金融随机分析

第6章 Brown运动

陈昱

cyu@ustc.edu.cn

管理科研楼 1003

63602243

2019 年 5 月

Suppose a fair coin toss outcome is $\omega = (\omega_1, \omega_2, \omega_3, \dots)$. Let

$$X_i = \begin{cases} 1 & \text{if } \omega_i = \textit{Head} \text{ with probability } \frac{1}{2}, \\ -1 & \text{if } \omega_i = \textit{Tail} \text{ with probability } \frac{1}{2}. \end{cases}$$

Define

$$M_t = \begin{cases} 0 & \text{if } t = 0, \\ \sum_{i=1}^{t} X_i & \text{for } t = 1, 2, 3, \dots. \end{cases}$$

The process $M = (M_t)$ is a symmetric random walk.

The process $M = (M_t)$ is a martingale.

1. M is adapted.
2. $|M_t| \leq t$ for all t.
3. For $0 \leq s \leq t$,

$$\begin{aligned}\mathbb{E}(M_t \mid \mathcal{F}_s) &= \mathbb{E}(M_t - M_s + M_s \mid \mathcal{F}_s) \\ &= \mathbb{E}(M_t - M_s) + M_s \\ &= M_s.\end{aligned}$$

Consider a time partition $\Pi = \left\{t_0, t_1, t_2, \ldots, t_{m-1}, t_m\right\}$ where $0 = t_0 < t_1 < t_2 < \cdots < t_{m-1} < t_m$.

The r.v.

$$M_{t_1} - M_{t_0}, M_{t_2} - M_{t_1}, \ldots, M_{t_m} - M_{t_{m-1}}$$

are independent.

In addition,

$$\mathbb{E}\left(M_{t_i} - M_{t_{i-1}}\right) = 0,$$

$$\mathbb{V}\mathrm{ar}\left(M_{t_i} - M_{t_{i-1}}\right) = \sum_{j=t_{i-1}+1}^{t_i} \mathbb{V}\mathrm{ar}\left(X_j\right) = t_i - t_{i-1}.$$

The quadratic variation of the symmetric random walk M up to time T is defined to be

$$[M, M]_T = \sum_{i=1}^{T} \Big(\underbrace{M_i - M_{i-1}}_{X_i}\Big)^2 = T.$$

Let $\Pi = \{t_0, t_1, t_2, \dots, t_{m-1}, t_m\}$ be a partition of $[0, T]$. The quadratic variation of a stochastic process Y up to time T is defined to be

$$[Y, Y]_T = \operatorname*{plim}_{\|\Pi\| \to 0} \sum_{i=1}^{m} \left(Y_{t_i} - Y_{t_{i-1}}\right)^2.$$

- $\|\Pi\| \equiv \max_i \{t_i - t_{i-1}\}$ is the maximum step size of the partition.
- $\operatorname*{plim}_{n \to \infty} X_n = X \equiv \lim_{n \to \infty} \mathbb{P}(|X_n - X| < \epsilon) = 1.$

Fix $n > 0$. Define

$$W^{(n)}(t) = \frac{1}{\sqrt{n}}\left(\alpha_t M_{[nt]+1} + (1-\alpha_t)M_{[nt]}\right)$$

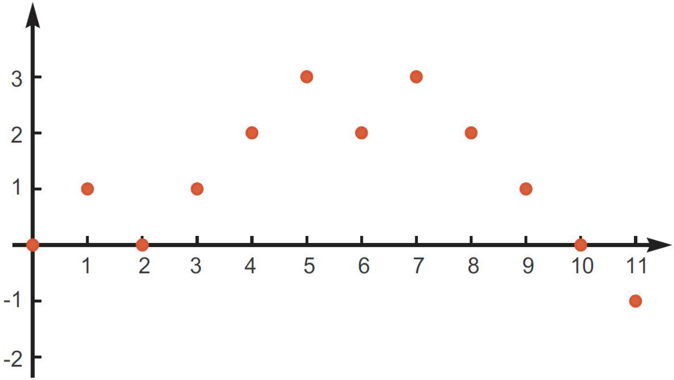
where $\alpha_t = nt - [nt]$.

Note that $W^{(n)}(t) = \frac{1}{\sqrt{n}} M_{nt} = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i$ if $nt \in \mathbb{Z}$.

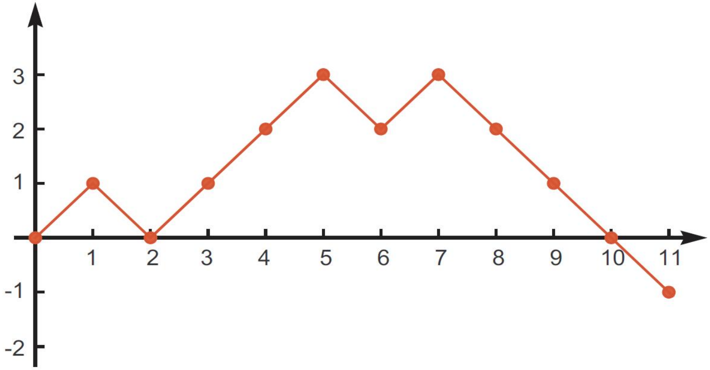
如果$nt \notin \mathbb{Z}$, 我们通过线性插值定义$W^{(n)}(t)$, i.e.,

$$W^{(n)}(t) = ([nt]+1-nt)W^{(n)}(\frac{[nt]}{n}) + (nt-[nt])W^{(n)}(\frac{[nt]+1}{n})$$

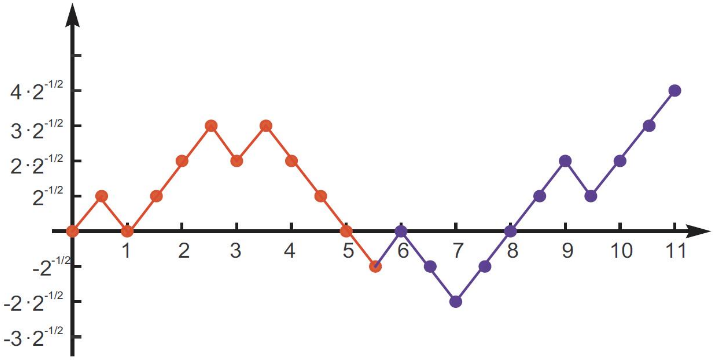
M_k只有整数点上有值:



$W^{(n)}(t)$想法则差不多, 上下跳动的大小为$1/\sqrt{n}$, 跳动的频率变为$1/n$, 中间用一直线连起来. 下图是$W^{(1)}(t)$: 将M_k连起来即可



$W^{(2)}(t)$, 上下跳动的大小为$1/\sqrt{2}$, 跳动的频率变为$1/2$, 中间一样用直线连起来. 下图是$W^{(2)}(t)$:



Suppose $nt_i \in \mathbb{Z}$ for all i. Show the following:

1. The r.v. $W^{(n)}(t_1) - W^{(n)}(t_0)$, $W^{(n)}(t_2) - W^{(n)}(t_1)$, ..., $W^{(n)}(t_n) - W^{(n)}(t_{n-1})$ are independent.

