金融随机分析

第8章 Black-Scholes 公式

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A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t), W_2(t), \ldots, W_d(t))^\top,$$

for $t \geq 0$, such that:

1. Each $W_i(t)$ is a 1-dimensional Brownian motion.
2. $W_i(t)$ and $W_j(t)$ are independent for $i \neq j$.

Associated with $\{W(t)\}_{t \geq 0}$ is a filtration $\{\mathcal{F}(t)\}_{t \geq 0}$ such that:

3. $\mathcal{F}(s) \subset \mathcal{F}(t)$ for $0 \leq s < t$.
4. $W(t)$ is $\mathcal{F}(t)$-measurable for each $t \geq 0$.
5. $W(t) - W(s)$ is independent of $\mathcal{F}(s)$ for $0 \leq s < t$.

For a multivariate Brownian motion W,

$$dW_i(t)\,dW_j(t) = \begin{cases} dt & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $f(t,x,y)$ be a $C^{1,2,2}$ function, $W=(W_1,W_2)^\top$ be a two dimensional Brownian motion, and X and Y be Itô processes where

$$X(t)=X(0)+\int_0^t \alpha_1(u)\,du+\int_0^t \sigma_{11}(u)\,dW_1(t)+\int_0^t \sigma_{12}(u)\,dW_2(t)$$

$$Y(t)=Y(0)+\int_0^t \alpha_2(u)\,du+\int_0^t \sigma_{21}(u)\,dW_1(t)+\int_0^t \sigma_{22}(u)\,dW_2(t).$$

多维Ito过程

Then

$$
\begin{aligned}
& f(t, X(t), Y(t)) \\
& = f(0, X(0), Y(0)) + \int_0^t f_t(u, X(u), Y(u))\, du \\
& \quad + \int_0^t f_x(u, X(u), Y(u))\, dX(u) + \int_0^t f_y(u, X(u), Y(u))\, dY(u) \\
& \quad + \frac{1}{2}\int_0^t f_{xx}(u, X(u), Y(u))\, d[X, X](u) \\
& \quad + \frac{1}{2}\int_0^t f_{yy}(u, X(u), Y(u))\, d[Y, Y](u) \\
& \quad + \int_0^t f_{xy}(u, X(u), Y(u))\, d[X, Y](u).
\end{aligned}
$$

多维Ito-Doeblin公式

In differential form,

$$dX(t) = \alpha_1(t)\,dt + \sigma_{11}(t)\,dW_1(t) + \sigma_{12}(t)\,dW_2(t),$$
$$dY(t) = \alpha_2(t)\,dt + \sigma_{21}(t)\,dW_1(t) + \sigma_{22}(t)\,dW_2(t)$$

and

$$\begin{aligned} df\big(t, X(t), Y(t)\big) &= f_t\big(t, X(t), Y(t)\big)\,dt + f_x\big(t, X(t), Y(t)\big)\,dX(t) \\ &\quad + f_y\big(t, X(t), Y(t)\big)\,dY(t) + \frac{1}{2} f_{xx}\big(t, X(t), Y(t)\big)\,d[X, X](t) \\ &\quad + \frac{1}{2} f_{yy}\big(t, X(t), Y(t)\big)\,d[Y, Y](t) + f_{xy}\big(t, X(t), Y(t)\big)\,d[X, Y](t). \end{aligned}$$

多维Ito-Doeblin公式

$$
\begin{aligned}
df(\cdot) = \Big[& f_t(\cdot) + \alpha_1(t) f_x(\cdot) + \alpha_2(t) f_y(\cdot) \\
& + \frac{1}{2}\left(\sigma_{11}^2(t) + \sigma_{12}^2(t)\right) f_{xx}(\cdot) + \frac{1}{2}\left(\sigma_{21}^2(t) + \sigma_{22}^2(t)\right) f_{yy}(\cdot) \\
& + \left(\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right) f_{xy}(\cdot) \Big] \, dt \\
& + \left(\sigma_{11}(t) f_x(\cdot) + \sigma_{21}(t) f_y(\cdot)\right) dW_1(t) \\
& + \left(\sigma_{12}(t) f_x(\cdot) + \sigma_{22}(t) f_y(\cdot)\right) dW_2(t).
\end{aligned}
$$

Let $f(t,\mathbf{x}) : \mathbb{R}^+ \times \mathbb{R}^m \mapsto \mathbb{R}$ be a function with continuous partial derivatives f_t, f_{x_i} and $f_{x_i x_j}$, for $1 \le i, j \le m$, and $\mathbf{X}$ be a m-dimensional Itô process with

$$d\mathbf{X}(t) = \boldsymbol{\alpha}(t, \mathbf{X}(t))\, dt + \boldsymbol{\sigma}(t, \mathbf{X}(t))\, d\mathbf{W}(t)$$

where $\boldsymbol{\alpha} = (\alpha_i)_{m\times 1}$, $\boldsymbol{\sigma} = (\sigma_{ij})_{m\times d}$ and $\mathbf{W} = (W_i)_{d\times 1}$ is a d-dimensional Brownian motion. Then

$$df = \left(f_t + \sum_{i=1}^{m} \alpha_i f_{x_i} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=1}^{d} \sigma_{ik}\sigma_{jk} f_{x_i x_j} \right) dt + \sum_{i=1}^{m} \sum_{k=1}^{d} \sigma_{ik} f_{x_i} dW_k$$

$$= \left(f_t + (\nabla_{\mathbf{x}} f)^\top \boldsymbol{\alpha} + \frac{1}{2} \mathrm{Tr}\left(\boldsymbol{\sigma}^\top \mathbf{H}_{\mathbf{x}} f\, \boldsymbol{\sigma} \right) \right) dt + (\nabla_{\mathbf{x}} f)^\top \boldsymbol{\sigma}\, d\mathbf{W}(t).$$

例子

Let the stochastic processes M_i be martingales, for $i = 1, 2, \ldots, d$. Suppose $M_i(0) = 0$, $M_i(t)$ has continuous paths, and

$$[M_i, M_j](t) = \begin{cases} t & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases}$$

for $1 \leq i, j \leq d$ and $t \geq 0$. Then $M = (M_i)$ is a d-dimensional Brownian motion.

