Some New Results in Sasakian Geometry

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Introduction

- This talk will consider new results on Sasakian manifolds.
- We will give a uniqueness result on canonical Sasakian metrics:
  constant scalar curvature Sasakian (cscS) metrics, and more generally Sasaki-extremal metrics.

A Sasakian manifold is a special type of metric contact manifold, which can be considered as an odd dimensional version of a Kähler manifold.

In the past 20 years there has been much research from two sources:

- In differential geometry Sasakian manifolds have provided many new examples of compact Einstein manifolds.
  C. P. Boyer, K. Galicki, J. Kollár, A. Futaki, and others.
- In physics Sasaki-Einstein manifolds are an important ingredient in the string theory duality AdS/CFT
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- A proof of the convexity of the K-energy along weak geodesics, (following ideas of R. Berman and B. Berndtson, 2014),
- Uniqueness of cscS metrics (and Sasaki-extremal metrics) for a fix transversal holomorphic structure,
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This is a metric contact structure (M, η, ξ, Φ, g) with an additional integrability condition. One has

- a contact structure
$$\eta = d^c \log r^2 = \frac{1}{2} Jd \log r^2$$
with Reeb vector field $\xi = Jr\partial_r$, a Killing field, and
- a strictly pseudoconvex CR structure (D, I), $D = \ker \eta$.
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The transversal Kähler metrics in $\mathcal{S}(\xi, \bar{J})$ are

$$\{\omega_\phi^T = \omega^T + dd^c\phi \mid \phi \in C_b^\infty(M) \text{ and } (\omega^T + dd^c\phi)^m \wedge \eta > 0\}.$$

We will consider the space of potentials

$$\mathcal{H}_{(\xi,\bar{J})} = \{\phi \in C_b^\infty(M) \mid (\omega^T + dd^c\phi)^m \wedge \eta > 0\}$$

$\phi \in \mathcal{H}_{(\xi,\bar{J})}$ defines a new Sasakian structure $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$:

$$\eta_\phi = \eta + 2d^c\phi,\ \Phi_\phi = \Phi - d\phi \otimes \xi,\ g_\phi = \frac{1}{2}d\eta_\phi \circ (\mathbb{1} \otimes \Phi_\phi) + \eta_\phi \otimes \eta_\phi$$

The complex structure on the cone $C(M)$, and transversal complex structure $(\mathscr{F}_\xi, \bar{J})$ are unchanged.

- $\mathcal{H}_{(\xi,\bar{J})}$ has a natural Riemannian metric and connection (T. Mabuchi 1986)
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Canonical metrics

Given a Sasakian manifold (M, η, ξ, Φ, g) is there a best Sasakian structure in $\mathcal{H}_{(\xi,\bar{J})}$?

Sasaki-Einstein Satisfy the Einstein equation $\mathrm{Ric}_g = \lambda g$ $(\lambda = 2m)$.
Sasakian structure must satisfy $a\omega^T \in c_1(\mathscr{F}_\xi, \bar{J}),\ a > 0$.

cscS More generally we require

$$s_g = \mathrm{const}\left(= \frac{\int_M 4m\pi c_1(\mathscr{F}_\xi, \bar{J}) \wedge (\omega^T)^{m-1} \wedge \eta}{\int_M (\omega^T)^m \wedge \eta} - 2m\right)$$

Sasaki-extremal cscS metrics do not always exist. The Futaki invariant is a well-known obstruction.

Sasaki-extremal metrics are critical points of the Calabi functional:

$$\begin{array}{ccc} \mathcal{H}_{(\xi,\bar{J})} & \xrightarrow{\mathcal{C}} & \mathbb{R} \\ \phi & \mapsto & \int_M s_{g_\phi}^2 \, d\mu_\phi \end{array}$$

Critical points are those structures with the gradient of s_{g_ϕ} real transversely holomorphic.

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I have generalized these uniqueness results to prove uniqueness of cscS structures and more generally Sasaki-extremal structures.

K-energy

Given a Sasakian manifold M the K-energy is a functional on $\mathcal{H}_{(\xi,\bar{J})}$:

$$\mathcal{M}(\phi) = -\int_0^1 \int_M \dot{\phi}_t (S(\phi_t) - \bar{S})(\omega^T_{\phi_t})^m \wedge \eta \, dt, \quad \bar{S} = \frac{2n\pi c_1(\mathscr{F}_\xi) \cup [\omega^T]^{m-1}}{[\omega^T]^m}$$

X. X. Chen 2000 rewrote this formula to extend $\mathcal{M}$ to weak $C^{1,1}$ structures

$$\mathcal{M}(\phi) = \frac{\bar{S}}{m+1}\mathcal{E}(\phi) - \mathcal{E}^{\mathrm{Ric}}(\phi) + \int_M \log\left(\frac{\omega_\phi^m \wedge \eta}{\omega^m}\right)\omega_\phi^m \wedge \eta$$

$$\mathcal{E}(\phi) := \sum_{j=0}^{m} \int_M \phi \omega_\phi^{m-j} \wedge \omega^j \wedge \eta,$$

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Convexity of K-energy

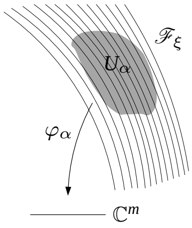

Figure : Transversally complex foliation

The *transversely holomorphic structure* on a foliation $\mathscr{F}_\xi$ is given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M

- $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^m$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

- There is a Kähler structure ω_α on $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^m$.

