金融随机分析

第7章 随机积分

陈昱

cyu@ustc.edu.cn

管理科研楼 1003

63602243

2019 年 5 月

第七章 随机积分

Definition (定义 7.1 Riemann 积分)

设$f(x)$为定义在[0,1]区间上的实值函数，考虑分割

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_n = 1$$

和 $mesh(\tau_n) = \max_{1 \leq j \leq n}(t_j - t_{j-1})$. 当$mesh(\tau_n) \to 0$ 时，如果下面的极限对某个拆分σ_n(由满足$t_{i-1} \leq y_i \leq t_i$的任意取值$y_i$给出的)

$$S_n = \sum_{j=1}^{n} f(y_j)\,(t_j) - t_{j-1}))\,,$$

存在，记为S且与分割和拆分σ_n都无关，就称S为f在[0,1]区间上的Riemann积分. 记为

$$S = \int_0^1 f(t)dt.$$

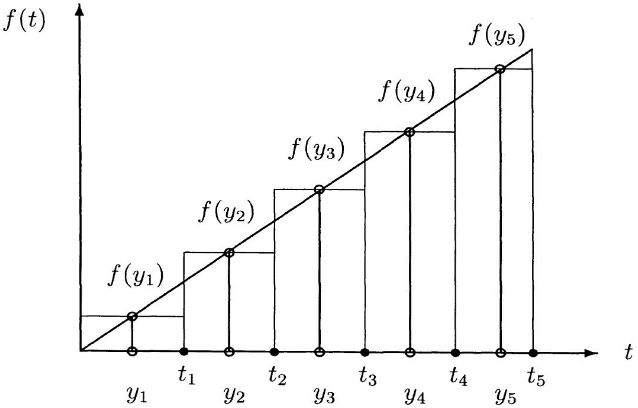

Figure 2.1.1 *An illustration of a Riemann sum with partition* (t_i) *and intermediate partition* (y_i). *The sum of the rectangular areas approximates the area between the graph of* $f(t)$ *and the* t*-axis, i.e.* $\int_0^{t_5} f(t)\,dt$.

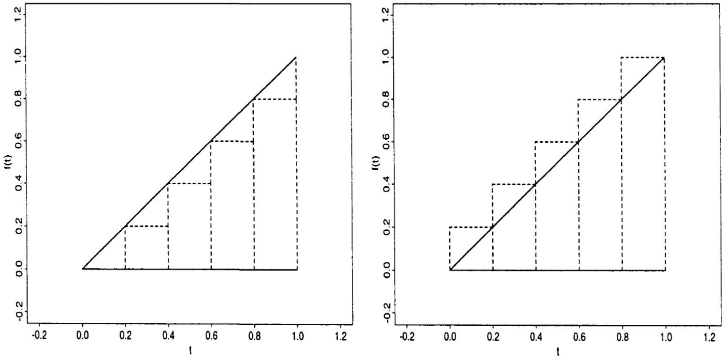

Figure 2.1.2 *Two Riemann sums for the integral* $\int_0^1 t\,dt$*; see Example 2.1.3. The partition is given by* $t_i = i/5$, $i = 0, \ldots, 5$. *The left* (left figure) *and the right* (right figure) *end points of the intervals* $[(i-1)/5, i/5]$ *are taken as points* y_i *of the intermediate partition.*

Riemann积分最为一个基本模型，一个新的积分有类似的性质．下面我们列举性质如下．对[0,1]上Riemann可积函数f, f_1, f_2,

1. Riemann积分的线性性，对任意的常数c_1, c_2有

$$\int_0^1 [c_1 f_1(t) + c_2 f_2(t)]dt = c_1 \int_0^1 f_1(t)dt + c_2 \int_0^1 f_2(t)dt.$$

2. Riemann积分在相邻区间的线性可加性:

$$\int_0^1 f(t)dt = \int_0^a f(t)dt + \int_a^1 f(t)dt, 0 \le a \le 1.$$

3. Riemann积分不定积分定义为积分上限函数

$$\int_0^s f(t)dt = \int_0^1 f(t) I_{[0,s]}(t)dt.$$

Definition (定义 7.2 Riemann-Stieltjes积分)

设$f(x)$和$g(x)$为定义在[0,1]区间上的实值函数，考虑分割

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_n = 1$$

和 $mesh(\tau_n) = \max_{1\leq j\leq n}(t_j - t_{j-1})$. 当$mesh(\tau_n) \to 0$ 时，如果对某个拆分σ_n(由满足$t_{i-1} \leq y_i \leq t_i$的任意取值$y_i$给出的)

$$S =: \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=1}^{n} f(y_j)\left(g(t_j) - g(t_{j-1})\right),$$

存在，记为S且与分割和拆分σ_n都无关，就称S为f关于g在[0,1]区间上的Riemann积分. 记为

$$S = \int_0^1 f(t)\mathrm{d}g(t).$$

我们给出Riemann-Stieltjes积分$\int_0^1 f(t)\mathrm{d}g(t)$可积性的一个充分条件，非常接近必要性条件.

- 函数f和g在同一点不能都不连续.
- 假设f是连续的，g有有界变差,即

$$\sup_{\tau} \sum_{j=1}^{n} |g(t_j) - g(t_{j-1})| < \infty,$$

其中的上确界是对所有的分割τ取得.

众所周知$B(t)$是不存在有限的有界变差，那如何定义积分$\int_0^1 f(t)dB(t)$?

