Lie Groups, Problem set 1

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Problem 1

Let $X \in \mathcal{X}(M)$ be a smooth vector field on a manifold M, and let Φ_t be the 1-parameter group integrating X. (Recall that Φ is actually only locally defined unless X is complete: every $p \in M$ has a neighborhood U on which Φ_t is defined for $t \in (-\epsilon, \epsilon)$. But here we are concerned with a local result.)
If $(\Phi_t)_* Y$ is the push-forward for $Y \in \mathcal{X}(M)$, then we defined the Lie derivative

$$\mathcal{L}_X Y = -\frac{d}{dt}(\Phi_t)_* Y|_{t=0}.$$

Prove that $\mathcal{L}_X Y = [X, Y]$, where $[X, Y]$ is the bracket of vector fields

$$[X, Y]_p(f) = X_p Y(f) - Y_p X(f) \text{ for } p \in M \text{ and } f \in C^\infty(M).$$

Problem 2

Let $G \subset \mathrm{GL}(m, \mathbb{C})$ be a closed Lie subgroup. Recall that we defined the Lie algebra of G to be the vector space $\mathfrak{g} \subseteq \mathfrak{gl}(m, \mathbb{C})$, where

$$\mathfrak{g} = \{\dot{A}(0) | A : (-\epsilon, \epsilon) \to G \text{ is smooth as funct. into } \mathrm{GL}(m, \mathbb{C})\},$$

and we proved that $\mathfrak{g}$ has a Lie algebra structure given by the bracket of matrices

$$[A, B] = AB - BA, \text{ for } A, B \in \mathfrak{g}.$$

Prove that this is the same as the Lie bracket defined by considering $\mathfrak{g}$ as the space of left-invariant vector fields on G.

Problem 3

Let $X, Y \in \mathcal{X}(M)$ have flows Φ_t and Ψ_t respectively.
a) Prove that

$$\frac{d}{dt}\Psi_{-\sqrt{t}} \circ \Phi_{-\sqrt{t}} \circ \Psi_{\sqrt{t}} \circ \Phi_{\sqrt{t}}(p)|_{t=0^+} = [X, Y]_p.$$

Hint: Define $g(s, t) = f(\Psi_{-s} \circ \Phi_{-t} \circ \Psi_s \circ \Phi_t(p))$ and show that

$$\frac{\partial^2 g}{\partial s \partial t}(0, 0) = [X, Y]_p(f).$$

Apply the two variable mean value theorem to identify this with the required limit.

b) Let G be a Lie group with Lie algebra $\mathfrak{g}$. Define $q(s) = g(s, s)$. Show that $\frac{dq}{ds}(0) = 0$ and

$$\frac{d^2 q}{ds^2}(0) = 2[X, Y]_p(f).$$

From the Taylor series conclude that

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(t^2[X,Y] + O(t^3)),$$

where $O(t^3)$ denotes a smooth function of t into $\mathfrak{g}$ so that $\frac{O(t^3)}{t^3}$ is bounded as $t \to 0$.

Problem 4

Recall that the quaternions $\mathbb{H}$ is a division algebra over $\mathbb{R}$ with $\mathbb{R}$ basis $\{1, i, j, k\}$ satisfying $i^2 = j^2 = k^2 = -1$, $ij = k, jk = i$, and $ki = j$. (See A. Knapp p. 57 for more details).

a) Groups of complex matrices can be identified with groups of real matrices.

The complex vector space $\mathbb{C}^n$ is $\mathbb{R}$-isomorphic with $\mathbb{R}^{2n}$. Define $\mathrm{Re} : \mathbb{C}^n \to \mathbb{R}^n$ and $\mathrm{Im} : \mathbb{C}^n \to \mathbb{R}^n$ by $v = \mathrm{Re}(v) + i\,\mathrm{Im}(v)$ for $v \in \mathbb{C}^n$. Then

$$\mathbb{C}^n \ni v \mapsto \begin{bmatrix} \mathrm{Re}\, v \\ \mathrm{Im}\, v \end{bmatrix} \in \mathbb{R}^{2n}$$

is an $\mathbb{R}$-isomorphism.
Let $M \in \mathfrak{gl}(m, \mathbb{C})$ and write $M = \mathrm{Re}\, M + i\,\mathrm{Im}\, M$. Show that under the this isomorphism M is identified with

$$R(M) = \begin{bmatrix} \mathrm{Re}\, M & -\mathrm{Im}\, M \\ \mathrm{Im}\, M & \mathrm{Re}\, M \end{bmatrix}$$

and satisfies
i) $R(MN) = R(M)R(N)$
ii) $R(M^*) = R(M)^T$
iii) $\det R(M) = |\det M|^2$

b) Groups of quaternion matrices can be identified with groups of complex matrices.

Let $\mathbb{H}^n$ be the space of n-vectors with entries in $\mathbb{H}$. If $v \in \mathbb{H}^n$ is written as $v = a + ib + jc + kd$ for $a, b, c, d \in \mathbb{R}^n$ then write $X(v) = a + ib$ and $Y(v) = c - id$. So $v = X(v) + jY(v)$, and

$$\mathbb{H}^n \ni v \mapsto \begin{bmatrix} X(v) \\ Y(v) \end{bmatrix} \in \mathbb{C}^{2n}$$

is a $\mathbb{C}$-linear isomorphism if $\mathbb{H}^n$ is considered a right $\mathbb{C}$-vector space.
If $M \in \mathfrak{gl}(n, \mathbb{H})$ we define $X(M)$ and $Y(M)$ similarly, so $M = X(M) + jY(M)$. Show that under the above isomorphism left multiplication by M on $\mathbb{H}^n$ corresponds to left multiplication by

$$C(M) = \begin{bmatrix} X(M) & -\overline{Y(M)} \\ Y(M) & \overline{X(M)} \end{bmatrix}.$$

Show this satisfies
i) $C(MN) = C(M)C(N)$
ii) $C(M^*) = C(M)^*$

Problem 5

a) Let $J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$. Recall that the real symplectic group is

$$Sp(n, \mathbb{R}) := \{A \in GL(2n, \mathbb{R}) | A^T J A = J\}.$$

Use part a) of the previous problem to show that

$$U(n) = Sp(n, \mathbb{R}) \cap O(2n).$$

Here of course $U(n)$ is considered as a subgroup of real matrices.

b) The *compact symplectic group* is

$$Sp(n) := \{A \in GL(n, \mathbb{H}) | AA^* = I_n\}.$$

While the complex symplectic group is

$$Sp(n, \mathbb{C}) := \{A \in GL(2n, \mathbb{C}) | A^T J A = J\}.$$

Use part b) of the previous problem to show that

$$Sp(n) = Sp(n, \mathbb{C}) \cap U(2n).$$