Some Results Related to Stability in Sasakian Geometry

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Introduction

Some of this work is joint with Carl Tipler,
"Deformations of Constant Scalar Curvature Sasakian Metrics and K-Stability," I.M.R.N. 2015.

This talk will consider some results on constant scalar curvature Sasakian (cscS) metrics, and more generally Sasaki-extremal metrics.

We will consider results related to the K-energy and K-stability, familiar in the study of Kähler manifolds. We will consider

- A proof of the convexity of the K-energy along weak geodesics, (following ideas of R. Berman and B. Berndtson, 2014),
- Uniqueness of cscS metrics (and Sasaki-extremal metrics) for a fix transversal holomorphic structure,
- Existence of cscS metric $\Rightarrow$ K-energy bounded below.

Considering the larger space of Sasakian structures compatible with a contact structure (η, ξ)

- A small deformation of a cscS structure has K-energy bounded below (by an "adjacent" cscS structure),
- A small deformation of a cscS structure also admits a cscS structure if it is K-polystable.

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Introduction

Definition 1.1

A Riemannian manifold (M, g) is Sasakian if the metric cone $(C(M), \bar{g})$, $C(M) := \mathbb{R}_+ \times M$ and $\bar{g} = dr^2 + r^2 g$, is Kähler, i.e. $\bar{g}$ admits a compatible almost complex structure J so that $(C(M), \bar{g}, J)$ is a Kähler structure.

This is a metric contact structure (M, η, ξ, Φ, g) with an additional integrability condition. One has

- a contact structure
$$\eta = d^c \log r = Jd \log r$$
with Reeb vector field $\xi = Jr\partial_r$, a Killing field, and
- a strictly pseudoconvex CR structure (D, I), $D = \ker \eta$.
- I induces a transversely holomorphic structure on $\mathscr{F}_\xi$, the Reeb foliation, with Kähler form $\omega^T = \frac{1}{2} d\eta$.
- $(C(M), J)$ is an affine variety Y polarized by ξ. So (Y, ξ) is the analogue of a polarized Kähler manifold.
- $\mathcal{S}(\xi, \bar{J})$ is the space of Sasakian metrics with transversal complex structure $\bar{J}$. Analogue of the space of Kähler metrics in a polarization.

The transversal Kähler metrics in $\mathcal{S}(\xi, \bar{J})$ are

$$\mathcal{H}(\xi, \bar{J}) = \{\omega_{\phi_t} = \omega^T + dd^c\phi \mid (\omega^T + dd^c\phi)^m \wedge \eta > 0\}.$$

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Background on results

Uniqueness of cscS structures:

- K. Cho, A. Futaki, H Ono 2007 Proved uniqueness of toric cscS structures. The geodesic equation is just $\ddot{G} = 0$, in terms of symplectic potential G.
- Y. Nitta and K. Sekiya 2009 Proved uniqueness of Sasaki-Einstein structures, extending arguments of S. Bando and T. Mabuchi.

This talk considers K-polystability and deformations of constant scalar curvature Sasakian metrics.

- Tristan C. Collins, Gábor Székelyhidi 2012 defined K-polystability for Sasakian manifolds by defining the Donaldson-Futaki invariant in terms of the Hilbert series on the affine cone $C(M)$.
- and they proved
$$\text{cscS} \Rightarrow \text{K-semistable}.$$
- The deformation result we give only requires polystability with respect to smooth degenerations, where this is the usual Futaki invariant on central fiber.

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Given a Sasakian manifold M the K-energy is a functional on $\mathcal{H}(\xi, \bar{J})$:

$$\mathcal{M}(\phi) = -\int_0^1 \int_M \dot{\phi}_t (S(\phi_t) - \bar{S})(\omega^T)^m \wedge \eta \, dt, \quad S = \frac{2n\pi c_1(\mathscr{F}_\xi)[\omega^T]^{m-1}}{[\omega^T]^m}$$

X. X. Chen 2000 rewrote this formula to extend $\mathcal{M}$ to weak $C^{1,1}$ structures

$$\mathcal{M}(\phi) = \frac{\bar{S}}{m+1}\mathcal{E}(\phi) - \mathcal{E}^{\mathrm{Ric}}(\phi) + \int_M \log\left(\frac{\omega_\phi^m \wedge \eta}{\omega^m}\right)\omega_\phi^m \wedge \eta$$

