ALGEBRAIC SURFACE BLENDING USING WU'S METHOD

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In this paper, we apply Wu's method in blending algebraic surfaces with (piecewise) algebraic surfaces— an important problem in geometric design. This new approach can find the lowest degree algebraic blending surfaces in most cases, and it is much more efficient than Gröbner basis method. Examples are provided to illustrate the algorithm.

Keywords Wu's method, algebraic surfaces, blending.

1 Introduction

Blending is a method to construct a smooth transitional surface between given surfaces, and the study of constructing blending surfaces is one of the important problems in Geometric Modeling. In the past decades, a lot of research literature has focused on this problem, and many methods such as the rolling ball method[9], Liming skill[8] and potential method[5] are proposed to solve the problem. However, these methods generally produce high degree algebraic blending surfaces which are unsatisfactory in subsequent geometric operations. The algebraic surface blending problem can also be tackled by Gröbner basis[6], and generally this method can find the lowest degree algebraic blending surfaces. However Gröbner basis are very expensive to compute. In [12], an entirely different approach based on irreducible ascending sets and pseudo division algorithm is put forward to solve the algebraic surface blending problem. This new approach is much more efficient than Gröbner basis method and it can find the lowest degree algebraic blending surfaces in most cases. However, this method has still some drawbacks. First, the idea in that paper is not easy to generalize to the case where the blending surface and the initial surface have a high order contact. Second, the intersection curves at which the blending surface and the initial surfaces meet are required to be irreducible. Third, a single piece of algebraic surface is used to blend given surfaces, which may result in high degree algebraic blending surfaces.

In this paper, we put forward the idea in [12] one step further. We can get rid of many of the restraints imposed by the method in [12]. Namely, the order of contact of the blending surface and the initial surfaces can be arbitrarily high; the intersection curves at which the blending surface touches the initial surface can be reducible; and low degree piecewise algebraic surfaces can also be used as the blending surfaces, if necessary. These properties make our method more flexible and useful in constructing blending surfaces.

The paper is organized as follows. In the next section, we recall some basic knowledge about ascending sets and pseudo division algorithm which are the core parts of Wu's method. In section 3, the main theoretic foundation and the algorithm is developed. In section 4, several examples are given to illustrate the algorithm. Finally, in section 5, we make some discussions and conclude the paper.

2 Preliminaries

We first recall some fundamental knowledge in Wu's method, and a basic concept about geometric continuity of two algebraic surfaces.

Let K be the basic field of characteristic 0, and $x_1, \dots, x_n$ be independent indeterminate. $K[x_1, \dots, x_n]$ denotes the polynomial ring over K.

Given two polynomials $f, g \in K[x_1, \dots, x_n]$, write them in the form:

$$f = f_{d_1} x_n^{d_1} + \ldots + f_1 x_n + f_0$$
$$g = g_{d_2} x_n^{d_2} + \ldots + g_1 x_n + g_0$$

where $f_i, g_i \in K[x_1, \dots, x_{n-1}]$. x_n is called the *leading variable*, and f_{d_1}, g_{d_2} the *initial term* of f and g respectively. Using x_n as the main variable, divide g by f results in

$$f_{d_1}^s g = qf + r \tag{1}$$

where $q, r \in K[x_1, \dots, x_n]$, $deg_{x_n}(r) < deg_{x_n}(f)$ and s is a nonnegative integer. r is called the *pseudo remainder* of g divided by f, denote it by $r = \text{Prem}(g, f, x_n)$. (1) is called the pseudo division formula [10]. Pseudo division algorithm is a key part of Wu's method. Another important concept in Wu's elimination method is ascending set.

Definition 1[10,11] Let $AS = \{P_1, \dots, P_m\}$ be a set of polynomials in $K[x_1, \dots, x_n]$, and

$$P_i = I_{c_i,0} x_{c_i}^{d_i} + I_{c_i,1} x_{c_i}^{d_i - 1} + \cdots + I_{c_i,d_i}, \quad i = 1, \dots, m,$$

which satisfy the following conditions:

1. $0 < c_1 < \cdots < c_m$,

2. $I_{c_i,j} \in K[x_1, \dots, x_{c_i - 1}]$,

3. $\forall i > j$, $\deg_{x_j}(P_i) < \deg_{x_j}(P_j)$.