Suppose $d = 1$. Let $f(t, x) = e^{ux - \frac{1}{2}u^2 t}$. Then

$$df(t, M(t)) = \overbrace{f_t(t, M(t))}^{= -\frac{1}{2}u^2 f(t,x)} dt + \overbrace{f_x(t, M(t))}^{= u f(t,x)} dM(t) + \frac{1}{2}\overbrace{f_{xx}(t, M(t))}^{= u^2 f(t,x)} \overbrace{d[M, M](t)}^{= dt},$$
$$= u f(t, M(t))\, dM(t),$$

$$f(t, M(t)) = 1 + \underbrace{\int_0^t u f(s, M(s))\, dM(s)}_{\text{Martingale}}.$$

Hence,

$$\mathbb{E}\left(e^{u M(t)}\right) = e^{\frac{1}{2}u^2 t}\, \mathbb{E}\left(f(t, M(t))\right) = e^{\frac{1}{2}u^2 t}.$$

Let $0 \le \tau < t$. Then

$$d\left(e^{uM(t)}\right) = u\ e^{u\,M(t)}\,dM(t) + \frac{1}{2}u^2\,e^{u\,M(t)}\,dt$$

$$e^{u\,M(t)} = e^{u\,M(\tau)} + \underbrace{u\int_\tau^t e^{u\,M(s)}\,dM(s)}_{\text{Martingale}} + \frac{1}{2}u^2\int_\tau^t e^{u\,M(s)}\,ds.$$

Therefore,

$$\mathbb{E}\left(e^{u\left(M(t)-M(\tau)\right)}\,\middle|\,\mathcal{F}(\tau)\right) = 1 + \frac{1}{2}u^2\int_\tau^t \mathbb{E}\left(e^{u\left(M(s)-M(\tau)\right)}\,\middle|\,\mathcal{F}(\tau)\right)ds$$

$$\frac{d}{dt}\left(\mathbb{E}\left(e^{u\left(M(t)-M(\tau)\right)}\,\middle|\,\mathcal{F}(\tau)\right)\right) = \frac{1}{2}u^2\,\mathbb{E}\left(e^{u\left(M(t)-M(\tau)\right)}\,\middle|\,\mathcal{F}(\tau)\right).$$

Hence,

$$\mathbb{E}\left(e^{u\left(M(t)-M(\tau)\right)}\,\middle|\,\mathcal{F}(\tau)\right) = e^{\frac{1}{2}u^2(t-\tau)}.$$

例子

Let $0 \le \tau_1 < t_1 \le \tau_2 < t_2$. Then

$$\begin{aligned}
&\mathbb{E}\left(e^{u_2\left(M(t_2)-M(\tau_2)\right)+u_1\left(M(t_1)-M(\tau_1)\right)}\right) \\
&= \mathbb{E}\left(\mathbb{E}\left(e^{u_2\left(M(t_2)-M(\tau_2)\right)}\,\middle|\,\mathcal{F}(\tau_2)\right)e^{u_1\left(M(t_1)-M(\tau_1)\right)}\right) \\
&= e^{\frac{1}{2}u_2^2(t_2-\tau_2)}\,\mathbb{E}\left(\mathbb{E}\left(e^{u_1\left(M(t_1)-M(\tau_1)\right)}\,\middle|\,\mathcal{F}(\tau_1)\right)\right) \\
&= e^{\frac{1}{2}\left(u_2^2(t_2-\tau_2)+u_1^2(t_1-\tau_1)\right)}
\end{aligned}$$

Hence increments over non-overlapping intervals are independent.

Suppose $d = 2$. From the one-dimensional result, both M_1 and M_2 are Brownian motions. It remains to show that they are independent. Let $f(t,x,y) = e^{u_1 x + u_2 y - \frac{1}{2}(u_1^2+u_2^2)t}$. Then

$$
\begin{aligned}
df\left(t, M_1(t), M_2(t)\right) = {} & \overbrace{f_t\left(t, M_1(t), M_2(t)\right)}^{=-\frac{1}{2}(u_1^2+u_2^2)f(t,x,y)} dt \\
& + \overbrace{f_x\left(t, M_1(t), M_2(t)\right)}^{=u_1 f(t,x,y)} dM_1(t) + \overbrace{f_y\left(t, M_1(t), M_2(t)\right)}^{=u_2 f(t,x,y)} dM_2(t) \\
& + \frac{1}{2}\overbrace{f_{xx}\left(t, M_1(t), M_2(t)\right)}^{=u_1^2 f(t,x,y)} \overbrace{d[M_1, M_1](t)}^{=dt} + \overbrace{f_{xy}\left(t, M_1(t), M_2(t)\right)}^{=u_1 u_2 f(t,x,y)} \overbrace{d[M_1, M_2](t)}^{=0} \\
& + \frac{1}{2}\overbrace{f_{yy}\left(t, M_1(t), M_2(t)\right)}^{=u_2^2 f(t,x,y)} \overbrace{d[M_2, M_2](t)}^{=dt}, \\
= {} & f\left(t, M(t)\right)\Big(u_1\, dM_1(t) + u_2\, dM_2(t)\Big).
\end{aligned}
$$