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Convexity of K-energy

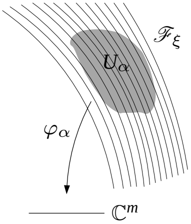

Figure : Transversally complex foliation

The *transversely holomorphic structure* on a foliation $\mathscr{F}_\xi$ is given by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\mathcal{A}}$ where $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M

- $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ covers M,
- the $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{C}^m$ has fibers the leaves of $\mathscr{F}_\xi$ locally on U_α,
- holomorphic isomorphism $g_{\alpha\beta} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ such that

$$\varphi_\alpha = g_{\alpha\beta} \circ \varphi_\beta \quad \text{on } U_\alpha \cap U_\beta.$$

- There is a Kähler structure ω_α on $\varphi_\alpha(U_\alpha) \subset \mathbb{C}^m$.

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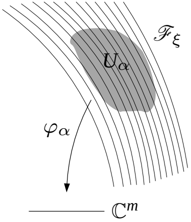

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Convexity of K-energy

Analysis is done on the foliation charts.

Let T_α be a closed degree (k, k) current defined on V_α so that $g^*_{\alpha\beta} T_\alpha = T_\beta$.

$$\mathrm{PSH}(M, \omega) := \{\phi \mid \phi \text{ u.s.c. inv. under } \xi \text{ and plurisubharmonic on each chart} V_\alpha\}$$

Given $\phi_1, \ldots, \phi_{m-k} \in \mathrm{PSH}(M, \omega)$, in each V_α we define (E. Bedford and B. Taylor 1976):

$$\omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T_\alpha$$

a positive Borel measure on V_α, and we take the product measure on each chart which is easily seen to be invariant of the chart by Fubini's theorem, defining

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Convexity of K-energy

The following will be useful

Proposition 2.1

Let $\phi \in \mathrm{PSH}(M, \omega) \cap C^0(M)$*. Then there exists a sequence* $\phi_i \in \mathrm{PSH}(M, \omega) \cap C^\infty(M)$ *with* $\phi_i \searrow \phi$ *as* $i \to \infty$*.*

We have weak continuity of the Monge-Ampère measure.

Given decreasing sequences $\phi_1^i \to \phi_1, \ldots, \phi_{m-k}^i \to \phi_{m-k}$ in $\mathrm{PSH}(M, \omega)$ we have

$$\omega_{\phi_1^i} \wedge \cdots \wedge \omega_{\phi_{m-k}^i} \wedge T \wedge \eta \to \omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T \wedge \eta$$

weak convergence of Borel measures.

The following will be useful

Proposition 2.1

Let $\phi \in \mathrm{PSH}(M, \omega) \cap C^0(M)$*. Then there exists a sequence* $\phi_i \in \mathrm{PSH}(M, \omega) \cap C^\infty(M)$ *with* $\phi_i \searrow \phi$ *as* $i \to \infty$*.*

We have weak continuity of the Monge-Ampère measure.

Given decreasing sequences $\phi_1^i \to \phi_1, \ldots, \phi_{m-k}^i \to \phi_{m-k}$ in $\mathrm{PSH}(M, \omega)$ we have

$$\omega_{\phi_1^i} \wedge \cdots \wedge \omega_{\phi_{m-k}^i} \wedge T \wedge \eta \to \omega_{\phi_1} \wedge \cdots \wedge \omega_{\phi_{m-k}} \wedge T \wedge \eta$$

weak convergence of Borel measures.

Convexity of K-energy

Let $D \subset \mathbb{C}$ then we have the Homogeneous Monge-Ampère equation

$$(\pi^*\omega + dd^c U_\tau)^{m+1} = 0 \quad \text{for } U_\tau \in \mathrm{PSH}(M \times D, \pi^*\omega),$$

P. Guan and X. Zhang 2012 solved it for $D = \{\tau \in \mathbb{C} \mid 1 \leq |\tau| \leq e\}$ and $U(\cdot, 1) = \phi_0, U(\cdot, e) = \phi_1 \in C_b^\infty(M)$ on ∂D, and showed U is weak $C^{1,1}$, meaning

$$\pi^*\omega + dd^c U_\tau \geq 0 \quad \text{is } L^\infty(M \times D).$$

Then

$$\omega + dd^c u_t \geq \text{ is weak } C^{1,1} \text{ geodesic connecting } \omega_{\phi_0}, \omega_{\phi_1}, 0 \leq t \leq 1.$$

$t = \log \tau$.

