金融随机分析

第四章概率基础—有限概率空间

陈昱

cyu@ustc.edu.cn

东区管理科研楼1003

2019 年 5 月

Outline

概率基础—有限概率空间

§ 3.3 简单随机变量期望定义

定义(示性随机变量) 设

$$I_A(\omega) = \begin{cases} 0, & if\ \omega \notin A \\ 1, & if\ \omega \in A \end{cases}$$

称$I_A(\omega)$为A的示性函数，简记为I_A。

---

(1) $I_A(\omega)$ 为随机变量当且仅当$A \in \mathcal{F}$.
(2) 设 X 为简单随机变量，若(A_i不一定互斥)

$$X(\omega) = \sum_{i=1}^{\infty} b_i I_{A_i}(\omega), \quad A_i \in \mathcal{F}.$$

(3) 对简单随机变量X(如(2)定义)，则定义X的期望为

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} b_i \mathbb{P}(A_i)$$

例题：设$(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}_1, m)$，设

$$X = \sum_{i=1}^{\infty} \frac{1}{2^i} I_{[0,2^{-i})}.$$

则

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbb{P}([0, 2^{-i})) = 1/3.$$

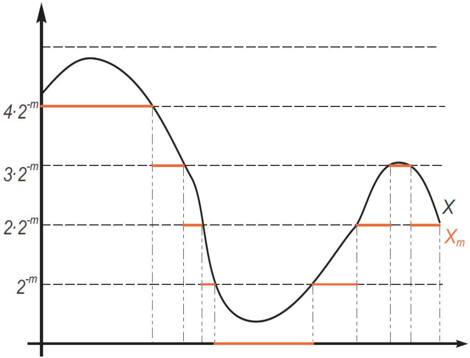
定义(非负随机变量期望) 设X非负，定义

$$A_{mn} = \{\omega : \frac{n}{2^m} \le X(\omega) < \frac{n+1}{2^m}\} \in \mathcal{F}, m, n \in \mathbb{N}$$

令

$$X_m = \sum_{n=0}^{\infty} \frac{n}{2^m} I_{A_{mn}},$$

X_m与X



可见$X_m(\omega)\uparrow$且

$$\lim_{m\to\infty} X_m(\omega) = X(\omega)$$

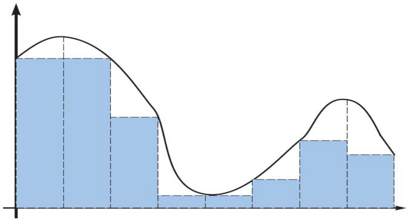
---

(1) 如果$\mathbb{E}[X_m] = +\infty$,对某个$m$, 则定义$\mathbb{E}[X] = +\infty$

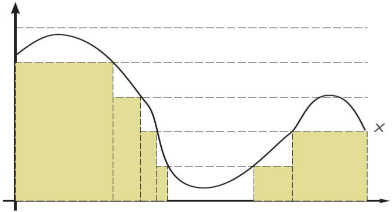
(2) 如果$\mathbb{E}[X_m] < +\infty$,对所有的$m$, 则定义

$$\mathbb{E}[X] = \lim_{m\to\infty} \mathbb{E}[X_m] = \lim_{m\to\infty} \sum_{n=0}^{\infty} \frac{n}{2^m} \mathbb{P}\left(\frac{n}{2^m} \le X(\omega) < \frac{n+1}{2^m}\right)$$

Riemann积分



Lebesgue积分



定义(期望) 考虑一般随机变量，则记

$$X = X^+ - X^-, \quad X^+ = X \vee 0, X^- = (-X) \vee 0$$

---

(1) 如果$\mathbb{E}[X^+] = \mathbb{E}[X^-] = \infty$，定义$\mathbb{E}[X] = \infty$.
(2) 如果$\mathbb{E}[X^+]$和$\mathbb{E}[X^-]$不都为∞，定义

$$\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-].$$

(3) 如果$\mathbb{E}[|X|] = \mathbb{E}[X^+] + \mathbb{E}[X^-] < \infty$，则定义

$$\mathbb{E}[X] = \int_\Omega X \mathrm{d}\mathbb{P}(\omega) = \int_\Omega X(\omega)\mathbb{P}(\mathrm{d}\omega)$$

定义 1.7 设 X 为随机变量，则其 (数学) 期望定义为

$$\mathbb{E}[X] = \int_{\Omega} X \mathrm{d}\mathbb{P}(\omega).$$

(1) 设 X 为离散型随机变量，有概率质量函数 $p_i = \mathbb{P}(X = a_i)$, $i = 1, 2, \ldots$, 则其期望为

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} a_i p_i.$$

(2) 设 X 为连续型随机变量，有概率密度函数 f，则其期望为

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x) \mathrm{d}x.$$

随机变量的数学期望的性质

- 单调性. 如果 $X(\omega) \leq Y(\omega), \forall \omega \in \Omega$, 则

$$\mathbb{E}[X] \leq \mathbb{E}[Y].$$

- 线性性. 对任意

$$\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c.$$

- Jense 不等式. 若 $\phi : \mathbb{R} \to \mathbb{R}$ 是一个凸函数, 则

$$\mathbb{E}[\phi(X)] \geq \phi(\mathbb{E}[X]).$$

例 3.6 考虑例 3.5 中期权收益 V_3, 其中 $p = 1/2$. 由

$$\mathbb{P}(V_3 = 27) = p^3,$$
$$\mathbb{P}(V_3 = 3) = 3p^2(1-p),$$
$$\mathbb{P}(V_3 = 0) = 1 - p^2(3-2p).$$

则

$$\mathbb{E}[V_3] = 27 \times p^3 + 3 \times 3p^2(1-p) = 4.5.$$

The *expectation* or *expected value* of a function g of a r.v. X is

$$\mathbb{E}(g(X)) = \int_{\Omega} g(X(\omega))\, d\mathbb{P}(\omega) = \int_{-\infty}^{\infty} g(x)\, dF_X(x).$$

For discrete r.v. X,

$$\mathbb{E}(g(X)) = \sum_x g(x) p_x = \sum_x g(x) f_X(x).$$

For continuous r.v. X,

$$\mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx.$$

条件概率：考虑$(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{D} = \{D_1, \ldots, D_n\}$ 为样本空间Ω的一个划分, 对$A \in \mathcal{F}$, 定义A关于划分$\mathcal{D}$的条件概率如下：

$$\mathbb{P}(A|\mathcal{D}) = \sum_{i=1}^{n} \mathbb{P}(A|D_i) I_{D_i}.$$

- 如果$\mathcal{D} = \{\Omega\}$, 则

$$\mathbb{P}(A|\mathcal{D}) = \mathbb{P}(A|\Omega) I_\Omega = \mathbb{P}(A).$$

- $\mathbb{P}(A|\mathcal{D})$是个随机变量. 如果$\omega \in \mathcal{D}_i$, 则$\mathbb{P}(A|\mathcal{D}) = \mathbb{P}(A|\mathcal{D}_i)$.
- 如果$A, B \in \mathcal{F}$, $AB = \emptyset$, 则

$$\mathbb{P}(A \cup B|\mathcal{D}) = \mathbb{P}(A|\mathcal{D}) + \mathbb{P}(B|\mathcal{D})$$

考虑$(\Omega, \mathcal{F}, \mathbb{P})$上的随机变量$Y$,

$$Y = \sum_{i=1}^{n} y_i I_{D_i}$$

其中$y_1, \ldots, y_n$ 是不同的常数值, 假定$\mathcal{D} = \{D_1, \ldots, D_n\}$ 为样本空间Ω的一个划分并满足$\mathbb{P}(D_i) > 0$. 则

$$D_i = \{Y = y_i\}.$$

定义:

- 称$\mathcal{D}_Y = \{D_1, \ldots, D_n\}$为由$Y$导出的划分.
- 称$\mathbb{P}(A|Y) := \mathbb{P}(A|\mathcal{D}_Y)$是关于随机变量$Y$的条件概率.