Since

$$f\left(t, M_1(t), M_2(t)\right) = 1 + \overbrace{\int_0^t u_1\, f\left(s, M_1(s), M_2(s)\right) dM_1(s)}^{\text{Martingale}}$$
$$+ \underbrace{\int_0^t u_2\, f\left(s, M_1(s), M_2(s)\right) dM_2(s)}_{\text{Martingale}},$$

thus,

$$\mathbb{E}\left(e^{u_1 M_1(t)+u_2 M_2(t)}\right) = e^{\frac{1}{2}(u_1^2+u_2^2)t}\, \mathbb{E}\left(f\left(t, M_1(t), M_2(t)\right)\right)$$
$$= e^{\frac{1}{2}(u_1^2+u_2^2)t}.$$

Let $M(t)$, $t \geq 0$, denotes the risk-free asset price process that is modeled by

$$dM(t) = r\,M(t)\,dt$$

where r is a constant. The ordinary differential equation has the solution

$$M(t) = e^{r\,t}$$

assuming $M(0) = 1$.

Let $S(t)$, $t \geq 0$, denotes the risky asset price process that is modeled by a geometric Brownian motion

$$dS(t) = S(t)\Big(\mu\, dt + \sigma\, dW(t)\Big)$$

where W is a Brownian motion, and μ and $\sigma > 0$ are constants. The stochastic differential equation has the solution

$$S(t) = S(0)\, e^{\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)}$$

and

$$\ln\left(\frac{S(t)}{S(0)}\right) \sim \mathcal{N}\Big(\big(\mu - \tfrac{1}{2}\sigma^2\big)\, t, \sigma^2 t\Big).$$

Consider a portfolio strategy $\big(h(t), v(t)\big)$, $t \geq 0$, where $h(t)$ and $v(t)$ represent the unit holdings at time t in the risky and risk-free assets respectively.

Let $X(t)$, $t \geq 0$, denotes the portfolio value process of the above strategy. Then

$$X(t) = h(t)\, S(t) + v(t)\, M(t)$$

and

$$\begin{aligned} dX(t) &= h(t)\, dS(t) + v(t)\, dM(t) \\ &= \underbrace{X(t)\, r\, dt}_{\text{Riskless Return}} + \underbrace{h(t)\, S(t)\, (\mu - r)\, dt}_{\text{Risk Compensation}} + \underbrace{h(t)\, S(t)\, \sigma\, dW(t)}_{\text{Risk / Volatility}}. \end{aligned}$$

证明:

Since

$$X(t+\Delta t)-X(t) = h(t)\underbrace{\Big(S(t+\Delta t)-S(t)\Big)}_{=S(t)\left(\mu\Delta t+\sigma\Delta W(t)\right)} + v(t)\underbrace{\Big(M(t+\Delta t)-M(t)\Big)}_{=r\,M(t)\Delta t},$$

letting $\Delta t \to 0$, we have

$$\begin{aligned} dX(t) &= h(t)\,dS(t) + v(t)\,dM(t) \\ &= h(t)\,S(t)\Big(\mu\,dt + \sigma\,dW(t)\Big) + \underbrace{v(t)\,M(t)}_{=X(t)-h(t)S(t)}\; r\,dt \\ &= X(t)\,r\,dt + h(t)\,S(t)\,(\mu - r)\,dt + h(t)\,S(t)\,\sigma\,dW(t). \end{aligned}$$

The continuous-time self-financing condition

$$dX(t) = h(t)\,dS(t) + v(t)\,dM(t)$$

can equivalently be stated as

$$S(t)\,dh(t) + dS(t)\,dh(t) + M(t)\,dv(t) + \underbrace{dM(t)\,dv(t)}_{=0} = 0.$$

Using Itô-Doeblin formula,

$$\underbrace{dX(t)}_{=h(t)\,dS(t)+v(t)\,dM(t)} = h(t)\,dS(t) + S(t)\,dh(t) + d[S,h](t) + v(t)\,dM(t) + M(t)\,dv(t) + d[M,v](t),$$

hence

$$S(t)\,dh(t) + dS(t)\,dh(t) + M(t)\,dv(t) + dM(t)\,dv(t) = 0.$$

证明

Since

$$\Big(h(t+\Delta t)-h(t)\Big)S(t+\Delta t)+\Big(v(t+\Delta t)-v(t)\Big)M(t+\Delta t)=0,$$

$$\begin{aligned}
S(t)\Big(h(t+\Delta t)-h(t)\Big)&+\Big(S(t+\Delta t)-S(t)\Big)\Big(h(t+\Delta t)-h(t)\Big)\\
+M(t)\Big(v(t+\Delta t)-v(t)\Big)&+\Big(M(t+\Delta t)-M(t)\Big)\Big(v(t+\Delta t)-v(t)\Big)\\
&=0,
\end{aligned}$$

letting $\Delta t\to 0$, we have

$$S(t)\,dh(t)+dS(t)\,dh(t)+M(t)\,dv(t)+dM(t)\,dv(t)=0.$$

For $t \geq 0$, the value of a portfolio at time t is

$$X(t) = \sum_{i=0}^{m} h_i(t)\, S_i(t),$$

where $S_i(t)$ is the price of asset i and $h_i(t)$ is the unit of asset i in the portfolio at time t, for $i = 0, 1, \ldots, m$.

