Lie Groups, Problem set 2, solutions

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Problem 1

a) This can be proved by induction on $\text{level}(\beta) := \sum_i n_i$ for $\beta = \sum_i n_i \alpha_i \in \Delta^+$. Here $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$. We have

$$0 < |\beta|^2 = \sum_i n_i \langle \alpha_i, \beta \rangle,$$

so there is a j so that $\langle \alpha_j, \beta \rangle > 0$. Consider the α_j-string containing β,

$$\beta + n\alpha_j, \quad -p \leq n \leq p, p, q \geq 0.$$

Since

$$p \geq p - q = \frac{2\langle \beta, \alpha_j \rangle}{|\alpha_j|^2},$$

$p \geq 1$ and $\beta, \beta - \alpha_j, \dots, \beta - p\alpha_j$ are roots. The result then follows by induction on $\text{level}(\beta)$.
b) This follows from part a) and the fact that the space of root vectors $\mathfrak{g}_\alpha, \alpha \in \Delta$, satisfies $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$. And if $\alpha, \beta \in \Delta$ with $\alpha + \beta \neq 0$, then

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}.$$

Problem 2

We have the representation $\mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{gl}(W)$ where $W = \Lambda^2 \mathbb{C}^4$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{C}^4$. Note that $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ spans $\Lambda^4 \mathbb{C}^4$ and gives an identification $\Lambda^4 \mathbb{C}^4 \cong \mathbb{C}$. We define a complex symmetric bilinear form $S : W \times W \to \mathbb{C}$ by

$$S(\theta, tau) := \theta \wedge \tau \in \Lambda^4 \mathbb{C}^4 \cong \mathbb{C},$$

for $\theta, \tau \in W$. Note that W has a basis

$$\tau^1_\pm = e_1 \wedge e_2 \pm e_3 \wedge e_4$$
$$\tau^2_\pm = e_1 \wedge e_3 \mp e_2 \wedge e_4$$
$$\tau^3_\pm = e_1 \wedge e_4 \pm e_2 \wedge e_3$$

and one easily sees that $\frac{\sqrt{2}}{2}\tau^1_+, \frac{\sqrt{2}}{2}\tau^2_+, \frac{\sqrt{2}}{2}\tau^3_+, \frac{\sqrt{2}i}{2}\tau^1_-, \frac{\sqrt{2}i}{2}\tau^2_-, \frac{\sqrt{2}i}{2}\tau^3_-$ is an orthonormal basis for S.
One checks that S is invariant with respect to $\mathfrak{sl}(4, \mathbb{C})$: for $A \in \mathfrak{sl}(4, \mathbb{C})$

$$S(A\theta, \tau) + S(\theta, A\tau) = A\theta \wedge \tau + \theta \wedge A\tau = A(\theta \wedge \tau) = 0.$$

Note that $\mathfrak{sl}(4, \mathbb{C})$ acts trivially on $\Lambda^4 \mathbb{C}^4$.
So we have a Lie algebra homomorphism $\mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{so}(W, S) \cong \mathfrak{so}(6, \mathbb{C})$. This must be an isomorphism since both Lie algebras have the same dimension and $\mathfrak{sl}(4, \mathbb{C})$ is simple.

Problem 3

Recall that $\mathfrak{sp}(2,\mathbb{C}) = \{X \in \mathfrak{sl}(4,\mathbb{C}) | X^t J + JX = 0\}$ where $J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$. Note that the skew-symmetric matrix J can be identified with $\tau = e_1 \wedge e_3 + e_2 \wedge e_4 \in \Lambda^2\mathbb{C}^4$. Under this identification, the action of $A \in \mathfrak{sl}(4,\mathbb{C})$ on J, $AJ + JA^t$, is identified with the action of A on τ as in the last problem. Then $\mathfrak{sp}(2,\mathbb{C})$ is the Lie subalgebra of all $A \in \mathfrak{sl}(4,\mathbb{C})$ with $A\tau = 0$.
Consider the restriction of the representation in problem 2, $\mathfrak{sp}(2,\mathbb{C}) \to \mathfrak{so}(W,S)$. Let $V := \{\beta \in W | S(\beta,\tau) = 0\} \cong \mathbb{C}^5$. Suppose $\beta \in V$ and $A \in \mathfrak{sp}(2,\mathbb{C})$, then

$$S(A\beta,\tau) = -S(\beta, A\tau) = 0,$$

so $V \subset W$ is invariant under $\mathfrak{sp}(2,\mathbb{C})$. We have a homomorphism of Lie algebras

$$\mathfrak{sp}(2,\mathbb{C}) \to \mathfrak{so}(V,S) \cong \mathfrak{so}(5).$$

This must be an isomorphism since $\mathfrak{sp}(2,\mathbb{C})$ is simple and both Lie algebras have dimension 10.