Riemann-Stieltjes积分$\int_0^1 f(t)\mathrm{d}g(t)$ 什么时候存在? 如果以下两个条件之一存在则$\int_0^1 f(t)\mathrm{d}g(t)$存在:

Definition (定义 有界p变差)

存在[0,1]区间上的实值函数h对某个$p>0$满足条件,

$$\sup_{\tau}\sum_{j=1}^{n}|h(t_j)-h(t_{j-1})|^p<\infty,$$

其中的上确界是对所有的分割τ取得, 称h有有界p变差 $p=1$, h有有界变差

Riemann-Stieltjes积分 $\int_0^1 f(t)\mathrm{d}g(t)$ 什么时候存在?

Definition

如果以下两个条件满足, Riemann-Stieltjes积分 $\int_0^1 f(t)\mathrm{d}g(t)$ 存在

- 函数f和g在同一点不能都不连续(在区间上不同有相同的不连续点)
- 存在常数$p>0$和$q>0$满足$p^{-1}+q^{-1}>1$, 使得函数f有有界p变差, 函数g有有界q变差.

Definition (定义 7.2 简单过程的Ito积分)

如果随机过程$C = \{C_t, t \in [0, T]\}$满足：存在分割

$$\tau_n : 0 = t_0 < t_1 < \cdots < t_n = T$$

和一个适应于$\{\mathcal{F}_{t_{i-1}}, i = 1, 2, \cdots, n\}$的随机变量序列$\{Z_i, 1 \le i \le n\}$，对所有的$i$，$EZ_i^2 < \infty$，使得

$$C_t = \begin{cases} Z_n, & t = T \\ Z_i, & t_{i-1} \le t < t_i, i = 1, 2, \ldots, n \end{cases}$$

则过程$C = \{C_t, t \in [0, T]\}$为简单过程.

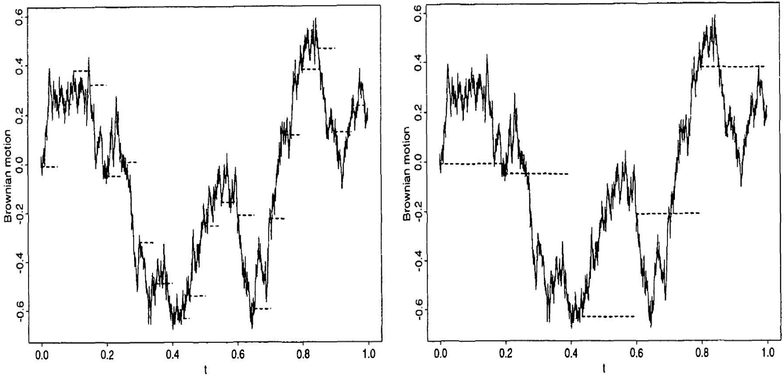

Figure 2.2.2 *Two approximations of a Brownian sample path by simple processes* C *(dashed lines), given by* (2.10).

Definition (定义 7.2 简单过程的Ito积分)

如果随机过程$C=\{C_t, t\in[0,T]\}$为简单过程，它的Ito随机积分由下式给出

$$\int_0^T C_s dB_s=\sum_{i=1}^n C_{t_{i-1}}(B_{t_i}-B_{t_{i-1}})=\sum_{i=1}^n Z_i\Delta_i B.$$

设$t_{k-1}\le t\le t_k$，在区间$[0,T]$上，简单过程C的随机积分定义为

$$\int_0^t C_s \mathrm{d}B_s:=\int_0^T C_s I_{[0,t]}(s)dB_s=\sum_{i=1}^{k-1} Z_i(B_{t_i}-B_{t_{i-1}})+Z_k(B_t-B_{t_{k-1}})$$

记为$I_t(C)=\int_0^t C_s \mathrm{d}B_s$.

其实$\{\int_0^{t_k} C_s \mathrm{d}B_s,\}$是一个关于$\sigma$域流$(\mathcal{F}_{t_k}, k=0,\ldots,n)$的鞅变换.

简单过程Ito积分示意图

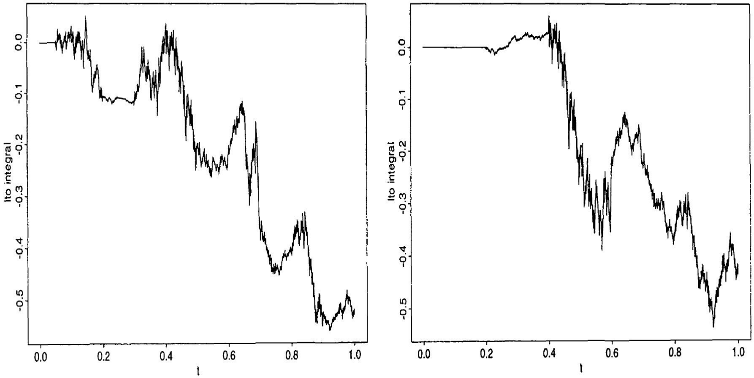

Figure 2.2.3 *The Itô stochastic integral* $\int_0^t C_s\,dB_s$ *corresponding to the paths of* C *and* B *given in Figure 2.2.2.*

- $I_t(C)=\int_0^t C_s \mathrm{d}B_s$是关于Brown运动的自然$\sigma$域流$(\mathcal{F}_t, t\in[0,T])$的鞅.
- $EI_t(C)=0$.
- 满足等距性质:

$$E\Big(\int_0^t C_s \mathrm{d}B_s\Big)^2=\int_0^t EC_s^2 \mathrm{d}s, t\in[0,T].$$

