Lie Groups, Problem set 1, solutions

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November, 2014

Problem 1

Let $X, Y \in \mathfrak{X}(M)$ and Φ_t the local flow of X. Then we have

$$\mathcal{L}_X Y = -\frac{d}{dt}(\Phi_t)_* Y|_{t=0}.$$

Let $f \in C^\infty(U)$ where U is a neighborhood of $p \in M$. Then

$$\begin{aligned}
\big(\mathcal{L}_X Y\big)_p f &= -\frac{d}{dt}\big(\Phi_{t*}Y\big)_p f|_{t=0} \\
&= -\frac{d}{dt}\big(\Phi_{t*}Y_{\Phi_t^{-1}(p)}\big) f|_{t=0} \\
&= -\frac{d}{dt}\big(Y_{\Phi_t^{-1}(p)}(f\circ\Phi_t)\big) f|_{t=0} \\
&= X_p Y(f) - Y_p X(f)
\end{aligned}$$

Problem 2

Suppose $G \subset \mathrm{GL}(m,\mathbb{R})$ be a closed Lie subgroup. Let $\mathfrak{X}(G)_{L-I}$ denote the left-invariant vector fields on M, which is isomorphic to the tangent space at the identity.

$$\begin{array}{ccc}
\mathfrak{X}(G)_{L-I} & \xrightarrow{\sim} & T_e G \\
X & \mapsto & X_e
\end{array}$$

Let $X, Y \in \mathfrak{X}(G)_{L-I}$. Suppose $X_e = A$ and $Y_e = B$ where $A, B \in \mathfrak{gl}(m,\mathbb{R})$.
Claim : $[X,Y]_e = AB - BA$.

Let $x_{ij} : \mathrm{GL}(m,\mathbb{R}) \to \mathbb{R}$ be the (i,j)-th entry coordinate function. At any point of G a subset of these functions restrict to local coordinate functions on G. At $g \in G$ we have $X_g = L_{g*}X_e$. So

$$X_g(x_{ij}) = X_g(x_{ij}\circ L_g) = x_{ij}(gA),$$

and similarly $Y_g(x_{ij}) = x_{ij}(gB)$. So

$$\begin{aligned}
[X,Y]_e(x_{ij}) &= \big(X_e Y - Y_e X\big)(x_{ij}) \\
&= X_e\big(x_{ij}(gB)\big) - Y_e\big(x_{ij}(gA)\big) \\
&= x_{ij}\big(AB - BA\big)
\end{aligned}$$

Thus

$$[X,Y]_e = AB - BA.$$

The case with $G \subset \mathrm{GL}(m,\mathbb{C})$ is nearly identical with coordinates $\operatorname{Re} z_{ij}$ and $\operatorname{Im} z_{ij}$. Or when G is a complex subgroup z_{ij} restrict to holomorphic coordinates.

Problem 3

a) We differentiate $g(s,t) = f(\Psi_{-s}\circ\Phi_{-t}\circ\Psi_s\circ\Phi_t(p))$ with respect to t

$$\frac{\partial g}{\partial t} = -\big(\Psi_{-s*}X_{\Phi_{-t}\circ\Psi_s\circ\Phi_t(p)}\big)(f) + \big(\Psi_{-s}\circ\Phi_{-t}\circ\Phi_s\big)_* X_{\Phi_t(p)}(f).$$

At $t=0$

$$\begin{aligned}\frac{\partial g}{\partial t}(s,0) &= -\big(\Psi_{-s*}X_{\Psi_s(p)}\big)(f) + X_p(f)\\ &= -\big(\Psi_{-s*}X\big)_p(f) + X_p(f)\end{aligned}$$

So

$$\frac{\partial^2 g}{\partial s\partial t}(0,0) = \big(\mathcal{L}_{-Y}X\big)_p(f) = [X,Y]_p(f).$$

By the two variable mean value theorem (See Rudin,*Principles of Math. Analysis*) we have

$$g(h,k) - g(h,0) - g(k,0) - g(0,0) = \frac{\partial^2 g}{\partial s\partial t}(\hat{h},\hat{k})hk,$$

for $0<\hat{h}<h,\ 0<\hat{k}<k$. In our case this gives

$$f(\Psi_{-h}\circ\Phi_{-k}\circ\Psi_h\circ\Phi_k(p)) - f(p) = \frac{\partial^2 g}{\partial s\partial t}(\hat{h},\hat{k})hk.$$

Substitute $h=k=\sqrt{t}$ to get

$$\lim_{t\to 0^+}\frac{f(\Psi_{-\sqrt{t}}\circ\Phi_{-\sqrt{t}}\circ\Psi_{\sqrt{t}}\circ\Phi_{\sqrt{t}}(p)) - f(p)}{t} = \frac{\partial^2 g}{\partial s\partial t}(0,0) = [X,Y]_p(f).$$

b) Note that $g(s,0) = g(0,t) = f(p)$. So

$$\frac{dq}{ds}(0) = \frac{\partial g}{\partial s}(0,0) + \frac{\partial g}{\partial t}(0,0) = 0.$$

And it is easy to see that

$$\frac{d^2 q}{ds^2}(0) = 2\frac{\partial^2 g}{\partial s\partial t}(0,0) = 2[X,Y]_p(f).$$