The trading strategy $\left(h_0(t), h_1(t), \ldots, h_m(t)\right)_{t \geq 0}$ is *self-financing* if

$$dX(t) = \sum_{i=0}^{m} h_i(t)\, dS_i(t),$$

for $t \geq 0$.

Consider a simple option that pays $g(S(T))$ at time T.

Suppose the value of the option at time t is given by $C(t, S(t))$, for $0 \le t \le T$. Then

$$
\begin{aligned}
& dC(t, S(t)) \\
& \quad = \left(C_t(t, S(t)) + \mu S(t) C_S(t, S(t)) + \frac{1}{2}\sigma^2 S^2(t) C_{SS}(t, S(t)) \right) dt \\
& \qquad + \sigma S(t) C_S(t, S(t))\, dW(t).
\end{aligned}
$$

证明:

Since

$$dS(t) = S(t)\Big(\mu\, dt + \sigma\, dW(t)\Big) \text{ and } d[S,S](t) = \sigma^2 S^2(t)\, dt,$$

from the Itô-Doeblin formula,

$$\begin{aligned}
& dC\big(t, S(t)\big) \\
&= C_t\big(t, S(t)\big)\, dt + C_S\big(t, S(t)\big)\, dS(t) + \frac{1}{2} C_{SS}\big(t, S(t)\big)\, d[S,S](t) \\
&= \Big(C_t\big(t, S(t)\big) + \mu S(t)\, C_S\big(t, S(t)\big) + \frac{1}{2}\sigma^2 S^2(t)\, C_{SS}\big(t, S(t)\big)\Big)\, dt \\
&\qquad + \sigma\, S(t)\, C_S\big(t, S(t)\big)\, dW(t).
\end{aligned}$$

Consider a hedging portfolio that consists of a short one unit of option and $h(t)$ units of the underlying at time t, $0 \le t \le T$.

The portfolio value at time t is

$$X(t) = h(t)\, S(t) - C\big(t, S(t)\big)$$

and

$$\begin{aligned}
dX(t) &= h(t)\, dS(t) + \overbrace{S(t)\, dh(t) + dS(t)\, dh(t)}^{=0 \text{ Self-Financing}} - dC\big(t, S(t)\big) \\
&= \Big(\mu\, h(t)\, S(t) - C_t\big(t, S(t)\big) \\
&\qquad - \mu\, S(t)\, C_S\big(t, S(t)\big) - \frac{1}{2}\sigma^2 S^2(t)\, C_{SS}\big(t, S(t)\big)\Big)\, dt \\
&\quad + \sigma\, S(t)\, \Big(h(t) - C_S\big(t, S(t)\big)\Big)\, dW(t).
\end{aligned}$$

证明:

Choose $h(t) = C_S\big(t, S(t)\big)$. Then

$$dX(t) = \left(-C_t\big(t, S(t)\big) - \frac{1}{2}\sigma^2 S^2(t) C_{SS}\big(t, S(t)\big)\right) dt$$
$$= r\,X(t)\,dt.$$

Therefore

$$-C_t\big(t, S(t)\big) - \frac{1}{2}\sigma^2 S^2(t) C_{SS}\big(t, S(t)\big)$$
$$= r\left(S(t)\,C_S\big(t, S(t)\big) - C\big(t, S(t)\big)\right).$$

证明:

The value of the option $C(t,s)$ satisfies the Black-Scholes-Merton partial differential equation

$$C_t(t,s) + r\,s\,C_s(t,s) + \frac{1}{2}\sigma^2 s^2 C_{ss}(t,s) = r\,C(t,s),$$

for $0 \le t < T$ and $0 \le s < \infty$, with terminal condition

$$C(T,s) = g(s).$$

Outline

预付远期价格 $\text{PV}(S_T)$ 是 S_T 在时刻 0 的价格. 由下面几种可能算法:

- 无股息:
$$\text{PV}(S_T) = S_0.$$
- 离散股息:
$$\text{PV}(S_T) = S_0 - \text{PV}(\text{Div}).$$
- 连续股息 (比率 δ):
$$\text{PV}(S_T) = e^{-\delta T} S_0.$$

Definition (正态分布)

如果一个随机变量 X 具有概率密度函数

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{(x-\mu)^2}{2\sigma^2} \right\}, \quad -\infty < x < \infty,$$

其中 $-\infty < \mu < \infty$, $\sigma > 0$, 则称 X 为一正态随机变量，记为 $X \sim N(\mu, \sigma^2)$.

$X \sim N(\mu, \sigma^2)$, 则 $(X-\mu)/\sigma \sim N(0, 1)$;

Definition (二元正态分布)

设一二元随机向量 $\mathbf{X} = (X_1, X_2)$ 的密度函数为

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-a)^2}{\sigma_1^2} - 2\rho\frac{(x-a)(y-b)}{\sigma_1\sigma_2} + \frac{(y-b)^2}{\sigma_2^2}\right]\right\}$$

其中$-\infty < a, b < \infty$, $0 < \sigma_1, \sigma_2 < \infty$, $-1 \le \rho \le 1$，则称(X_1, X_2) 服从参数为$a, b, \sigma_1^2, \sigma_2^2, \rho$ 的二元正态分布，记为$\mathbf{X} \sim N(a, b, \sigma_1^2, \sigma_2^2, \rho)$.