- 积分线性性质: 对任意的常数c_1, c_2和$[0,T]$上的简单过程C_1和C_2有

$$\int_0^t [c_1C_1(s)+c_2C_2(s)]\mathrm{d}B_s=c_1\int_0^t C_1(s)\mathrm{d}B_s+c_2\int_0^t C_2(s)\mathrm{d}B_s.$$

- Ito积分在相邻区间的线性可加性: $t\in[0,T]$

$$\int_0^T C_s\mathrm{d}B_s=\int_0^t C_s\mathrm{d}B_s+\int_t^T C_s\mathrm{d}B_s.$$

- $[I_t, I_t](t) = \int_0^t C_s^2 \mathrm{d}B_s$

Definition (定义 7.5 Ito 积分)

设适应过程C_t满足

$$E\int_0^T C_s^2 \mathrm{d}t < \infty,$$

选择简单过程$C_n^2(t)$满足

$$\lim_{n\to\infty} E\int_0^T |C(s) - C_n(s)|^2 \mathrm{d}t = 0.$$

则一般的随机过程$C(t)$的Ito 积分 定义为

$$\int_0^T C(t)\mathrm{d}B(t) = \lim_{n\to\infty}\int_0^T C(t)\mathrm{d}B(t),$$

其中 $C(t) : t \in [0, T]$ 是一个适应过程.

Definition (定义 7.5 Ito 积分)

设 $0 = t_0 < t_1 < \cdots < t_n = T$ 和 $\Delta t = \max_{1\le j\le n}(t_j - t_{j-1})$. Ito 积分 定义为

$$\int_0^T X(t)\mathrm{d}W(t) = \lim_{\Delta t\to 0}\sum_{j=1}^{n} X(t_{j-1})\left(W(t_j) - W(t_{j-1})\right),$$

其中 $X(t,\omega) : [0,T]\times\Omega \to \mathbb{R}$ 是一个适应过程.

Proposition (性质 7.4)

$$\mathbb{E}\left[\int_0^T X(t)\mathrm{d}W(t)\right] = 0$$

和

$$\mathbb{E}\left[\left(\int_0^T X(t)\mathrm{d}W(t)\right)^2\right] = \int_0^T \mathbb{E}\left[X^2(t)\right]\mathrm{d}t.$$

例 7.4 (Ito 积分) 特别地，我们有

$$\mathbb{E}\left[\int_0^T W(t)\mathrm{d}W(t)\right] = 0$$

和

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T W(t)\mathrm{d}W(t)\right)^2\right] &= \int_0^T \mathbb{E}\left[W^2(t)\right]\mathrm{d}t \\ &= \int_0^T \mathrm{Var}\left[W(t)\right]\mathrm{d}t \\ &= \int_0^T t\mathrm{d}t \\ &= \frac{1}{2}T^2.\end{aligned}$$

Fix $T > 0$. Let h be an adapted and square integrable process. Then the Itô integral $I(t) = \int_0^t h(u)\,dW(u)$ as defined earlier has the following properties:

1. (Continuity) Paths of $I(t)$, as a function of t, are continuous.
2. (Adaptivity) $I(t)$ is $\mathcal{F}(t)$-measurable for all t.
3. (Linearity) Itô integral of a linear combination is the linear combination of Itô integrals.
4. (Martingale) Itô integral is a martingale.
5. (Itô isometry) $\mathbb{E}\left(I^2(t)\right) = \mathbb{E}\left(\int_0^t h^2(u)\,du\right)$ for all $t \geq 0$.
6. (Quadratic variation) $[I, I](t) = \int_0^t h^2(u)\,du$ for all $t \geq 0$.

Compute

$$\int_0^T W(u)\,dW(u).$$

Hint: Let $n \in \mathbb{Z}^+$ and $t_i = \frac{T}{n} i$, for $i = 0, 1, 2, \ldots, n$. Consider the simple process

$$h_n(t) = \sum_{i=1}^{n} W(t_{i-1})\,\mathbf{1}_{[t_{i-1}, t_i)}(t).$$

$$
\begin{aligned}
\int_0^T W(u)\,dW(u) &= \lim_{n\to\infty}\int_0^T h_n(u)\,dW(u) \\
&= \lim_{n\to\infty}\sum_{i=1}^{n} W(t_{i-1})\Big(W(t_i)-W(t_{i-1})\Big) \\
&= \lim_{n\to\infty}\frac{1}{2}\left(W^2(T)-\sum_{i=1}^{n}\Big(W(t_i)-W(t_{i-1})\Big)^2\right) \\
&= \frac{1}{2}\Big(W^2(T)-[W,W](T)\Big) \\
&= \frac{1}{2}\Big(W^2(T)-T\Big).
\end{aligned}
$$

Ito-Doeblin公式

Let $f(t,x)$ be a $C^{1,2}$ function and W be a Brownian motion. Then, for every $T \geq 0$,

$$
\begin{aligned}
f(T, W(T)) =& f(0, W(0)) + \int_0^T f_t(t, W(t))\, dt \\
&+ \underbrace{\int_0^T f_x(t, W(t))\, dW(t)}_{\text{Itô integral}} + \frac{1}{2}\int_0^T f_{xx}(t, W(t))\, dt;
\end{aligned}
$$

and in differential form

$$
\begin{aligned}
df(T, W(T)) =& f_t(T, W(T))\, dT \\
&+ f_x(T, W(T))\, dW(T) + \frac{1}{2} f_{xx}(T, W(T))\, dT.
\end{aligned}
$$

