Stability in Sasakian geometry

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Centre de Recherches Mathématiques
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Introduction

This talk considers K-polystability and deformations of constant scalar curvature Sasakian metrics. Most of this talk is joint work with Carl Tippler.

We will consider deforming constant scalar curvature Sasakian (cscS) manifolds from the perspective of GIT stability, using the approach of S. K. Donaldson and others.

- Tristan C. Collins, Gábor Székelyhidi 2012 defined K-semistability for Sasakian manifolds and proved
$$\text{cscS} \Rightarrow \text{K-semistable}.$$
- As with Kähler manifolds one has the conjecture
$$\text{Existence of cscS metric} \Leftrightarrow \text{K-polystable}.$$
- At least $\Rightarrow$ should be provable.
- We prove that if (M, η, Φ_0) is cscS and Φ is a contact complex structure close to Φ_0, then
$$(M, \eta, \Phi) \text{ K-polystable} \Rightarrow \exists\, \text{cscS}(M, \eta, \Phi') \text{ with same trans. complex structure}.$$

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Sasakian manifolds

Definition 1.1

A Riemannian manifold (M, g) is Sasakian if the metric cone $(C(M), \bar{g})$, $C(M) := \mathbb{R}_+ \times M$ and $\bar{g} = dr^2 + r^2 g$, is Kähler, i.e. $\bar{g}$ admits a compatible almost complex structure J so that $(C(M), \bar{g}, J)$ is a Kähler structure.

This is a metric contact structure with an additional integrability condition. One has

- a contact structure
$$\eta = d^c \log r = Jd \log r$$
with Reeb vector field $\xi = Jr\partial_r$, a Killing field, and
- a strictly pseudoconvex CR structure (D, I), $D = \ker \eta$.
- I induces a transversely holomorphic structure on $\mathscr{F}_\xi$, the Reeb foliation, with Kähler form $\omega^T = \frac{1}{2} d\eta$.

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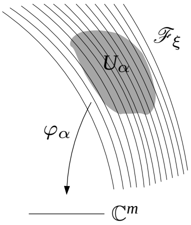

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- $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to \mathbb{C}^{m-1}$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- there are holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

It is well known that the cone $Y := C(M) \cup \{o\}$ of a Sasakian manifold is an affine algebraic variety with an algebraic action of some $T^r = (\mathbb{C}^*)^r$ and can be embedded

$$Y \hookrightarrow \mathbb{C}^N,$$

with T^r acting diagonally.

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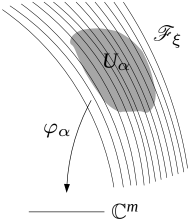

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Space of Sasakian structures

Fix a contact manifold (M, η, ξ).

Definition 2.1

A $(1,1)$-tensor field $\Phi : TM \to TM$ on a contact manifold (M, η, ξ) is called an almost contact-complex structure if

$$\Phi\xi = 0, \ \Phi^2 = -id + \xi \otimes \eta.$$

An almost contact-complex structure is called K-contact if in addition, $\mathcal{L}_\xi \Phi = 0$.

Definition 2.2

An almost contact-complex structure Φ on a contact manifold (M, η, ξ) is compatible with η if

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \text{ and } d\eta(X, \Phi X) > 0 \text{ for } X \in ker(\eta), \ X \neq 0.$$

If Φ is compatible with η, (M, η, Φ) defines a Riemannian metric

$$g_\Phi(X, Y) = \frac{1}{2} d\eta(X, \Phi Y) + \eta(X)\eta(Y),$$

and $(\eta, \xi, \Phi, g_\Phi)$ is called a contact metric structure on M.
This metric structure is called a K-contact metric structure if $\mathcal{L}_\xi \Phi = 0$.
Furthermore, it is a Sasakian structure if in addition

$$N_\Phi(X, Y) := [X, Y] + \Phi([\Phi X, Y] + [X, \Phi Y]) - [\Phi X, \Phi Y] = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

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Moment map

The space $\mathcal{K}$ of K-contact metric structures is an infinite dimensional Kähler manifold.
The subspace of Sasakian, transversely integrable, structures $\mathcal{K}^{int} \subset \mathcal{K}$ is an analytic subvariety.

