{2D}$, $t = x/l$ and

$$W(t) = \frac{b}{a}\{-t^2 + t - \frac{2}{a} + \frac{1}{acosh(\sqrt{a})}[\sqrt{a}sinh(\sqrt{a}t) + 2cosh(\sqrt{a}(1-t))]\}.$$

3. Consider the following convection-diffusion problem (boundary layer):

$$(-u' + bu)' = 0, \quad x \in (0,1), \quad u(0) = 0, \quad u(1) = 1, \quad b = 10, \quad \text{and} \quad b = 20\ .$$

The solution of this problem is $u(x) = (e^{b(x-1)} - e^{-b})/(1 - e^{-b})$.