Deformation of the Killing spinor equation on Sasaki-Einstein and 3-Sasaki manifolds

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Introduction

Let (M, g) be spin with spin bundle Σ.

- We will consider variations of the Killing Spinor equation

$$\nabla_X \psi = cX \cdot \psi, \quad \psi \in \Gamma(\Sigma),\ c \in \mathbb{R} \setminus \{0\}, \tag{1}$$

- In particular, we will consider variations on Sasaki-Einstein and 3-Sasaki manifolds, where (1) has a 2 and $m+1$ ($\dim M = 4m-1$) space of solutions respectively.
- In this case the Reeb vector field ξ generates a transversally holomorphic, and Kähler, foliation $\mathscr{F}_\xi$. The holomorphic structure on $\mathscr{F}_\xi$ has a versal deformation space, with tangent space

$$H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}), \quad \text{where } \mathcal{A}^{0,k} = \Gamma(\Lambda_b^{0,k} \otimes T_b^{1,0})$$

and

$$0 \to \mathcal{A}^{0,0} \xrightarrow{\bar\partial_b} \mathcal{A}^{0,1} \xrightarrow{\bar\partial_b} \cdots$$

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- If (M, g) is 3-Sasaki, $\dim M = 4m - 1$, then of the $m + 1$ Killing spinors $\sigma_0, \ldots, \sigma_m$ the 2 determined by the Sasaki structure ξ, σ_0, σ_m are preserved by h^β, while $\sigma_1, \ldots, \sigma_{m-1}$ never are.
- If (M, g) is Sasaki-Einstein, then clearly not all h^β integrate to Sasaki-Einstein deformations.

But by considering the deformation theory of transversally extremal metric analogous to the Kähler case (due to Y. Rollin, S. Simanca, C. Tipler 2010) we get:

Theorem 1.1

If $T \subseteq \mathrm{Aut}(M, g, \xi)$ is a maximal torus, then a neighborhood of zero $\mathcal{U} \subset H^1_{\bar{\partial}_b}(\mathcal{A}^{0,\bullet})^T$ parametrizes a family of Sasaki-Einstein metrics g_t on M with $g_0 = g$.

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- If (M, g) is 3-Sasaki, then there is a diffeomorphism $\varsigma : M \to M$ which an anti-holomorphic automorphism on $\mathscr{F}_\xi$. Thus $H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$ has a real structure.

Theorem 1.2

A neighborhood of zero $\mathcal{V} \subset \operatorname{Re} H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$ parametrizes a family of Sasaki-Einstein metrics g_t with $g_0 = g$.

Theorem 1.2 has the following consequences:

- It is well known that 3-Sasaki structures are rigid (H. Pedersen and Y. S. Poon 1999). Thus there is a neighborhood $\mathcal{N}$ of $0 \in \mathcal{V}$ so that $g_t,\ t \in \mathcal{N} \setminus \{0\}$ does not admit a 3-Sasaki structure.
- In other words if $\dim M = 4m - 1$, then $g = g_0$ admits $m + 1$ Killing spinors while $g_t,\ t \in \mathcal{N} \setminus \{0\}$ has 2.
- We give examples where this happens. We apply Theorems 1.1 and 1.2 to toric 3-Sasaki 7-manifolds. Toric means it admits a T^2 group of automorphisms.

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Let (M, g) be a toric 3-Sasaki 7-manifold. Then $\dim_{\mathbb{C}} H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) = b_2(M) - 1$ *and there is a neighborhood* $\mathcal{U} \subset H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) = \mathbb{C}^{b_2-1}$ *of 0 parametrizing Sasaki-Einstein metrics such that* $g_t,\ t \in \mathcal{U} \setminus \{0\}$*, are not 3-Sasaki.*

- Therefore, a toric 3-Sasaki 7-manifold (M, g) with $b_2(M) > 1$ has Einstein deformations to metrics g_t which are Sasaki-Einstein but not 3-Sasaki.
- They give examples of manifolds with a metric with 3 Killing spinors with deformations to metrics admitting only 2. So there is no analogue of the following theorem of M. Wang for Killing spinors.