Define $\mathcal{G}$ to be the group of strict contactomorphism of (M, η, ξ). $\mathcal{G}$ acts on $\mathcal{K}$, for $g \in \mathcal{G}$

$$(g, \Phi) \mapsto g_*^{-1} \Phi g_* .$$

The Lie algebra of $\mathcal{G}$ is

$$\begin{array}{rcl} \mathrm{Lie}(\mathcal{G}) = \big(\{X \in \Gamma(TM) : \ \mathcal{L}_X \eta = 0\}, [\cdot, \cdot]\big) & \cong & \big(C^\infty_b(M), \{\cdot, \cdot\}\big) \\ X & \mapsto & H_X = \eta(X) \end{array} .$$

Theorem 2.3 (W. He 2011, S. K. Donaldson 1997)

The action of $\mathcal{G}$ on $\mathcal{K}$ is Hamiltonian with equivariant moment map $\mu : \mathcal{K} \to \mathcal{G}^*$

$$\mu(\Phi) = s^T(\Phi) - s^T_0 .$$

$$\mathcal{P} : C^\infty_b(M) \to \Gamma(T_{\Phi_0}\mathcal{K}), \quad \mathcal{P}(f) = \mathcal{L}_{X_f} \Phi_0$$

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Then

$$\langle \mathcal{Q}(A), H \rangle_{L^2} = \Omega(A, \mathcal{P}(H)), \quad A \in T_{\Phi_0}\mathcal{K}, \ H \in C^\infty_b(M).$$

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Define $\mathcal{G}$ to be the group of strict contactomorphism of (M, η, ξ). $\mathcal{G}$ acts on $\mathcal{K}$, for $g \in \mathcal{G}$

$$(g, \Phi) \mapsto g_*^{-1} \Phi g_* .$$

The Lie algebra of $\mathcal{G}$ is

$$\begin{array}{rcl} \mathrm{Lie}(\mathcal{G}) = \big(\{X \in \Gamma(TM) : \ \mathcal{L}_X \eta = 0\}, [\cdot,\cdot]\big) & \cong & \big(C_b^\infty(M), \{\cdot,\cdot\}\big) \\ X & \mapsto & H_X = \eta(X) \end{array} .$$

Theorem 2.3 (W. He 2011, S. K. Donaldson 1997)

The action of $\mathcal{G}$ on $\mathcal{K}$ is Hamiltonian with equivariant moment map $\mu : \mathcal{K} \to \mathcal{G}^*$

$$\mu(\Phi) = s^T(\Phi) - s_0^T .$$

$$\mathcal{P} : C_b^\infty(M) \to \Gamma(T_{\Phi_0}\mathcal{K}), \quad \mathcal{P}(f) = \mathcal{L}_{X_f} \Phi_0$$

$$\mathcal{Q} : T_{\Phi_0}\mathcal{K} \to C_b^\infty(M), \quad \mathcal{Q}(A) = ds^T(A)$$

Then

$$\langle \mathcal{Q}(A), H \rangle_{L^2} = \Omega(A, \mathcal{P}(H)), \quad A \in T_{\Phi_0}\mathcal{K}, \ H \in C_b^\infty(M).$$

- s^T is the scalar curvature of the connection on $\Lambda_b^{m,0}$ induced by the Chern connection.
- But when $\Phi \in \mathcal{K}^{int}$, $s^T(\Phi) - s_0^T = s(\Phi) - s_0$, the normalized scalar curvature of g_Φ.
- Although, a complexification $\mathcal{G}^{\mathbb{C}}$ does not exist, the Lie algebra $\mathrm{Lie}(\mathcal{G}) \otimes \mathbb{C} \cong C_b^\infty(M, \mathbb{C})$ acts on $\mathcal{K}$.
- If $\Phi \in \mathcal{K}^{int}$ and $f \in C_b^\infty(M)$, then $\sqrt{-1}f$ acts on ω^T by

$$\mathcal{L}_{\Phi X_f} \omega^T = -\sqrt{-1}\partial\bar{\partial}f.$$

Thus $\mathcal{G}^{\mathbb{C}}$ induces a holomorphic foliation on $\mathcal{K}^{int}$ whose leaves are transversal Kähler classes.