Proposition 2.2

If $u \in \mathrm{PSH}(M, \omega) \cap C^0$ then the first variations of the functionals $\mathcal{E}$ and $\mathcal{E}^{\mathrm{Ric}}$ are

$$d\mathcal{E}|_u = (m+1)\omega_u^m \wedge \eta, \quad d\mathcal{E}^{\mathrm{Ric}}|_u = m\omega_u^{m-1} \wedge \mathrm{Ric}_\omega \wedge \eta.$$

And second variations

$$d_\tau d_\tau^c \mathcal{E}(U_\tau) = \int_M (\pi^*\omega + dd^c U_\tau)^{m+1} \wedge \eta \quad d_\tau d_\tau^c \mathcal{E}^{\mathrm{Ric}}(U_\tau) = \int_M (\pi^*\omega + dd^c U_\tau)^m \wedge \pi^* \mathrm{Ric}_\omega \wedge \eta.$$

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Convexity of K-energy

Theorem 2.3

Let u_τ be a weak $C^{1,1}$ geodesic connecting two points in $\mathcal{H}(\xi, \bar{J})$. Then $\mathcal{M}(u_\tau)$ is subharmonic with respect to $\tau \in D$. Thus $\mathcal{M}(u_t),\ 0 \leq t \leq 1, t = \log \tau$, is convex.

$\omega_{u_\tau}^m$ defines a singular metric e^Ψ on the transversal canonical bundle $\mathbf{K}_{\mathscr{F}_\xi}$,
The second variation is the current

$$d_\tau d_\tau^c \mathcal{M}(U_\tau) = \int_M T, \quad T := dd^c(\Psi(\pi^*\omega + dd^c U)^m) \wedge \eta$$

But the main problem is to show that T defines a non-negative current on $M \times D$, i.e. a Borel measure.

This is done as in the Kähler case with a local Bergman kernel approximation as in

R. Berman and B. Berndtsson, arXiv: 1405.0401, 2014.

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Some ideas of the proof

Bergman kernel for holomorphic functions on the ball $B \subset \mathbb{C}^m$ with weight ϕ.

$$\beta_k = \frac{m!}{k^m} K_{k\phi} e^{-k\phi}$$

$$K_{k\phi}(x) = \sup_{s \in H^0(B, K_B)} \frac{s \wedge \bar{s}(x)}{\int_B s \wedge \bar{s} e^{-k\phi}}.$$

$$\beta_k \to (dd^c \phi)^m \quad \text{in total variation.}$$

Choose local psh Φ so that $dd^c\Phi = \pi^*\omega + dd^c U$, $\phi_\tau = \Phi(\cdot, \tau)$. Define

$$T_k = dd^c \Psi_k \wedge (dd^c \Phi)^m \wedge \eta, \quad \Psi_k = \log \beta_k.$$

Then $\lim_{k\to\infty} T_k = T$.

(B. Berndtsson 2006) Plurisubharmonic variation of Bergman kernels

$$dd^c \log K_{k\phi_\tau} \geq 0 \quad \text{on } B \times D$$

So

$$dd^c \log \beta_k \geq -k dd^c \Phi,$$

and

$$\begin{aligned} T_k &= dd^c \log \beta_k \wedge (dd^c\Phi)^m \wedge \eta \\ &\geq -k(dd^c\Phi)^{m+1} \wedge \eta \\ &\geq 0 \end{aligned}$$

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Consequences of convexity

For $\phi_0, \phi_1 \in \mathcal{H}(\xi, \bar{J})$ we have

$$\mathcal{M}(\phi_1) - \mathcal{M}(\phi_0) \geq -d(\phi_1, \phi_0)\big(\mathrm{Cal}(\phi_0)\big)^{\frac{1}{2}},$$

$$\text{Calabi Energy} \quad \mathrm{Cal}(\phi) := \int_M (S(\phi) - \bar{S})^2 \omega_\phi^m \wedge \eta.$$

Corollary 2.4

Suppose that $(\eta_0, \xi, \omega_0^T), (\eta_1, \xi, \omega_1^T) \in \mathcal{S}(\xi, \bar{J})$ are two cscS structures. Then there is a $a \in \mathrm{Aut}(\mathscr{F}_\xi, \bar{J})$, diffeomorphisms preserving the transversely holomorphic foliation, so that $a^*\omega_1^T = \omega_0^T$.