证明

Let $\Pi = \{t_0, t_1, t_2, \ldots, t_n\}$ be a partition of $[0, T]$. Then

$$f(T, W(T)) - f(0, W(0)) = \sum_{i=1}^{n} \Big(f(t_i, W(t_i)) - f(t_{i-1}, W(t_{i-1})) \Big).$$

By Taylor's Theorem,

$$\begin{aligned}
& f(t_i, W(t_i)) - f(t_{i-1}, W(t_{i-1})) \\
&\quad = f_t(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1}) \\
&\qquad + f_x(t_{i-1}, W(t_{i-1}))(W(t_i) - W(t_{i-1})) \\
&\qquad + \frac{1}{2} f_{tt}(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1})^2 \\
&\qquad + f_{tx}(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1})(W(t_i) - W(t_{i-1})) \\
&\qquad + \frac{1}{2} f_{xx}(t_{i-1}, W(t_{i-1}))(W(t_i) - W(t_{i-1}))^2 + o(\text{higher order}).
\end{aligned}$$

证明续:

Since

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n} f_t(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1}) = \int_0^T f_t(t, W(t))\,dt,$$

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n} f_x(t_{i-1}, W(t_{i-1}))(W(t_i) - W(t_{i-1})) = \int_0^T f_x(t, W(t))\,dW(t),$$

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n} f_{tt}(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1})^2 = 0,$$

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n} f_{tx}(t_{i-1}, W(t_{i-1}))(t_i - t_{i-1})(W(t_i) - W(t_{i-1})) = 0,$$

$$\lim_{\|\Pi\|\to 0}\sum_{i=1}^{n} f_{xx}(t_{i-1}, W(t_{i-1}))(W(t_i) - W(t_{i-1}))^2 = \int_0^T f_{xx}(t, W(t))\,dt,$$

we have the Itô-Doeblin formula.

Ito-Doeblin公式例子

Let $f(x) = \frac{1}{2}x^2$. Then

$$\begin{aligned}
\frac{1}{2}W^2(T) &= f(W(T)) - f(W(0)) \\
&= \int_0^T \underbrace{f_x(W(t))}_{W(t)} dW(t) + \frac{1}{2}\int_0^T \underbrace{f_{xx}(W(t))}_{1} dt \\
&= \int_0^T W(t)\, dW(t) + \frac{1}{2}T.
\end{aligned}$$

Ito-Doeblin公式例子

Let s, r and σ be positive constants, and $\left(W(t)\right)_{t\geq 0}$ be a Brownian motion. Define

$$f\left(t, W(t)\right) = s\ \mathrm{e}^{(r-\frac{1}{2}\sigma^2)t+\sigma W(t)}, \quad \text{for } t \geq 0.$$

Show that

$$f\left(t, W(t)\right) = s + \int_0^t r\, f\left(u, W(u)\right) du + \int_0^t \sigma\, f\left(u, W(u)\right) dW(u).$$

Ito-Doeblin公式例子

Consider $f(t,x) = s\, e^{(r-\frac{1}{2}\sigma^2)t+\sigma x} \in C^{1,2}$. Since

$$f_t(t,x) = \left(r - \tfrac{1}{2}\sigma^2\right) f(t,x),$$

$$f_x(t,x) = \sigma f(t,x), \quad \text{and} \quad f_{xx}(t,x) = \sigma^2 f(t,x),$$

the Itô-Doeblin formula says

$$\begin{aligned} & df(u, W(u)) \\ & = \underbrace{\left(r - \tfrac{1}{2}\sigma^2\right) f(u, W(u))}_{=f_t(u,W(u))} du + \underbrace{\sigma f(u, W(u))}_{=f_x(u,W(u))} dW(u) + \frac{1}{2} \underbrace{\sigma^2 f(u, W(u))}_{=f_{xx}(u,W(u))} du \\ & = r f(u, W(u))\, du + \sigma f(u, W(u))\, dW(u). \end{aligned}$$

Definition (定义7.5 Ito 过程)

随机过程 $X(t)$ 称为 *Ito 过程*, 如果

$$X(t) = X(0) + \int_0^t \mu(s)\mathrm{d}s + \int_0^t \sigma(s)\mathrm{d}W(s),$$

其中 $\mu(s)$ 和 $\sigma(s)$ 是两个适应过程.

$X(t)$ 的*随机微分方程 (SDE)* 是

$$\mathrm{d}X(t) = \mu(t)\mathrm{d}t + \sigma(t)\mathrm{d}W(t).$$

- e.g. 线性布朗运动 $\mathrm{d}X(t) = \mu\mathrm{d}t + \sigma\mathrm{d}W(t)$
- e.g. 几何布朗运动 $\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}W(t)$
- e.g. 扩散过程 $\mathrm{d}X(t) = \mu(t, X(t))\mathrm{d}t + \sigma(t, X(t))\mathrm{d}W(t)$

Ito过程的随机积分

Let X be an Itô process where

$$X(t) = X(0) + \int_0^t g(u)\,du + \int_0^t h(u)\,dW(u)$$

and Γ be an adapted process. Define the integral with respect to an Itô process

$$\int_0^t \Gamma(u)\,dX(u) = \underbrace{\int_0^t \Gamma(u)g(u)\,du + \int_0^t \Gamma(u)h(u)\,dW(u)}_{\text{Itô process}}.$$