then AS is called an *ascending set.*

Given a polynomial $g \in K[x_1, \ldots, x_n]$, g can be divided by AS as follows:

$$\begin{aligned} I_{c_m,0}^{s_m} g &= q_m P_m + R_{m-1} \\ I_{c_{m-1},0}^{s_{m-1}} R_{m-1} &= q_{m-1} P_{m-1} + R_{m-2} \\ &\ldots\ldots \\ I_{c_1,0}^{s_1} R_1 &= q_1 P_1 + R_0 \end{aligned}$$

From above equation, we get

$$I_{c_1,0}^{s_1} \cdots I_{c_m,0}^{s_m} \cdot g = Q_1 P_1 + \cdots + Q_m P_m + R \tag{2}$$

where s_i are some nonnegative integers, $Q_i \in K[x_1, \ldots, x_{c_i}]$, $i = 1, \ldots, m$. $R \in K[x_1, \ldots, x_n]$ is called the *pseudo remainder* of g divided by AS, and we write it as $R = \mathrm{Prem}(g/AS)$. It is easy to know R has the following property:

$$deg_{x_{c_i}}(R) < deg_{x_{c_i}}(P_i), \quad i = 1, 2, \ldots, m.$$

Given a set of polynomials $A = \{A_1, \ldots, A_k\}$, we can always transform it into an ascending set by pseudo division algorithm (2). The details can be found in [10,11].

Next we review a definition about the geometric continuity of two algebraic surfaces meeting at a common space curve.

An algebraic surface in $\mathbb{R}^3$ is defined as the zero set of a polynomial $f \in \mathbb{R}[x, y, z]$, whose coefficients are over the real numbers. We often write it as $f = 0$ or $\mathbf{V}(f)$. Given two algebraic surfaces$f = 0$, $g = 0$, the intersection of the two algebraic surfaces is an algebraic space curve $C = \mathbf{V}(f = 0, g = 0)$. If there exists some $(x_0, y_0, z_0) \in C$ such that $f = 0$, $g = 0$ have different normals at (x_0, y_0, z_0), then we say $f = 0$ and $g = 0$ *intersect transversally* along curve C.

Definition 2[4] Suppose two algebraic surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ intersect transversally along a space curve C. Then $f = 0$ and $g = 0$ meet with GC^k continuity along C if and only if there exist two polynomials $\alpha(x, y, z)$ and $\beta(x, y, z)$ such that

1. α and β are not identically zero on curve C;
2. All derivatives up to order k of $\alpha f - \beta g$ equal zero at all points along C.

The above definition of geometric continuity coincides with our geometric intuition. For example, GC^1 continuity is the same as tangential continuity.

3 The Problem and the Method

The surface blending problem can be stated as follows.

Problem

Given m initial algebraic surfaces $f_i = 0$ and another m auxiliary surfaces $h_i = 0$, where $f_i, h_i \in \mathbb{R}[x, y, z]$, $i = 1, 2, \ldots, m$. Suppose $f_i = 0$ and $h_i = 0$ intersect transversally at curve C_i, $i = 1, 2, \ldots, m$. Find a (piecewise) algebraic surface $g = 0$ such that $g = 0$ meets $f_i = 0$ with GC^k continuity along curve C_i, $i = 1, 2, \ldots, m$.

The main idea of our method to solve the above problem is based on pseudo division algorithm. We describe the algorithm as follows.

Algorithm

Step 1 Express polynomial g in the form:

$$g = \sum_{0 \leqslant \lambda+\mu+\nu \leqslant d} g_{\lambda\mu\nu} x^\lambda y^\mu z^\nu$$

where $g_{\lambda\mu\nu}$ are unknowns, d is the degree of g.

Step 2 Choose an appropriate monomial order of variables x, y, z. Construct ascending sets AS_i from polynomial sets $A_i := \{f_i, h_i^{k+1}\}$, $i = 1, 2, \ldots, m$.

Step 3 Divide g by each of the ascending sets AS_i using pseudo division algorithm. Let $R_i = \mathrm{Prem}(g/AS_i)$. $R_i \in \mathbb{R}[x, y, z]$ is a polynomial whose coefficients are linear combinations of $g_{\lambda\mu\nu}$.

Step 4 Set $R_i = 0$, $i = 1, 2, \ldots, m$, i.e., set all the coefficients of R_i to be zero. A system of linear equations with $g_{\lambda\mu\nu}$ being unknowns is thus obtained.