Equation (1) with $c = 0$ is just the equation for a parallel spinor.

Theorem 1.4 (M. Y. Wang, 1991)

Let (M, g) is a compact simply connected spin manifold with irreducible holonomy admitting a nonzero parallel spinor. Then there is a neighborhood $\mathcal{W}$ of g in the Einstein moduli space such that each $\bar g \in \mathcal{W}$ admits the same number of independent parallel spinors.

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Killing spinors

Let (M,g) a Riemannian manifold with a spin structure and spin bundle Σ.

Definition 2.1

A Killing spinor is a nonzero section $\psi \in \Gamma(\Sigma)$ which satisfies

$$\nabla_X \psi = cX \cdot \psi,$$

where c is a constant and $X \cdot \psi$ is Clifford multiplication.

Note that c can be rescaled by rescaling g, so we denote by $\mathcal{N}_+$ (resp. $\mathcal{N}_-$) the $\mathbb{C}$-dimension of the space of Killing spinors with $c > 0$ (resp. $c < 0$).

Existence of a Killing spinor $\psi \in \Gamma(\Sigma)$ has the following consequences (T. Friedrich 1980):

- (M,g) is an Einstein manifold with $\mathrm{Ric}_g = 4(n-1)c^2 g$.
- So c is either 0, in which case ψ is parallel; c is imaginary, in which case M is noncompact; or c is real, in which case M is compact and irreducible.
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classification

Simply connected manifolds admitting a non-zero Killing spinor were classified by C. Bär, 1992:

$\dim M$	N^+	N^-	$\mathrm{Hol}(C(M))$	geometry
n	$2^{\lfloor \frac{n}{2} \rfloor}$	$2^{\lfloor \frac{n}{2} \rfloor}$	Id	n-sphere
$4m-1$	2	0	$\mathrm{SU}(2m)$	Sasaki-Einstein
$4m+1$	1	1	$\mathrm{SU}(2m+1)$	Sasaki-Einstein
$4m-1$	m+1	0	$\mathrm{Sp}(m)$	3-Sasaki
6	1	1	G_2	nearly Kähler
7	1	0	$\mathrm{Spin}(7)$	weak G_2

- The connection $\hat{\nabla}_X\psi = \nabla_X\psi - cX\cdot\psi,\ c = \pm\frac{1}{2}$, on $\Sigma(M)$, naturally identifies with with that induced by the Levi-Civita connection on $\Sigma(C(M))$ (or $\Sigma_\pm(C(M))$ when $\dim M$ is odd).
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Definition 2.2

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Thus is a particular metric contact structure. We have

- a contact structure η with Reeb vector field $\xi = Jr\partial_r$, a Killing field, and
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- I induces a transversely holomorphic structure on $\mathscr{F}_\xi$, with Kähler form $\omega^T = \frac{1}{2} d\eta$.
- The tensor $\phi = \nabla \xi$, with $\phi|_D = I$ and $\phi(\xi) = 0$ defines the CR structure.

We say that the Sasaki structure is

- *quasi-regular* if the orbits of ξ are closed (orbit space is an orbifold),
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Sasaki-Einstein manifolds

Suppose (M, g) is Sasaki $\dim M = n = 2m - 1$. We are interested in Sasaki-Einstein structures

$$\mathrm{Ric}_g = (n-1)g, \tag{2}$$

The Sasaki condition forces the Einstein constant to be $n - 1$.