2. For $0 \le i < j \le m$,

$$\mathbb{E}\left(W^{(n)}(t_j) - W^{(n)}(t_i)\right) = 0,$$

$$\mathbb{V}\mathrm{ar}\left(W^{(n)}(t_j) - W^{(n)}(t_i)\right) = t_j - t_i,$$

$$\mathbb{E}\left(W^{(n)}(t_j) \,\middle|\, \mathcal{F}_{t_i}\right) = W^{(n)}(t_i).$$

For $t \geq 0$ such that $nt \in \mathbb{Z}$,

$$[W^{(n)}, W^{(n)}]_t = \sum_{i=1}^{nt} \Bigg(\underbrace{W^{(n)}\left(\tfrac{i}{n}\right) - W^{(n)}\left(\tfrac{i-1}{n}\right)}_{\frac{1}{\sqrt{n}} X_i} \Bigg)^2$$

$$= \sum_{i=1}^{nt} \frac{1}{n} X_i^2$$

$$= t.$$

Fix $t \geq 0$. As n approaches infinity, the distribution of the scaled random walk $W^{(n)}(t)$ evaluated at time t converges to the normal distribution with mean 0 and variance t, i.e.

$$W^{(n)}(t) \xrightarrow{\mathcal{D}} \mathcal{N}(0, t) \quad \text{as} \quad n \to \infty.$$

Since

$$\lim_{n\to\infty} f_n(u) = t \lim_{n\to\infty} \frac{\ln\left(\frac{1}{2}\,\mathrm{e}^{-\frac{u}{\sqrt{n}}} + \frac{1}{2}\,\mathrm{e}^{\frac{u}{\sqrt{n}}}\right)}{\frac{1}{n}} = \frac{1}{2}u^2 t,$$

the m.g.f. of $W^{(n)}(t)$,

$$\mathbb{E}\left(\mathrm{e}^{u\,W^{(n)}(t)}\right) \longrightarrow \mathrm{e}^{\frac{1}{2}u^2 t} \quad \text{as} \quad n \to \infty.$$

- We obtain

$$S^{(n)}(t) = S_0 e^{\frac{1}{2}\sqrt{n}(\log u_n - \log d_n)W^{(n)}(t) + \frac{1}{2}(\log u_n + \log d_n)nt} \quad (3.4)$$

- We want to construct $S(t)$ via taking $n \to \infty$. This works if we have limits

$$W^{(n)}(t) \to W(t)$$
$$\frac{1}{2}\sqrt{n}(\log u_n - \log d_n) \to \sigma$$
$$\frac{1}{2}(\log u_n + \log d_n)n \to c$$

for $n \to \infty$. In this case we obtain

$$S(t) = S_0 e^{\sigma W(t) + ct}$$

Now suppose we know that $W^{(n)}(t) \to W(t)$ for $n \to \infty$.
(We can hope that we get convergence since $Var[W^{(n)}(t)] = t$ for all n.)

Define

$$u_n = 1 + \frac{\sigma}{\sqrt{n}} + \frac{\alpha}{n}, \quad d_n = 1 - \frac{\sigma}{\sqrt{n}} + \frac{\alpha}{n} \tag{3.5}$$

with $\sigma > 0$, $\alpha \in \mathbb{R}$. Using $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$ we find

$$\frac{1}{2}(\log u_n - \log d_n) = \frac{\sigma}{\sqrt{n}} + O(n^{-\frac{3}{2}}),$$

$$\frac{1}{2}(\log u_n + \log d_n) = \frac{1}{n}\left(\alpha - \frac{1}{2}\sigma^2\right) + O(n^{-\frac{3}{2}}).$$

So from (3.4)

$$\begin{aligned}
&\log S^{(n)}(t)\\
&= \log S_0 + \tfrac{1}{2}\sqrt{n}(\log u_n - \log d_n)W^{(n)}(t) + \tfrac{1}{2}(\log u_n + \log d_n)nt\\
&= \log S_0 + \sigma W^{(n)}(t) + O(n^{-1})W^{(n)}(t) + \left(\alpha - \tfrac{1}{2}\sigma^2\right)t + O(n^{-\frac{1}{2}})\\
&\to \log S_0 + \sigma W(t) + \left(\alpha - \tfrac{1}{2}\sigma^2\right)t.
\end{aligned}$$

Hence the random variables $S^{(n)}$ with u_n, d_n in (3.5) converge to

$$S(t) = S_0 e^{\sigma W(t) + \left(\alpha - \frac{1}{2}\sigma^2\right)t} \tag{3.6}$$

in distr. for $n \to \infty$ (*geometric Brownian motion*, see below).

The scaled random walk $W^{(n)}$ has

- some natural time step;
- is linear between these time step;
- and is approximately normal.

Note that $W^{(\infty)} = \lim_{n\to\infty} W^{(n)}$ is not differentiable.

$$\frac{W^{(\infty)}(t+h) - W^{(\infty)}(t)}{h} \overset{d}{=} \frac{Z}{\sqrt{h}} \longrightarrow \infty \quad \text{as} \quad h \to 0.$$

定义6.1 随机过程 $W=\{W(t):t\geq 0\}$ 称为布朗运动, 如果

1. $W(0)=0$
2. 对任意的 $\omega\in\Omega$, $W(t,\omega)$ 关于 $t\geq 0$ 连续.
3. 对 $t>s\geq 0$, 增量

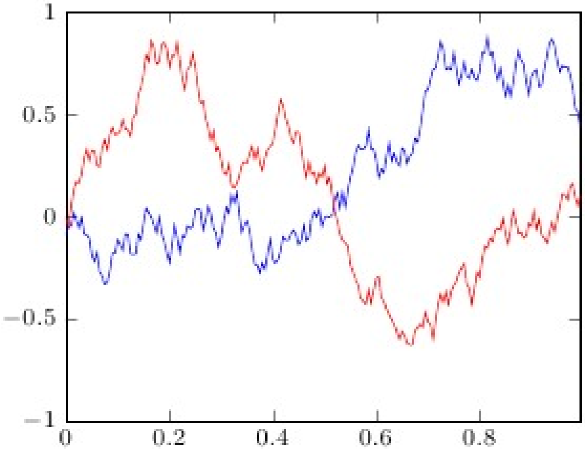
$$W(t)-W(s)\sim N(0,t-s),$$

   其中 $N(0,t-s)$ 服从均值为 0 方差为 $t-s$ 的正态分布.

4. 对 $0=t_0<t_1<\cdots<t_m$, 增量

$$W(t_1),W(t_2)-W(t_1),\ldots,W(t_m)-W(t_{m-1})$$

   相互独立.