Step 5 Solve the system of the linear equation. If there is just one trivial solution $g_{\lambda\mu\nu} = 0$, then increase the degree of polynomial g and go to Step 1. Otherwise, select a suitable solution among all the solutions.

We claim the above algorithm produces a solution to the given problem.

Theorem *Let g be a solution obtained by the above algorithm. Then algebraic surface $g = 0$ meets with $f_i = 0$ with GC^k continuity.*

Proof: For any i $(1 \leqslant i \leqslant m)$, let $AS_i = \{A_{i1}, A_{i2}\}$, where $A_{i1}, A_{i2} \in \mathbb{R}[x, y, z]$. By the algorithm to compute the ascending set of a polynomial set, we know there exist polynomials $I_{ij}, \alpha_{ij}, \beta_{ij} \in \mathbb{R}[x,y,z]$, $j=1,2$ such that

$$I_{i1}^{s_{i1}} A_{i1} = \alpha_{i1} f_i + \beta_{i1} h_i^{k+1}, \\ I_{i2}^{s_{i2}} A_{i2} = \alpha_{i2} f_i + \beta_{i2} h_i^{k+1}, \tag{3}$$

where s_{ij}, $j = 1,2$ are nonnegative integers. On the other hand, by $\mathrm{Prem}(g/AS_i) = 0$, one has

$$I_i^{s_i} g = a_i A_{i1} + b_i A_{i2} \tag{4}$$

where $I_i, a_i, b_i \in \mathbb{R}[x,y,z]$. From (3) and (4), we get

$$Ig = \alpha f_i + \beta h_i^{k+1} \tag{5}$$

from some polynomial $I, \alpha, \beta \in \mathbb{R}[x,y,z]$. From Eq.(5), we immediately have $d^l(Ig - \alpha f_i)|_{C_i} = 0$, $l = 0,1,\ldots,k$. If I and α are not identically zero along the intersection curve C_i, then by the definition of geometric continuity, g meets with $f_i = 0$ with GC^k continuity along C_i. This completes the proof.

4 Examples

In this section, we provide several examples to demonstrate our method in constructing algebraic blending surfaces. In all the examples, $f_i = 0$'s represent the initial surfaces to blend, $h_i = 0$'s are the auxiliary surfaces, and $g = 0$ is the blending surface.

Example 1 Given two circular cylinders whose axes are perpendicular to each other:

$$f_1 := y^2 + z^2 - 1, \qquad f_2 := x^2 + y^2 - 1 = 0,$$

and two auxiliary planes $h_1 := x - 2 = 0$ and $h_2 := z - 2 = 0$, we are interested in finding a cubic algebraic surface $g = 0$ which meets f_i with GC^1 continuity along circular sections $C_i := \mathbf{V}(f_i, h_i)$, $i = 1,2$.

It is easy to check $AS_1 := \{h_1^2, f_1\}$ and $AS_2 := \{f_2, h_2^2\}$ are the ascending sets under ordering $x > y > z$.

Now let $\mathrm{Prem}(g/AS_1) = 0$ and $\mathrm{Prem}(g/AS_2) = 0$, we obtain 24 homogenous linear equations with 20 unknowns. The system of equations have a unique solution up to a constant factor:

$$g = 4 + 3x + 3z - 4x^2 - 4y^2 - 4z^2 - 4xz + x^3 + x^2 z + xy^2 + xz^2 + y^2 z + z^3.$$

The corresponding figure is shown in Fig.(1).

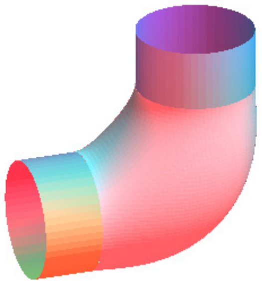

Figure 1. Blending surfaces between two cylinders

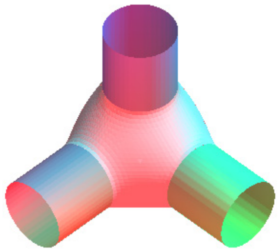

Figure 2. Blending three cylinders.