§ 3 关于离散随机变量的条件期望

考虑$(\Omega, \mathcal{F}, \mathbb{P})$上的随机变量$X$,

$$X = \sum_{i=1}^{m} x_i I_{A_i}$$

其中$x_1, \ldots, x_n$ 是不同的常数值, 假定$\{A_1, \ldots, A_n\}$ 为样本空间Ω的不相交的区间.

定义: 随机变量X关于划分$\mathcal{D} = \{D_1, \ldots, D_n\}$的条件期望定义为

$$\mathbb{E}[X|\mathcal{D}] = \sum_{i=1}^{m} x_i \mathbb{P}(A_i|\mathcal{D})$$

- $\mathbb{E}[aX + bY|\mathcal{D}] = a\mathbb{E}[X|\mathcal{D}] + b\mathbb{E}[Y|\mathcal{D}]$
- $\mathbb{E}[c|\mathcal{D}] = c, \mathbb{E}[I_A|\mathcal{D}] = \mathbb{P}(A|\mathcal{D})$.
- $\mathbb{E}[\mathbb{E}[X|\mathcal{D}]] = \mathbb{E}[X]$

例子：设$(\Omega, \mathcal{F}, \mathbb{P}) = ([0,1], \mathcal{B}[0,1], m)$,
$\mathcal{D} = \{[0,1/2),[1/2,1]\} = \{D_1, D_2\}$. 令

$$X = I_{[0,1/3]} + 2I_{(1/3,2/3)} + 3I_{[2/3,1]}$$

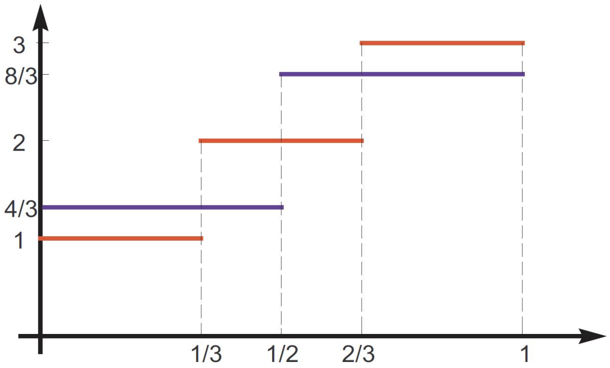
计算可得

$$\mathbb{E}[X|\mathcal{D}_1] = \frac{4}{3}, \mathbb{E}[X|\mathcal{D}_2] = \frac{8}{3}$$

从而

$$\mathbb{E}[X|\mathcal{D}] = \frac{4}{3} I_{D_1} + \frac{8}{3} I_{D_2}$$

随机变量X和它的条件期望$\mathbb{E}[X|\mathcal{D}]$.



- 考虑$(\Omega, \mathcal{F}, \mathbb{P})$上的连续随机变量$Y$, 怎么定义关于$Y$的条件期望$\mathbb{E}[X|Y]$?
- 子σ代数$\mathcal{G} \subset \mathcal{F}$, X是非负的随机变量, 如何定义$\mathbb{E}[X|\mathcal{G}]$?

定义: $(\Omega, \mathcal{F}, \mathbb{P})$上非负随机变量$X$关于$\sigma$代数$\mathcal{G} \subset \mathcal{F}$的条件期望, 记为$\mathbb{E}[X|\mathcal{G}]$, 若满足

- $\mathbb{E}[X|\mathcal{G}] \in \mathcal{G}$. (该条件期望是根据$\mathcal{G}$算出来的)
- 对所有的$A \in \mathcal{F}$, 有(一样表示平均高度)

$$\int_A \mathbb{E}[X|\mathcal{G}] d\mathbb{P} = \int_A X d\mathbb{P}.$$

条件期望定义另一种说法:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X a r.v. with $\mathbb{E}(|X|) < \infty$, and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.

Define the conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}(X \mid \mathcal{G})$, to be any r.v. Y such that

1. Y is $\mathcal{G}$-measurable, i.e. $Y \in \mathcal{G}$,
2. $\int_G Y \, d\mathbb{P} = \int_G X \, d\mathbb{P}$ for each set $G \in \mathcal{G}$.

If a r.v. $\widetilde{Y}$ has the above properties, then $\widetilde{Y}$ is a *version* of the conditional expectation of X given $\mathcal{G}$, i.e. $\widetilde{Y} = \mathbb{E}(X \mid \mathcal{G})$ a.s..

随机变量函数的条件期望

Let X and Y be continuous r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with conditional density function $f_{X|Y}(x \mid y)$.

Let h be a function such that $\mathbb{E}(|h(X)|) < \infty$ and set

$$g(y) = \int_{-\infty}^{\infty} h(x) f_{X|Y}(x|y)\, dx.$$

Then $g(Y)$ is a version of the conditional expectation of $h(X)$ given $\sigma(Y)$, i.e.

$$g(Y) = \mathbb{E}(\, h(X) \mid \sigma(Y)) \;\; a.s..$$

随机变量函数的条件期望证明:

Let $G \in \sigma(Y)$. Then $Y(G) = A \subset \mathbb{R}$ and

$$\begin{aligned}\int_G h(X)\,d\mathbb{P} &= \int_\Omega h(X(\omega))\,\mathbf{1}_G(\omega)\,d\mathbb{P}(\omega)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\,\mathbf{1}_A(y) f_{X,Y}(x,y)\,dx\,dy\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\,\mathbf{1}_A(y) f_{X|Y}(x|y) f_Y(y)\,dx\,dy\\ &= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x) f_{X|Y}(x|y)\,dx\right)\mathbf{1}_A(y) f_Y(y)\,dy\end{aligned}$$

证明续:

Let $G \in \sigma(Y)$. Then $Y(G) = A \subset \mathbb{R}$ and

$$\begin{aligned}
\int_G h(X)\,d\mathbb{P} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} h(x) f_{X|Y}(x|y)\,dx \right) \mathbf{1}_A(y) f_Y(y)\,dy \\
&= \int_{\mathbb{R}} g(y)\,\mathbf{1}_A(y) f_Y(y)\,dy \\
&= \int_{\Omega} g(Y(\omega))\,\mathbf{1}_G(\omega)\,d\mathbb{P}(\omega) \\
&= \int_G g(Y)\,d\mathbb{P}.
\end{aligned}$$

条件期望的性质:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, a and b constants, X and Y integrable r.v., and $\mathcal{G}$ and $\mathcal{H}$ sub-σ-algebras of $\mathcal{F}$.

1. $\mathbb{E}(Y) = \mathbb{E}(X)$ if Y is a version of $\mathbb{E}(X \mid \mathcal{G})$.
2. $\mathbb{E}(X \mid \mathcal{G}) = X$ if X is $\mathcal{G}$-measurable.
3. $\mathbb{E}(YX \mid \mathcal{G}) = Y\,\mathbb{E}(X \mid \mathcal{G})$ if Y is $\mathcal{G}$-measurable.
4. $\mathbb{E}(X \mid \sigma(\mathcal{G}, \mathcal{H})) = \mathbb{E}(X \mid \mathcal{G})$ if $\mathcal{H}$ is independent of $\sigma(\sigma(X), \mathcal{G})$.
5. $\mathbb{E}(aX + bY \mid \mathcal{G}) = a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G})$ (Linearity Property).