Recall that on a Lie group the flow Φ_t of a left-invariant vector field X is given by right multiplication by $\exp(tX)$, i.e. $\Phi_t = R_{\exp(tX)}$. We can write

$$\exp(tX)\exp(tY)\exp(-tX)\exp(-tY) = \exp(Z(t)),$$

with $Z(t) : (-\epsilon,\epsilon)\to\mathfrak{g}\ C^\infty$, since $\exp:\mathfrak{g}\to G$ is a local diffeomorphism. Let $f\in C^\infty(U)$ with U a neighborhood of the identity $e\in G$. Then

$$q(t) = f(\exp(tX)\exp(tY)\exp(-tX)\exp(-tY)),$$

and by part a) the Taylor series gives

$$q(t) = f(e) + t^2[X,Y]_e(f) + O(t^3). \tag{1}$$

We want to find the terms $Z_1, Z_2\in\mathfrak{g}$ of the Taylor series of $Z(t)$

$$Z(t) = tZ_1 + t^2 Z_2 + O(t^3).$$

We compute the first few terms of the Taylor series of $f(\exp Z(t))$ (see A. Knapp, *Lie Groups Beyond an Intro.*, p. 80)

$$f(\exp Z(t)) = f(e) + t\tilde{Z}_1(f)(e) + t^2(\frac{1}{2}\tilde{Z}_1^2 + \tilde{Z}_2)(f)(e) + O(t^3), \tag{2}$$

where $\tilde{Z}_1$ is the left-invariant vector field associated to Z_1.
Since (1) and (2) are equal for all f, we see that $Z_1 = 0$ and $Z_2 = [X,Y]_e$ and

$$Z(t) = t^2[X,Y]_e + O(t^3).$$

Problem 4

For the first part, observe that if $M = A + iB$ for $A, B \in \mathfrak{gl}(n, \mathbb{R})$ the the real and imaginary components of

$$(A + iB)(u + iw), \quad \text{for } u, w \in \mathbb{R}^n,$$

are the same as components of

$$\begin{bmatrix} A & -B \\ B & A \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}.$$

Note that the image of $R : \mathfrak{gl}(n, \mathbb{C}) \to \mathfrak{gl}(2n, \mathbb{R})$ is precisely the set of matrices which commute with

$$J = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

i) This is just a direct computation.

ii)

$$\begin{aligned} R(M^*) = R(A^T - iB^T) &= \begin{bmatrix} A^T & B^T \\ -B^T & A^T \end{bmatrix} \\ &= R(M)^T \end{aligned}$$

iii) Consider $\begin{bmatrix} A & -B \\ B & A \end{bmatrix}$ as a transformation of $\mathbb{C}^{2n}$. Write the matrix it in the basis $\{e_k - ie_{k+n}, k = 1, \dots n, e_k + ie_{k+n}, k = 1, \dots n\}$ to get

$$\begin{bmatrix} A + iB & 0 \\ 0 & A - iB \end{bmatrix}$$

and the result follows.
b) Let $M = A + jB$ for $A, B \in \mathfrak{gl}(n, \mathbb{C})$. One checks that the components of

$$(A + jB)(X + jY), \quad \text{for } X, Y \in \mathbb{C}^n,$$

are the same as the components of

$$\begin{bmatrix} A & -\overline{B} \\ B & \overline{A} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}.$$

Note that the image of $R : \mathfrak{gl}(n, \mathbb{H}) \to \mathfrak{gl}(2n, \mathbb{C})$ consists of precisely the set of matrices $N \in \mathfrak{gl}(2n, \mathbb{C})$ so that $JN = \overline{N}J$.
i) This is a direct computation.
ii)

$$\begin{aligned} C(M^*) = C(A^* - jB^T) &= \begin{bmatrix} A^* & B^* \\ -B^T & A^T \end{bmatrix} \\ &= C(M)^* \end{aligned}$$

Problem 5

a) Suppose $N \in Sp(n, \mathbb{R}) \cap O(2n)$. So

$$N^T J N = J \quad \text{and} \quad N^T = N^{-1}$$

which implies that $JN = NJ$.
By 4a) there is a matrix $A + iB \in GL(n, \mathbb{C})$ so that $N = R(A + iB)$. We have

$$I_{2n} = R(A + iB)R((A + iB)^*) = R((A + iB)(A + iB)^*),$$

So $(A+iB)(A+iB)^* = I_n$, thus $A+iB \in U(n)$.
Conversely, if $M = A+iB \in U(n)$, then $N = R(M) = \begin{bmatrix} A & -N \\ B & A \end{bmatrix}$ satisfies

$$NN^T = R(M)R(M^*) = R(MM^*) = R(I_n) = I_{2n}$$

So $N \in O(2n)$. Since $JN = NJ$, $N^T JN = N^{-1}JN = J$ and $N \in Sp(n, \mathbb{R})$.
b) Suppose $N \in Sp(n, \mathbb{C}) \cap U(2n)$. Then

$$N^T JN = J \quad \text{and} \quad N^* N = I_{2n}$$

implies that $JN = \overline{N}J$. So there is a matrix $M = A + jB \in GL(n, \mathbb{H})$ so that $N = C(A + jB)$. Then using 4b) we have

$$C(M^*M) = C(M^*)C(M) = C(M)^*C(M) = N^*N = I_{2n}.$$

So $M^*M = I_n$ and $M \in Sp(n)$.
Conversely, if $M \in Sp(n)$ then the above computation shows that $N = C(M) \in U(2n)$. Since $JN = \overline{N}J$ we have

$$N^T JN = \overline{N}^{-1} JN = \overline{N}^{-1}\overline{N}J = J.$$

So $N \in Sp(n\mathbb{C})$.