1
0.5
0
−0.5
−1
0
0.2
0.4
0.6
0.8

Let W be a Brownian motion. Suppose $s < t$. Then

$$\begin{aligned}\mathbb{E}\left(W(s)W(t)\right) &= \mathbb{E}\left(W(s)\left(W(t) - W(s)\right) + W^2(s)\right) \\ &= \mathbb{E}\left(W(s)\right)\mathbb{E}\left(\left(W(t) - W(s)\right)\right) + \mathbb{E}\left(W^2(s)\right) \\ &= s.\end{aligned}$$

Therefore,

$$\mathbb{C}\text{ov}\left(W(s), W(t)\right) = s \wedge t.$$

例 6.1 计算方差 $\mathrm{Var}\left[W(1)+W(2)+W(3)\right]$.

解:

$$
\begin{aligned}
& \mathrm{Var}\left[W(3)+W(2)+W(1)\right] \\
= \; & \mathrm{Var}\left[W(3)-W(2)+2\left(W(2)-W(1)\right)+3W(1)\right] \\
= \; & \mathrm{Var}\left[W(3)-W(2)\right]+4\mathrm{Var}\left[W(2)-W(1)\right]+9\mathrm{Var}\left[W(1)\right] \\
= \; & 1+4+9 \\
= \; & 14.
\end{aligned}
$$

1，若$W(0)=x$称之为始于x的布朗运动，记为$W^x(t)$，则

$$W^x(t)-x=W^0(t)$$

即为始于0的布朗运动，即标准布朗运动。

2，$EW_t=0$,

$$\forall s\le t,\ \operatorname{cov}(W_s,W_t)=\operatorname{cov}(W_s,W_s)+\operatorname{cov}(W_s,W_t-W_s)=s$$

$$\therefore \operatorname{cov}(W_s,W_t)=E(W_sW_t)=s\wedge t=\min(s,t)$$

3，平移不变性：　　若$\{W_t, t \geq 0\}$为布朗运动，则$\{W_{t+a} - W_a, t \geqslant 0\}$（$a$为常数）也是布朗运动

4，尺度不变性：若$\{W_t, t \geqslant 0\}$为布朗运动，则

$$\left\{\frac{W_{ct}}{\sqrt{c}}, t \geq 0\right\}, (c > 0)$$

也是布朗运动。

5，标准布朗运动$\{W_t, t \geqslant 0\}$在时刻t的概率密度函数为

$$f(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

6，布朗运动是一马尔可夫过程。

因为布朗运动是独立增量过程，所以，$B(t+s)-B(s)$ 与过程在时刻s之前的值独立。

$$\therefore \forall s,t>0$$

$$P\{B(t+s)\le a \mid B(s)=x, B(u), 0\le u<s\}$$

$$=P\{B(t+s)-B(s)\le a-x \mid B(s)=x, B(u), 0\le u<s\}$$

$$=P\{B(t+s)-B(s)\le a-x\}$$

$$=P\{B(t+s)\le a \mid B(s)=x\}$$

设$B(t)$是标准布朗运动，对任意的$t_1<t_2<\cdots<t_n$，有

$\left(B(t_1),\cdots,B(t_n)\right)$ 的联合密度函数为

$$f(x_1,x_2,\cdots,x_n)=f_{t_1}(x_1)f_{t_2-t_1}(x_2-x_1)\cdots f_{t_n-t_{n-1}}(x_n-x_{n-1})$$

其中 $$f_t(x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$$

由此可以看出 $\left(B(t_1),\cdots,B(t_n)\right)$ 服从 n 维正态分布。

这是因为在$B(t_1)=x_1$的条件下，$B(t_2)$的条件密度函数为

$$f_{B(t_2)|B(t_1)}(x_2|x_1)=\frac{1}{\sqrt{2\pi(t_2-t_1)}}e^{-\frac{(x_2-x_1)^2}{2(t_2-t_1)}}$$

$$=f_{t_2-t_1}(x_2-x_1)$$

即 $$B(t_2)\Big|_{|B(t_1)=x_1} \sim N(x_1, t_2 - t_1)$$

$$\because P(B(t_2) \le x_2 \,|B(t_1) = x_1) = P(B(t_2) - x_1 \le x_2 - x_1 \,|B(t_1) = x_1)$$

$$= P(B(t_2) - B(t_1) \le x_2 - x_1 \,|B(t_1) = x_1)$$

$$= P(B(t_2) - B(t_1) \le x_2 - x_1)$$

$$= \int_{-\infty}^{x_2 - x_1} \frac{1}{\sqrt{2\pi(t_2 - t_1)}} e^{-\frac{y^2}{2(t_2 - t_1)}} dy$$

$B(t_2)-B(t_1)$ 与 $B(t_1)$ 独立

所以 $$E[B(t_2)\,|B(t_1) = x_0] = x_0$$

$$Var\left(B(t_2)\,|B(t_1) = x_0\right) = t_2 - t_1$$

$$E[B(t_2)\,|B(t_1)] = B(t_1)$$

$$Var\left(B(t_2)\,|B(t_1)\right) = t_2 - t_1$$

布朗运动的性质

$$
\begin{aligned}
& f\left(x_1, x_2, \cdots, x_n\right) \\
& = f_{B(t_1)}(x_1) f_{B(t_2)|B(t_1)}(x_2 | x_1) f_{B(t_3)|B(t_2),B(t_1)}(x_3 | x_2, x_1) \\
& \cdots f_{B(t_n)|B(t_{n-1}),\cdots,B(t_1)}(x_n | x_{n-1}, \cdots, x_1) \\
& = f_{B(t_1)}(x_1) f_{B(t_2)|B(t_1)}(x_2 | x_1) f_{B(t_3)|B(t_2)}(x_3 | x_2) \cdots f_{B(t_n)|B(t_{n-1})}(x_n | x_{n-1}) \\
& = f_{t_1}(x_1) f_{t_2-t_1}(x_2 - x_1) \cdots f_{t_n-t_{n-1}}(x_n - x_{n-1})
\end{aligned}
$$

在$B(t_0)=x_0$的条件下，$B(t_0+t)$的条件密度函数为

$$f_{B(t_0+t)|B(t_0)}\left(x|x_0\right)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-x_0)^2}{2t}}$$

$$\therefore P\left\{B\left(t_0+t\right)>x_0\,\middle|\,B\left(t_0\right)=x_0\right\}=\int_{x_0}^{+\infty}f_{B(t_0+t)|B(t_0)}\left(x|x_0\right)dx$$

$$=P\left\{B\left(t_0+t\right)\le x_0\,\middle|\,B\left(t_0\right)=x_0\right\}=\frac{1}{2}$$