条件期望的性质:

6. $\mathbb{E}(X \mid \mathcal{G}) \geq \mathbb{E}(Y \mid \mathcal{G})$ if $X \geq Y$ (Monotonicity).
7. $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H})$ if $\mathcal{H} \subset \mathcal{G}$ (Tower Property).
8. $\mathbb{E}(\psi(X) \mid \mathcal{G}) \geq \psi(\mathbb{E}(X \mid \mathcal{G}))$ if ψ is a convex function (Jensen's Inequality).
9. $\liminf_{n\to\infty} \mathbb{E}(X_n \mid \mathcal{G}) \geq \mathbb{E}(\liminf_{n\to\infty} X_n \mid \mathcal{G})$ if $X_n \geq 0$ (Fatou's Lemma).
10. $\mathbb{E}(X_n \mid \mathcal{G}) \uparrow \mathbb{E}(X \mid \mathcal{G})$ if $X_n \geq 0$ and $X_n \uparrow X$ (Monotone Convergence Theorem).
11. $\mathbb{E}(X_n \mid \mathcal{G}) \to \mathbb{E}(X \mid \mathcal{G})$ if $|X_n| \leq Y$ for all n and $X_n \to X$ (Dominated Convergence Theorem).

条件期望的性质与证明:

(Linearity) $\mathbb{E}(aX + bY \mid \mathcal{G}) = a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G})$.

Proof. Since

1. $a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G}) \in \mathcal{G}$,

we need only to verify the condition

3. $\int_G a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G})\, d\mathbb{P} = \int_G aX + bY\, d\mathbb{P}$ for $G \in \mathcal{G}$.

条件期望的性质与证明:

(Linearity) $\mathbb{E}(aX+bY \mid \mathcal{G}) = a\,\mathbb{E}(X \mid \mathcal{G}) + b\,\mathbb{E}(Y \mid \mathcal{G})$.

Proof. Let $G \in \mathcal{G}$.

$$\begin{aligned}
&\int_G a\,\mathbb{E}(X \mid \mathcal{G}) + b\,\mathbb{E}(Y \mid \mathcal{G})\, d\mathbb{P} \\
&= a\int_G \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} + b\int_G \mathbb{E}(Y \mid \mathcal{G})\, d\mathbb{P} \\
&= a\int_G X\, d\mathbb{P} + b\int_G Y\, d\mathbb{P} \\
&= \int_G aX + bY\, d\mathbb{P}.
\end{aligned}$$

条件期望的性质与证明:

(Monotonicity) $\mathbb{E}(X \mid \mathcal{G}) \geq \mathbb{E}(Y \mid \mathcal{G})$ a.s. if $X \geq Y$.

Proof. For any $G \in \mathcal{G}$,

$$\int_G \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} = \int_G X\, d\mathbb{P} \geq \int_G Y\, d\mathbb{P} = \int_G \mathbb{E}(Y \mid \mathcal{G})\, d\mathbb{P},$$

$$\int_G \mathbb{E}(Y \mid \mathcal{G}) - \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} \leq 0.$$

Let $\epsilon > 0$. Then

$$A_\epsilon = \left\{ \mathbb{E}(Y \mid \mathcal{G}) - \mathbb{E}(X \mid \mathcal{G}) > \epsilon \right\} \in \mathcal{G}$$

and

$$\mathbb{P}(A_\epsilon) = 0.$$

条件期望的性质与证明:

(Jensen's Inequality) $\mathbb{E}\left(\psi(X) \mid \mathcal{G}\right) \geq \psi(\mathbb{E}\left(X \mid \mathcal{G}\right))$ a.s. if $\psi : \mathbb{R} \to \mathbb{R}$ is a convex function.

Proof. Let $\mathcal{L} = \left\{(a, b) \in \mathbb{Q}^2 \mid ax + b \leq \psi(x)\ \forall\ x\right\}$. Then

$$\psi(x) = \sup\left\{ax + b \mid (a, b) \in \mathcal{L}\right\}.$$

If $\psi(x) \geq ax + b$,

$$\mathbb{E}\left(\psi(X) \mid \mathcal{G}\right) \geq a\,\mathbb{E}\left(X \mid \mathcal{G}\right) + b.$$

Taking the supremum over $(a, b) \in \mathcal{L}$ gives

$$\mathbb{E}\left(\psi(X) \mid \mathcal{G}\right) \geq \psi\left(\mathbb{E}\left(X \mid \mathcal{G}\right)\right).$$

条件期望的性质与证明:

If Y is a version of $\mathbb{E}(X \mid \mathcal{G})$, then $\mathbb{E}(Y) = \mathbb{E}(X)$.

Proof. For any $A \in \mathcal{G}$,

$$\int_\Omega \mathbf{1}_A Y \, d\mathbb{P} = \int_A Y \, d\mathbb{P} = \int_A X \, d\mathbb{P} = \int_\Omega \mathbf{1}_A X \, d\mathbb{P},$$

$$\mathbb{E}(\mathbf{1}_A Y) = \mathbb{E}(\mathbf{1}_A X).$$

In particular, for $A = \Omega \in \mathcal{G}$,

$$\mathbb{E}(Y) = \mathbb{E}(X).$$

条件期望的性质与证明:

$\mathbb{E}(YX \mid \mathcal{G}) = Y\,\mathbb{E}(X \mid \mathcal{G})$ a.s. if Y is $\mathcal{G}$-measurable.

Proof. Since

1. $Y\,\mathbb{E}(X \mid \mathcal{G}) \in \mathcal{G}$,

we need only to verify the condition

3. $\int_G Y\,\mathbb{E}(X \mid \mathcal{G})\,d\mathbb{P} = \int_G YX\,d\mathbb{P}$ for $G \in \mathcal{G}$.

条件期望的性质与证明:

$\mathbb{E}(YX \mid \mathcal{G}) = Y\,\mathbb{E}(X \mid \mathcal{G})$ a.s. if Y is $\mathcal{G}$-measurable.

Proof. Let $G \in \mathcal{G}$. Suppose $Y = \mathbf{1}_B$ for some $B \in \mathcal{G}$.

$$\int_G \mathbf{1}_B \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} = \int_{G\cap B} \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} = \int_{G\cap B} X\, d\mathbb{P}$$

$$= \int_G \mathbf{1}_B X\, d\mathbb{P}.$$

If $Y \geq 0$, let $\{Y_n\}$ be a sequence of simple r.v. that converges to Y, and use the monotone convergence theorem to show that

$$\int_G Y\,\mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} = \int_G Y X\, d\mathbb{P}.$$

条件期望的性质与证明:

$\mathbb{E}(Y X \mid \mathcal{G}) = Y\, \mathbb{E}(X \mid \mathcal{G})$ a.s. if Y is $\mathcal{G}$-measurable.

Proof. If $Y \geq 0$, let $\{Y_n\}$ be a sequence of simple r.v. that converges to Y, and use the monotone convergence theorem to conclude that

$$\int_G Y\, \mathbb{E}(X \mid \mathcal{G})\, d\mathbb{P} = \int_G Y X\, d\mathbb{P}.$$

To prove the result in general, split Y into positive and negative parts.

条件期望的性质与证明:

(Tower Property) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H})$ a.s. if $\mathcal{H} \subset \mathcal{G}$.