Example 2 In this example, we use a quartic surface to GC^1 blend three cylinders whose axes are perpendicular to each other:

$$\begin{aligned} C_1 &: f_1 = y^2 + z^2 - 1, \quad h_1 = x - 2, \\ C_2 &: f_2 = z^2 + x^2 - 1, \quad h_2 = y - 2, \\ C_3 &: f_3 = x^2 + y^2 - 1, \quad h_3 = z - 2. \end{aligned}$$

Obviously, $AS_1 = \{h_1^2, f_1\}$, $AS_2 = \{h_2^2, f_2\}$ and $AS_3 = \{h_3^2, f_3\}$ are ascending sets with respect to variable ordering $x > y > z$, $y > z > x$ and $z > x > y$ respectively. By letting $\mathrm{Prem}(g/AS_i) = 0$, $i = 1, 2, 3$, we obtain 48 linear equations with 35 unknowns. The system of linear equations has 2 families of solutions. One such solution is:

$$\begin{aligned} g =& 4y - 28 + xz^3 + y^3x + x^3y + x^2yz + 12y^2 - 4xyz + 12x^2 - 4x^3 \\ &+ 12z^2 + yz^3 + y^3z + x^3z - 4xy^2 - 4z^2y - 4x^2y + 3xy + 3xz - 4x^2z \\ &+ 3yz - 4xz^2 - 4y^2z - 4y^3 - 4z^3 + z^2xy + xy^2z + 4z + 4x \end{aligned}$$

Fig.2 is the corresponding figure.

Example 3 Suppose

$$\begin{aligned} C_1 &: f_1 = y^2 + z^2 - 1, \quad h_1 = x - 2, \\ C_2 &: f_2 = y^2 + z^2 - 1, \quad h_2 = x + 2, \\ C_3 &: f_3 = x^2 + z^2 - 1, \quad h_3 = y - 2, \\ C_4 &: f_4 = x^2 + z^2 - 1, \quad h_4 = y + 2. \end{aligned}$$

Now both $AS_1 = \{h_1^2, f_1\}$ and $AS_2 = \{h_2^2, f_2\}$ are ascending sets with respect to $x > y > z$, and $AS_3 = \{h_3^2, f_3\}$ and $AS_4 = \{h_4^2, f_4\}$ ascending sets with respect to $y > z > x$. A degree four algebraic surface $g = 0$ which GC^1 blends $f_i = 0$ along C_i, $i = 1, 2, 3, 4$ is given by

$$g = x^2z^2 + y^2z^2 + x^2y^2 + 47 - 17x^2 - 17y^2 - 14z^2 + 2x^4 + 2y^4 - z^4,$$

Fig.3 shows the corresponding blending surface.

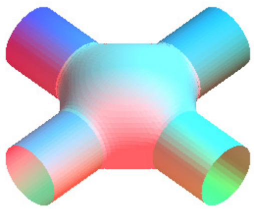

Figure 3. Blending four cylinders.

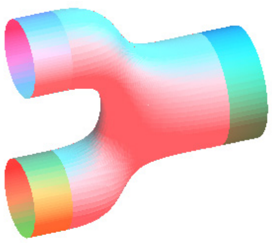

Figure 4. Blending along reducible curves.

Example 4 Now we consider the case where one of the initial surface is reducible. Let

$$\begin{aligned} C_1 : f_1 &= (y^2 + (z-2)^2 - 1)(y^2 + (z+2)^2 - 1), & h_1 &= x - 2, \\ C_2 : f_2 &= y^2 + z^2 - 2, & h_2 &= x + 2. \end{aligned}$$

We are going to find a quintic surface $g = 0$ such that $g = 0$ blends $f_i = 0$, $i = 1, 2$ with GC^1 continuity. For simplicity, we assume $g = 0$ is symmetric about planes xy and xz. Note that $f_1 = 0$ is the union of two cylinders and thus in reducible. The intersection curve $\mathbf{V}(f_1, h_1)$ is the union of two circles and is also reducible.

Again, $AS_1 = \{h_1^2, f_1\}$ and $AS_2 = \{h_2^2, f_2\}$ are ascending sets with respect to $x > y > z$.

The system of linear equations obtained has six families of solutions.

$$\begin{aligned} g = &- 88xy^4 - 1604x - 176xy^2z^2 + 7x^2z^2 + 7x^5 + 7x^2 - 420z^2 + 7x^4 - 1124y^2 \\ &-88xz^4 - 169x^2y^2 + 484xy^2 + 1188xz^2 - 28x^3y^2 - 28x^3z^2 + 1260 \end{aligned}$$

gives one solution. Fig.4 depicts the blending surface.