- This is equivalent to $(C(M), \bar{g})$ being Ricci-flat as

$$\mathrm{Ric}_{\bar{g}} = \mathrm{Ric}_g - (n-1)g. \tag{3}$$

- Also, the Ricci curvatures of $\bar{g}$ and g^T satisfy $(n = 2m - 1)$

$$\mathrm{Ric}_{\bar{g}} = \mathrm{Ric}_{g^T} - 2m\, g^T, \tag{4}$$

- From (3) and (4)

$$\mathrm{Ric}_g(X, Y) = \mathrm{Ric}^T_{g^T} - 2g^T, \quad \text{for basic vector fields } X, Y \in \Gamma(D), \tag{5}$$

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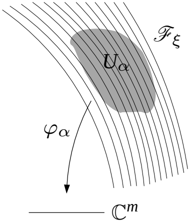
transversally holomorphic foliation



A *transversely holomorphic structure* on a foliation $\mathscr{F}_\xi$ is given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{A}}$ where $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers M

- $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to \mathbb{C}^{m-1}$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- there are holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

deformations of the foliation

There is a versal deformation space $\mathcal{V}$ of transversely holomorphic structures on $\mathscr{F}_\xi$ fixing it as a smooth foliation (A. El Kacimi Alaoui, M. Nicolau '89; Girbau 1993).

$\mathcal{V}$ is the germ of $\theta^{-1}(0)$ where θ is an analytic map

$$H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) \xrightarrow{\theta} H^2_{\bar\partial_b}(\mathcal{A}^{0,\bullet}).$$

We have:

- $$H^2_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) = H^{m-3}_{\bar\partial_b}(\Gamma(\Lambda_b^{1,\bullet} \otimes \Lambda_b^{m-1,0})) = 0,$$

  by Kodaira-Nakano vanishing, since $\Lambda_b^{m-1,0} < 0$ and $(m-3)+1 = m-2 < m-1$.
- Thus $\mathcal{V}$ is smooth. And after shrinking $\mathcal{V}$ we may assume (A. El Kacimi Alaoui, B. Gimira 1997) that for $t \in \mathcal{V}$ $\mathscr{F}_\xi^t$ admits a transversal Kähler structure (I_t, g_t^T, ω_t^T).
- The Kähler structure can be chosen so that it lifts to a Sasaki structure (g_t, ξ, η_t) with transversal holomorphic structure I_t and $\frac{1}{2}d\eta_t = \omega_t^T$. From (6), up to homothety, we will have

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3-Sasaki manifolds

Recall that a hyperkähler structure on a $4m$-dimensional manifold consists of a metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1 J_2 = -J_2 J_1 = J_3$ etc.

Definition 2.3

A Riemannian manifold (M, g) is 3-Sasaki if the metric cone $(C(M), \bar{g})$ is hyperkähler, i.e. $\bar{g}$ admits a compatible almost complex structures $J_\alpha,\ \alpha = 1, 2, 3$ such that $(C(M), \bar{g}, J_1, J_2, J_3)$ is a hyperkähler structure. Equivalently, $\mathrm{Hol}(C(M)) \subseteq \mathrm{Sp}(m)$.

A consequence of the definition is that (M, g) is equipped with three Sasaki structures $(\xi_i, \eta_i, \phi_i),\ i = 1, 2, 3$. The Reeb vector fields $\xi_k,\ k = 1, 2, 3$ are orthogonal and satisfy $[\xi_i, \xi_j] = 2\varepsilon^{ijk}\xi_k$, where ε^{ijk} is anti-symmetric in the indices $i, j, k \in \{1, 2, 3\}$ and $\varepsilon^{123} = 1$. The tensors $\phi_i,\ i = 1, 2, 3$ satisfy the identities

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Recall that a hyperkähler structure on a $4m$-dimensional manifold consists of a metric g which is Kähler with respect to three complex structures J_1, J_2, J_3 satisfying the quaternionic relations $J_1 J_2 = -J_2 J_1 = J_3$ etc.