- The metrics $g_0 = \omega_0^T(\cdot, \bar{J}\cdot) + \eta_0 \otimes \eta_0$ and $a^*g_1 = \omega_0^T(\cdot, \bar{J}\cdot) + \hat{\eta} \otimes \hat{\eta}$, where $\hat{\eta} = a^*\eta_1$, are not necessarily equal. But $\hat{\eta} - \eta_0$ is a closed basic form. So uniqueness is up changing the contact form by $H^1(M, \mathbb{R}) = \mathcal{H}^1_{g^T}$.
- The proof extends to prove uniqueness of Sasaki-extremal structures, $\partial^\#_{g^T} S_g$ transversely holomorphic.
  One considers a bluerelative K-energy $\mathcal{M}_V$, V extremal vector field.
- Since $\mathcal{M}$ is not known to be strictly convex the argument involves an approximation with

$$\mathcal{M}_s := \mathcal{M} + s\mathcal{F}_\mu, \quad \mathcal{F}_\mu(u) = \int_M u\, d\mu - \mathcal{E}(u),$$

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Space of Sasakian structures

Consider a fixed contact structure (M, η, ξ).

Definition 3.1

- *A $(1,1)$-tensor field $\Phi : TM \to TM$ on a contact manifold (M, η, ξ) is called an almost contact-complex structure if*

$$\Phi\xi = 0, \quad \Phi^2 = -id + \xi \otimes \eta.$$

- *An almost contact-complex structure Φ on a contact manifold (M, η, ξ) is compatible with η if*

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \text{ and } d\eta(X, \Phi X) > 0 \text{ for } X \in ker(\eta), \ X \neq 0.$$

- *Φ is called K-contact if in addition, $\mathcal{L}_\xi \Phi = 0$.*

It is a Sasakian structure if in addition

$$N_\Phi(X, Y) := [X, Y] + \Phi([\Phi X, Y] + [X, \Phi Y]) - [\Phi X, \Phi Y] = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

The subspace of Sasakian, transversely integrable, structures $\mathcal{K}^{int} \subset \mathcal{K}$ is an analytic subvariety.

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$$\Phi\xi = 0, \quad \Phi^2 = -id + \xi \otimes \eta.$$

- *An almost contact-complex structure Φ on a contact manifold (M, η, ξ) is compatible with η if*

$$d\eta(\Phi X, \Phi Y) = d\eta(X, Y), \text{ and } d\eta(X, \Phi X) > 0 \text{ for } X \in ker(\eta), \ X \neq 0.$$

- *Φ is called K-contact if in addition, $\mathcal{L}_\xi \Phi = 0$.*

It is a Sasakian structure if in addition

$$N_\Phi(X, Y) := [X, Y] + \Phi([\Phi X, Y] + [X, \Phi Y]) - [\Phi X, \Phi Y] = 0 \quad \text{for all } X, Y \in \Gamma(TM).$$

The subspace of Sasakian, transversely integrable, structures $\mathcal{K}^{int} \subset \mathcal{K}$ is an analytic subvariety.

Proposition 3.2

Given a K-contact complex structure Φ_0, the manifold $\mathcal{K}$ is parameterized by operators $P : T^{1,0}(\Phi_0) \to T^{0,1}(\Phi_0)$ satisfying the following:

- *(i) After lowering an index $P^\flat \in \Gamma(S^2(\Lambda_b^{1,0}))$, basic symmetric tensors, and*
- *(ii) $Id - \bar{P}P > 0$.*

And one has

$$T^{1,0}(\Phi) = \mathrm{Im}(Id - P), \quad T^{0,1}(\Phi) = \mathrm{Im}(Id - \bar{P}),$$

where $\Phi = \Phi_0(Id + Q)(Id - Q)^{-1},\ Q = \frac{1}{2}(P + \bar{P})$.

The subspace $\mathcal{K}^i \subseteq \mathcal{K}$ of Sasakian structures is the subvariety which in the complex parametrization is given by

$$N(P) = \bar{\partial}_b P + [P, P] = 0.$$

The space $\mathcal{K}$ of K-contact metric structures is an infinite dimensional Kähler manifold with Kähler form

$$\Omega_{\mathcal{K}}(A, B) = \int_M \mathrm{tr}(\Phi AB)\, d\mu_\eta.$$

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Space of Sasakian structures