上式表明，给定初始条件$B(t_0)=x_0$，对于任意的t>0，布朗运动在t_0+t时刻的位置高于或低于初始位置的概率相等。这种性质称为布朗运动的对称性。

布朗运动的联合密度

设 $s<t$，在给定 $B(t)=x_0$ 的条件下，$B(s)$ 的条件密度函数为

$$f_{B(s)|B(t)}\left(x|x_0\right)=\frac{f_{B(s),B(t)}(x,x_0)}{f_{B(t)}(x_0)}=\frac{f_s\left(x\right)f_{t-s}\left(x_0-x\right)}{f_t\left(x_0\right)}$$

$$=\frac{1}{\sqrt{2\pi s}}\frac{1}{\sqrt{2\pi(t-s)}}\frac{1}{f_t\left(x_0\right)}e^{-\frac{x^2}{2s}-\frac{(x_0-x)^2}{2(t-s)}}$$

$$=\frac{1}{\sqrt{2\pi s}}\frac{1}{\sqrt{2\pi(t-s)}}\frac{1}{f_t\left(x_0\right)}e^{-\frac{x_0^2}{2t}}e^{-\frac{t(x-\frac{sx_0}{t})^2}{2s(t-s)}}$$

$$=\sqrt{\frac{t}{s}}\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{t(x-\frac{sx_0}{t})^2}{2s(t-s)}}=\frac{1}{\sqrt{2\pi\frac{s(t-s)}{t}}}e^{-\frac{(x-\frac{sx_0}{t})^2}{2s(t-s)/t}}$$

所以，在给定$B(t)=x_0$的条件下，$B(s)$ ($s<t$) 的条件分布为正态分布, 即

$$B(s)\Big|_{B(t)=x_0} \sim N\left(\frac{sx_0}{t}, \frac{s(t-s)}{t}\right)$$

所以

$$E[B(s)|B(t)=x_0]=\frac{sx_0}{t}$$

$$Var\left(B(s)|B(t)=x_0\right)=\frac{s(t-s)}{t}$$

$$E[B(s)|B(t)]=\frac{s}{t}B(t)$$

$$Var\left(B(s)|B(t)\right)=\frac{s(t-s)}{t}$$

注意：若$s<t$，在给定$B(s)=x_0$的条件下，$B(t)$的条件密度函数为

$$f_{B(t)|B(s)}(x|x_0)=\frac{1}{\sqrt{2\pi(t-s)}}e^{-\frac{(x-x_0)^2}{2(t-s)}}$$

$$=f_{t-s}(x-x_0)$$

即 $B(t)\big|_{B(s)=x_0}\sim N(x_0,(t-s))$

$$E[B(t)|B(s)=x_0]=x_0$$

$$Var(B(t)|B(s)=x_0)=(t-s)$$

Let $W^{(\Pi)} = \big(W(t_1), W(t_2), \ldots, W(t_m)\big)^\top$. Then $W^{(\Pi)}$ is a multi-variate normal r.v. and

$$\mathbb{V}ar\left(W^{(\Pi)}\right) = (t_i \wedge t_j) = \begin{pmatrix} t_1 & t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & t_2 & \ldots & t_2 \\ t_1 & t_2 & t_3 & \ldots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \ldots & t_m \end{pmatrix},$$

$$\mathbb{E}\left(e^{u^\top W^{(\Pi)}}\right) = e^{\frac{1}{2}u^\top \Sigma u} = e^{\frac{1}{2}\sum_{i=1}^m \left(\left(\sum_{j=i}^m u_j\right)^2 (t_i - t_{i-1})\right)}.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which a Brownian motion W is defined.

A filtration for the Brownian motion is a collection of σ-field $\{\mathcal{F}(t)\}_{t\geq 0}$ satisfying

1. (Information accumulation) For $0 \leq s < t$, $\mathcal{F}(s) \subset \mathcal{F}(t)$.
2. (Adaptivity) For each $t \geq 0$, $W(t)$ is $\mathcal{F}(t)$-measurable.
3. For $0 \leq s < t$, the increment $W(t) - W(s)$ is independent of $\mathcal{F}(s)$.

Definition (定义 6.2 条件期望)

对随机过程 $X(t)$, 记

$$\mathbb{E}_s[X(t)]$$

为 $X(t)$ 在给定到时刻 s 所有信息下的条件期望.

条件期望的性质:

- $\mathbb{E}_0[X(t)] = \mathbb{E}[X(t)]$, $\mathbb{E}_t[X(t)] = X(t)$
- 线性性: $\mathbb{E}_s[aX(t) + bY(t)] = a\mathbb{E}_s[X(t)] + b\mathbb{E}_s[Y(t)]$.
- 提取已知量: 如果 Y 给定到时刻 s 所有信息时候是已知的, 则$\mathbb{E}_s[X(t)Y] = Y\mathbb{E}_s[X(t)]$.
- 独立性:如果 Y 与给定到时刻 s 所有信息是独立的, 则$\mathbb{E}_s[Y] = \mathbb{E}[Y]$.
- 迭代法则: $\mathbb{E}\left[\mathbb{E}_s[X]\right] = \mathbb{E}\left[X\right]$.

Proposition (性质 6.1 鞅)

布朗运动 $W(t)$ 是一个鞅, 即对任意的 $0 \le s \le t$,

$$\mathbb{E}_s\left[W(t)\right] = W(s).$$

特别地,

$$\mathbb{E}\left[W(t)\right] = W(0) = 0.$$

证: 对 $0 \le s \le t$, 我们有

$$\begin{aligned}
\mathbb{E}_s\left[W(t)\right] &= \mathbb{E}_s\left[W(t) - W(s)\right] + \mathbb{E}_s\left[W(s)\right] \\
&= \mathbb{E}\left[W(t) - W(s)\right] + W(s) \\
&= 0 + W(s) \\
&= W(s).
\end{aligned}$$

一次变差

The first order variation of a function f up to time T is defined by

$$FV_T(f) = \lim_{\|\Pi\|\to 0} \sum_{i=1}^{m} \left| f(t_i) - f(t_{i-1}) \right|$$

where $\Pi = \left\{t_0, t_1, t_2, \ldots, t_{m-1}, t_m\right\}$ is a partition of $[0, T]$.

Proposition (性质 6.2 二次变差)

设 $W(t)$ 是布朗运动. 对任意的 $T \geq 0$, 其二次变差为

$$[W,W](T) := \lim_{\Delta t \to 0} \sum_{j=1}^{n} \left(W(t_j) - W(t_{j-1})\right)^2 = T, \qquad (1)$$

其中 $0 = t_0 < t_1 < \cdots < t_n = T$ 和 $\Delta t = \max_{1 \leq j \leq n}(t_j - t_{j-1})$.

(1) 的微分形式是

$$\mathrm{d}W(t) \cdot \mathrm{d}W(t) = \mathrm{d}t.$$

Proposition (性质 6.3 二次协变差)

设 $W(t)$ 为布朗运动且 $\mathrm{I}(t)=t$. 对任意的 $T\geq 0$, 二次协变差

$$[W,\mathrm{I}](T)=\lim_{\Delta t\to 0}\sum_{j=1}^{n}(W(t_j)-W(t_{j-1}))(t_j-t_{j-1})=0, \qquad (2)$$

其中 $0=t_0<t_1<\cdots<t_n=T$, $\Delta t=\max_{1\leq j\leq n}(t_j-t_{j-1})$.