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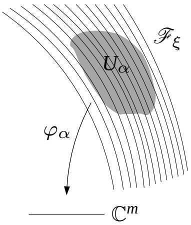

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Analysis is done on the foliation charts.

Let T_α be a closed degree (k,k) current defined on V_α so that $g_{\alpha\beta}^* T_\alpha = T_\beta$.

$$\mathrm{PSH}(M,\omega) := \{\phi \mid \phi \text{ u.s.c. inv. under } \xi \text{ and plurisubharmonic on each chart} V_\alpha\}$$

Given $\phi_1, \ldots, \phi_{m-k} \in \mathrm{PSH}(M,\omega)$, in each V_α we define (E. Bedford and B. Taylor 1976):

$$\omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T_\alpha$$

a positive Borel measure on V_α, and we take the product measure on each chart which is easily seen to be invariant of the chart by Fubini's theorem, defining

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The following will be useful

Proposition 2.1

Let $\phi \in \mathrm{PSH}(M, \omega) \cap C^0(M)$. Then there exists a sequence $\phi_i \in \mathrm{PSH}(M, \omega) \cap C^\infty(M)$ with $\phi_i \searrow \phi$ as $i \to \infty$.

We have weak continuity of the Monge-Ampère measure.

Given decreasing sequences $\phi_1^i \to \phi_1, \ldots, \phi_{m-k}^i \to \phi_{m-k}$ in $\mathrm{PSH}(M, \omega)$ we have

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Let $D \subset \mathbb{C}$ then we have the Homogeneous Monge-Ampère equation

$$(\pi^*\omega + dd^c U_\tau)^{m+1} = 0 \quad \text{for } U_\tau \in \mathrm{PSH}(M \times D, \pi^*\omega),$$

P. Guan and X. Zhang 2012 solved it for $D = \{\tau \in \mathbb{C} \mid 1 \leq |\tau| \leq e\}$ and $U(\cdot, 1) = \phi_0, U(\cdot, e) = \phi_1 \in C_b^\infty(M)$ on ∂D, and showed U is weak $C^{1,1}$, meaning

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Then

$$\omega + dd^c u_t \geq \text{ is weak } C^{1,1} \text{ geodesic connecting } \omega_{\phi_0}, \omega_{\phi_1}, 0 \leq t \leq 1.$$

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If $u \in \mathrm{PSH}(M, \omega) \cap C^0$ then the first variations of the functionals $\mathcal{E}$ and $\mathcal{E}^{\mathrm{Ric}}$ are

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And second variations

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Theorem 2.3

Let u_τ be a weak $C^{1,1}$ geodesic connecting two points in $\mathcal{H}_{(\xi,\bar{J})}$. Then $\mathcal{M}(u_\tau)$ is subharmonic with respect to $\tau \in D$. Thus $\mathcal{M}(u_t)$, $0 \leq t \leq 1$, $t = \log \tau$, is convex.

$\omega_{u_\tau}^m$ defines a singular metric e^Ψ on the transversal canonical bundle $\mathbf{K}_{\mathscr{F}_\xi}$,
The second variation is the current

$$d_\tau d_\tau^c \mathcal{M}(U_\tau) = \int_M T, \quad T := dd^c(\Psi(\pi^*\omega + dd^c U)^m) \wedge \eta$$

But the main problem is to show that T defines a non-negative current on $M \times D$, i.e. a Borel measure.

This is done as in the Kähler case with a local Bergman kernel approximation as in

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Some ideas of the proof

Bergman kernel for holomorphic functions on the ball $B \subset \mathbb{C}^m$ with weight ϕ.

$$\beta_k = \frac{m!}{k^m} K_{k\phi} e^{-k\phi}$$

$$K_{k\phi}(x) = \sup_{s \in H^0(B, K_B)} \frac{s \wedge \bar{s}(x)}{\int_B s \wedge \bar{s} e^{-k\phi}}.$$

$$\beta_k \to (dd^c \phi)^m \quad \text{in total variation.}$$

Choose local psh Φ so that $dd^c\Phi = \pi^*\omega + dd^c U$, $\phi_\tau = \Phi(\cdot, \tau)$. Define