Proof. For any r.v. Y and any $A \in \mathcal{H}$,

$$\mathbb{E}(\mathbf{1}_A \mathbb{E}(Y \mid \mathcal{H})) = \mathbb{E}(\mathbb{E}(\mathbf{1}_A Y \mid \mathcal{H})) = \mathbb{E}(\mathbf{1}_A Y).$$

Since $A \in \mathcal{H} \subset \mathcal{G}$,

$$\mathbb{E}\left(\mathbf{1}_A \mathbb{E}\left(\underbrace{\mathbb{E}(X \mid \mathcal{G})} \mid \mathcal{H}\right)\right) = \mathbb{E}\left(\mathbf{1}_A \underbrace{\mathbb{E}(X \mid \mathcal{G})}\right) = \mathbb{E}(\mathbf{1}_A X).$$

条件期望

考虑连续抛掷 n 次硬币的概率空间 $(\Omega_n, \mathcal{F}_n, \mathbb{P})$ 中，求给定 $S_n = s$ 条件下，S_{n+1} 的条件期望.

$$\mathbb{E}[S_{n+1}|S_n = s] = usp_u + dsp_d.$$

故有

$$\mathbb{E}[S_{n+1}|S_n] = uS_np_u + dS_np_d.$$

更一般地，我们有

$$\mathbb{E}[S_{n+m}|S_n = s] = \sum_{k=0}^{m} \binom{m}{k} u^k d^{m-k} sp_u^k p_d^{m-k}.$$

$$\mathbb{E}[S_{n+m}|S_n] = \sum_{k=0}^{m} \binom{m}{k} u^k d^{m-k} p_u^k p_d^{m-k} S_n.$$

条件期望

考虑连续抛掷 n 次硬币的概率空间 $(\Omega_n, \mathcal{F}_n, \mathbb{P})$ 中，求给定前 n 次抛掷结果的条件下 S_{n+1} 的条件期望

$$\mathbb{E}_n[S_{n+1}] = uS_n p_u + dS_n p_d.$$

更一般地，我们有

$$\mathbb{E}_n[X] = ?$$

条件期望的性质

- 线性性. 对任意

$$\mathbb{E}_n(aX + bY + c) = a\mathbb{E}_n(X) + b\mathbb{E}_n(Y) + c.$$

- Jense 不等式. 若 $\phi : \mathbb{R} \to \mathbb{R}$ 是一个凸函数, 则

$$\mathbb{E}_n[\phi(X)] \geq \phi(\mathbb{E}_n[X]).$$

- 提取已知量. 如果 X 依赖于前 n 次抛掷结果, 则

$$\mathbb{E}_n[XY] = X\mathbb{E}_n[Y].$$

- 平滑公式. 对 $0 \leq n \leq m \leq N$, 则

$$\mathbb{E}_n[\mathbb{E}_m[X]] = \mathbb{E}_n[X].$$

风险中性概率：

$$\tilde{p} = \frac{1 + r_f - d}{u - d}, \quad \tilde{q} = 1 - \tilde{p} = \frac{u - (1 + r_f)}{u - d}$$

容易验证：

$$\frac{\tilde{p}u + \tilde{q}d}{1 + r_f} = 1$$

所以，在时刻$t = n$，对于每一个给定的投掷序列$\omega_1\omega_2\ldots\omega_n$，有：

$$S_n(\omega_1\omega_2\ldots\omega_n) = \frac{1}{1 + r_f}[\tilde{p}S_{n+1}(\omega_1\omega_2\ldots\omega_n H) + \tilde{q}S_{n+1}(\omega_1\omega_2\ldots\omega_n T)]$$

在时刻$t = n$的股票价格是时刻$t = n + 1$股票价格加权平均的折现，定义

$$\widetilde{E}_n[S_{n+1}](\omega_1\omega_2\ldots\omega_n) = \tilde{p}S_{n+1}(\omega_1\omega_2\ldots\omega_n H) + \tilde{q}S_{n+1}(\omega_1\omega_2\ldots\omega_n T)$$

省略投掷序列$\omega_1\omega_2\ldots\omega_n$，则有：

$$S_n = \frac{1}{1 + r_f}\widetilde{E}_n[S_{n+1}]$$

称$\widetilde{E}_n[S_{n+1}]$依据时刻$t = n$所具有的信息，对股票价格在$t = n + 1$时刻的条件期望。

Definition 3.1

令$1 \le n \le N$，给定$\omega_1\omega_2\ldots\omega_n$，时刻$n$固定。在时刻$n$后的投掷结果的序列$\omega_{n+1}\omega_{n+2}\ldots\omega_N$，有$2^{N-n}$个。记$\#H(\omega_{n+1}\omega_{n+2}\ldots\omega_N)$是投掷结果序列$\omega_{n+1}\omega_{n+2}\ldots\omega_N$出现$H$的数目，记$\#T(\omega_{n+1}\omega_{n+2}\ldots\omega_N)$是投掷结果序列$\omega_{n+1}\omega_{n+2}\ldots\omega_N$出现$T$的数目。定义：

$$\widetilde{E}_n[X](\omega_1\omega_2\ldots\omega_n) = \sum_{\omega_{n+1}\omega_{n+2}\ldots\omega_N} \widetilde{p}^{\#H(\omega_{n+1}\omega_{n+2}\ldots\omega_N)}\widetilde{q}^{\#T(\omega_{n+1}\omega_{n+2}\ldots\omega_N)}X(\omega_1\ldots\omega_n\omega_{n+1}\ldots\omega_N)$$

称$\widetilde{E}_n[X]$依据$t=n$时刻信息，对随机变量X的条件期望。特别地：

$$\widetilde{E}_0[X] = \widetilde{E}[X], \quad \widetilde{E}_N[X] = X$$

记号:

Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a *filtration* is a nested family $\{\mathcal{F}_t \mid t \in \mathbb{N}\}$ of sub-σ-algebras of $\mathcal{F}$ where

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}.$$

The probability space with the filtration $\{\mathcal{F}_t \mid t \in \mathbb{N}\}$ is called a *filtered probability space*

$$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P}\right).$$

记号:

A process $X = \{X_t\}$ is *adapted* to the filtration $\{\mathcal{F}_t\}$ if X_t is $\mathcal{F}_t$-measurable for each t, i.e. $X_t \in \mathcal{F}_t$.

Recall that X_t is $\mathcal{F}_t$-measurable means

$$X_t^{-1}(B) = \{\omega \in \Omega \mid X_t(\omega) \in B\} \in \mathcal{F}_t \ \forall B \in \mathcal{B}(\mathbb{R}).$$

鞅的定义

A process $X = \{X_t\}$ is a *martingale* relative to a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ if

1. X is adapted,
2. $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$ a.s. for $t \geq s$.

鞅的第2条另一种描述:

For discrete time process $X = \{X_t\}$, condition 2 for a martingale can be restated as

2. $\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) = X_{t-1}$ a.s. for $t \geq 1$.

$$
\begin{aligned}
\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) & & = X_{t-1} \\
\mathbb{E}(X_t \mid \mathcal{F}_{t-2}) &= \mathbb{E}(\mathbb{E}(X_t \mid \mathcal{F}_{t-1}) \mid \mathcal{F}_{t-2}) & \\
&= \mathbb{E}(X_{t-1} \mid \mathcal{F}_{t-2}) & = X_{t-2} \\
&\vdots & \\
\mathbb{E}(X_t \mid \mathcal{F}_{t-k}) &= \mathbb{E}(\mathbb{E}(X_t \mid \mathcal{F}_{t-k+1}) \mid \mathcal{F}_{t-k}) & \\
&= \mathbb{E}(X_{t-k+1} \mid \mathcal{F}_{t-k}) & = X_{t-k}
\end{aligned}
$$

上下鞅的定义

A process $X = \{X_t\}$ is a *supermartingale* relative to a filtered space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ if

1. X is adapted,
2. $\mathbb{E}(X_t \mid \mathcal{F}_s) \leq X_s$ a.s. for $t \geq s$.

And X is a *submartingale* if condition 2 is replaced with

2. $\mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s$ a.s. for $t \geq s$.