(2) 的微分形式是

$$\mathrm{d}W(t)\cdot\mathrm{d}t=0$$

Proposition (性质 6.1,2,3)

总结. 我们有如下的三种微分形式:

$$\begin{cases} \mathrm{d}W(t)\cdot\mathrm{d}W(t)=\mathrm{d}t \\ \mathrm{d}W(t)\cdot\mathrm{d}t=0 \\ \mathrm{d}t\cdot\mathrm{d}t=0. \end{cases}$$

最后的微分形式 $\mathrm{d}t\cdot\mathrm{d}t=0$ 来自

$$[\mathrm{I},\mathrm{I}](T)=\lim_{\Delta t\to 0}\sum_{j=1}^{n}(t_j-t_{j-1})^2\le\lim_{\Delta t\to 0}\Delta t\cdot T=0,$$

其中 $0=t_0<t_1<\cdots<t_n=T$, $\Delta t=\max_{1\le j\le n}(t_j-t_{j-1})$.

样本路径的光滑与不光滑

大多数初等函数都是可微的，即通常都是光滑的。但Brown运动是一个极端，处处不可导。怎样来衡量一个函数的光滑程度？从中心极限定理（Donsker定理），Brown运动连续，它及其不光滑的一种，无法用笔描述出来，是因为我们的手不可能自然的抖到那种程度。那怎么来描述这种不光滑程度？

函数图像的长度： 考虑$[0,1]$上的连续函数：$f = f(t)$, 利用无穷小分析方法，考虑一任意划分：

$$\pi : 0 = t_0 < t_1 < \cdots < t_n = 1$$

按照直线最短原则，函数f的长度函数$L(f)$在分割的区间$[t_{i-1}, t_i)$有

$$L(f)[t_{i-1}, t_i) \geq \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

在$[0,1]$区间上的长度为

$$L(f) \geq \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}.$$

在[0, 1]区间上的长度定义为

$$L(f) = \sup_{\pi} \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}.$$

函数长度无限，可以想象函数本身一定很不光滑， 正常的人不可能画出一条长度无限的曲线来。由不等式

$$\begin{aligned} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| &\leq \sum_{i=1}^{n} \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2} \\ &\leq 1 + \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| \end{aligned}$$

最后一个不等式来自于两边之和大于第三边。
所以$L(f) < \infty \Leftrightarrow FV(f) < \infty$.

Outline

Definition (定义6.3 对数正态分布)

随机变量 X 称为是服从对数正态分布, 如果

$$X = \mathrm{e}^{Y} \text{ 和 } Y \sim N(\mu, \sigma^2).$$

- 记为 $X \sim \ln N(\mu, \sigma^2)$, 或等价地, $\ln X \sim N(\mu, \sigma^2)$.
- 如果 $X \sim \ln N(\mu, \sigma^2)$, 我们有

$$\mathbb{E}[X] = \mathrm{e}^{\mu + \frac{1}{2}\sigma^2}.$$

Let W be a Brownian motion and $S(0)$ be a positive constant. For $t \geq 0$, define

$$S(t) = S(0)\, e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

Let $0 \leq T_1 < T_2$ and $\Pi = \{t_0, t_1, t_2, \dots, t_{m-1}, t_m\}$ be a partition of the interval $[T_1, T_2]$. Then

$$\ln\left(\frac{S(t_i)}{S(t_{i-1})}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)(t_i - t_{i-1}) + \sigma\big(W(t_i) - W(t_{i-1})\big).$$

$$\sum_{i=1}^{m}\left(\ln\left(\frac{S(t_i)}{S(t_{i-1})}\right)\right)^2$$
$$=\left(\mu-\frac{1}{2}\sigma^2\right)^2\sum_{i=1}^{m}(t_i-t_{i-1})^2$$
$$+2\sigma\left(\mu-\frac{1}{2}\sigma^2\right)\sum_{i=1}^{m}(t_i-t_{i-1})\left(W(t_i)-W(t_{i-1})\right)$$
$$+\sigma^2\sum_{i=1}^{m}\left(W(t_i)-W(t_{i-1})\right)^2$$
$$\longrightarrow\sigma^2(T_2-T_1)\quad\text{as}\quad\|\Pi\|\to 0.$$

$$\sigma^2 \approx \underbrace{\frac{1}{T_2 - T_1}\sum_{i=1}^{m}\left(\ln\left(\frac{S(t_i)}{S(t_{i-1})}\right)\right)^2}_{\text{Realized volatility}}.$$

Definition (定义6.4 对数正态股票价格模型)

对数正态股票价格模型, 也称为几何布朗模型 (GBM), 是

$$S(t) = S(0)\mathrm{e}^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma W(t)},$$

其中 α 是股票的增值/升值比率, δ 是股息, σ 是波动率.

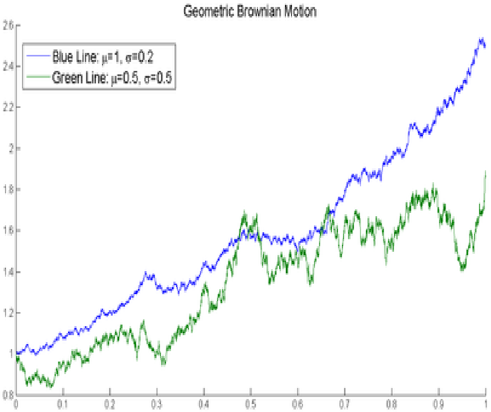
- 对固定的 t,

$$S(t) \sim S(0)\ln N\left((\alpha-\delta-\frac{1}{2}\sigma^2)t, \sigma^2 t\right)$$

- 对固定的 t,

$$\mathbb{E}\left[S(t)\right] = S(0)\mathrm{e}^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\frac{1}{2}\sigma^2 t} = S(0)\mathrm{e}^{(\alpha-\delta)t}.$$

几何布朗运动的样本路径:



例6.2 (GBM模型) 设股票价格服从对数正态分布, $\alpha = 3\%$, $\delta = 0$ 和$\sigma = 20\%$. 求$\mathrm{Var}\left[\ln\sqrt{S(1)S(2)}\right]$.

解: 由 $S(t) = S(0)e^{(0.03-\frac{1}{2}0.2^2)t+0.2W(t)} = S(0)e^{0.01t+0.2W(t)}$, 有

$$\sqrt{S(1)S(2)} = \sqrt{S^2(0)e^{0.01+0.2W(1)+0.02+0.2W(2)}} = S(0)e^{0.015+0.1W(1)+0.1W(2)},$$

这意味着

$$\ln\sqrt{S(1)S(2)} = \ln S(0) + 0.015 + 0.1W(1) + 0.1W(2).$$

因此,

$$\begin{aligned}\mathrm{Var}\left[\ln\sqrt{S(1)S(2)}\right] &= 0.1^2\mathrm{Var}\left[W(1)+W(2)\right] \\ &= 0.01\mathrm{Var}\left[2W(1)+W(2)-W(1)\right] \\ &= 0.01\,(4+1) \\ &= 0.05\end{aligned}$$

例6.3 (GBM模型) 设股票价格 $S(t)$ 服从对数正态分布 $S(0)=100$, $\alpha=10\%$, $\delta=0$ 和 $\sigma=30\%$. 求 $S(2)$ 95% 的置信区间.