设 $(X,Y)\sim N(a,b,\sigma_1^2,\sigma_2^2,\rho)$.

- 则两者的相关系数$\mathrm{corr}(X,Y)=\rho$.
- $\left(\frac{X-a}{\sigma_1},\frac{Y-b}{\sigma_2}\right)\sim N(0,0,1,1,\rho)$;
- $[Y|X=x]\sim N(\mu_2+\frac{\sigma_2}{\sigma_1}(x-\mu_1),\sigma_2^2(1-\rho^2))$
- 则X 与Y 独立当且仅当$\rho=0$.
- 设随机向量 $X\sim N(0,1)$, $Y\sim N(0,1)$, 且相互独立, 则 X 与 $\rho X+(1-\rho)Y$ 的相关系数为ρ.

模型框架:

- 无风险债券 $B(t) = \mathrm{e}^{rt}$. 故,

$$\mathrm{d}B(t) = r\mathrm{e}^{rt}\mathrm{d}t = rB(t)\mathrm{d}t.$$

- 对数正态分布股票价格模型

$$S(t) = S(0)\mathrm{e}^{(\alpha-\delta-\frac{1}{2}\sigma^2)t+\sigma W(t)}$$

满足随机微分方程

$$\mathrm{d}S(t) = (\alpha - \delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t).$$

目的: 考虑一欧式看涨期权, 行权价 K, 收益为

$$C(S(T), T) = (S(T) - K)_+.$$

其在时刻 0 的无套利价格 $C(S(0), 0)$ 是多少?

Black-Scholes 模型的局限:

- 无交易费用 (No transaction cost)
- 股票收益率服从正态分布
- 股票价格连续
- 无风险利率是已知常数
- 波动率是已知常数
- 股息收益是已知常数
- ...

自筹资金交易策略

- 考虑一交易策略 (x, y)
  - $x(t)$ 为在时刻 t 持有股票份额
  - $y(t)$ 为在时刻 t 持有无风险债券份额
- 在时刻 t 投资组合 (x, y) 的价值为

$$V(t) = x(t)S(t) + y(t)B(t).$$

Definition (定义 8.1 自筹资金交易策略)

交易策略 (x, y) 称为是自筹资金的, 如果对任意的 $t \in (0, T)$,

$$\mathrm{d}V(t) = x(t)\left[\mathrm{d}S(t) + \delta S(t)\mathrm{d}t\right] + y(t)\mathrm{d}B(t).$$

Definition (定义 8.2 复制投资组合)

自筹资金投资组合 $V(t)$ 称为是看涨期权 $C(S(t),t)$ 的复制投资组合, 如果

$$V(t) = C(S(t),t) \text{ 对任意的 } t,$$

即

$$\mathrm{d}V(t) = \mathrm{d}C(S(t),t) \quad \text{对任意的 } t.$$

- 自筹资金投资组合价值的动态变化

$$\mathrm{d}V(t) = x(t)\left[\mathrm{d}S(t) + \delta S(t)\mathrm{d}t\right] + y(t)\mathrm{d}B(t).$$

- 由 Ito 公式, 期权价格的动态变化

$$\mathrm{d}C(S(t),t) = C_t\mathrm{d}t + C_S\mathrm{d}S(t) + \frac{1}{2}C_{SS}\left[\mathrm{d}S(t)\right]^2.$$

注意到

$$\mathrm{d}S(t) = (\alpha - \delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t),$$
$$\mathrm{d}B(t) = rB(t)\mathrm{d}t,$$

我们有

$$\begin{aligned}\mathrm{d}V(t) &= x(t)\left[\mathrm{d}S(t) + \delta S(t)\mathrm{d}t\right] + y(t)\mathrm{d}B(t)\\ &= \left[\alpha x(t)S(t) + ry(t)B(t)\right]\mathrm{d}t + \sigma S(t)x(t)\mathrm{d}W(t),\end{aligned}$$

和

$$\begin{aligned}\mathrm{d}C(S(t),t) &= C_t\mathrm{d}t + C_S\mathrm{d}S + \frac{1}{2}C_{SS}\left(\mathrm{d}S\right)^2\\ &= \left(C_t + (\alpha - \delta)SC_S + \frac{1}{2}\sigma^2S^2C_{SS}\right)\mathrm{d}t + \sigma SC_S\mathrm{d}W(t).\end{aligned}$$

令 $\mathrm{d}V = \mathrm{d}C$ ($\mathrm{d}t$ 和 $\mathrm{d}W(t)$), 得到

$$\begin{cases} x(t) = C_S \\ y(t) = \frac{1}{rB(t)}\left(C_t - \delta S C_S + \frac{1}{2}\sigma^2 S^2 C_{SS}\right). \end{cases}$$

- 将 $(x(t), y(t))$ 带入 $C(t) = x(t)S(t) + y(t)B(t)$,

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + (r-\delta) S C_S - rC = 0,$$

  边界条件为 $C(S, T) = (S-K)_+$, $S > 0$.

- 等式独立于 α!

Theorem (定理 8.1 欧式期权 Black-Scholes 定价公式)

在 Black-Scholes 模型, 欧式看涨期权价格为

$$C(S(0),0) = S(0)\mathrm{e}^{-\delta T}\Phi(d_1) - Ke^{-rT}\Phi(d_2),$$

其中

$$d_1 = \frac{\ln(S(0)/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

和 $\Phi(x)$ 标准正态分布的(累积)分布函数.

