Stability in Sasakian geometry

Craig van Coevering
craigvan@ustc.edu.cn

University of Science and Technology of China, Hefei

Hefei, P. R. China
November 17, 2012

Introduction

This talk considers deformations of Sasakian manifolds with Einstein or constant scalar curvature metrics.

There are two parts:

- We will consider variations of the Killing Spinor equation

$$\nabla_X \psi = cX \cdot \psi, \quad \psi \in \Gamma(\Sigma),\ c \in \mathbb{R} \setminus \{0\}, \tag{1}$$

  on a Sasaki-Einstein manifold.
- We will consider deforming constant scalar curvature Sasaki manifolds from the perspective of GIT stability, using the approach of S. K. Donaldson and others. (joint work with Carl Tippler)
  - Tristan C. Collins, Gábor Székelyhidi 2012 defined K-semistability for Sasaki manifolds and proved

    $$\text{cscS} \Rightarrow \text{K-semistable}.$$

  - As with Kähler manifolds one has the conjecture

    $$\text{Existence of cscS metric} \Leftrightarrow \text{K-polystable}.$$

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Sasaki manifolds

Definition 1.1

A Riemannian manifold (M,g) is Sasaki if the metric cone $(C(M),\bar{g})$, $C(M) := \mathbb{R}_+ \times M$ and $\bar{g} = dr^2 + r^2 g$, is Kähler, i.e. $\bar{g}$ admits a compatible almost complex structure J so that $(C(M),\bar{g},J)$ is a Kähler structure. Equivalently, $\mathrm{Hol}(C(M),\bar{g}) \subseteq \mathrm{U}(m)$.

Thus is a metric contact structure with an additional integrability condition. One has

- a contact structure
$$\eta = d^c \log r = Jd \log r$$
with Reeb vector field $\xi = Jr\partial_r$, a Killing field, and
- a strictly pseudoconvex CR structure (D,I), $D = \ker \eta$.
- I induces a transversely holomorphic structure on $\mathscr{F}_\xi$, the Reeb foliation, with Kähler form $\omega^T = \frac{1}{2} d\eta$.

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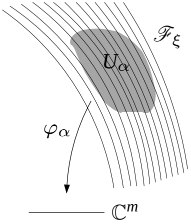
Transversely holomorphic foliation



A *transversely holomorphic structure* on a foliation $\mathscr{F}_\xi$ is given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers M

- $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to \mathbb{C}^{m-1}$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- there are holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

3-Sasaki manifolds

Definition 1.2

A Riemannian manifold (M,g) *is* 3-Sasaki *if the metric cone* $(C(M),\bar{g})$ *is hyperkähler, i.e.* $\bar{g}$ *admits a compatible almost complex structures* $J_\alpha,\ \alpha = 1,2,3$ *such that* $(C(M),\bar{g},J_1,J_2,J_3)$ *is a hyperkähler structure. Equivalently,* $\mathrm{Hol}(C(M)) \subseteq \mathrm{Sp}(m)$.

(M,g) is equipped with three Sasaki structures $(\xi_i,\eta_i,\phi_i),\ i = 1,2,3$. The Reeb vector fields $\xi_k,\ k = 1,2,3$ satisfy $[\xi_i,\xi_j] = -2\varepsilon^{ijk}\xi_k$, and thus generate $\mathfrak{sp}(1)$.

A 3-Sasaki manifold M comes with a family of related geometries. The maps are labeled with their generic fibers.

$$\begin{tikzcd}
 & C(M) \arrow[dl, "\mathbb{R}_+"'] \arrow[dr, "\mathbb{C}^*"] & \\
M \arrow[rr, "S^1"] \arrow[dr, "\mathrm{Sp}(1)/\mathrm{SO}(3)"'] & & \mathcal{Z} \arrow[dl, "\mathbb{C}P^1"] \\
 & \mathcal{M} &
\end{tikzcd}$$

- $\mathcal{Z}$, the *twistor space*, is the orbifold leaf space $\mathscr{F}_{\xi_1}$ with a complex contact structure $\theta \in \Omega^1(\mathbf{L})$.
- $\mathcal{M}$ is a *quaternionic-Kähler* orbifold.
- The *LeBrun-Salamon conjecture* proposes that $\mathcal{M}$ is smooth only if it is symmetric, only proved in dim $= 4$ and 8.