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Moment map

Define $\mathcal{G}$ to be the group of strict contactomorphism of (M, η, ξ). $\mathcal{G}$ acts on $\mathcal{K}$ by holomorphic isometries

$$(g, \Phi) \mapsto g_* \Phi g_*^{-1}.$$

The Lie algebra of $\mathcal{G}$ is

$$\begin{array}{rcl} \mathrm{Lie}(\mathcal{G}) = \big(\{X \in \Gamma(TM) : \ \mathcal{L}_X \eta = 0\}, [\cdot,\cdot]\big) & \cong & \big(C_b^\infty(M), \{\cdot,\cdot\}\big) \\ X & \mapsto & H_X = \eta(X) \end{array}.$$

Theorem 3.3 (W. He 2011, S. K. Donaldson 1997)

The action of $\mathcal{G}$ on $\mathcal{K}$ is Hamiltonian with equivariant moment map $\mu : \mathcal{K} \to \mathcal{G}^*$

$$\mu(\Phi) = s^T(\Phi) - s_0^T.$$

$$\mathcal{P} : C_b^\infty(M) \to \Gamma(T_{\Phi_0}\mathcal{K}), \quad \mathcal{P}(f) = \mathcal{L}_{X_f}\Phi_0$$

$$\mathcal{Q} : T_{\Phi_0}\mathcal{K} \to C_b^\infty(M), \quad \mathcal{Q}(A) = ds^T(A)$$

Then

$$\langle \mathcal{Q}(A), H\rangle_{L^2} = \Omega(A, \mathcal{P}(H)), \quad A \in T_{\Phi_0}\mathcal{K}, \ H \in C_b^\infty(M).$$

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Slice argument

- Although, a complexification $\mathcal{G}^{\mathbb{C}}$ does not exist, the Lie algebra $\mathrm{Lie}(\mathcal{G}) \otimes \mathbb{C} \cong C_b^\infty(M, \mathbb{C})$ acts on $\mathcal{K}$. Complexify $\mathcal{P}$

$$\mathcal{P} : C_b^\infty(M, \mathbb{C}) \to \Gamma(T\mathcal{K}).$$

  Then $\Phi(t)$ is in the orbit of $\mathcal{G}^{\mathbb{C}}$ iff

$$\dot{\Phi}(t) \in \mathrm{Im}\,\mathcal{P} \quad \forall t.$$

- If $\Phi \in \mathcal{K}^{int}$ and $f \in C_b^\infty(M)$, then $\sqrt{-1}f$ acts on ω^T by

$$\mathcal{L}_{\Phi X_f}\omega^T = -\sqrt{-1}\partial\bar{\partial}f.$$

  Thus $\mathcal{G}^{\mathbb{C}}$ induces a holomorphic foliation on $\mathcal{K}^{int}$ whose leaves are transversal Kähler classes.
- Moser's argument shows that the $\mathcal{G}^{\mathbb{C}}$-orbit of $\Phi \in \mathcal{K}^{int}$ is the space of all $(\eta_\phi, \xi, \Phi_\phi, g_\phi)$ with

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We consider a finite dimensional slice for the action of $\mathcal{G}^{\mathbb{C}}$ on $\mathcal{K}$.

Let (M, η, ξ, Φ_0) be a cscS structure and $G = \mathrm{Aut}(M, \eta, \xi, \Phi_0)$.

We have the transversally elliptic $\mathcal{B}^\bullet$:

$$0 \to C_b^\infty(M, \mathbb{C}) \xrightarrow{\mathcal{P}} T_{\Phi_0}\mathcal{K} \xrightarrow{\bar{\partial}_b} \mathcal{B}^2 \to \cdots . \qquad (1)$$

$$H^1(\mathcal{B}^\bullet) \cong ker((\bar{\partial}^* \bar{\partial})^2 + \mathcal{P}\mathcal{P}^*)$$

is the space of first order deformations of Φ_0 modulo the action of $\mathcal{G}^{\mathbb{C}}$.

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Slice argument

Proposition 3.4

There exists a G-equivariant C^2 map $\hat{S}$ from a neighborhood B of 0 *in* $H^1(\mathcal{B}^\bullet)$ *to* $\mathcal{K}$ *with* $\hat{S}(0) = \Phi_0$*, such that* $\mu \circ \hat{S} = (s^T - s_0^T) \circ \hat{S}$ *takes value in* $\mathrm{Lie}(G)$.
Furthermore, the $\mathcal{G}^{\mathbb{C}}$ *orbit of every integrable smooth* Φ *close to* Φ_0 *intersects the image of* $\hat{S}$*. If* x *and* x' *lie in the same* $G^{\mathbb{C}}$ *orbit in B and* $\hat{S}(x) \in \mathcal{K}^i$*, then* $\hat{S}(x)$ *and* $\hat{S}(x')$ *are in the same* $\mathcal{G}^{\mathbb{C}}$ *orbit in* $\mathcal{K}$*. Moreover,* $\hat{S}$ *is tangent to S at 0 to first order.*

This idea has been used by S. Donaldson, G. Székelyhidi, T. Brönnle but there are technical difficulties in applying the implicit function theorem, therefore only C^2 regularity is proved.