The following initial value problem on $(-\infty, \infty)$:

$$\begin{aligned} & f_\tau(\tau, x) = a f_{xx}(\tau, x), \quad 0 < \tau < \infty, -\infty < x < \infty, \\ & f(0, x) = g(x); \end{aligned}$$

has the solution

$$f(\tau, x) = \frac{1}{\sqrt{4\pi a\tau}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a\tau}} g(y)\, dy.$$

The following initial value problem on $[0, \infty)$ with homogenous Dirichlet boundary conditions:

$$\begin{aligned} f_\tau(\tau, x) &= a\, f_{xx}(\tau, x), \quad 0 < \tau < \infty, 0 \le x < \infty, \\ f(0, x) &= g(x), \\ f(\tau, 0) &= 0; \end{aligned}$$

has the solution

$$f(\tau, x) = \frac{1}{\sqrt{4\pi a \tau}} \int_0^\infty \left(e^{-\frac{(x-y)^2}{4a\tau}} - e^{-\frac{(x+y)^2}{4a\tau}} \right) g(y)\, dy.$$

The value of a European call option $C(t,s)$, with strike price K and maturity time T, satisfies the Black-Scholes-Merton partial differential equation

$$C_t(t,s) + r\,s\,C_s(t,s) + \frac{1}{2}\sigma^2 s^2 C_{ss}(t,s) - r\,C(t,s) = 0$$

for $0 \le t < T$ and $0 \le s < \infty$, with terminal condition

$$C(T,s) = (s-K)^+.$$

Under the following transformation:

$$x = \ln s \quad \text{and} \quad \tau = T - t;$$

the Black-Scholes-Merton partial differential equation becomes

$$C_\tau(\tau, x) - \left(r - \frac{1}{2}\sigma^2\right) C_x(\tau, x) - \frac{1}{2}\sigma^2 C_{xx}(\tau, x) + r\,C(\tau, x) = 0$$

for $0 < \tau \leq T$ and $-\infty < x < \infty$, with initial condition

$$C(0, x) = (\,e^x - K)^+.$$

Let

$$C(\tau, x) = \mathrm{e}^{\alpha\tau+\beta x} f(\tau, x)$$

where

$$\alpha = -r - \frac{1}{2\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 \quad \text{and} \quad \beta = \frac{1}{2} - \frac{r}{\sigma^2}.$$

Then the Black-Scholes-Merton partial differential equation reduces to the heat equation

$$f_\tau(\tau, x) = \frac{1}{2}\sigma^2 f_{xx}(\tau, x)$$

for $0 < \tau \leq T$ and $-\infty < x < \infty$, with initial condition

$$f(0, x) = \mathrm{e}^{-\beta x}(\mathrm{e}^x - K)^+.$$

$$f(\tau, x) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \int_{\ln K}^{\infty} e^{-\frac{(x-y)^2}{2\sigma^2\tau}} \left(e^{(1-\beta)y} - K\, e^{-\beta y} \right) dy$$

and

$$\begin{aligned} C(\tau, x) &= e^{\alpha\tau+\beta x} f(\tau, x) \\ &= e^{x} N(d_+) - e^{-r\tau} K\, N(d_-) \end{aligned}$$

where

$$d_{\pm} = \frac{x - \ln K + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

- 在风险中性测度 $\mathbb{Q}$ 下，我们应该有

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[S(t)] &= e^{rt}\mathrm{PV}_{0,t}(S(t)) \\ &= e^{(r-\delta)t}S(0).\end{aligned}$$

- 事实上，由对数正态分布的均值

$$\mathbb{E}^{\mathbb{Q}}\left[S(0)e^{(r-\delta-\sigma^2/2)t+\sigma W^{\mathbb{Q}}(t)}\right] = e^{(r-\delta)t}S(0),$$

其中，$W^{\mathbb{Q}}$ 在 $\mathbb{Q}$ 下为标准布朗运动.

- 因此，我们有

$$S(0)e^{(\alpha-\delta-\sigma^2/2)t+\sigma W(t)} = S(t) = S(0)e^{(r-\delta-\sigma^2/2)t+\sigma W^{\mathbb{Q}}(t)},$$

这意味着

$$W^{\mathbb{Q}}(t) = \frac{\alpha-r}{\sigma}t + W(t).$$

- 两个概率测度 $\mathbb{P}$ 和 $\mathbb{Q}$ 的关系可由下式完全决定

$$W^{\mathbb{Q}}(t) = \frac{\alpha - r}{\sigma} t + W(t).$$

- 我们有

$$\begin{aligned} S(t) &= S(0)e^{(\alpha-\delta-\sigma^2/2)t+\sigma W(t)} \\ &= S(0)e^{(r-\delta-\sigma^2/2)t+\sigma W^{\mathbb{Q}}(t)} \end{aligned}$$

或等价地,

$$\begin{aligned} \mathrm{d}S(t) &= (\alpha-\delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t) \\ &= (r-\delta)S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W^{\mathbb{Q}}(t) \end{aligned}$$

注意 $S(t) = S(0)e^{(r-\delta-\sigma^2/2)t+\sigma W^{\mathbb{Q}}(t)}$, 我们有

$$
\begin{aligned}
C(0) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}[C(T)] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S(T)-K)_+\right] \\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\left(S(0)e^{(r-\delta-\sigma^2/2)T+\sigma W^{\mathbb{Q}}(T)}-K\right)_+\right] \\
&= e^{-rT}\int_{-\infty}^{\infty}\left(S(0)e^{(r-\delta-\sigma^2/2)T+\sigma\sqrt{T}x}-K\right)_+\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\mathrm{d}x \\
&= \text{计算}\ldots \\
&= S(0)e^{-\delta T}\Phi(d_1)-Ke^{-rT}\Phi(d_2),
\end{aligned}
$$

其中, $d_1 = \frac{\ln(S(0)/K)+(r-\delta+\sigma^2/2)T}{\sigma\sqrt{T}}$ 和 $d_2 = d_1 - \sigma\sqrt{T}$.