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Killing spinors

Simply connected manifolds admitting a non-zero Killing spinor were classified by C. Bär, 1992:

$\dim M$	N^+	N^-	$\mathrm{Hol}(C(M))$	geometry
n	$2^{\lfloor \frac{n}{2} \rfloor}$	$2^{\lfloor \frac{n}{2} \rfloor}$	Id	n-sphere
$4m-1$	2	0	$\mathrm{SU}(2m)$	Sasaki-Einstein
$4m+1$	1	1	$\mathrm{SU}(2m+1)$	Sasaki-Einstein
$4m-1$	m+1	0	$\mathrm{Sp}(m)$	3-Sasaki
6	1	1	G_2	nearly Kähler
7	1	0	$\mathrm{Spin}(7)$	weak G_2

We consider a deformation of the Killing spinor equation.
Let σ_t be Killing spinors for metrics g_t satifying

$$\nabla^{g_t}_X \sigma_t = cX \cdot_t \sigma_t. \qquad (2)$$

Note that $c = \pm\frac{1}{2}$ when (M, g) is Sasakian.

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Infinitesimal deformations

We first consider first order deformations of (2).

Let $h = \frac{d}{dt} g_t|_{t=0}$ and $\dot{\sigma}_t = \frac{d}{dt}\sigma_t|_{t=0}$ and define $\beta : TM \to TM$ g-symmetric by $h(X,Y) = g(\beta(X),Y)$.

Proposition 2.1 (M. Wang 1991)

If $\mathrm{tr}_g \beta = \delta\beta = 0$, then $(\beta, \dot{\sigma})$ satisfies (2) to first order if and only if

$$\nabla_X \dot{\sigma} = cX\dot{\sigma},$$
$$\mathcal{D}\Psi^{(\dot{\alpha},\sigma_0)} = nc\Psi^{(\dot{\alpha},\sigma_0)}.$$

Here $\Psi^{(\beta,\sigma)}$ with $\Psi^{(\beta,\sigma)}(X) = \beta(X)\sigma$ is a spinor values 1-form and eigenvalue of the twisted Dirac operator:

$$\mathcal{D} : \Gamma(\Sigma \otimes TM_{\mathbb{C}}) \to \Gamma(\Sigma \otimes TM_{\mathbb{C}}).$$

If these conditions are satisfied by $(\beta, \dot{\sigma})$, then

$$\big(\nabla^*\nabla + 2L\big)h = 0 \text{ where } (Lh)_{ij} = R_{i\ j}^{\ k\ l} h_{kl}.$$

So $h \in \Gamma\big(S^2 T^*M\big)$ is an infinitesimal Einstein deformation.

Definition 2.2 (M. Wang 1991)

An infinitesimal deformation of the Killing spinor σ_0 is a pair (β,σ) satisfying:

(i) σ *is a Killing spinor with constant c,*
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Here $\Psi^{(\beta,\sigma)}$ with $\Psi^{(\beta,\sigma)}(X) = \beta(X)\sigma$ is a spinor values 1-form and eigenvalue of the twisted Dirac operator:

$$\mathcal{D} : \Gamma(\Sigma \otimes TM_{\mathbb{C}}) \to \Gamma(\Sigma \otimes TM_{\mathbb{C}}).$$

If these conditions are satisfied by $(\beta, \dot{\sigma})$, then

$$\big(\nabla^*\nabla + 2L\big)h = 0 \text{ where } (Lh)_{ij} = R_{i\ j}^{\ k\ l} h_{kl}.$$

So $h \in \Gamma\big(\mathrm{S}^2\, T^*M\big)$ is an infinitesimal Einstein deformation.