Ito过程的随机积分Ito-Doeblin公式

Let $f(t,x)$ be a $C^{1,2}$ function, W be a Brownian motion and X be an Itô process where

$$X(t) = X(0) + \int_0^t g(u)\,du + \int_0^t h(u)\,dW(u).$$

Then for every $T \geq 0$

$$\begin{aligned} f(T,X(T)) &= f(0,X(0)) + \int_0^T f_t(t,X(t))\,dt \\ &\quad + \int_0^T f_x(t,X(t))\,dX(t) + \frac{1}{2}\int_0^T f_{xx}(t,X(t))\,d[X,X](t) \\ &= f(0,X(0)) + \int_0^T f_t(t,X(t))\,dt + \int_0^T f_x(t,X(t))g(t)\,dt \\ &\quad + \int_0^T f_x(t,X(t))h(t)\,dW(t) + \frac{1}{2}\int_0^T f_{xx}(t,X(t))h^2(t)\,dt. \end{aligned}$$

Ito过程的随机积分的微分表示

In differential form,

$$dX(T) = g(T)\,dT + h(T)\,dW(T)$$

and

$$\begin{aligned} df(T, X(T)) &= f_t(T, X(T))\,dT \\ &\quad + f_x(T, X(T))\,dX(T) + \frac{1}{2} f_{xx}(T, X(T))\,d[X, X](T) \\ &= f_t(T, X(T))\,dT + f_x(T, X(T)) g(T)\,dT \\ &\quad + f_x(T, X(T)) h(T)\,dW(T) + \frac{1}{2} f_{xx}(T, X(T)) h^2(T)\,dT. \end{aligned}$$

关于Ito过程的随机积分分部积分

Let X and Y be Itô processes. Then for every $T \geq 0$

$$X(T)Y(T) = X(0)Y(0) + \int_0^T X(t)\,dY(t) + \int_0^T Y(t)\,dX(t) + [X,Y](T).$$

In particular,

$$X^2(T) = X^2(0) + 2\int_0^T X(t)\,dX(t) + [X,X](T).$$

In differential form,

$$d\Big(X(T)Y(T)\Big) = X(T)\,dY(T) + Y(T)\,dX(t) + dX(T)\,dY(T)$$

and

$$d\Big(X^2(T)\Big) = 2X(T)\,dX(T) + dX(T)\,dX(T)$$

Using the Itô-Doeblin formula for $f(x) = x^2$, we have

$$X^2(T) = X^2(0) + 2\int_0^T X(t)\,dX(t) + [X, X](T).$$

The integration by parts formula then follows from

$$X(T)Y(T) = \frac{1}{2}\Big(\big(X(T) + Y(T)\big)^2 - X^2(T) - Y^2(T)\Big).$$

Theorem (定理 7.1 Ito 公式)

考虑 Ito 过程 $\mathrm{d}X(t) = g(t)\mathrm{d}t + h(t)\mathrm{d}W(t)$. 设函数 $f(t,x)$ 满足 f_t 和 f_{xx} 是连续的. 我们有

$$
\begin{aligned}
& \mathrm{d}f(t,X(t)) \\
= \;& f_t(t,X(t))\mathrm{d}t + f_x(t,X(t))\mathrm{d}X(t) + \frac{1}{2}f_{xx}(t,X(t))\,[\mathrm{d}X(t)]^2 \\
= \;& f_t(t,X(t))\mathrm{d}t + f_x(t,X(t))g(t)\mathrm{d}t + \frac{1}{2}f_{xx}(t,X(t))h^2(t)\mathrm{d}t \\
& + f_x(t,X(t))h(t)\mathrm{d}W(t).
\end{aligned}
$$

Theorem (定理 7.2 Ito 公式)

考虑 Ito 过程 $X(T) = X(0) + \int_0^T g(t)\mathrm{d}t + \int_0^T h(t)\mathrm{d}W(t)$ *和函数* $f(t,x)$ *满足* f_t *和* f_{xx} *是连续的. 则*

$$\begin{aligned}
& f(T, X(T)) - f(0, X(0)) \\
= \;& \int_0^T f_t(t, X(t))\mathrm{d}t + \int_0^T f_x(t, X(t))\mathrm{d}X(t) \\
& + \frac{1}{2}\int_0^T f_{xx}(t, X(t))\,[\mathrm{d}X(t)]^2 \\
= \;& \int_0^T f_t(t, X(t))\mathrm{d}t + \int_0^T f_x(t, X(t))g(t)\mathrm{d}t \\
& + \frac{1}{2}\int_0^T f_{xx}(t, X(t))h^2(t)\mathrm{d}t + \int_0^T f_x(t, X(t))h(t)\mathrm{d}W(t).
\end{aligned}$$

Ito 公式的主要用途:

- 获得随机微分方程 (SDE)
- 求解随机微分方程 (SDE)

例7.5 (Ito 公式) 求 $W^3(t)$ 的 SDE.