Proposition 3.5

Suppose that $x \in U$*, after possibly shrinking U, is polystable for the* $G^{\mathbb{C}}$*-action on* $H^1(\mathcal{B}^\bullet)$*. Then there is y in the* $G^{\mathbb{C}}$*-orbit of x such that* $s^T(\hat{S}(y)) - s_0^T = 0$*. If in addition* $\hat{S}(x)$ *is integrable, then the corresponding cscS manifold* $(M, \eta, \hat{S}(y))$ *is in* $\mathcal{S}(M, \xi, \hat{S}(x))$.

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There exists a G-equivariant C^2 map $\hat{S}$ from a neighborhood B of 0 in $H^1(\mathcal{B}^\bullet)$ to $\mathcal{K}$ with $\hat{S}(0) = \Phi_0$, such that $\mu \circ \hat{S} = (s^T - s_0^T) \circ \hat{S}$ takes value in $\mathrm{Lie}(G)$.
Furthermore, the $\mathcal{G}^{\mathbb{C}}$ orbit of every integrable smooth Φ close to Φ_0 intersects the image of $\hat{S}$. If x and x' lie in the same $G^{\mathbb{C}}$ orbit in B and $\hat{S}(x) \in \mathcal{K}^i$, then $\hat{S}(x)$ and $\hat{S}(x')$ are in the same $\mathcal{G}^{\mathbb{C}}$ orbit in $\mathcal{K}$. Moreover, $\hat{S}$ is tangent to S at 0 to first order.

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Suppose that $x \in U$, after possibly shrinking U, is polystable for the $G^{\mathbb{C}}$-action on $H^1(\mathcal{B}^\bullet)$. Then there is y in the $G^{\mathbb{C}}$-orbit of x such that $s^T(\hat{S}(y)) - s_0^T = 0$. If in addition $\hat{S}(x)$ is integrable, then the corresponding cscS manifold $(M, \eta, \hat{S}(y))$ is in $\mathcal{S}(M, \xi, \hat{S}(x))$.

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Consequences

Theorem 3.6

Let (M, η, ξ, Φ_0) be a cscS manifold and (M, η, ξ, Φ) a nearby Sasakian manifold with transverse complex structure $\bar{J}$. Then if (M, η, ξ, Φ) is K-polystable, there is a constant scalar curvature Sasakian structure in the space $\mathcal{S}(\xi, \bar{J})$, unique up to automorphisms.

Theorem 3.7

Let (M, η, ξ, Φ_0) be a cscS manifold. Then any small deformation (M, η, ξ, Φ) which is Sasakian is K-semistable.

Proof.

For any test configuration of (Y, ξ) we have

$$\inf_{g \in \mathcal{S}(\xi, \bar{J})} (\mathrm{Cal}(g))^{\frac{1}{2}} \|v\|_\xi \geq c(n)\,\mathrm{Fut}(Y_0, \xi, v). \tag{2}$$

Slice argument shows that unstable orbits are adjacent to cscS structures. □

The next result was prove by V. Tosatti 2010 in the Käher case.

Theorem 3.8

Let (M, η, ξ, Φ_0) be a cscS manifold. Then any small deformation (M, η, ξ, Φ) which is Sasakian has K-energy bounded below. The lower bound is given $\lim_{t \to \infty} \mathcal{M}(\phi_t)$ where $\omega + dd^c \phi_t$ approaches an adjacent cscS structure.

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Let (M, η, ξ, Φ_0) be a cscS manifold and (M, η, ξ, Φ) a nearby Sasakian manifold with transverse complex structure $\bar{J}$. Then if (M, η, ξ, Φ) is K-polystable, there is a constant scalar curvature Sasakian structure in the space $\mathcal{S}(\xi, \bar{J})$, unique up to automorphisms.

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Further work

- One should be able to prove that the non-polystable orbits do not admit csc Sasakian structures.
  This is proved in Kähler case by X. Chen and S. Sun 2010 using the Calabi flow.
- Then the space of csc Sasakian structures compatible with (η, ξ) is a complex space with a singular Kähler structure.

Thank you