鞅的例子:

Suppose it is equally likely to get a head or a tail when a coin is toss. Let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th toss results in a head,} \\ -1 & \text{if the } i\text{th toss results in a tail.} \end{cases}$$

Consider the stochastic process $\{M_n\}$ defined by $M_0 = 0$ and

$$M_n = \sum_{i=1}^{n} X_i, \quad \text{for } n \geq 1.$$

1. Show that $\{M_n\}$ is a martingale, i.e. $\mathbb{E}(M_n \mid M_{n-1}) = M_{n-1}$ for $n \geq 1$.

鞅的例子构造:

Let ζ be an integrable r.v. define on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{\mathcal{F}_t\}$ be a filtration.

Define

$$M_t = \mathbb{E}(\zeta \mid \mathcal{F}_t) \quad \text{for all } t.$$

Show that the stochastic process $\{M_t\}$ is a martingale.

Theorem 3.2 (条件期望的基本性质)

N为正整数，X和Y为依赖于前n次投掷的随机变量，给定$n(1 \le n \le N)$，以下性质成立。

(i) 线性：对于任意的常数c_1和c_2，有$E_n[c_1X + c_2Y] = c_1E_n[X] + c_2E_n[Y]$

(ii) 提取已知项：X只依赖于前n次投掷，有$E_n[XY] = XE_n[Y]$

(iii) 迭代条件：如果$0 \le n \le m \le N$，则$E_n[E_m[X]] = E_n[X]$

(iv) 独立性：如果X只依赖于从$t = n+1$到$t = N$ 的投掷，则$E_n[X] = E[X]$

(v) 条件 Jensen 不等式：如果$\varphi(x)$是关于哑变量x的凸函数，则：

$$E_n[\varphi(X)] \ge \varphi(E_n[X])$$

在风险中性概率下：

$$S_n = \widetilde{E}_n[\frac{S_{n+1}}{1+r_f}] \Rightarrow \frac{S_n}{(1+r_f)^n} = \widetilde{E}_n[\frac{S_{n+1}}{(1+r_f)^{n+1}}]$$

Definition 3.3 (鞅)

考虑二项式资产定价模型，令$M_0, M_1, \ldots, M_N$是随机变量序列，每一个M_n仅依赖于前n次投掷的结果(M_0是常数)。称此随机变量序列为适应性随机过程。

(i) 如果$M_n = E_n[M_{n+1}], n = 0, 1, \ldots, N-1$，称此随机过程为鞅

(ii) 如果$M_n \leq E_n[M_{n+1}], n = 0, 1, \ldots, N-1$，称此随机过程为下鞅

(iii) 如果$M_n \geq E_n[M_{n+1}], n = 0, 1, \ldots, N-1$，称此随机过程为上鞅

Remark of Definition 3.3

"one-step-ahead" Martingale:

$$M_{n+1} = E_{n+1}[M_{n+2}], 0 \leq n \leq N-2$$

Taking conditional expectations on both sides based on the information at time n:

$$E_n[M_{n+1}] = E_n[E_{n+1}[M_{n+2}]] = E_n[M_{n+2}], 0 \leq n \leq N-2$$

"two-step-ahead" Martingale:

$$M_n = E_n[M_{n+2}], 0 \leq n \leq N-2$$

"multi-step-ahead" Martingale:

$$M_n = E_n[M_m], 0 \leq n \leq m \leq N$$

Specially, $M_0 = E[M_n]$

Theorem 3.4 (折现股价是鞅)

在二项式模型中，$0 < d < 1 + r_f < u$，风险中性概率如下所示：

$$\tilde{p} = \frac{1 + r_f - d}{u - d}, \quad \tilde{q} = 1 - \tilde{p} = \frac{u - (1 + r_f)}{u - d}$$

则在风险中性概率下，对于任意时刻n以及任意的投掷序列$\omega_1\omega_2\ldots\omega_n$，折现股价是个鞅。

$$\frac{S_n}{(1 + r_f)^n} = \tilde{E}_n[\frac{S_{n+1}}{(1 + r_f)^{n+1}}]$$

Theorem 3.4 证明(1).

$$
\begin{aligned}
&\widetilde{E}_n[\frac{S_{n+1}}{(1+r_f)^{n+1}}](\omega_1\omega_2\ldots\omega_n) \\
&= \frac{1}{(1+r_f)^n}\frac{1}{1+r_f}[\widetilde{p}S_{n+1}(\omega_1\omega_2\ldots\omega_n H)+\widetilde{q}S_{n+1}(\omega_1\omega_2\ldots\omega_n T)] \\
&= \frac{1}{(1+r_f)^n}\frac{1}{1+r_f}[\widetilde{p}uS_n(\omega_1\omega_2\ldots\omega_n)+\widetilde{q}dS_n(\omega_1\omega_2\ldots\omega_n)] \\
&= \frac{S_n}{(1+r_f)^n}(\omega_1\omega_2\ldots\omega_n)\frac{\widetilde{p}u+\widetilde{q}d}{1+r_f} \\
&= \frac{S_n}{(1+r_f)^n}(\omega_1\omega_2\ldots\omega_n)
\end{aligned}
$$

□

Theorem 3.4 证明(2).

$$
\begin{aligned}
\widetilde{E}_n[\frac{S_{n+1}}{(1+r_f)^{n+1}}] &= \widetilde{E}_n[\frac{S_n}{(1+r_f)^{n+1}}\frac{S_{n+1}}{S_n}] \\
&= \frac{S_n}{(1+r_f)^n}\widetilde{E}_n[\frac{1}{1+r_f}\frac{S_{n+1}}{S_n}] \\
&\text{(Taking out what is known)} \\
&= \frac{S_n}{(1+r_f)^n}\frac{1}{1+r_f}\widetilde{E}[\frac{S_{n+1}}{S_n}] \\
&\text{(Independence)} \\
&= \frac{S_n}{(1+r_f)^n}\frac{\widetilde{p}u+\widetilde{q}d}{1+r_f} \\
&= \frac{S_n}{(1+r_f)^n}
\end{aligned}
$$

□

Theorem 3.5 (折现财富是鞅)

考虑一个N期的二项式定价模型，$\Delta_0, \Delta_1, \ldots, \Delta_{n-1}$是适应性随机过程。令$X_0$是实数。令$X_1, X_2, \ldots, X_N$由财富过程依次产生。那么，折现财富过程$\frac{X_n}{(1+r_f)^n}, n = 1, 2, \ldots, N$在风险中性概率下是鞅。

Theorem 3.5 证明.

$$
\begin{aligned}
\widetilde{E}_n[\frac{X_{n+1}}{(1+r_f)^{n+1}}] &= \widetilde{E}_n[\frac{\Delta_n S_{n+1}}{(1+r_f)^{n+1}} + \frac{X_n - \Delta_n S_n}{(1+r_f)^n}] \\
&= \widetilde{E}_n[\frac{\Delta_n S_{n+1}}{(1+r_f)^{n+1}}] + \widetilde{E}_n[\frac{X_n - \Delta_n S_n}{(1+r_f)^n}] \text{(Linearity)} \\
&= \Delta_n \widetilde{E}_n[\frac{S_{n+1}}{(1+r_f)^{n+1}}] + \frac{X_n - \Delta_n S_n}{(1+r_f)^n} \text{(Taking out what is known)} \\
&= \Delta_n \frac{S_n}{(1+r_f)^n} + \frac{X_n - \Delta_n S_n}{(1+r_f)^n} = \frac{X_n}{(1+r_f)^n}
\end{aligned}
$$

□

Theorem 3.6 (风险中性定价公式)

考虑一个N期的二项式定价模型，并且$0 < d < 1 + r < u$，风险中性概率测度为$\widetilde{\mathbb{P}}$。令V_N(衍生证券在时刻N的支付)是取决于投掷结果的随机变量。由后向前定义随机变量$V_{N-1}, V_{N-2}, \ldots, V_0$:

$$V_n = \frac{1}{1+r_f}[\widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T)]$$