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K-semistability

The cone $Y = C(M) \cup \{0\}$ over a Sasakian manifold (M, g, η, ξ, Φ) is an affine variety.

The Reeb field ξ generates a torus $T \subset \mathrm{Aut}(Y)$. And we have a grading by weights

$$H^0(Y, \mathcal{O}_Y) = \sum_{\alpha \in \mathcal{W}_T} H^0(Y, \mathcal{O}_Y)_\alpha \tag{1}$$

Definition 2.4

A T-equivariant test configuration for Y is a set of T-homogeneous elements $\{f_1, \ldots, f_k\}$ that generate $H^0(Y, \mathcal{O}_Y)$ in sufficiently high degrees together with a set of integers $\{w_1, \ldots, w_k\}$.

Geometrically, $\{f_1, \ldots, f_k\}$ generate an embedding

$$Y \hookrightarrow \mathbb{C}^k,$$

and consider the $\mathbb{C}^*$-action on $\mathbb{C}^k$ with weights $\{w_1, \ldots, w_k\}$. Then the flat limit Y_0 of the $\mathbb{C}^*$-orbit over 0 defines a flat family of schemes over $\mathbb{C}$. In addition to the T-action Y_0 has the induced $\mathbb{C}^*$-action.

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Making use of the Hilbert series of (1) T. Collins and G. Székelyhidi defined Donaldson-Futaki invariant of a test configuration.

Definition 2.5
A polarized affine variety (Y, ξ) is K-semistable if, for every torus T, $\xi \in Lie(T)$, and every T-equivariant test configuration with central fiber Y_0, the Donaldson-Futaki invariant satisfies

$$Fut(Y_0, \xi, \upsilon) \geq 0$$

with υ a generator of the induced $\mathbb{C}^*$ action on the central fiber.
It is K-polystable if the equality holds if and only if the T-equivariant test configuration is a product configuration.

The following extends the results of S. K. Donaldson 2005 for Kähler manifolds and J. Ross and R. Thomas 2008 for quasi-regular Sasakian manifolds.

Theorem 2.6 (T. Collins and G. Székelyhidi 2012)
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Complex deformations

We consider $\mathcal{K}$ up to the action of $\mathcal{G}^{\mathbb{C}}$ and the relevant complexes.

There is a deformation theory of the foliation $(\mathscr{F}_\xi, \bar{J}_0)$, where $\bar{J}_0$ is the transversal complex structure to $\Phi_0 \in \mathcal{K}^{int}$. Let $\mathcal{A}^k := \Gamma(\Lambda_b^{0,k} \otimes \nu(\mathscr{F})^{1,0})$ then

$$0 \to \mathcal{A}^0 \xrightarrow{\bar{\partial}_b} \mathcal{A}^1 \xrightarrow{\bar{\partial}_b} \mathcal{A}^2 \to \cdots . \tag{2}$$

Then $H^1(\mathcal{A})$ is the space of first order deformations.