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The Reeb vector fields $\xi_k,\ k = 1, 2, 3$ generate an action of $\mathrm{Sp}(1)$ or $\mathrm{SO}(3)$.
A 3-Sasaki manifold M comes with a family of related geometries. The maps are labeled with their generic fibers.

$$\begin{tikzcd}
 & C(M) \arrow[dl, "\mathbb{R}_+"'] \arrow[dr, "\mathbb{C}^*"] & \\
M \arrow[rr, "S^1"] \arrow[dr, "\begin{array}{c}\mathrm{Sp}(1)\\ \mathrm{SO}(3)\end{array}"'] & & \mathcal{Z} \arrow[dl, "\mathbb{C}P^1"] \\
 & \mathcal{M} &
\end{tikzcd}$$

- $\mathcal{Z}$, the *twistor space*, is the orbifold leaf space $\mathscr{F}_{\xi_1}$ with a complex contact structure $\theta \in \Omega^1(\mathbf{L})$.
- $\mathcal{M}$ is a *quaternionic-Kähler* orbifold.
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Killing spinor deformations

We will need the machinery due to J.P. Bourguignon and P. Gauduchon 1991 for describing spinors under metric variations.

Let P be the bundle of oriented orthonormal frames on (M, g). A spin structure is a double cover $\tilde{P}$. Given a symmetric, w.r.t. g, automorphism $\alpha : TM \to TM$ we have a new metric

$$g^\alpha(X, Y) = g(\alpha^{-1}X, \alpha^{-1}Y).$$

If P^α is the bundle of g^α-orthonormal oriented frames, $\alpha : P \to P^\alpha$ is $\mathrm{SO}(n)$-equivariant, and gives an isomorphism

$$\Sigma = \tilde{P} \times_{\mathrm{Spin}(n)} \Delta_n \xrightarrow{\tilde{\alpha}} \Sigma^\alpha = \tilde{P}^\alpha \times_{\mathrm{Spin}(n)} \Delta_n.$$

Let $\alpha(t)$ be a smooth path of symmetric automorphisms with $\alpha(0) = Id_{TM}$, and $\hat{\sigma}_t$ Killing spinors for g^α,

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A *deformation of the Killing spinor* σ_0 is a path $(\alpha(t), \sigma_t)$ satisfying

$$\mathcal{L}^c(\alpha(t), \sigma_t)(X) := \bar{\nabla}_X^{\alpha(t)} \sigma_t - c\alpha(t)^{-1}(X) \cdot \sigma_t = 0. \tag{10}$$

We will make use of

- Twisted Dirac operator:

$$\mathcal{D} : \Gamma(\Sigma \otimes TM_{\mathbb{C}}) \to \Gamma(\Sigma \otimes TM_{\mathbb{C}})$$

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Differentiating (10) at (Id_{TM}, σ_0):

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$$d\mathcal{L}^c(\dot{\alpha}, \dot{\sigma})(X) = \nabla \dot{\sigma}_X - cX\dot{\sigma} + c\dot{\alpha}(X)\sigma_0 - \frac{1}{2}\sum_i e_i(\nabla_i \dot{\alpha})(X)\sigma_0 + \frac{1}{2}g(\delta\dot{\alpha}, X)\sigma_0.$$

If $\operatorname{tr}_g(\dot{\alpha}) = \delta\dot{\alpha} = 0$*, then* $d\mathcal{L}^c(\dot{\alpha}, \dot{\sigma}) = 0$ *if and only if* $\nabla_X \dot{\sigma} = cX\dot{\sigma}$ *and* $\mathcal{D}\Psi^{(\dot{\alpha},\sigma_0)} = nc\Psi^{(\dot{\alpha},\sigma_0)}$.

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For $\beta : TM \to TM$ g-symmetric, define $h(X,Y) = -2g(\beta(X),Y)$

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If $\mathrm{tr}_g\,\beta = \delta\beta = 0$ *and* $\mathcal{D}\Psi^{(\beta,\sigma_0)} = nc\Psi^{(\beta,\sigma_0)}$, *then* $\left(\nabla^*\nabla + 2L\right)h = 0$ *where* $(Lh)_{ij} = R_{i\ j}^{\ k\ l}h_{kl}$.