例 8.1 (Black-Scholes 公式) 设 $S(0)=20$, $\sigma=24\%$, $r=5\%$, $\delta=3\%$. 在Black-Scholes 模型下计算执行价格为 25 的三月期的欧式看涨、看跌期权价格.

解: 我们有

$$
\begin{aligned}
d_1 &= \frac{\ln(20/25)+(0.05-0.03+\frac{1}{2}0.24^2)/4}{0.24\sqrt{1/4}} = -1.75786 \\
d_2 &= d_1-\sigma\sqrt{T} = -1.87786.
\end{aligned}
$$

因此由正态分布表 $\Phi(d_1)=0.03939$ 和 $\Phi(d_2)=0.0302$,

$$
C = 20e^{-0.03/4}\Phi(d_1)-25e^{-0.05/4}\Phi(d_2)=0.03629.
$$

利用看跌看涨平价公式, $C-P=S(0)e^{-\delta T}-Ke^{-rT}$, 我们有

$$
P = 0.03629-20e^{-0.03/4}+25e^{-0.05/4}=4.87518.
$$

Outline

- 设 $W_1(t)$ 和 $W_2(t)$ 是两个布朗运动，且相关系数为

$$\operatorname{corr}(W_1(t), W_2(t)) = \frac{\operatorname{Cov}(W_1(t), W_2(t))}{\operatorname{sd}(W_1(t))\operatorname{sd}(W_2(t))} = \rho(t) \in [-1, 1].$$

- 特别地，如果 $\rho(t) \equiv \rho \in [-1, 1]$ 对任意的 $t \geq 0$，则我们有

$$\begin{aligned} \mathbb{E}[W_1(t)W_2(t)] &= \operatorname{Cov}(W_1(t), W_2(t)) \\ &= \operatorname{corr}(W_1(t), W_2(t))\operatorname{sd}(W_1(t))\operatorname{sd}(W_2(t)) \\ &= \rho t. \end{aligned}$$

Theorem (定理 8.2)

对 $t \geq 0$，如果 $\operatorname{corr}(W_1(t), W_2(t)) = \rho$，则二次协变差为

$$[W_1, W_2](t) = \rho t$$

和相应的微分形式为

$$\mathrm{d}W_1(t)\mathrm{d}W_2(t) = \rho\mathrm{d}t.$$

Theorem (定理 8.3 Ito 乘积法则)

设 $X(t)$ 和 $Y(t)$ 为两个 Ito 过程, 则

$$\mathrm{d}(X(t)Y(t)) = X(t)\mathrm{d}Y(t) + Y(t)\mathrm{d}X(t) + \mathrm{d}X(t)\mathrm{d}Y(t).$$

例 8.2 (Ito 乘积) 给定 $\mathrm{d}X(t) = (\alpha - \beta X(t))\,\mathrm{d}t + \sigma \mathrm{d}W(t)$. 求 $\mathrm{d}\left(e^{\beta t}X(t)\right)$ 的 SDE.

解: 由 Ito 乘积法则, 我们有

$$\begin{aligned}
\mathrm{d}\left(e^{\beta t}X(t)\right) &= X(t)\mathrm{d}e^{\beta t} + e^{\beta t}\mathrm{d}X(t) + \mathrm{d}e^{\beta t}\mathrm{d}X(t) \\
&= \beta e^{\beta t}X(t)\mathrm{d}t + e^{\beta t}\left[(\alpha - \beta X(t))\,\mathrm{d}t + \sigma \mathrm{d}W(t)\right] \\
&= \alpha e^{\beta t}\mathrm{d}t + \sigma e^{\beta t}\mathrm{d}W(t).
\end{aligned}$$

例 8.3 (Ito 乘积) 设

$$\begin{aligned} dS_1(t) &= \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \\ dS_2(t) &= \alpha_2 S_2(t)dt + \sigma_2 S_2(t)dW_2(t), \end{aligned}$$

其中 $dW_1(t)dW_2(t) = \rho dt$. 求 $d\frac{S_1(t)}{S_2(t)}$ 的 SDE.