Definition 2.2 (M. Wang 1991)

An infinitesimal deformation of the Killing spinor σ_0 *is a pair* (β, σ) *satisfying:*

- *(i)* σ *is a Killing spinor with constant c,*
- *(ii)* $\mathrm{tr}_g\,\beta = \delta\beta = 0$,
- *(iii)* $\mathcal{D}\Psi^{(\beta,\sigma_0)} = nc\Psi^{(\beta,\sigma_0)}$.

Infinitesimal deformations on Sasakian manifold

For a Sasaki manifold (M, g) the transversal holomorphic structure on $\mathscr{F}_\xi$ has a versal deformation space, with tangent space

$$H^1(\mathcal{A}^\bullet), \quad \text{where } \mathcal{A}^k = \Gamma(\Lambda_b^{0,k} \otimes T_b^{1,0})$$

and

$$0 \to \mathcal{A}^0 \xrightarrow{\bar{\partial}_b} \mathcal{A}^1 \xrightarrow{\bar{\partial}_b} \cdots$$

is the *basic* Dolbeault complex with values in the transverse holomorphic tangent bundle $T_b^{1,0}$ to $\mathscr{F}_\xi$.

If $\mathrm{Ric}_g > -2$ (equivalent to $\mathrm{Ric}^T > 0$), then $H^2(\mathcal{A}^\bullet) = \{0\}$.

Proposition 2.3

Let $\alpha \in H^1(\mathcal{A}^\bullet)$ be Harmonic, $\bar{\partial}_b \alpha = \bar{\partial}_b^* \alpha = 0$. Then

$$h^\alpha(X, Y) = g(\alpha X, Y),$$

is an infinitesimal Einstein deformation of g, that is

$$\operatorname{tr} h^\alpha = \delta h^\alpha = 0$$
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Infinitesimal deformations on Sasakian manifold

Proposition 2.4

Let (M,g) be a spin Sasaki-Einstein manifold admitting the 2 defining Killing spinors $\sigma_j,\ j=0,1$. If $\beta \in \mathcal{H}^1_{\Delta_{\bar{\partial}_b}}(\mathcal{A}^{0,\bullet})$, then h^β is an infinitesimal Einstein deformation of g, and $(h^\beta, 0)$ is an infinitesimal deformation of the Killing spinors σ_j for $j=0,1$.

- $(h^\beta, 0)$ is integrable to actual deformation $\Leftrightarrow$ $\mathrm{Ric}^T = (n+1)g^T$ can be solved for the deformation of $(\mathscr{F}_\xi, \bar{J})$.
- Recall the category of polarized Kähler manifolds $(X, \mathbf{L})$ is contained in category of Sasaki manifolds.
- Sufficient conditions for small deformations of $(X, \mathbf{L})$ to admit a K-E metric are known (G. Székelyhidi 2010), necessary conditions are almost known.
- Similar techniques apply to Sasakian metrics.

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Deformations of 3-Sasaki manifolds

Proposition 2.5

Let (M,g), $\dim M = 4m-1$, be a 3-Sasaki manifold with Killing spinors σ_j, $j = 0,\dots,m$. If $\beta \in \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^\bullet)$ is non-zero, then $(h^\beta, 0)$ is an infinitesimal deformation of the Killing spinors σ_j for $j = 0, m$, but never for $j = 1,\dots,m-1$.

The real structure on the twistor space induces a real structure

$$\varsigma : H^1(\mathcal{A}^\bullet) \to H^1(\mathcal{A}^\bullet).$$

Theorem 2.6

The subspace of infinitesimal Einstein deformations h^β of g for $\beta \in \operatorname{Re}\mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})$ integrates to a family g_t, $t \in \mathcal{N} \subset \mathbb{R}^d$, $d = \dim_{\mathbb{C}} H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$ of Einstein deformations of g preserving only σ_0 and σ_m.