Suppose $\Phi_0 \in \mathcal{K}^{int}$ and let

$$\mathcal{P} : C_b^\infty(M) \to \Gamma(T_{\Phi_0}\mathcal{K})$$

be the operator representing the infinitesimal action of $\mathcal{G}$ on $\mathcal{K}$, so $\mathcal{P}(f) = \mathcal{L}_{X_f}\Phi_0$ for $f \in C_b^\infty(M)$. We extend it by complex linearity to

$$\mathcal{P} : C_b^\infty(M, \mathbb{C}) \to \Gamma(T_{\Phi_0}\mathcal{K}).$$

We have the transversally elliptic subcomplex $\mathcal{B}$ of (2):

$$0 \to C_b^\infty(M, \mathbb{C}) \xrightarrow{\mathcal{P}} T_{\Phi_0}\mathcal{K} \xrightarrow{\bar{\partial}_b} \mathcal{B} \subset \mathcal{A}^2 \to \cdots . \tag{3}$$

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Finite dimensional slice

Proposition 2.7

There is a natural homomorphism $H^1(\mathcal{B}) \to H^1(\mathcal{A})$ which is injective if $H_b^{0,1}(M) = 0$ and is surjective if $H_b^{0,2}(M) = 0$.

Let $G \subset \mathcal{G}$ be the stabilizer of Φ_0, $G = \mathrm{Aut}(M, g_{\Phi_0}, \eta, \xi, \Phi_0)$ with $\mathfrak{g} = \mathrm{Lie}(G)$.

The following technique of reducing to finite dimensions was developed by G. Székelyhidi and T. Brönnle.

Proposition 2.8

There exists a G-equivariant map Ψ from a neighborhood U of 0 in $H^1(\mathcal{B})$ to a neighborhood of Φ_0 in $\mathcal{K}$ such that the $\mathcal{G}^c$ orbit of every integrable Φ close to Φ_0 intersects the image of Ψ. If x and x' lie in the same $G^{\mathbb{C}}$ orbit in U then $\Psi(x)$ and $\Psi(x')$ are in the same $\mathcal{G}^c$ orbit in $\mathcal{K}$. Moreover, we can assume that

$$\mu \circ \Psi = (s^T - s_0^T) \circ \Psi$$

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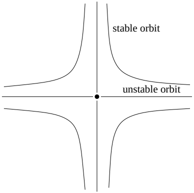

Figure : $\mathbb{C}^*$ acting on $\mathbb{C}^2$

Suppose $(M, g_{\Phi_0}, \eta, \xi, \Phi_0)$ is cscS.

The perturbed slice $\Psi : H^1(\mathcal{B}) \supset U \to \mathcal{K}$ is no longer holomorphic, but the moment map restricts

$$\mu_0 = \mu \circ \Psi : U \to \mathfrak{g}^*,$$

and is close enough to the moment map of the flat Kähler structure (U, ω_0, J_0) one can still apply the Kempf-Ness theorem to show:

$$v \in U \text{ polystable for } G^{\mathbb{C}} \Rightarrow \exists v_0 \in G^{\mathbb{C}} \cdot v \ : \ \mu_0(v_0) = 0.$$

Theorem 2.9

Let (M, η, ξ, Φ) be a cscS manifold and (M, η, ξ, Φ') a nearby Sasakian manifold with transverse complex structure $\bar{J}'$. Then if (M, η, ξ, Φ') is K-polystable, then there is a cscS structure in the space $\mathcal{S}(\xi, \bar{J}')$.

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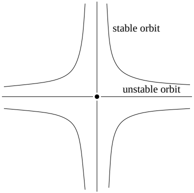

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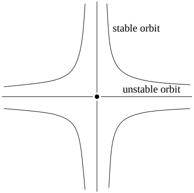

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Mukai-Umemura manifold

This is a Kähler example, or regular Sasaki-Einstein, but it gives a good illustration of the above.

Let V be a 7-dim complex vector space, and $\mathbf{U} \to \mathrm{Gr}_3(V)$ the tautological bundle.
$\Omega \in \Lambda^2(V^*)$ defines a section σ_Ω of $\Lambda^2\mathbf{U}^*$.
Now a 3-plane $\Pi \in \mathrm{Gr}_3(\Lambda^2 V^*)$ defines a subvariety $X_\Pi \subset \mathrm{Gr}_3(V)$ by

$$X_\Pi = \{z \in \mathrm{Gr}_3(V) \ : \ \sigma_\Omega(z) = 0, \forall \Omega \in \Pi\}.$$

X_Π is a smooth Fano 3-fold for $\Pi \in \mathrm{Gr}_3(\Lambda^2 V^*)$ generic.