那么对任意的$0 \le n \le N$，衍生证券在时刻n的价格由以下风险中性定价公式给出:

$$V_n = \widetilde{\mathbb{E}}_n[\frac{V_N}{(1+r_f)^{N-n}}]$$

并且，折现的衍生证券的价格在风险中性概率测度下为鞅，即:

$$\frac{V_n}{(1+r_f)^n} = \widetilde{\mathbb{E}}_n[\frac{V_{n+1}}{(1+r_f)^{n+1}}]$$

Theorem 3.7 (现金流估值)

考虑一个N期的二项式定价模型，并且$0 < d < 1 + r < u$，风险中性概率测度为$\widetilde{\mathbb{P}}$。衍生证券的支付$C_0, C_1, \ldots, C_N$是随机变量序列，并且C_n只取决于$\omega_1\omega_2\ldots\omega_n$。衍生证券在$t = n$时刻的价格由如下公式给出：

$$V_n = \widetilde{\mathbb{E}}_n[\sum_{k=n}^{N} \frac{C_k}{(1+r_f)^{k-n}}], \quad n = 0, 1, 2, \ldots, N$$

价格过程$V_n, \quad n = 1, 2, \ldots, N$，满足：

$$C_n = V_n - \frac{1}{1+r_f}[\widetilde{p}V_{n+1}(H) + \widetilde{q}V_{n+1}(T)]$$

定义：

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}, \quad 0 \leq n \leq N-1$$

Theorem 3.7 (现金流估值)

令$X_0 = V_0$，并且根据以下方程依次产生$X_1, X_2, \ldots, X_N$

$$X_{n+1} = \Delta_n S_{n+1} + (1 + r_f)(X_n - C_n - \Delta_n S_n)$$

则任选n和$\omega_1 \ldots \omega_n$，均有：

$$X_n(\omega_1 \ldots \omega_n) = V_n(\omega_1 \ldots \omega_n)$$

Remark

$$V_n = \widetilde{\mathbb{E}}_n[\sum_{k=n}^{N} \frac{C_k}{(1 + r_f)^{k-n}}], \quad n = 0, 1, 2, \ldots, N-1$$

$$V_n = C_n + \widetilde{\mathbb{E}}_n[\sum_{k=n+1}^{N} \frac{C_k}{(1 + r_f)^{k-n}}] \quad n = 0, 1, 2, \ldots, N-1$$

Theorem 3.7 证明.

(1)当$n=0$时，依据定理，有$X_0=V_0$

(2)当$n+1$时$(0\le n\le N-1)$

$$
\begin{aligned}
V_n &= \widetilde{\mathbb{E}}_n[\sum_{k=n}^{N}\frac{C_k}{(1+r_f)^{k-n}}] \\
&= C_n+[\frac{1}{1+r_f}\widetilde{E}_{n+1}[\sum_{k=n+1}^{N}\frac{C_k}{(1+r_f)^{k-(n+1)}}]] \\
&= C_n+\frac{1}{1+r_f}E_n[V_{n+1}]
\end{aligned}
$$

$$
V_n-C_n=\frac{1}{1+r_f}[\widetilde{p}V_{n+1}(H)+\widetilde{q}V_{n+1}(T)]
$$

□

Theorem 3.7 证明(续).

$$
\begin{aligned}
X_{n+1}(H) &= \Delta_n S_{n+1}(H) + (1+r_f)(X_n - C_n - \Delta_n S_n) \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)}(S_{n+1}(H) - (1+r_f)S_n) + (1+r_f)(V_n - C_n) \\
&= \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}(uS_n - (1+r_f)S_n) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (V_{n+1}(H) - V_{n+1}(T))\frac{u-(1+r_f)}{u-d} + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (V_{n+1}(H) - V_{n+1}(T))\tilde{q} + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= V_{n+1}(H)
\end{aligned}
$$

同理可证: $X_{n+1}T = V_{n+1}(T)$ □

记号(测度等价):

A probability measure $\mathbb{Q}$ is *absolutely continuous* w.r.t. measure $\mathbb{P}$, denoted by $\mathbb{Q} \ll \mathbb{P}$, if

$$\mathbb{P}(A) = 0 \Longrightarrow \mathbb{Q}(A) = 0 \quad \forall A \in \mathcal{F}.$$

The probability measures $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent measure*, denoted by $\mathbb{P} \sim \mathbb{Q}$, if

$$\mathbb{Q} \ll \mathbb{P} \text{ and } \mathbb{P} \ll \mathbb{Q}.$$

等价鞅测度:

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $Z \geq 0$ a.s. with $\mathbb{E}^{\mathbb{P}}(Z) = 1$. For $A \in \mathcal{F}$, define

$$\mathbb{Q}(A) = \int_A Z(\omega)\, d\mathbb{P}(\omega).$$

Then $\mathbb{Q}$ is a probability measure. Furthermore,

$$\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{P}}(ZX)$$

for any integrable r.v. X. If $Z > 0$ a.s., we also have

$$\mathbb{E}^{\mathbb{P}}(Y) = \mathbb{E}^{\mathbb{Q}}\left(\frac{Y}{Z}\right)$$

for any integrable r.v. Y.

Let X be an integrable r.v. defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{Q} \sim \mathbb{P}$.

1. $\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{P}}\left(X \dfrac{d\mathbb{Q}}{d\mathbb{P}}\right)$.

2. $\mathbb{E}^{\mathbb{Q}}(X \,|\, \mathcal{F}_t) = \dfrac{\mathbb{E}^{\mathbb{P}}\left(X \dfrac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right)}{\mathbb{E}^{\mathbb{P}}\left(\dfrac{d\mathbb{Q}}{d\mathbb{P}} \,\middle|\, \mathcal{F}_t\right)}$.

证明:

1. $\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{P}}\left(X\,\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$

$$\mathbb{E}^{\mathbb{Q}}(X) = \int_{\Omega} X\,d\mathbb{Q} = \int_{\Omega} X\frac{d\mathbb{Q}}{d\mathbb{P}}\,d\mathbb{P} = \mathbb{E}^{\mathbb{P}}\left(X\,\frac{d\mathbb{Q}}{d\mathbb{P}}\right).$$

证明:

2. $$\mathbb{E}^{\mathbb{P}}\left(X\,\frac{d\mathbb{Q}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right) = \mathbb{E}^{\mathbb{Q}}(X\,|\,\mathcal{F}_t)\,\mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right).$$

Let $A \in \mathcal{F}_t$.

$$\begin{aligned}
\int_A X\frac{d\mathbb{Q}}{d\mathbb{P}}\,d\mathbb{P} &= \int_A X\,d\mathbb{Q} = \int_A \mathbb{E}^{\mathbb{Q}}(X\,|\,\mathcal{F}_t)\,d\mathbb{Q} \\
&= \int_A \mathbb{E}^{\mathbb{Q}}(X\,|\,\mathcal{F}_t)\frac{d\mathbb{Q}}{d\mathbb{P}}\,d\mathbb{P} \\
&= \int_A \mathbb{E}^{\mathbb{P}}\left(\mathbb{E}^{\mathbb{Q}}(X\,|\,\mathcal{F}_t)\frac{d\mathbb{Q}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right)d\mathbb{P} \\
&= \int_A \mathbb{E}^{\mathbb{Q}}(X\,|\,\mathcal{F}_t)\,\mathbb{E}^{\mathbb{P}}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\,\middle|\,\mathcal{F}_t\right)d\mathbb{P}.
\end{aligned}$$