This has an analytic proof, but it also follows from GIT arguments.

Theorem 2.7 (M. Y. Wang, 1991)

Let (M,g) is a compact simply connected spin manifold with irreducible holonomy admitting a nonzero parallel spinor. Then there is a neighborhood $\mathcal{W}$ of g in the Einstein moduli space such that each $\bar g \in \mathcal{W}$ admits the same number of independent parallel spinors.

We will see by example that (2.7) is false for Killing spinors. The dimension of the space of Killing spinors is only upper semicontinuous under Einstein deformations.

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Space of Sasaki structures

Fix a contact structure (M, η, ξ).

Definition 3.1

A $(1,1)$-tensor field $\Phi : TM \to TM$ on a contact manifold (M, η, ξ) is called an almost contact-complex structure if

$$\Phi\xi = 0, \ \Phi^2 = -id + \xi \otimes \eta.$$

An almost contact-complex structure is called K-contact if in addition, $\mathcal{L}_\xi \Phi = 0$.

Definition 3.2

An almost contact-complex structure Φ on a contact manifold (M, η, ξ) is compatible with η if

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \ \text{and } d\eta(X, \Phi X) > 0 \text{ for } X \in ker(\eta), X \neq 0.$$

If Φ is compatible with η, (M, η, Φ) defines a Riemannian metric

$$g_\Phi(X, Y) = \frac{1}{2} d\eta(X, \Phi Y) + \eta(X)\eta(Y),$$

and $(\eta, \xi, \Phi, g_\Phi)$ is called a contact metric structure on M.
This metric structure is called a K-contact metric structure if $\mathcal{L}_\xi \Phi = 0$.
Furthermore, it is a Sasaki structure if in addition

$$N_\Phi(X, Y) := [X, Y] + \Phi([\Phi X, Y] + [X, \Phi Y]) - [\Phi X, \Phi Y] = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

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Moment map

The space $\mathcal{K}$ of K-contact metric structures is an infinite dimensional Kähler manifold.
The subspace of Sasaki, transversely integrable, structures $\mathcal{K}^{int} \subset \mathcal{K}$ is an analytic subvariety.

Define $\mathcal{G}$ to be the group of strict contactomorphism of (M, η, ξ). $\mathcal{G}$ acts on $\mathcal{K}$, for $g \in \mathcal{G}$

$$(g, \Phi) \mapsto g_*^{-1} \Phi g_* .$$

The Lie algebra of $\mathcal{G}$ is

$$\begin{array}{rcl} \mathrm{Lie}(\mathcal{G}) = \big(\{X \in \Gamma(TM) : \ \mathcal{L}_X \eta = 0\}, [\cdot,\cdot]\big) & \cong & \big(C_b^\infty(M), \{\cdot,\cdot\}\big) \\ X & \mapsto & H_X = \eta(X) \end{array} .$$

Theorem 3.3 (W. He 2011, S. K. Donaldson 1997)

The action of $\mathcal{G}$ on $\mathcal{K}$ is Hamiltonian with equivariant moment map $m : \mathcal{K} \to \mathcal{G}^*$

$$m(\Phi) = s^T(\Phi) - s_0^T .$$

- When $\Phi \in \mathcal{K}^{int}$ $s^T(\Phi) - s_0^T = s(\Phi) - s_0$, the normalized scalar curvature of g_Φ.
- Although, a complexification $\mathcal{G}^{\mathbb{C}}$ does not exist, the Lie algebra $\mathrm{Lie}(\mathcal{G}) \otimes \mathbb{C} \cong C_b^\infty(M, \mathbb{C})$ acts on $\mathcal{K}$.
- If $\Phi \in \mathcal{K}^{int}$, $\sqrt{-1}f \in C_b^\infty(M, \mathbb{C})$ acts on ω^T by

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Thus $\mathcal{G}^{\mathbb{C}}$ induces a holomorphic foliation on $\mathcal{K}^{int}$ whose leaves are transversal Kähler classes.