Problem 4

a) Suppose α_i and α_j are simple roots connected by a single edge on the Dynkin diagram, that is

$$\frac{2\langle\alpha_i,\alpha_j\rangle}{|\alpha_i|^2}\frac{2\langle\alpha_j,\alpha_i\rangle}{|\alpha_j|^2} = 1.$$

This implies that $\frac{2\langle\alpha_i,\alpha_j\rangle}{|\alpha_i|^2} = -1$. Then

$$S_{\alpha_i}(\alpha_j) = \alpha_j - \frac{2\langle\alpha_i,\alpha_j\rangle}{|\alpha_i|^2}\alpha_i = \alpha_j + \alpha_i.$$

And one easily checks that

$$S_{\alpha_i} \circ S_{\alpha_j}(\alpha_i) = \alpha_j.$$

b) By proposition 2.62 in *Lie groups beyond intro.* for any $\alpha \in \Delta$ there exists a $w \in W(\Delta)$ and a simple root $\alpha_i \in \Pi$ so that $w\alpha_i = \alpha$. For each irreducible Dynkin diagram any two simple roots of the same length can be connected by a string of single edges. Then apply part a).

Problem 5

a) Any invariant non-degenerate complex bilinear form $\hat{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ defines an isomorphism $\hat{B} : \mathfrak{g} \to \mathfrak{g}^*$. Invariance implies that this is an isomorphism of Lie algebra representations where $\mathfrak{g}$ has the adjoint action and $\mathfrak{g}^*$ has the coadjoint action. The Killing form defines another such isomorphism $B : \mathfrak{g} \to \mathfrak{g}^*$. Then

$$\hat{B}^{-1} \circ B : \mathfrak{g} \to \mathfrak{g}$$

is an isomorphism of adjoint representations. Since $\mathfrak{g}$ is simple this representation is irreducible. By Schur's lemma $\hat{B}^{-1} \circ B = c \in \mathbb{C}^*$. Thus $\hat{B} = c^{-1}B$.
b) By part a) we know that $\hat{B} = c^{-1}B$, so that we only need to evaluate both sides on a convenient element of $\mathfrak{sl}(n+1,\mathbb{C})$. Recall the formula for B on the Cartan subalgebra $\mathfrak{h}$

$$B(H,H') = \sum_{\alpha\in\Delta} \alpha(H)\alpha(H'), \quad H, H' \in \mathfrak{h}.$$

In this case $\Delta = \{e_i - e_j | i \neq j, 1 \leq i,j \leq n+1\}$. Consider

$$\begin{bmatrix} h & 0 & \cdots \\ 0 & -h & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \in \mathfrak{h}$$

only nonzero in the first two diagonal entries. Then

$$\begin{aligned} B(H,H) &= 2\big((e_1-e_2)(H)\big)^2 + 2\sum_{i=3}^{n+1}\big((e_1-e_i)(H)\big)^2 + 2\sum_{i=3}^{n+1}\big((e_2-e_i)(H)\big)^2 \\ &= 8h^2 + 2(n-1)h^2 + 2(n-1)h^2 \\ &= (4n+4)h^2 \end{aligned}$$

Since $\hat{B} = 2h^2$, we have $\hat{B} = \frac{1}{2n+2}B$.
The same argument will work with the other classical simple Lie algebras using the Cartan subalgebras given in the beginning of Ch.2 of *Lie groups beyond an intro.*.

For $\mathfrak{so}(2n+1,\mathbb{C}), n \geq 2, \hat{B} = \frac{1}{2n-1}B$.

For $\mathfrak{sp}(n,\mathbb{C}), n \geq 3, \hat{B} = \frac{1}{2n+2}B$.

For $\mathfrak{so}(2n,\mathbb{C}), n \geq 4, \hat{B} = \frac{1}{2n-2}B$.