Let $V = S^6(\mathbb{C}^2)$, symmetric product of $\mathbb{C}^2$ with $\mathrm{SL}(2,\mathbb{C})$ action. Then

$$\Lambda^2 V^* = S^{10}(\mathbb{C}^2) \oplus S^6(\mathbb{C}^2) \oplus S^2(\mathbb{C}^2),$$

and the 3-plane $\Pi_0 = S^2(\mathbb{C}^2) \subset \Lambda^2 V^*$ defines $X_0 = X_{\Pi_0}$ with a $\mathrm{PSL}(2,\mathbb{C})$-action, the Mukai-Umemura manifold.

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$\Omega \in \Lambda^2(V^*)$ defines a section σ_Ω of $\Lambda^2\mathbf{U}^*$.
Now a 3-plane $\Pi \in \mathrm{Gr}_3(\Lambda^2 V^*)$ defines a subvariety $X_\Pi \subset \mathrm{Gr}_3(V)$ by

$$X_\Pi = \{z \in \mathrm{Gr}_3(V) \ : \ \sigma_\Omega(z) = 0, \forall \Omega \in \Pi\}.$$

X_Π is a smooth Fano 3-fold for $\Pi \in \mathrm{Gr}_3(\Lambda^2 V^*)$ generic.

Let $V = S^6(\mathbb{C}^2)$, symmetric product of $\mathbb{C}^2$ with $\mathrm{SL}(2,\mathbb{C})$ action. Then

$$\Lambda^2 V^* = S^{10}(\mathbb{C}^2) \oplus S^6(\mathbb{C}^2) \oplus S^2(\mathbb{C}^2),$$

and the 3-plane $\Pi_0 = S^2(\mathbb{C}^2) \subset \Lambda^2 V^*$ defines $X_0 = X_{\Pi_0}$ with a $\mathrm{PSL}(2,\mathbb{C})$-action, the Mukai-Umemura manifold.

Mukai-Umemura manifold

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Deformations

The tangent space of $\mathrm{Gr}_3(\Lambda^2 V^*)$ at $\Pi_0 = S^2(\mathbb{C}^2)$ is

$$T\,\mathrm{Gr}_3(\Lambda^2 V^*) = S^2 \otimes \left(S^6 \oplus S^{10}\right) = S^{12} \oplus S^{10} \oplus 2S^8 \oplus S^6 \oplus S^4$$

The action of $\mathrm{SL}(V)$ gives a map

$$\mathfrak{sl}(V) \to T\,\mathrm{Gr}_3(\Lambda^2 V^*).$$

Taking the quotient gives

$$H^1(\mathcal{B}) = H^1(\mathcal{A}) = S^8(C^2).$$

We have $G^{\mathbb{C}} = PSL(2,\mathbb{C})$ and there are four orbit types, besides $\{0\}$, on $S^8(C^2)$:

1. The orbits of polynomials having no zero of multiplicity > 3 (closed).
2. The orbits of polynomials with 2 distinct zeros of multiplicity 4 (closed with stabilizer $\mathbb{C}^* \subset \mathrm{PSL}(2,\mathbb{C})$).
3. The orbits of polynomials with a zero of multiplicity 4 and other zeros of multiplicity <4 (not closed, contain type 2 orbits in closure).
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Small deformations of X_0 in orbits of types 1 and 2 admit K-E metrics while those in orbits of type 3 and 4 do not.

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Definition 3.2

A Riemannian manifold (M, g) is 3-Sasakian if the metric cone $(C(M), \bar{g})$ is hyperkähler, i.e. $\bar{g}$ admits a compatible almost complex structures $J_\alpha,\ \alpha = 1, 2, 3$ such that $(C(M), \bar{g}, J_1, J_2, J_3)$ is a hyperkähler structure.