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(B. Berndtsson 2006) Plurisubharmonic variation of Bergman kernels

$$dd^c \log K_{k\phi_\tau} \geq 0 \quad \text{on } B \times D$$

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Consequences of convexity

For $\phi_0, \phi_1 \in \mathcal{H}_{(\xi,\bar{J})}$ we have

$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq -d(\phi_1, \phi_0)\big(\mathrm{Cal}(\phi_0)\big)^{\frac{1}{2}},$$

$$\text{Calabi Energy} \quad \mathrm{Cal}(\phi) := \int_M (S(\phi) - \bar{S})^2 \omega_\phi^m \wedge \eta.$$

Corollary 2.4

Suppose that $(\eta_1, \xi, \omega_1^T), (\eta_2, \xi, \omega_2^T) \in \mathcal{S}(\xi, \bar{J})$ *are two cscS structures. Then there is a* $a \in \mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$*, diffeomorphisms preserving the transversely holomorphic foliation, so that* $a^*\omega_2^T = \omega_1^T$.

- The proof extends to prove uniqueness of Sasaki-extremal structures, $\partial^{\#}_{g^T} S_g$ transversely holomorphic.
  One considers a relative K-energy $\mathcal{M}_V$, $V := \partial^{\#}_{g^T} S_g$ extremal vector field on $\mathcal{H}^G_{(\xi,\bar{J})}$, potential invariant under a maximal compact $G \subset \mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$.
- Since $\mathcal{M}$ is not known to be strictly convex the argument involves an approximation with

$$\mathcal{M}_s := \mathcal{M} + s\mathcal{F}_\mu, \quad \mathcal{F}_\mu(u) = \int_M u \, d\mu - \mathcal{E}(u),$$

where μ is a smooth volume form.

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Suppose that $(\eta_1, \xi, \omega_1^T), (\eta_2, \xi, \omega_2^T) \in \mathcal{S}(\xi, \bar{J})$ are two cscS structures. Then there is a $a \in \mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$, diffeomorphisms preserving the transversely holomorphic foliation, so that $a^*\omega_2^T = \omega_1^T$.

- The proof extends to prove uniqueness of Sasaki-extremal structures, $\partial^{\#}_{g^T} S_g$ transversely holomorphic.
  One considers a relative K-energy $\mathcal{M}_V$, $V := \partial^{\#}_{g^T} S_g$ extremal vector field on $\mathcal{H}^G_{(\xi,\bar{J})}$, potential invariant under a maximal compact $G \subset \mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$.
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where μ is a smooth volume form.

Consequences of convexity

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$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq -d(\phi_1,\phi_0)\big(\mathrm{Cal}(\phi_0)\big)^{\frac{1}{2}},$$

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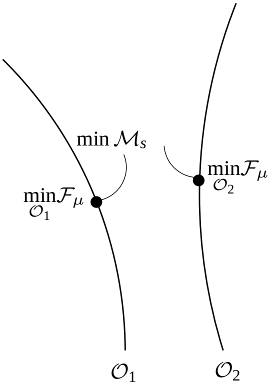
$\mathcal{F}_\mu(u) = \int_M u\, d\mu - \mathcal{E}(u)$ is strictly convex on geodesics.

Suppose $\phi_1, \phi_2 \in \mathcal{H}_{(\xi,\bar{J})}$ with both $\omega_1^T = \omega^T + dd^c\phi_1$ and $\omega_2^T = \omega^T + dd^c\phi_2$ constant scalar curvature.

$\mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$ acts on ω_1^T and ω_2^T with orbits $\mathcal{O}_1$ and $\mathcal{O}_2$.



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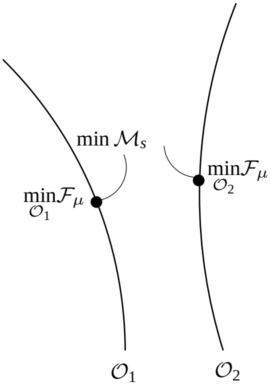

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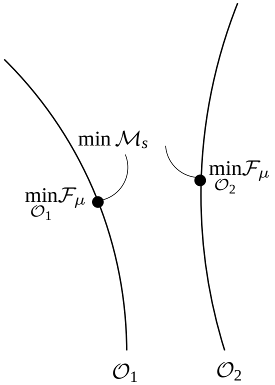

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- Since $\mathcal{F}_\mu$ is strictly convex, it is proper on $\mathcal{O}_1, \mathcal{O}_2$. So there is a minimum $\hat{\phi}_1, \hat{\phi}_2$ on each orbit.
- An implicit function theorem argument shows that there are paths $\psi_s^i, i = 1, 2$ with $\psi_0^i = \hat{\phi}_i$ which are local minimums of $\mathcal{M}_s$ for $s \in [0, \epsilon)$.
- For small s connect ψ_s^1 to ψ_s^2 by a $C^{1,1}$ geodesic. By the strict convexity of $\mathcal{M}_s$ $\psi_s^1 = \psi_s^2$, which implies $\hat{\phi}_1 = \hat{\phi}_2$ and $\mathcal{O}_1 = \mathcal{O}_2$.

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Thank you