解: 寻找 U 和 L 使得

$$\mathbb{P}\{S(2)<U\}=\mathbb{P}\{S(2)>L\}=0.975.$$

由 $S(2)=100e^{0.11+0.3W(2)}\sim 100\ln N(0.11,0.18)$, 我们有

$$\mathbb{P}\{S(2)<U\}=\mathbb{P}\left\{\ln\frac{S(2)}{100}<\ln\frac{U}{100}\right\}=\mathbb{P}\left\{N(0.11,0.18)<\ln\frac{U}{100}\right\}$$

$$=\mathbb{P}\left\{N(0,1)<\frac{\ln\frac{U}{100}-0.11}{\sqrt{0.18}}\right\}=0.975.$$

由正态分布表, $\frac{\ln\frac{U}{100}-0.11}{\sqrt{0.18}}=\Phi^{-1}(0.975)=1.96$, 有

$$U=256.3972.$$

类似地, $L=48.5995$.

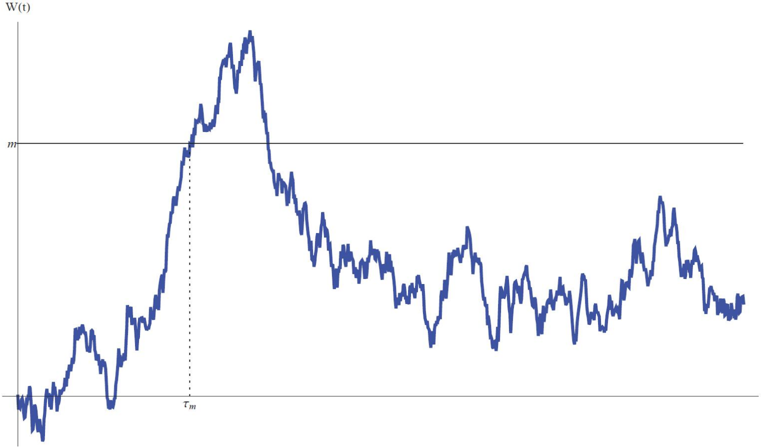
Outline

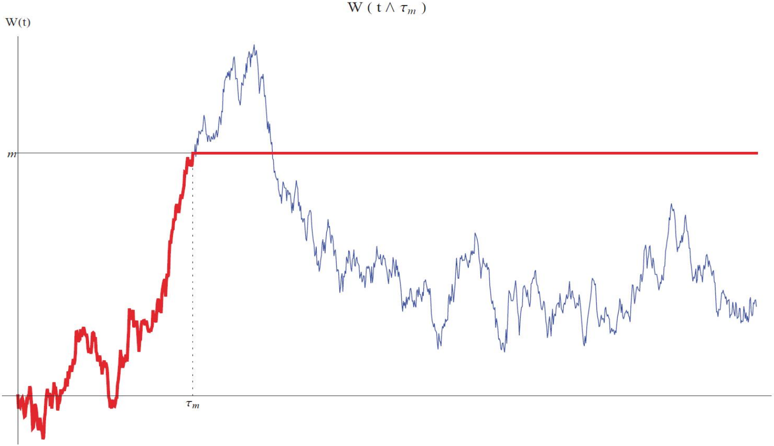
§6.3 First passage time

Let W be a Brownian motion and m be a constant. The first passage time to level m is

$$\tau_m = \min\left\{t \geq 0 \mid W(t) = m\right\}.$$

If W never reaches the level m, set $\tau_m = \infty$.


W(t)
m
τ_m


W (t ∧ τ_m)
W(t)
m
τ_m

Let W be a Brownian motion and, for $t \geq 0$, define the exponential process

$$Z(t) = e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}.$$

1. Show that the process $\{Z(t)\}_{t\geq 0}$ is a martingale.
2. Show that the stopped process $\{Z(t \wedge \tau_m)\}_{t\geq 0}$ is a martingale.

1. For $0 \le s \le t$, show that

$$\mathbb{E}(Z(t) \mid \mathcal{F}(s)) = e^{-\frac{1}{2}\sigma^2 t + \sigma W(s)}\, \mathbb{E}\left(e^{\sigma(W(t)-W(s))} \right) = Z(s).$$

2. Let $0 \le s \le t$. Consider a partition $\Pi = \{t_0, t_1, \ldots, t_n\}$ of $[0, t]$ such that it contains all those values $s \wedge \tau_m$ and $t \wedge \tau_m$ can take. First show that

$$Z(t \wedge \tau_m) - Z(s \wedge \tau_m) = \sum_{i=1}^{n} \mathbf{1}_{s\wedge\tau_m < t_i \le t\wedge\tau_m} \left(Z(t_i) - Z(t_{i-1}) \right)$$

and

$$\{s \wedge \tau_m < t_i \le t \wedge \tau_m\} \in \mathcal{F}(t_{i-1})$$

for $i = 1, 2, \ldots, n$.

2. Then show that

$$
\begin{aligned}
&\mathbb{E}\left(\mathbf{1}_A\big(Z(t\wedge\tau_m)-Z(s\wedge\tau_m)\big)\right)\\
&\quad=\sum_{i=1}^{n}\mathbb{E}\left(\mathbf{1}_A\,\mathbf{1}_{s\wedge\tau_m<t_i\le t\wedge\tau_m}\big(Z(t_i)-Z(t_{i-1})\big)\right)\\
&\quad=\sum_{i=1}^{n}\mathbb{E}\left(\mathbf{1}_A\,\mathbf{1}_{s\wedge\tau_m<t_i\le t\wedge\tau_m}\,\mathbb{E}\big(Z(t_i)-Z(t_{i-1})\,|\,\mathcal{F}(t_{i-1})\big)\right)\\
&\quad=0,
\end{aligned}
$$

for any $A\in\mathcal{F}(s)$.

The first passage time of Brownian motion to level $m \in \mathbb{R}$ is finite almost surely and the Laplace transform of its distribution is

$$\mathbb{E}\left(e^{-\alpha\tau_m}\right) = e^{-|m|\sqrt{2\alpha}}$$

for all $\alpha > 0$.