Example 5 Our last example comes from C. Bajaj[1]. Let

$$\begin{aligned}
C_1 : f_1 &= 10yz - 25z^2 + 40z - y^2 + 10xy - 8y - 25x^2 + 40x - 16, \\
h_1 &= y - 1, \\
C_2 : f_2 &= 4z^2 + 4xz - 12z + 4y^2 + 4xy - 12y + x^2 - 6x + 9, \\
h_2 &= x - 1, \\
C_3 : f_3 &= (x-1)^2 + (y-1)^2 - 1, \\
h_3 &= (z-1)^2.
\end{aligned}$$

where $f_1 = 0$ and $f_2 = 0$ are two cones and $f_3 = 0$ is a cylinder. In [1], Bajaj etal use a quintic algebraic surface to GC^1 blend the three surfaces $f_i = 0$, and it is shown that the quintic surface has the lowest degree. Here we are going to use a piecewise quartic surface to GC^1 blend $f_i = 0$, $i = 1, 2, 3$.

Let $p_1 = x - y$, $p_2 = z - x$, $p_3 = y - z$. We want to construct three quartic algebraic surfaces g_1, g_2 and g_3 such that

1. $g_i = 0$ meets $f_i = 0$ along C_i with GC^1 continuity, $i = 1, 2, 3$, and

2. $g_i = 0$ meets $g_{i+1} = 0$ on $p_i = 0$ with GC^1 continuity, $i = 1, 2, 3$, where the subscripts are moduled by 3.

By condition 2, we can assume

$$\begin{aligned}
g_2 &= g_1 + \alpha_1 p_1^2, \\
g_3 &= g_2 + \alpha_2 p_2^2, \\
g_1 &= g_3 + \alpha_3 p_3^2,
\end{aligned}$$

where $\alpha_i = b_{0,i} + b_{1,i}x + b_{2,i}y + b_{3,i}z$. From these three equations, a consistency condition

$$\alpha_1 p_1^2 + \alpha_2 p_2^2 + \alpha_3 p_3^2 = 0. \tag{6}$$

must be hold[3].

From $\{f_i, h_i^2\}$, $i = 1, 2, 3$, we compute three ascending sets $AS_1 = \{f_1 + h_1, h_1\}$ under ordering $z > y > x$, $AS_2 = \{f_2 - h_2, h_2\}$ under $y > x > z$, $AS_3 = \{f_3, h_3\}$ under $x > z > y$. By combining $\mathrm{Prem}(g_i/AS_i) = 0$, $i = 1, 2, 3$ and Eqn.(6), we get a system of linear equations with unknowns. The solution of the equations has ten free parameters. A suitable choice of free parameters gives

$$\begin{aligned}
g_1 =& 2z^2xy + 2x^2y - 580x^2yz - 6496yz - 4306xy - 4842xz + 645x^2z \\
&+1088xy^2 + 1358xz^2 + 3265y^2z + 1708z^2y + 1090x^3y + 1425x^3z + 1068y^3x \\
&-538y^3z - 737x^2y^2 - 1295x^2z^2 + 546y^2z^2 + 1425xz^3 - 528yz^3 \\
&-14146 - 15395x^2 - 4657y^2 - 9319z^2 + 5710x^3 - 2684y^3 + 853z^3 \\
&-1295x^4 + 534y^4 + 1471xy^2z - 4274xyz + 19190x + 14538y + 16924z, \\
g_2 =& g_1 + 2(606x + 606y - 1212z + 5x^2 - y^2 - 7z^2 + 9xy + 5yz - 11xz)(x-y)^2, \\
g_3 =& g_1 - 2(606x - 606y + 13x^2 + y^2 - z^2 - 11xy - 3yz + xz)(y-z)^2
\end{aligned}$$

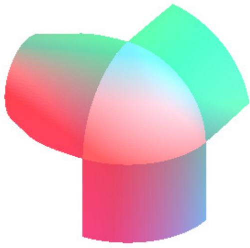

Figure 5. Piecewise algebraic blending surfaces.

5 Conclusions

In this paper, we made some attempts to apply Wu's method in constructing (piecewise) algebraic surfaces to smoothly blend given algebraic surfaces. Unlike previous approaches, Wu's method can find the lowest degree blending surfaces, and it is much more efficient than Gröbner basis method. Examples suggest this new approach is powerful in constructing algebraic blending surfaces for various situations.

Currently, we are relying on a trial-and-error approach to determine the degree of the blending surface. How to determine the degree of the blending surface in advance is a problem worthy further research. On the other hand, how to choose suitable parameters in the final solutions such that the resulting blending surface is satisfactory in practice needs further study.

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