(M, g) is equipped with three Sasaki structures $(\xi_i, \eta_i, \phi_i),\ i = 1, 2, 3$. The Reeb vector fields $\xi_k,\ k = 1, 2, 3$ satisfy $[\xi_i, \xi_j] = -2\varepsilon^{ijk}\xi_k$, and thus generate $\mathfrak{sp}(1)$.

Acting by $e^{\frac{\pi}{2}j} \in \mathrm{Sp}(1)$ defines a real structure on $H^1(\mathcal{A}) = H^1(\mathcal{B})$.

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Any decomposable element of

$$H^1(\mathcal{A}) = \mathrm{Re}\, H^1(\mathcal{A}) \otimes_{\mathbb{R}} \mathbb{C}$$

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- toric means there is a 2-torus $T^2 \subset \mathrm{Aut}(M, g, \xi_1, \xi_2, \xi_3)$, preserving all 3 Sasakian structures.
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Let (M, g) be a toric 3-Sasakian 7-manifold. Then g is in an effective complex $b_2(M) - 1$-dimensional family $\{g_t\}_{t \in \mathcal{U}}$, $\mathcal{U} \subset \mathbb{C}^{b_2(M)-1}$ with $g_0 = g$, of Sasaki-Einstein metrics where g_t is not 3-Sasakian for $t \neq 0$.

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See figure 2 for the isometry groups of the metrics.

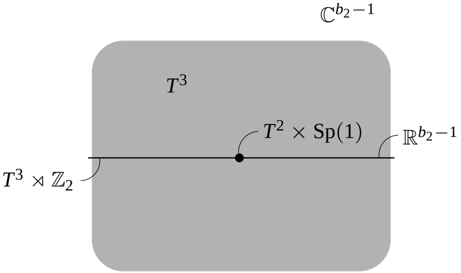

Figure : Space of Sasaki-Einstein metrics

Further results

Of course one would like to strengthen the result of T. Collins and G. Székelyhidi to

$$(M,\eta,\xi,\Phi)\ \text{cscS} \ \Rightarrow\ (Y,\xi) \text{ is K-polystable}.$$

I do not have a proof, but one can prove:

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One defines W. Ding and G. Tian's energy functional F_{ω^T} on

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Corollary 4.2

Let (M,η) be a contact manifold. Then the moduli space of compatible Sasaki-Einstein structures $\Phi\in\mathcal{K}^{int}$ modulo the contactomorphism group $\mathcal{G}$ is a complex space.

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There exists transversal holomorphic foliations of Sasaki type, with $c_1(\mathscr{F}_\xi)=ad\eta,\ a>0$, which do not admit Sasaki-Einstein structures.

Consider the explicit Sasaki-Einstein manifold of J. Gauntlett, D. Martelli, J. Sparks, D. Waldram $Y^{p,q}(X_0)$, $\frac{p}{2}<q<p\ \gcd(p,q)=1$, with X_0 the Mukai-Umemura 3-fold. Then deformations $Y^{p,q}(X_\Pi)$ with Π in orbits of type 3 or 4 provide the counter examples.

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Corollary 4.2

Let (M, η) be a contact manifold. Then the moduli space of compatible Sasaki-Einstein structures $\Phi \in \mathcal{K}^{int}$ modulo the contactomorphism group $\mathcal{G}$ is a complex space.

Corollary 4.3

There exists transversal holomorphic foliations of Sasaki type, with $c_1(\mathscr{F}_\xi) = a d\eta, \ a > 0$, which do not admit Sasaki-Einstein structures.

Consider the explicit Sasaki-Einstein manifold of J. Gauntlett, D. Martelli, J. Sparks, D. Waldram $Y^{p,q}(X_0)$, $\frac{p}{2} < q < p$ $\gcd(p,q) = 1$, with X_0 the Mukai-Umemura 3-fold. Then deformations $Y^{p,q}(X_\Pi)$ with Π in orbits of type 3 or 4 provide the counter examples.

Further results

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