分布证明:

Let $\sigma > 0$ and $m > 0$. Consider the martingale

$$Z(t \wedge \tau_m) = e^{-\frac{1}{2}\sigma^2 (t\wedge\tau_m) + \sigma W(t\wedge\tau_m)}.$$

Then

$$0 \leq Z(t \wedge \tau_m) \leq e^{\sigma m}$$

and

$$\lim_{t\to\infty} Z(t \wedge \tau_m) = \mathbf{1}_{\tau_m < \infty} e^{-\frac{1}{2}\sigma^2 \tau_m + \sigma m}.$$

分布证明:

Since $\mathbb{E}(Z(t\wedge\tau_m)) = Z(0) = 1$, taking $t\to\infty$ and applying the Dominated Convergence Theorem gives

$$\mathbb{E}\left(\mathbf{1}_{\tau_m<\infty}\, e^{-\frac{1}{2}\sigma^2\tau_m}\right) = e^{-\sigma m}.$$

Taking $\sigma\to 0^+$ gives

$$\mathbb{P}(\tau_m<\infty) = 1.$$

Hence,

$$\mathbb{E}\left(e^{-\frac{1}{2}\sigma^2\tau_m}\right) = e^{-\sigma m},$$

for $\sigma>0$ and $m>0$.

分布证明续:

For $m > 0$, set $\sigma = \sqrt{2\alpha}$ to get

$$\mathbb{E}\left(e^{-\alpha\tau_m}\right) = e^{-m\sqrt{2\alpha}}.$$

For $m < 0$, note that $\tau_m \overset{\mathcal{D}}{=} \tau_{|m|}$, thus

$$\mathbb{E}\left(e^{-\alpha\tau_m}\right) = \mathbb{E}\left(e^{-\alpha\tau_{|m|}}\right) = e^{-|m|\sqrt{2\alpha}}.$$

For $m = 0$, the result follows since $\tau_m = 0$. Hence,

$$\mathbb{E}\left(e^{-\alpha\tau_m}\right) = e^{-|m|\sqrt{2\alpha}}$$

for all $\alpha > 0$.

分布证明续:

Although $\tau_m < \infty$ a.s.,

$$\mathbb{E}(\tau_m) = \infty$$

if $m \neq 0$.

镜面发射原理:

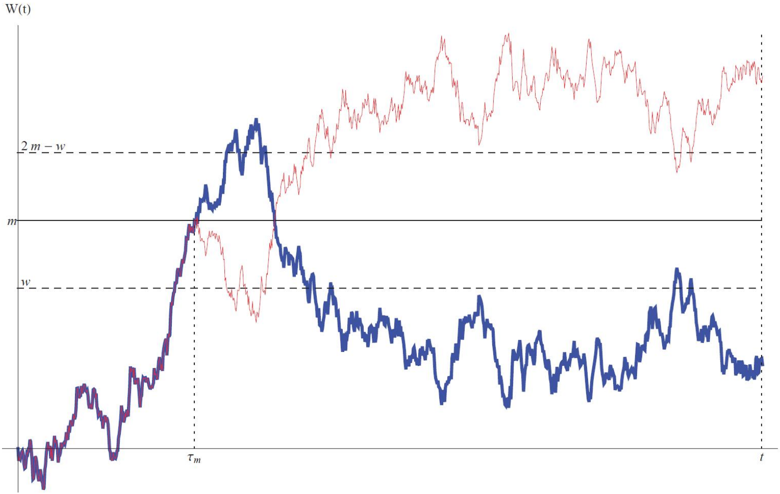
For a fixed positive level m and time t, there are two possibilities for $\tau_m \leq t$:

1. $\tau_m \leq t$ and $W(t) \leq w$;
2. $\tau_m \leq t$ and $W(t) \geq w$;

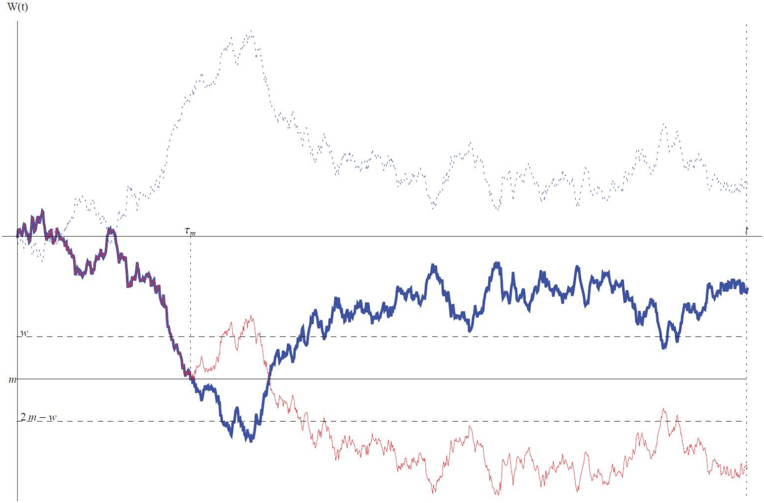
where $w \leq m$.

Note that for the first case each $\left(\tau_m \leq t, W(t) \leq w\right)$ path has a reflected $\left(W(t) \geq 2m - w\right)$ path.

反射原理示意图:



反射原理示意图:



For $m > 0$ and $w \leq m$,

$$\mathbb{P}\left(\tau_m \leq t, W(t) \leq w\right) = \mathbb{P}\left(W(t) \geq 2m - w\right).$$

For $m < 0$ and $w \geq m$,

$$\mathbb{P}\left(\tau_m \leq t, W(t) \geq w\right) = \mathbb{P}\left(W(t) \leq 2m - w\right).$$

For all $m \neq 0$, the random variable τ_m has the distribution function

$$F_{\tau_m}(t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} e^{-\frac{1}{2}z^2}\, dz$$

and the density function

$$f_{\tau_m}(t) = \frac{|m|}{\sqrt{2\pi t^3}}\, e^{-\frac{m^2}{2t}}$$

for all $t \geq 0$.

证明:

Suppose $m > 0$. Then by the reflection formula

$$\mathbb{P}\left(\tau_m \leq t, W(t) \leq m\right) = \mathbb{P}\left(W(t) \geq m\right).$$

In addition,

$$\mathbb{P}\left(\tau_m \leq t, W(t) \geq m\right) = \mathbb{P}\left(W(t) \geq m\right).$$

Therefore,

$$\mathbb{P}\left(\tau_m \leq t\right) = 2\,\mathbb{P}\left(W(t) \geq m\right) = 2\int_{\frac{m}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz.$$

证明:

Suppose $m < 0$. Then by the reflection formula

$$\mathbb{P}\left(\tau_m \le t, W(t) \ge m\right) = \mathbb{P}\left(W(t) \le m\right)$$

In addition,

$$\mathbb{P}\left(\tau_m \le t, W(t) \le m\right) = \mathbb{P}\left(W(t) \le m\right).$$

Therefore,

$$\mathbb{P}\left(\tau_m \le t\right) = 2\,\mathbb{P}\left(W(t) \le m\right) = 2\int_{-\infty}^{\frac{m}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

$$= 2\int_{\frac{|m|}{\sqrt{t}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz.$$