解: 令 $f(x) = x^3$, 由 Ito 公式,

$$\begin{aligned} \mathrm{d}W^3(t) &= \mathrm{d}f(W(t)) \\ &= f_x(W(t))\mathrm{d}W(t) + \frac{1}{2}f_{xx}(W(t))\left[\mathrm{d}W(t)\right]^2 \\ &= 3W^2(t)\mathrm{d}W(t) + \frac{1}{2}6W(t)\mathrm{d}t \\ &= 3W^2(t)\mathrm{d}W(t) + 3W(t)\mathrm{d}t. \end{aligned}$$

例7.6 (Ito 公式) 考虑一对数正态股票价格

$$S(t) = S(0)\mathrm{e}^{(\mu-\sigma^2/2)t+\sigma W(t)}.$$

求 $S(t)$ 的 SDE.

解: 设 $f(x) = \mathrm{e}^x$ 和 $X(t) = (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)$.

由 Ito 公式,

$$\begin{aligned}
\mathrm{d}S(t) &= S(0)\mathrm{d}f(X(t)) \\
&= S(0)\left(f_x(X(t))\mathrm{d}X(t) + \frac{1}{2}f_{xx}(X(t))(\mathrm{d}X(t))^2\right) \\
&= S(0)\mathrm{e}^{X(t)}\left((\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma\mathrm{d}W(t) + \frac{1}{2}\sigma^2\mathrm{d}t\right) \\
&= \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t).
\end{aligned}$$

例 7.7 (Ito 公式) 求解 $\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t)$, 即 $S(t)$ 的显式表达.

解: 由 Ito 公式,

$$\begin{aligned}
\mathrm{d}\ln S(t) &= \frac{1}{S(t)}\mathrm{d}S(t) - \frac{1}{2S^2(t)}\left[\mathrm{d}S(t)\right]^2 \\
&= \mu\mathrm{d}t + \sigma\mathrm{d}W(t) - \frac{1}{2S^2(t)}\sigma^2 S^2(t)\mathrm{d}t \\
&= (\mu - \frac{1}{2}\sigma^2)\mathrm{d}t + \sigma\mathrm{d}W(t)
\end{aligned}$$

对两边求积分得到

$$\begin{aligned}
\int_0^t \mathrm{d}\ln S(s) &= \int_0^t (\mu - \frac{1}{2}\sigma^2)\mathrm{d}s + \int_0^t \sigma\mathrm{d}W(s) \\
\ln S(t) - \ln S(0) &= (\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)
\end{aligned}$$

因此,

$$S(t) = S(0)\mathrm{e}^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}.$$

例 7.8 (Ito 公式) 用 $X(0)$ 求解微分方程 $\mathrm{d}X(t) = (\alpha - \beta X(t))\,\mathrm{d}t + \sigma \mathrm{d}W(t)$, 其中 α, β, σ 是正的常数.

解: 由 Ito 公式和 $f(t,x) = e^{\beta t}x$, 我们有

$$\begin{aligned}
\mathrm{d}\left(e^{\beta t}X(t)\right) &= \beta e^{\beta t}X(t)\mathrm{d}t + e^{\beta t}\mathrm{d}X(t) \\
&= \beta e^{\beta t}X(t)\mathrm{d}t + e^{\beta t}(\alpha - \beta X(t))\,\mathrm{d}t + e^{\beta t}\sigma \mathrm{d}W(t) \\
&= \alpha e^{\beta t}\mathrm{d}t + \sigma e^{\beta t}\mathrm{d}W(t).
\end{aligned}$$

对两边求积分得到

$$\begin{aligned}
e^{\beta t}X(t) - X(0) &= \alpha \int_0^t e^{\beta s}\mathrm{d}s + \sigma \int_0^t e^{\beta s}\mathrm{d}W(s) \\
&= \frac{\alpha}{\beta}(e^{\beta t} - 1) + \sigma \int_0^t e^{\beta s}\mathrm{d}W(s).
\end{aligned}$$

从而

$$X(t) = e^{-\beta t}X(0) + \frac{\alpha}{\beta}(1 - e^{-\beta t}) + \sigma \int_0^t e^{-\beta(t-s)}\mathrm{d}W(s).$$

关于Ito过程的随机积分分布

Let h be a non-random function of time t and

$$I(t) = \int_0^t h(u)\, dW(u).$$

For each $t \geq 0$, the random variable

$$I(t) \sim \mathcal{N}\left(0, \int_0^t h^2(u)\, du\right).$$

Let $u \in \mathbb{R}$, and define

$$X(t) = \underbrace{-\frac{1}{2}\int_0^t \Big(u\, h(s)\Big)^2 ds}_{=u^2\, \mathbb{V}\mathrm{ar}(I(t))} + \underbrace{\int_0^t u\, h(s)\, dW(s)}_{=u\, I(t)}.$$

From the Itô-Doeblin formula,

$$d\left(e^{X(t)}\right) = u\, h(t)\, e^{X(t)}\, dW(t),$$

$$e^{X(t)} = 1 + \int_0^t u\, h(s)\, e^{X(s)}\, dW(s),$$

therefore,

$$\mathbb{E}\left(e^{X(t)}\right) = 1 \quad \Rightarrow \quad \mathbb{E}\left(e^{u\, I(t)}\right) = e^{\frac{1}{2}u^2\, \mathbb{V}\mathrm{ar}(I(t))}.$$