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The action of $\mathcal{G}$ on $\mathcal{K}$ is Hamiltonian with equivariant moment map $m : \mathcal{K} \to \mathcal{G}^*$

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K-semistability

The cone $Y = C(M) \cup \{0\}$ over a Sasaki manifold (M, g, η, ξ, Φ) is an affine variety.

The Reeb field ξ generates a torus $T \subset \mathrm{Aut}(Y)$. And we have a grading by weights

$$H^0(Y, \mathcal{O}_Y) = \sum_{\alpha \in \mathcal{W}_T} H^0(Y, \mathcal{O}_Y)_\alpha \tag{3}$$

Definition 3.4

A T-equivariant test configuration for Y is a set of T-homogeneous elements $\{f_1, \ldots, f_k\}$ that generate $H^0(Y, \mathcal{O}_Y)$ in sufficiently high degrees together with a set of integers $\{w_1, \ldots, w_k\}$.

Geometrically, $\{f_1, \ldots, f_k\}$ generate an embedding

$$Y \hookrightarrow \mathbb{C}^k,$$

and consider the $\mathbb{C}^*$-action on $\mathbb{C}^k$ with weights $\{w_1, \ldots, w_k\}$. Then the flat limit Y_0 of the $\mathbb{C}^*$-orbit over 0 defines a flat family of schemes over $\mathbb{C}$. In addition to the T-action Y_0 has the induced $\mathbb{C}^*$-action.

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Making use of the Hilbert series of (3) T. Collins and G. Székelyhidi defined Donaldson-Futaki invariant of a test configuration.

Definition 3.5

A polarized affine variety (Y, ξ) is K-semistable if, for every torus T, $\xi \in Lie(T)$, and every T-equivariant test configuration with central fiber Y_0, the Donaldson-Futaki invariant satisfies

$$Fut(Y_0, \xi, \upsilon) \geq 0$$

with υ a generator of the induced $\mathbb{C}^*$ action on the central fiber.
It is K-polystable if the equality holds if and only if the T-equivariant test configuration is a product configuration.

The following extends the results of S. K. Donaldson 2005 for Kähler manifolds and J. Ross and R. Thomas 2008 for quasi-regular Sasaki manifolds.

Theorem 3.6 (T. Collins and G. Székelyhidi 2012)

If $(Y = C(M) \cup \{0\}, \xi)$ admits a constant scalar curvature Sasaki cone metric, equivalently $(M, \xi, \bar{J})$ admits a cscS metric, then (Y, ξ) is K-semistable.

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Finite dimensional slice

Suppose $\Phi_0 \in \mathcal{K}^{int}$ and let

$$\mathcal{P} : C_b^\infty(M, \mathbb{C}) \to \Gamma(T_{\Phi_0}\mathcal{K})$$

be the infinitesimal action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{K}$.
We have the transversally elliptic complex:

$$0 \to C_b^\infty(M, \mathbb{C}) \xrightarrow{\mathcal{P}} T_{\Phi_0}\mathcal{K} \xrightarrow{\bar{\partial}_b} \mathcal{B} \subset \mathcal{A}^2 \to \cdots . \tag{4}$$

Denote H^1_η the first cohomology of (4). Let G be the stabilizer of Φ_0,
$G = \mathrm{Aut}(M, g_{\Phi_0}, \eta, \xi, \Phi_0)$ with $\mathrm{Lie}(G) = \mathfrak{g}$.