假如一年365天能观察到的股票的价格如下：$S_0, S_1, \ldots, S_{365}$，除了当前时刻$t=0$ 的价格S_0是已知的之外，其他时刻的股票价格$S_t, t=1,\ldots,365$都是随机变量。每天的收益这样如下：

$$(\frac{S_1}{S_0}), (\frac{S_2}{S_1}), \cdots, (\frac{S_{365}}{S_{364}})$$

每天的收益率R_t可以表示成：

$$1+\frac{R_t}{365}=\frac{S_t}{S_{t-1}}$$

连续计息的年收益率：

$$1+R=\frac{S_{365}}{S_0}=(\frac{S_1}{S_0})(\frac{S_2}{S_1})\cdots(\frac{S_{365}}{S_{364}})=\prod_{t=1}^{365}(1+\frac{R_t}{365})$$

r_t表示第t日的连续复利计息的年收益率，就应有：

$$1+\frac{R_t}{365}=\lim_{n\to\infty}(1+\frac{r_t/365}{n})^n=e^{\frac{r_t}{365}}\Rightarrow\frac{r_t}{365}=\ln(1+\frac{R_t}{365})$$

连续计息的年收益率是：

$$r = \ln(1+R) = \ln\prod_{t=1}^{365}(1+\frac{R_t}{365}) = \sum_{t=1}^{365}\ln(1+\frac{R_t}{365}) = \frac{1}{365}\sum_{t=1}^{365} r_t$$

按照任意的时间间隔计算，有：

$$r = \frac{1}{\tau}(r_1 + r_2 + \cdots + r_\tau)$$

股价运动的基本假设：

1 所有的r_t都是独立同分布

2 股票的价格变化是连续的

当时间间隔取得很小，即τ可以去很大的数值时，根据中心极限定理，随机变量r(连续复利收益率)服从正态分布，价格的相对收益率$1+R=\frac{S_{365}}{S_0}$就服从对数正态分布

$$\ln(\frac{S_T}{S_t}) \sim \mathcal{N}(\mu(T-t), \sigma^2(T-t))$$

对二叉树的价格变化概率作如下假设：

$$\frac{S_t}{S_{t-1}} = \begin{cases} u, & q \\ \\ d, & 1-q \end{cases}$$

对数均值(数学期望)：$E[\ln(\frac{S_t}{S_{t-1}})] = q\ln(u) + (1-q)\ln(d)$
方差为：$var[\ln(\frac{S_t}{S_{t-1}})] = q(1-q)[\ln(\frac{u}{d})]^2$
如果从时刻$t=0$到时刻$t=T$的阶段数位τ，因为各个阶段之间价格变化是相互独立的，其均值和方差就应为：

$$E[\ln(\frac{S_T}{S_0})] = \tau[q\ln(u) + (1-q)\ln(d)]$$
$$var[\ln(\frac{S_T}{S_0})] = \tau q(1-q)[\ln(\frac{u}{d})]^2$$

当$\tau \to \infty$，选取适当的参数(u, d, q)，使股票价格的变化趋向于对数正态分布，并且使其均值和方差满足前述要求。因为二叉树各个阶段股票价格的变化是互相独立的，而且变化的概率分布是同分布的，因此满足条件(1)；如果选择u和d使得他们都能够足够快地趋于1，则股票价格的变化趋于连续。于是，由中心极限定理可知，当所分的阶段数$\tau \to \infty$时，股票收益率的变化服从正态分布(价格变化趋于对数正态分布)。选则参数u，d和q为：

$$u = \exp\{\sigma\sqrt{\frac{T-t}{\tau}}\}$$

$$d = \frac{1}{u} = \exp\{-\sigma\sqrt{\frac{T-t}{\tau}}\}$$

$$q = \frac{1}{2} + \frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{T-t}{\tau}}$$

当$\tau \to \infty$时，$\ln(\frac{S_T}{S_t}) \sim \mathscr{N}(\mu(T-t), \sigma^2(T-t))$

$$
\begin{aligned}
&\tau[q\ln(u) + (1-q)\ln(d)] \\
=&\tau\{[\frac{1}{2} + \frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{T-t}{\tau}}]\sigma\sqrt{\frac{T-t}{\tau}} + [\frac{1}{2} - \frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{T-t}{\tau}}](-\sigma\sqrt{\frac{T-t}{\tau}})\} \\
=&\mu(T-t) \\
&\tau q(1-q)[\ln(\frac{u}{d})]^2 \\
=&\tau[\frac{1}{2} + \frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{T-t}{\tau}}][\frac{1}{2} - \frac{1}{2}(\frac{\mu}{\sigma})\sqrt{\frac{T-t}{\tau}}](2\sigma\sqrt{\frac{T-t}{\tau}})^2 \\
=&\sigma^2(T-t) - \frac{[\mu(T-t)]^2}{\tau} \to \sigma^2(T-t)
\end{aligned}
$$

$$\frac{V_n}{(1+r_f)^n} = \widetilde{\mathbb{E}}_n[\frac{V_{n+1}}{(1+r_f)^{n+1}}] = \widetilde{\mathbb{E}}_n[\frac{V_N}{(1+r_f)^N}]$$

重新定义符号，期权的价格c_t；$R = e^{r_f}$，连续复利；风险中性概率π^*；到期日T；$B_t = R^{-(T-t)} = \frac{R^t}{R^T}$

$$c_t = \frac{B_t}{B_T} E_t^*(c_T) = R^{-(T-t)} E_t^*(c_T) = e^{-r_f(T-t)} E_t^*(c_T)$$

假设现在是t时期，标的股票的价格是S_t，则到期日T时刻，股票价格的可能取值是$S_t u^n d^{\tau-n}$，$n = 0, 1, \cdots, \tau$，并且$\tau \equiv T - t$。在概率测度下，股票价格取值为$S_t u^n d^{\tau-n}$的概率是：

$$\frac{\tau!}{n!(\tau-n)!}(\pi^*)^n(1-\pi^*)^{\tau-n}$$

于是得到

$$E_t^*(c_T) = \sum_{n=0}^{\tau} \frac{\tau!}{n!(\tau-n)!}(\pi^*)^n(1-\pi^*)^{\tau-n}\max(S_t u^n d^{\tau-n} - X, 0)$$
$$= S_t \sum_{n=0}^{\tau} \frac{\tau!}{n!(\tau-n)!}(\pi^*)^n(1-\pi^*)^{\tau-n}\max(u^n d^{\tau-n} - X/S_t, 0)$$

令j是满足$u^n d^{\tau-n} - X/S_t \geq 0$中最小的一个，数学表达式可写成$j = \arg\min_n(u^n d^{\tau-n} - X/S_t \geq 0)$，由$u^n d^{\tau-n} - X/S_t \geq 0$可知：

$$j \geq \frac{\ln[X/(d^\tau S_t)]}{\ln(u/d)}$$

在计算期权价格时，$n < j$的项可以不予以考虑：

$$
\begin{aligned}
E_t^*(c_T) &= S_t \sum_{n=j}^{\tau} \frac{\tau!}{n!(\tau-n)!}(\pi^*)^n(1-\pi^*)^{\tau-n}(u^n d^{\tau-n} - X/S_t) \\
&= S_t \sum_{n=j}^{\tau} \frac{\tau!}{n!(\tau-n)!}(u\pi^*)^n[d(1-\pi^*)]^{\tau-n} \\
&- X \sum_{n=j}^{\tau} \frac{\tau!}{n!(\tau-n)!}
\end{aligned}
$$

$$
\begin{aligned}
c_t &= R^{-\tau} E_t^*(c_T) \\
&= S_t \sum_{n=j}^{\tau} \frac{\tau!}{n!(\tau-n)!}(u\pi^*/R)^n[d(1-\pi^*)/R]^{\tau-n} \\
&- R^{-\tau} X \sum_{n=j}^{\tau} \frac{\tau!}{n!(\tau-n)!} \qquad (9)
\end{aligned}
$$