So $h \in \Gamma\left(S^2\,T^*M\right)$ is an infinitesimal Einstein deformation.

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An infinitesimal deformation of the Killing spinor σ_0 *is a pair* (β,σ) *satisfying:*

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Recall the transverse metric g^T on $\mathscr{F}_\xi$ is Kähler-Einstein,

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An *harmonic* representative $\beta \in H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$ satisfies

$$\bar\partial_b \beta = 0, \text{ and } \bar\partial_b^* \beta = 0. \tag{11}$$

Since $c_1^b(\mathscr{F}_\xi) > 0$, there are no non-zero harmonic sections of $\Lambda_b^{0,2}$ so $h_T^\beta(X,Y) = -2g(\beta(X),Y)$ is symmetric and of type $(0,2)$.

Remark 4.1. By abuse of notation h_T^β will also denote the real component of the previous object. So h_T^β is a real anti-Hermitian symmetric tensor.

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Remark 4.1. By abuse of notation h_T^β will also denote the real component of the previous object. So h_T^β is a real anti-Hermitian symmetric tensor.

Infinitesimal Einstein deformations of Kähler-Einstein metrics

We will make use of some results of N. Koiso 1983 on the Einstein deformations of a Kähler-Einstein metric.

Recall the transverse metric g^T on $\mathscr{F}_\xi$ is Kähler-Einstein,

$$\mathrm{Ric}_{g^T} = 2mg^T.$$

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Proposition 4.2 (N. Koiso 1983)

The space $\mathcal{E}^T$ is infinitesimal Einstein deformations of g^T, i.e.

$$\mathcal{E}^T := \{h \in \Gamma\left(S^2\, T_b^* M\right) | \operatorname{tr}_{g^T} h = \delta_{g^T} h = 0, \quad \left((\nabla^T)^* \nabla^T + 2L^T\right) h = 0\},$$

splits into Hermitian and anti-Hermitian components

$$\mathcal{E}^T = \mathcal{E}^T_H \oplus \mathcal{E}^T_A.$$

An anti-Hermitian $h \in \Gamma\left(S^2\, T_b^* M\right)$ is an element of $\mathcal{E}^T_A$ if and only if

$$\nabla^T_\alpha h_{\beta\gamma} - \nabla^T_\beta h_{\alpha\gamma} = 0 \tag{12}$$

$$(\nabla^T)^\alpha h_{\alpha\beta} = 0 \tag{13}$$

Therefore, we have an isomorphism: $\mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet}) \cong \mathcal{E}^T_A$.

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The next result follows from computations using the O'Niell tensor. Recall that the local projection onto the leaf space of $\mathscr{F}_\xi$ is a Riemannian submersion.

Lemma 4.3

Let (M, g) be a Sasaki-Einstein manifold. Suppose $h^T \in \Gamma\left(S^2\, T_b^* M\right)$ is an anti-Hermitian infinitesimal Einstein deformation of g^T. Then $h = \pi^* h^T$ is an infinitesimal Einstein deformation of g.

Proposition 4.4

Let (M, g) be a spin Sasaki-Einstein manifold admitting the 2 defining Killing spinors σ_j, $j = 0, 1$. If $\beta \in \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})$ and h_T^β the corresponding basic anti-Hermitian symmetric tensor, then $h^\beta := \pi^* h_T^\beta$ is an infinitesimal Einstein deformation of g, and $(h^\beta, 0)$ is an infinitesimal deformation of the Killing spinors σ_j for $j = 0, 1$.