Let W be a Brownian motion, and α, β, γ and σ be deterministic functions. The linear stochastic differential equation

$$dX(t) = \Big(\alpha(t) + \beta(t)X(t)\Big)\,dt + \Big(\gamma(t) + \sigma(t)X(t)\Big)\,dW(t)$$

has the solution

$$\begin{aligned} X(t) = {} & e^{\int_0^t \left(\beta(u) - \frac{1}{2}\sigma^2(u)\right)\,du + \int_0^t \sigma(u)\,dW(u)} \times \Bigg[X(0) \\ & + \int_0^t e^{-\int_0^u \left(\beta(v) - \frac{1}{2}\sigma^2(v)\right)\,dv - \int_0^u \sigma(v)\,dW(v)} \Big(\alpha(u) - \gamma(u)\sigma(u)\Big)\,du \\ & + \int_0^t e^{-\int_0^u \left(\beta(v) - \frac{1}{2}\sigma^2(v)\right)\,dv - \int_0^u \sigma(v)\,dW(v)} \gamma(u)\,dW(u) \Bigg]. \end{aligned}$$

Consider solutions of the form $X(t) = X_1(t)\,X_2(t)$ where

$$
\begin{aligned}
dX_1(t) &= \beta(t)X_1(t)\,dt + \sigma(t)X_1(t)\,dW(t),\\
dX_2(t) &= a(t)\,dt + b(t)\,dW(t)
\end{aligned}
$$

for some functions $a(t)$ and $b(t)$. Then

$$
\begin{aligned}
dX(t) &= X_1(t)\,dX_2(t) + X_2(t)\,dX_1(t) + d[X_1, X_2](t)\\
&= \Big(\overbrace{\big(a(t) + \sigma(t)b(t)\big)X_1(t)}^{=\alpha(t)} + \beta(t)X(t)\Big)\,dt\\
&\qquad + \Big(\underbrace{b(t)X_1(t)}_{=\gamma(t)} + \sigma(t)X(t)\Big)\,dW(t)
\end{aligned}
$$

Since

$$a(t) = \frac{\alpha(t) - \gamma(t)\sigma(t)}{X_1(t)} \quad \text{and} \quad b(t) = \frac{\gamma(t)}{X_1(t)},$$

therefore,

$$X_1(t) = e^{\int_0^t \left(\beta(u) - \frac{1}{2}\sigma^2(u)\right) du + \int_0^t \sigma(u)\, dW(u)}$$

and

$$X_2(t) = X(0) + \int_0^t \frac{\alpha(u) - \gamma(u)\sigma(u)}{X_1(u)}\, du + \int_0^t \frac{\gamma(u)}{X_1(u)}\, dW(u).$$

Let W be a Brownian motion, and α, β and σ be positive constants. The stochastic differential equation

$$dR(t) = \beta\Big(\alpha - R(t)\Big)\, dt + \sigma\sqrt{R(t)}\, dW(t)$$

has the "solution"

$$R(t) = \mathrm{e}^{-\beta\, t} R(0) + \alpha\left(1 - \mathrm{e}^{-\beta\, t}\right) + \sigma\, \mathrm{e}^{-\beta\, t} \int_0^t \mathrm{e}^{\beta\, u} \sqrt{R(u)}\, dW(u),$$

and

$$\mathbb{E}\left(R(t)\right) \to \alpha \quad \text{and} \quad \mathbb{V}\mathrm{a}r\left(R(t)\right) \to \frac{\alpha\sigma^2}{2\beta}$$

as $t \to \infty$.

Let $f(t,x) = e^{\beta t}x$. Then

$$f_t(t,x) = \beta f(t,x),\ f_x(t,x) = e^{\beta t} \ \text{and} \ f_{xx}(t,x) = 0.$$

From the Itô-Doeblin formula,

$$df\big(t, R(t)\big) = f_t(t, R(t))\,dt + f_x(t, R(t))\,dR(t),$$

$$d\big(e^{\beta t} R(t)\big) = \alpha\beta\, e^{\beta t}\,dt + \sigma\, e^{\beta t}\sqrt{R(t)}\,dW(t).$$

Hence,

$$e^{\beta t} R(t) = R(0) + \alpha\left(e^{\beta t} - 1\right) + \sigma \int_0^t e^{\beta u}\sqrt{R(u)}\,dW(u).$$

Since

$$R(t) = e^{-\beta t} R(0) + \alpha \left(1 - e^{-\beta t}\right) + \sigma e^{-\beta t} \underbrace{\int_0^t e^{\beta u} \sqrt{R(u)}\, dW(u)}_{\text{Itô integral}},$$

thus

$$\mathbb{E}\left(R(t)\right) = e^{-\beta t} R(0) + \alpha \left(1 - e^{-\beta t}\right)$$

and

$$\lim_{t \to \infty} \mathbb{E}\left(R(t)\right) = \alpha.$$

$$\begin{aligned}\mathbb{V}\mathrm{ar}\,(R(t)) &= \mathbb{E}\left(\left(\sigma\,\mathrm{e}^{-\beta\,t}\int_0^t \mathrm{e}^{\beta\,u}\sqrt{R(u)}\,dW(u)\right)^2\right)\\ &= \sigma^2\,\mathrm{e}^{-2\beta\,t}\,\mathbb{E}\left(\int_0^t \mathrm{e}^{2\beta\,u}R(u)\,du\right)\\ &= \sigma^2\,\mathrm{e}^{-2\beta\,t}\int_0^t \mathrm{e}^{2\beta\,u}\,\mathbb{E}\,(R(u))\,du\\ &= \frac{\alpha\sigma^2}{2\beta}\left(1-2\,\mathrm{e}^{-\beta\,t}+\mathrm{e}^{-2\beta\,t}\right)+R(0)\frac{\sigma^2}{\beta}\left(\mathrm{e}^{-\beta\,t}-\mathrm{e}^{-2\beta\,t}\right)\end{aligned}$$

and

$$\lim_{t\to\infty}\mathbb{V}\mathrm{ar}\,(R(t))=\frac{\alpha\sigma^2}{2\beta}.$$