Definition 4.1

$$\Phi(j;\tau;*)=\sum_{n=j}^{\tau}\frac{\tau!}{n!(\tau-n)!}(*)^n(1-*)^{\tau-n}$$

☞ $u\pi^*/R+d(1-\pi^*)/R=1$

$$\Phi(j;\tau;u\pi^*/R)=\sum_{n=j}^{\tau}\frac{\tau!}{n!(\tau-n)!}(u\pi^*/R)^n[d(1-\pi^*)/R]^{\tau-n} \tag{10}$$

$$\Phi(j;\tau;\pi^*)=\sum_{n=j}^{\tau}\frac{\tau!}{n!(\tau-n)!}(\pi^*)^n(1-\pi^*)^{\tau-n} \tag{11}$$

将等式(10)和(11)代入等式(9) 中，得到：

$$\begin{aligned} c_t &= S_t \Phi(j;\tau;u\pi^*/R) - R^{-\tau} X \Phi(j;\tau;\pi^*) \\ &= S_t \Phi(j;\tau;u\pi^*/R) - e^{-r_f(T-t)} X \Phi(j;\tau;\pi^*) \end{aligned} \tag{12}$$

在定义(4.1)中，$\Phi(j;\tau;*)$实际上是τ个独立且同分布的随机变量之和大于j的概率($X_\tau = x_1 + x_2 + \cdots + x_\tau$)，任意$x_i, i = 1, 2, \ldots, \tau$只取0或1，取1的概率为$*$，取0的概率为$1 - *$。$\Phi(j;\tau;*) = prob(X_\tau \geq j)$。将二叉树无限细分，则从$t$时期到$T$时期的步长变得无限小，式(12)中的二项分布就会趋于正态分布，适当地选择u，d等参数，最终可以得到：

$$c_t(S_t;X;T-t) = S_t \mathcal{N}(z) - X e^{-r_f(T-t)} \mathcal{N}(z - \sigma\sqrt{T-t})$$

$$z = \frac{\ln(S_t / X e^{-r_f(T-t)})}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}$$

定义（Markov 过程）： 考虑二叉树资产定价模型。设$X_0, X_1, \ldots, X_N$为适应过程。如果对每个0到$N-1$之间的n以及每个函数$f(x)$，存在另一个函数$g(x)$(依赖于n和f)，使得：

$$\mathbb{E}_n[f(X_{n+1})] = g(X_n)$$

则称$X_0, X_1, \ldots, X_N$ 是一个马尔可夫（Markov）过程.

- $\mathbb{E}_n[f(X_{n+1})]$是随机变量，依赖于前n次抛硬币的结果. Markov性表明，对前n次抛硬币结果的依赖仅通过X_n体现。
- 比起g的公式，我们更关心它的存在性。下面的例子给出如何寻找g的方法

§ 3.5 Markov过程

例子 在二叉树模型中，时刻$n+1$的股价通过时刻n的股价给出：

$$S_{n+1}(\omega_1 \cdots \omega_n \omega_{n+1}) = \begin{cases} uS_n(\omega_1 \cdots \omega_n) & if \;\; \omega_{n+1} = H \\ uS_n(\omega_1 \cdots \omega_n) & if \;\; \omega_{n+1} = T \end{cases}$$

于是

$$\mathbb{E}_n[f(X_{n+1})](\omega_1 \cdots \omega_n) = pf(uS_n(\omega_1 \cdots \omega_n)) + qf(dS_n(\omega_1 \cdots \omega_n))$$

略去$\omega_1 \cdots \omega_n$，可重写为

$$\mathbb{E}_n[f(X_{n+1})] = g(S_n)$$

其中$g(x) = pf(ux) + qf(dx)$.
这表明，无论在真实概率还是风险中性概率测度下，股票价格过程都是Markov过程

引理（提取已知量）： 考虑N阶段的二叉树资产定价模型，n是0到N之间整数．设$X^1,\dots,X_K$只依赖于第1次到第n次的抛硬币过程，随机变量$Y^1,\dots,Y_L$只依赖于第$n+1$次到第N次的抛硬币过程．设$f(x^1,\dots,x^K,y^1,\dots,y^L)$为$K+L$维函数，定义

$$g(x^1,\cdots,x^K)=\mathbb{E}[f(x^1,\cdots,x^K,Y^1,\cdots,Y^L)]$$

则有:

$$\mathbb{E}_n[f(X^1,\cdots,X^K,Y^1,\cdots,Y^L)]=g(X^1,\cdots,X^K)$$

- 假设$K=L=1$,于是有

$$g(x)=\mathbb{E}f(x,Y)$$

上面的式子变为

$$g(X)=\mathbb{E}_nf(X,Y)$$

引理证明： 设$\omega_1\cdots\omega_n$任意固定. 通过条件期望的定义有

$$\begin{aligned}\mathbb{E}_n & [f(X,Y)](\omega_1\cdots\omega_n) \\ = & \sum_{\omega_{n+1}\cdots\omega_N} f(X(\omega_1\cdots\omega_n), Y(\omega_{n+1}\cdots\omega_N)p^{\#H(\omega_{n+1}\cdots\omega_N)}q^{\#T(\omega_{n+1}\cdots\omega_N)}\end{aligned}$$

然而

$$\begin{aligned}g(x) & = \mathbb{E}[f(x,Y)](\omega_1\cdots\omega_n) \\ = & \sum_{\omega_{n+1}\cdots\omega_N} f(x, Y(\omega_{n+1}\cdots\omega_N)p^{\#H(\omega_{n+1}\cdots\omega_N)}q^{\#T(\omega_{n+1}\cdots\omega_N)}\end{aligned}$$

显然有

$$\mathbb{E}_n[f(X,Y)](\omega_1\cdots\omega_n) = g(X(\omega_1\cdots\omega_n)). \qquad \square$$

遇到一个非马尔科夫过程是，通过增加一些所谓的状态变量来重新获得马尔科夫性质.

定义（多维Markov 过程）： 考虑二叉树资产定价模型。设$\{X_n^1, X_n^2, \ldots, X_n^K), n = 0, 1, \ldots, N\}$为一个$K$维适应过程。如果对每个0到$N-1$之间的$n$以及每个函数$f(x^1, \ldots, x^K)$，存在另一个函数$g(x^1, \ldots, x^K)$(依赖于$n$和$f$)，使得：

$$\mathbb{E}_n[f(X_{n+1}^1, \cdots, X_{n+1}^K)] = g(f(X_n^1, \cdots, X_n^K))$$

则称$\{X_n^1, X_n^2, \ldots, X_n^K), n = 0, 1, \ldots, N\}$为一个$K$维马尔可夫（Markov）过程.

$$\mathbb{E}_n[f(X_{n+1}^1, \cdots, X_{n+1}^K)] = g(X_n^{)}$$

例子: 在一个N时段的二叉树模型，考虑二维适应过程$\{(S_n, M_n), n = 0, 1, \ldots, N\}$，其中$S_n$为时刻$n$时股票价格，$M_n = \max_{0\le k\le n} S_k$为截止时刻$n$时的股价最大值. 可以验证$\{(S_n, M_n), n = 0, 1, \ldots, N\}$是一个马尔科夫过程.

关键点: $M_{n+1} = M_n \vee S_{n+1} = M_n \vee (S_n Y), Y = \frac{S_{n+1}}{S_n}$

$$\mathbb{E}_n[f(S_{n+1}, M_{n+1})] = \mathbb{E}_n f(S_n Y, M_n \vee (S_n Y))$$

记$g(s, m) = \mathbb{E}f(sY, m\vee(sY)) = pf(us, m\vee(us)) + qf(ds, m\vee(ds))$