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Some remarks on the proof:

- Since $\mathrm{tr}_g(h^\beta) = \delta_g h^\beta = 0$, Proposition 3.1 is satisfied if
$$\sum_i e_i(\nabla_i h)(X)\sigma_i = 2ch(X)\sigma_i, \quad \text{for all } X \in TM, \ i = 1, 2. \tag{14}$$
- Using the O'Neill tensor one puts (14) in form
$$\sum_{i=1}^{2m-2} e_i(\nabla_i^T h)(X)\sigma_j - \phi h(X)\xi\sigma_j = 2ch(X)\sigma_i. \tag{15}$$
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Proposition 4.5

Let (M,g), $\dim M = 4m-1$, be a 3-Sasaki manifold with Killing spinors σ_j, $j = 0,\dots,m$. If $\beta \in \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})$ is non-zero and h_T^β the corresponding basic anti-Hermitian symmetric tensor, then $h^\beta := \pi^* h_T^\beta$ is an infinitesimal Einstein deformation of g, and $(h^\beta, 0)$ is an infinitesimal deformation of the Killing spinors σ_j for $j = 0, m$, but never for $j = 1,\dots,m-1$.

- Take a local quaternionic frame

$$(e_1, e_2, \dots, e_{4m}) = (f_1, J_1 f_1, J_2 f_1, J_3 f_1, f_2, \dots, f_m, J_1 f_m, J_2 f_m, J_3 f_m),$$

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- Then we have an Hermitian, w.r.t. J_1, frame $\varepsilon_\alpha = \frac{1}{\sqrt{2}}(e_{2\alpha-1} - \sqrt{-1} e_{2\alpha})$, $\alpha = 1,\dots,2m$.
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- We have the “symplectic form”

$$\varpi = \sum_{\alpha=1}^{m} \varepsilon_{2\alpha-1} \wedge \varepsilon_{2\alpha}. \tag{16}$$

- And the Killing spinors on (M, g) are $\sigma_k = \frac{1}{k!}\varpi^k,\ k = 0, \ldots, m$.
- If $\theta \in \Omega_b^{1,0}(\mathbf{L})$ is the complex contact form, then one can show that $\psi_{\bar\beta} = h^{\gamma}_{\bar\beta}\theta_\gamma \in \Omega_b^{0,1}(\mathbf{L})$ is harmonic and thus zero, since $H^1_{\bar\partial_b}(\Gamma(\Lambda^{0,\bullet}(\mathbf{L}))) = 0$.
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The rest is involves computing as in the last proposition.

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Integrable deformations: Sasaki-Einstein

We consider cases in which the above infinitesimal Killing spinor deformations integrate.

- The infinitesimal Killing spinor deformations h^β for $\beta \in \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})$ of a Sasaki-Einstein metric (M, g) do not necessarily integrate.
- Note that this problem includes that of deforming Kähler-Einstein metrics. When M is regular the leaf space Z is a Kähler-Einstein manifold.
- We saw that $\mathscr{F}_\xi$ has a smooth Kuranishi space $\mathcal{V} \subset H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$. But $\mathscr{F}^t_\xi$, $t \in \mathcal{V} \setminus \{0\}$ may not admit a transversal Kähler-Einstein metric.

Let $T \subset \mathrm{Aut}(M, g, \xi, \phi)$ be a maximal torus of the automorphism group of the Sasaki structure.

Theorem 5.1 (van Coevering, arXiv:1204.1630)

If $\beta \in \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})^T$, then h^β integrates to a deformation of Sasaki-Einstein structures (g_t, I_t, ξ_t).
More generally, let $\mathfrak{t}$ be the Lie algebra of T. Then there is a neighborhood of $\mathcal{N}$ of $(0, \xi) \in \mathcal{V} \times \mathfrak{t}$, so that for $(t, \zeta) \in \mathcal{N}$ there is a Sasaki-Extremal metric $(g_{t,\zeta}, I_t, \zeta)$.