Proposition 3.7

There exists a G-equivariant map Ψ from a neighborhood U of 0 in H^1_η to a neighborhood of Φ_0 in $\mathcal{K}$ such that the $\mathcal{G}^c$ orbit of every integrable Φ close to Φ_0 intersects the image of Ψ. If x and x' lie in the same $G^{\mathbb{C}}$ orbit in U then $\Psi(x)$ and $\Psi(x')$ are in the same $\mathcal{G}^c$ orbit in $\mathcal{K}$. Moreover, we can assume that

$$m \circ \Psi = (s^T - s_0^T) \circ \Psi$$

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Finite dimensions

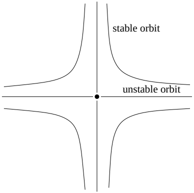

Figure : $\mathbb{C}^*$ acting on $\mathbb{C}^2$

Suppose $(M, g_{\Phi_0}, \eta, \xi, \Phi_0)$ is cscS.
The Kähler structure on $\mathcal{K}$ restricts to $U \subset H^1_\eta$, and we have a (local) action of $G^{\mathbb{C}}$ on U with moment map

$$m_0 = m \circ \Psi : U \to \mathfrak{g}^*.$$

The Kähler structure (U, ω_0, J_0) is not flat. But one can still apply the Kempf-Ness theorem to show:

$$v \in U \text{ polystable for } G^{\mathbb{C}} \Leftrightarrow \exists v_0 \in G^{\mathbb{C}} \cdot v \ : \ m_0(v_0) = 0.$$

Theorem 3.8

Let (M, η, ξ, Φ) be a cscS manifold and (M, η, ξ, Φ') a nearby Sasakian manifold with transverse complex structure $\bar{J}'$. Then if (M, η, ξ, Φ') is K-polystable, then there cscS structure in the space $\mathcal{S}(\xi, \bar{J}')$.

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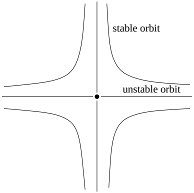

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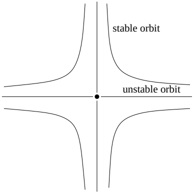

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Toric 3-Sasaki 7-manifolds

We consider toric 3-Sasaki 7-manifolds (C. Boyer, K. Galicki, B. Mann, E. Rees, 1998):

- toric means there is a 2-torus $T^2 \subset \mathrm{Aut}(M, g, \xi_1, \xi_2, \xi_3)$, preserving all 3 Sasakian structures.
- They are 3-Sasaki quotients, $S^{4n-1}//T^k$ by a torus $T^k \subset \mathrm{Sp}(n)$, $n - k = 2$.
- There are infinitely many examples for each $k = b_2(M) \geq 1$.

Lemma 4.1 (van Coevering, N.Y. Journal of Math. 2012)

If $\mathcal{Z}$ is the twistor space of a toric 3-Sasaki 7-manifold M, then $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})^{T^2}$,

$$\dim_{\mathbb{C}} H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = b_2(M) - 1 = k - 1.$$

And $\mathcal{Z}$ has a local $b_2(M) - 1$-dimensional space of deformations.

Theorem 4.2

Let (M, g) be a toric 3-Sasaki 7-manifold. Then g is in an effective complex $b_2(M) - 1$-dimensional family $\{g_t\}_{t \in \mathcal{U}}$, $\mathcal{U} \subset \mathbb{C}^{b_2(M)-1}$ with $g_0 = g$, of Sasaki-Einstein metrics where g_t is not 3-Sasaki for $t \neq 0$.

This can be proved by both GIT and analytic techniques, since the deformation is invariant by $T^3 \subset \mathrm{Aut}(M, g, \xi_1)$, a maximal torus.

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Toric 3-Sasaki 7-manifolds

Thus unlike the case of parallel spinors ($c = 0$) the dimension of the space of Killing spinors is not locally stable in general. See figure 2 for the isometry groups of the metrics.

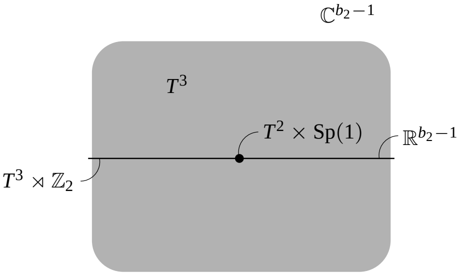

Figure : Space of Sasaki-Einstein metrics

Thank you