Definition (定义 Stratonovich积分)

如果随机过程$C=\{C_t, t\in[0,T]\}$定义为$C_t=f(B_t)$, 其中f是二次可微的函数，Riemann-Stieltjes积分和为

$$S_n=\sum_{i=1}^{n} f(B_{y_i})(B_{t_j}-B_{t_{j-1}}),\quad y_i=\frac{t_{j-1}+t_j}{2}$$

当$mesh(\tau_n)\to 0$, 上求和式均方极限存在，则称此极限为$f(B)$的Stratonovich积分。记为

$$\int_0^T f(B_s)\circ \mathrm{d}B_s.$$

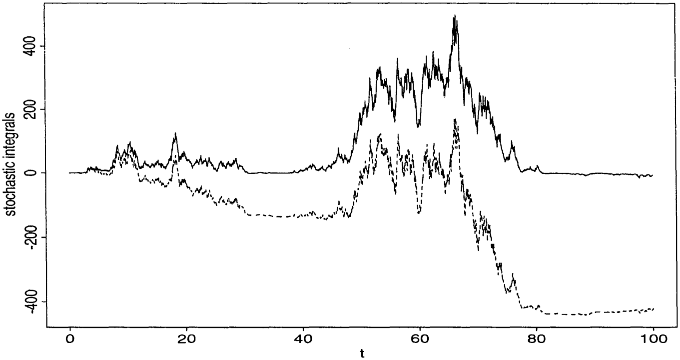

Figure 2.4.2 *One sample path of the process* $\int_0^t B_s^2\, dB_s$ (lower curve) *and of the process* $\int_0^t B_s^2 \circ dB_s$ (upper curve) *for the same Brownian sample path.*

可以计算

$$\int_0^T B_s \circ \mathrm{d}B_s = \frac{1}{2}B_T^2.$$

$$\begin{aligned}
&\sum_{i=1}^{n} f(B_{y_i})\,\Delta_i B \\
&= \sum_{i=1}^{n} f(B_{t_{i-1}})\,\Delta_i B + \sum_{i=1}^{n} f'(B_{t_{i-1}})\,(B_{y_i} - B_{t_{i-1}})\,\Delta_i B \ + \cdots \\
&= \sum_{i=1}^{n} f(B_{t_{i-1}})\,\Delta_i B + \sum_{i=1}^{n} f'(B_{t_{i-1}})\,(B_{y_i} - B_{t_{i-1}})^2 \\
&\quad + \sum_{i=1}^{n} f'(B_{t_{i-1}})(B_{y_i} - B_{t_{i-1}})(B_{t_i} - B_{y_i}) \ + \ \cdots \\
&= \widetilde{S}_n^{(1)} + \widetilde{S}_n^{(2)} + \widetilde{S}_n^{(3)} \ + \ \cdots,
\end{aligned}$$

显然，$\tilde{S}_n^{(1)}$的均方极限为Ito积分$\int_0^T f(B_s)\mathrm{d}B_s$，$\tilde{S}_n^{(2)}$ 均方极限为

$$\tilde{S}_n^{(2)} \to \frac{1}{2}\int_0^T f'(B_s)\mathrm{d}s$$

后面的项均方极限均为0. 于是有下面的定理

Theorem (定理)

若$\int_0^T Ef^2(B_t)\mathrm{d}t < \infty$, 且$\int_0^T E[f'(B_t)]^2\mathrm{d}t < \infty$

$$\int_0^T f(B_s)\circ \mathrm{d}B_s = \int_0^T f(B_s)\mathrm{d}B_s + \frac{1}{2}\int_0^T f'(B_s)\mathrm{d}s$$

Theorem (定理)

若$\int_0^T Ef^2(t, X_t)\mathrm{d}t < \infty$，且$X_t$ 为一Ito过程

$$X_t = X_0 + \int_0^T a(s, X_s)\mathrm{d}s + \int_0^T b(s, X_s)\mathrm{d}B_s.$$

于是一下的变换公式成立：

$$\int_0^T f(s, X_s) \circ \mathrm{d}B_s = \int_0^T f(t, X_s)\mathrm{d}B_s + \frac{1}{2}\int_0^T b(s, X_s) f_X(s, X_s)\mathrm{d}s.$$

利用Stratonovich积分求解微分方程

Theorem (定理)

若若X_t 为一Ito过程

$$X_t = X_0 + \int_0^T a(s, X_s)\mathrm{d}s + \int_0^T b(s, X_s)\mathrm{d}B_s.$$

于是以下的变换公式成立:

$$\int_0^T f(s, X_s) \circ \mathrm{d}B_s = \int_0^T f(t, X_s)\mathrm{d}B_s + \frac{1}{2}\int_0^T b(s, X_s) f_X(s, X_s)\mathrm{d}s.$$

带入上式, 取$f = b$, 则

$$X_T = X_0 + \int_0^T \tilde{a}(s, X_s)\mathrm{d}s + \int_0^T b(s, X_s) \circ \mathrm{d}B_s,$$

其中

$$\tilde{a}(s, X_s) = a(s, X_s) - \frac{1}{2} b(t, X_t) b_2(t, X_t).$$