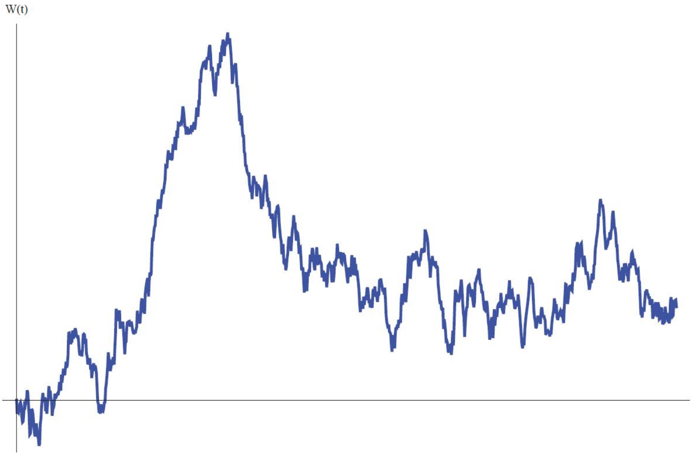
Brown运动的最大值过程:

The maximum to date for Brownian motion W is

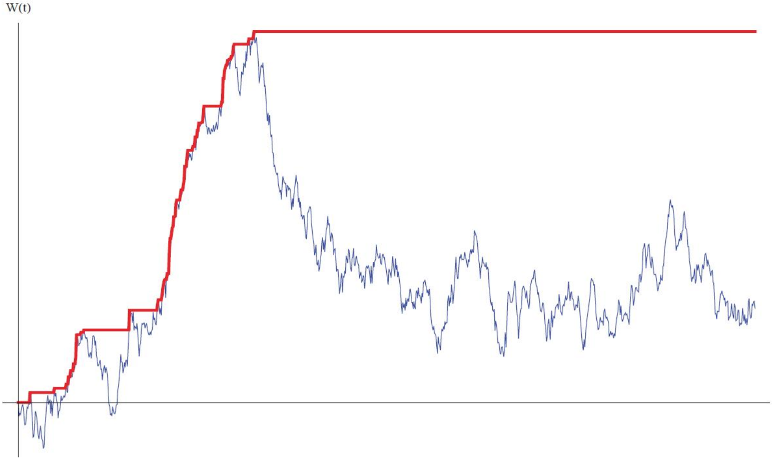
$$M(t) = \max_{0 \le s \le t} W(s).$$

Note that $M(t) \geq 0$ and $M(t) \geq W(t)$ for all $t \geq 0$.

Brown运动时序图:



Brown运动最大值序列时序图:



反射原理:

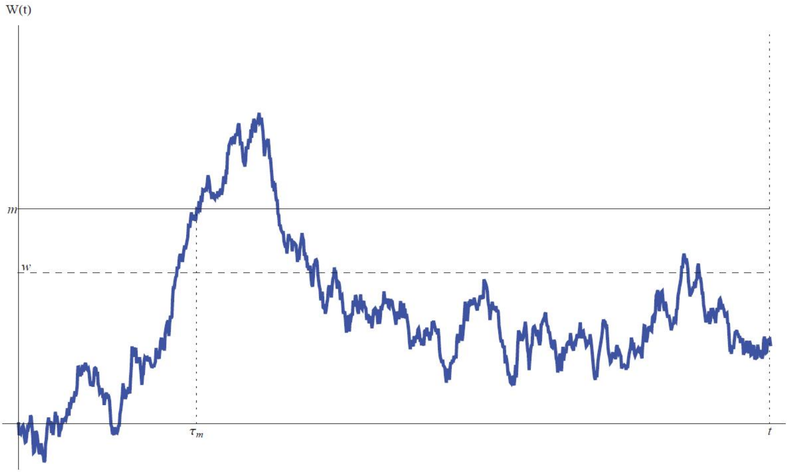
For $m > 0$, we have

$$M(t) \geq m \iff \tau_m \leq t.$$

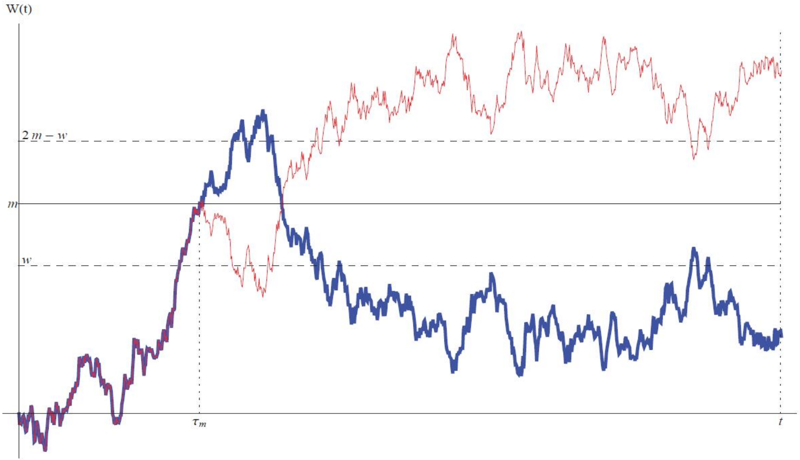
The reflection formula for $m > 0$ and $w \leq m$ becomes

$$\mathbb{P}\left(M(t) \geq m, W(t) \leq w\right) = \mathbb{P}\left(W(t) \geq 2m - w\right).$$

反射原理:



反射原理:



分布:

For $t > 0$, the joint density function of $\big(M(t), W(t)\big)$ is

$$f_{M(t),W(t)}(m, w) = \frac{2(2m - w)}{\sqrt{2\pi t^3}} e^{-\frac{(2m-w)^2}{2t}}$$

for $m > 0$ and $m \geq w$, and 0 otherwise; and the conditional distribution of $M(t)$ given $W(t) = w$ is

$$f_{M(t)|W(t)}(m|w) = \frac{2(2m - w)}{t} e^{-\frac{2m(m-w)}{t}}$$

for $m > 0$ and $m \geq w$, and 0 otherwise.

证明:

Since $\mathbb{P}\left(M(t) \geq m, W(t) \leq w\right) = \mathbb{P}\left(W(t) \geq 2m - w\right)$,

$$\int_m^\infty \int_{-\infty}^w f_{M(t),W(t)}(x,y)\,dy\,dx = \int_{2m-w}^\infty \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}}\,dz.$$

Differentiate w.r.t. m,

$$\int_{-\infty}^w f_{M(t),W(t)}(m,y)\,dy = \frac{2}{\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}.$$

Differentiate w.r.t. w,

$$f_{M(t),W(t)}(m,w) = \frac{2(2m-w)}{\sqrt{2\pi t^3}} e^{-\frac{(2m-w)^2}{2t}}.$$

证明续:

Therefore,

$$
\begin{aligned}
f_{M(t)|W(t)}(m|w) &= \frac{f_{M(t),W(t)}(m,w)}{f_{W(t)}(w)} \\
&= \frac{2(2m-w)}{t} e^{-\frac{2m(m-w)}{t}}.
\end{aligned}
$$