解: 由

$$\begin{aligned} d\frac{1}{S_2(t)} &= -\frac{1}{S_2(t)^2}dS_2(t) + \frac{1}{2}\frac{2}{S_2(t)^3}(dS_2(t))^2 \\ &= \frac{(\sigma_2^2-\alpha_2)}{S_2(t)}dt - \frac{\sigma_2}{S_2(t)}dW_2(t) \end{aligned}$$

由 Ito 乘积法则,

$$\begin{aligned} d\frac{S_1(t)}{S_2(t)} &= \frac{1}{S_2(t)}dS_1(t) + S_1(t)d\frac{1}{S_2(t)} + dS_1(t)d\frac{1}{S_2(t)} \\ &= \frac{S_1(t)}{S_2(t)}[(\alpha_1-\alpha_2+\sigma_2^2-\sigma_1\sigma_2\rho)dt + \sigma_1 dW_1(t) - \sigma_2 dW_2(t)]. \end{aligned}$$

Theorem (定理 8.4 欧式交换期权的 B-S 价格)

设在 Black-Scholes 模型下，有两只股票价格 $S_1(t)$ 和 $S_2(t)$，波动率分别为σ_1, σ_2，和相关系数为 ρ. 无风险利率为 r. 考虑在到期日 T 时用一份额 S_2 交换一份额 S_1 的欧式期权，其价格为

$$e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[(S_1(T)-S_2(T))_+\right] = PV(S_1(T))\Phi(d_1)-PV(S_2(T))\Phi(d_2),$$

其中，

$$\begin{aligned} d_1 &= \frac{\ln\left(PV(S_1(T))/PV(S_2(T))\right)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} \\ d_2 &= d_1-\sigma\sqrt{T} \\ \sigma &= \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}. \end{aligned}$$

Theorem (定理 8.5 欧式交换期权的 B-S 公式分解)

设在 Black-Scholes 模型下, 有两只股票价格 $S_1(t)$ 和 $S_2(t)$, 波动率分别为σ_1, σ_2, 和相关系数为 ρ. 无风险利率为 r. 我们有

$$\begin{aligned} e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_1(T)1_{\{S_1(T)>S_2(T)\}}\right] &= PV(S_1(T))\Phi(d_1) \\ e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[S_2(T)1_{\{S_1(T)>S_2(T)\}}\right] &= PV(S_2(T))\Phi(d_2) \end{aligned}$$

其中,

$$\begin{aligned} d_1 &= \frac{\ln\left(PV(S_1(T))/PV(S_2(T))\right)+\frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}, \\ d_2 &= d_1-\sigma\sqrt{T} \\ \sigma &= \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2} \end{aligned}$$

例 8.4 (B-S 公式推广) 设在 Black-Scholes 模型下，有两只无股息的股票价格为 $S_1(t)$ 和 $S_2(t)$. 给定 $S_1(0)=10$, $S_2(0)=20$, $\sigma_1=18\%$, $\sigma_2=25\%$, $\rho=-0.4$, $r=5\%$. 确定如下欧式期权价格，其在到期日 $T=1$ 时用一份额 S_2 交换两份额 S_1 的欧式期权.

解: 在到期日交换期权的收益为

$$C(1)=(2S_1(1)-S_2(1))_+.$$

由

$$\begin{aligned}
\sigma &= \sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}=0.361801\\
d_1 &= \frac{\ln(2S_1(0)/S_2(0))+\frac{1}{2}\sigma^2T}{\sigma\sqrt{T}}=0.1809, \quad \Phi(d_1)=0.57178\\
d_2 &= d_1-\sigma\sqrt{T}=-0.1809, \quad \Phi(d_2)=0.42822,
\end{aligned}$$

交换期权价格为

$$C(0)=2S_1(0)\Phi(d_1)-S_2(0)\Phi(d_2)=2.8712.$$

例 8.5 (B-S 公式推广) 设 $S(0) = 40$, $K = 40$, $\sigma = 30\%$, $r = 8\%$, $\delta = 0$ 和 $T = 0.25$. 计算如下期权价格: (1) 如果$S(T) > K$ 则收益 \$1 的期权价格 \$1.
(2) 如果$S(T) < K$ 则收益 \$1 的期权价格 \$1.

解: 我们令 $S_1(T) = S(T)$ 是第一个资产和 $S_2(T) = K$ 为第二个资产. 由广义的 B-S 公式,

$$C_1(0) = \frac{1}{K}\mathrm{PV}(K)\,\Phi(d_2),$$

其中, $d_2 = \frac{\ln(40/e^{-0.08\times0.25}40)+\frac{1}{2}0.3^2\times0.25}{0.3\sqrt{0.25}} - 0.3\sqrt{0.25} = 0.0583$. 则

$$C_1(0) = \mathrm{e}^{-0.08\times0.25}\Phi(0.0583) = 0.5129$$

和

$$C_2(0) = \mathrm{e}^{-0.08\times0.25} - C_1(0) = 0.4673.$$

例 8.6 (B-S 公式推广) 设 $S(0)=40$, $K=40$, $\sigma=30\%$, $r=8\%$, $\delta=0$ 和 $T=0.25$. 计算如果$S(T)>40$ 则收益 $S(T)-20$ 的看涨期权价格 $C(0)$.

解: 在到期日是期权收益为

$$S(T)1_{\{S(T)>40\}}-20\times 1_{\{S(T)>40\}}$$

由广义的 B-S 公式,

$$C(0)=\mathrm{PV}(S(T))\Phi(d_1)-20\mathrm{e}^{-rT}\Phi(d_2)=40\Phi(d_1)-20\mathrm{e}^{-0.08\times 0.25}\Phi(d_2)$$

其中,

$$\begin{aligned} d_1 &= \frac{\ln(40/e^{-0.08\times 0.25}40)+\frac{1}{2}0.3^2 0.25}{0.3\sqrt{0.25}}=0.20833 \\ d_2 &= d_1-0.3\sqrt{0.25}=0.05833 \end{aligned}$$

则我们有 $C(0)=40\Phi(d_1)-20\mathrm{e}^{-0.08\times 0.25}\Phi(d_2)=13.0427$.