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Integrable deformations: Sasaki-Einstein

A Sasaki-Extremal metric is a critical point of the Calabi functional.

$$\mathcal{S}(\xi, I) := \{\text{Sasaki structures with Reeb field } \xi \text{ and trans. holmorphic str. } I\}$$

$$\mathfrak{M}(\xi, I) := \{\text{metrics associated to structures in } \mathcal{S}(\xi, I)\}$$

The Calabi functional $\mathcal{C}$ is

$$\begin{array}{ccc} \mathfrak{M}(\xi, I) & \xrightarrow{\mathcal{C}} & \mathbb{R} \\ g & \mapsto & \int_M s_g^2 \, d\mu_g \end{array} \tag{17}$$

- Theorem 5.1 is a special case of results on deforming Sasaki-Extremal metrics.
- Proof uses implicit function theorem on the *reduced scalar curvature*.
- If the deformation does not preserve a maximal torus, then addition assumption is needed: Non-degeneracy of the relative Futaki invariant.
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Integrable deformations: 3-Sasaki

It is well known that 3-Sasaki structures are rigid (H. Pedersen and Y. S. Poon 1999).

Theorem 5.2

Let (M, g), $\dim M = 4m - 1$, *be a 3-Sasaki manifold with Killing spinors* σ_j, $j = 0, \ldots, m$. *Then any Einstein deformation* (M, g_t) *of* g *with compatible 3-Sasaki structures, i.e. preserving the existence of the* σ_j, $j = 0, \ldots, m$, *is trivial. That is, there exists a family* f_t *of diffeomorphisms of* M *with* $f_t^* g_t = g$.

The leaf space of $\mathscr{F}_\xi$ is a complex orbifold $\mathcal{Z}$, the *twistor space*. So there is an anti-holomorphic involution $\varsigma : \mathcal{Z} \to \mathcal{Z}$. So

- $H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) = H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})$, where $\Theta_{\mathcal{Z}}$ is the orbifold sheaf of holomorphic vector fields.
- We have a real structure $\varsigma : H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet}) \to H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$

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Let (M, g), $\dim M = 4m - 1$, *be a 3-Sasaki manifold, and denote by* σ_j, $j = 0, \ldots, m$ *the Killing spinors associated to the 3-Sasaki structure. Then the infinitesimal Einstein deformations* h^β *of* g *for* $\beta \in \operatorname{Re} \mathcal{H}^1_{\Delta_{\bar\partial_b}}(\mathcal{A}^{0,\bullet})$ *integrate to a family*

g_t, $t \in \mathcal{N} \subset \mathbb{R}^d$, $d = \dim_{\mathbb{C}} H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})$ *of Einstein deformations of* g *preserving only* σ_0 *and* σ_m.

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Idea of proof of theorem.

- We have transversal Kähler metrics ω_t^T, $t \in \mathcal{V}$ with $\omega_t^T \in \frac{\pi}{2m} c_1^b(\mathscr{F}_\xi^t)$.
- May assume that $\varsigma^* g_t = g_t$ and $\varsigma^* \omega_t^T = -\omega_t^T$.
- We want to solve $\mathrm{Ricci}(\omega_t^T + dd_t^c \varphi_t) - 4m\pi(\omega_t^T + dd_t^c \varphi_t) = 0$.
- Differentiating at $(t,\varphi) = (0,0)$ gives $\dot\phi \mapsto \left(-\Delta_{\bar\partial} + 4m\right)\dot\varphi$, which has kernel and cokernel the (normalized) holomorphy potentials $\mathcal{H}_g$ of holomorphic vector fields on Z.
- One can show that for $f \in \mathcal{H}_g$, $\varsigma^* f = -f$.
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A 3-Sasaki manifold (M, g), $\dim M = 4m - 1$, is toric if there is a $T^m \subseteq \mathrm{Aut}(M, g, \xi_1, \xi_2, \xi_3)$.

- Toric 3-Sasaki manifolds have been constructed from 3-Sasaki quotients by torus actions on S^{4n-1}, with the 3-Sasaki structure given by right multiplication by $\mathrm{Sp}(1)$.
- A result of R. Bielawski, 1999, is that this gives all of them.
- A subtorus $T^k \subset T^n$ is determined by a weight matrix $\Omega_{k,n} \in \mathrm{Mat}(k, n, \mathbb{Z})$. There are conditions on Ω (due to C. Boyer, K. Galicki, B. Mann, E. Rees, 1998) that imply the moment map $\mu : S^{4n-1} \to (\mathfrak{t}^k)^* \otimes \mathbb{R}^3$ is a submersion, and further that the quotient

$$M_{\Omega_{k,n}} = S^{4n-1} /\!/ T^k = \mu^{-1}(0)/T^k$$

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- When $n = k + 2$ the above authors showed there are infinitely many weight matrices in $\mathrm{Mat}(k, n, \mathbb{Z})$ for $k \geq 1$ giving infinitely many 7-manifolds $M_{\Omega_{k,n}}$ for each $b_2 = k \geq 1$.

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Toric 3-Sasaki manifolds

A 3-Sasaki manifold (M, g), $\dim M = 4m - 1$, is toric if there is a $T^m \subseteq \mathrm{Aut}(M, g, \xi_1, \xi_2, \xi_3)$.

- Toric 3-Sasaki manifolds have been constructed from 3-Sasaki quotients by torus actions on S^{4n-1}, with the 3-Sasaki structure given by right multiplication by $\mathrm{Sp}(1)$.
- A result of R. Bielawski, 1999, is that this gives all of them.
- A subtorus $T^k \subset T^n$ is determined by a weight matrix $\Omega_{k,n} \in \mathrm{Mat}(k, n, \mathbb{Z})$. There are conditions on Ω (due to C. Boyer, K. Galicki, B. Mann, E. Rees, 1998) that imply the moment map $\mu : S^{4n-1} \to (\mathfrak{t}^k)^* \otimes \mathbb{R}^3$ is a submersion, and further that the quotient
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Toric 3-Sasaki 7-manifolds

We assume now that $\dim M = 7$.

If $b_2(M) \geq 1$, then the maximal torus of *Sasaki* automorphisms, $T^3 \subset \mathrm{Aut}(M, \xi_1)$, is 3-dimensional.

Lemma 6.1 (van Coevering, arXiv:math.DG/0607721)

If $\mathcal{Z}$ is the twistor space of a toric 3-Sasaki 7-manifold M, then $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})^{T^2}$,

$$\dim_{\mathbb{C}} H^1(\mathcal{Z}, \Theta_{\mathcal{Z}}) = b_2(M) - 1 = k - 1.$$

And $\mathcal{Z}$ has a local $b_2(M) - 1$-dimensional space of deformations.

Theorem 6.2

Let (M, g) be a toric 3-Sasaki 7-manifold. Then g is in an effective complex $b_2(M) - 1$-dimensional family $\{g_t\}_{t \in \mathcal{U}}$, $\mathcal{U} \subset \mathbb{C}^{b_2(M)-1}$ with $g_0 = g$, of Sasaki-Einstein metrics where g_t is not 3-Sasaki for $t \neq 0$.

One applies Theorem 5.1; $H^1(\mathcal{Z}, \Theta_{\mathcal{Z}})^{T^2} = H^1_{\bar\partial_b}(\mathcal{A}^{0,\bullet})^{T^3}$, where T^2 is a maximal torus.

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Toric 3-Sasaki 7-manifolds

Thus unlike the case of parallel spinors ($c = 0$) the dimension of the space of Killing spinors is not locally stable in general. See figure 1 for the isometry groups of the metrics.

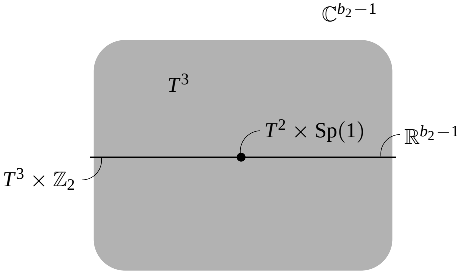

Figure: Space of Sasaki-Einstein metrics