FILLING HOLES WITH PIECEWISE ALGEBRAIC SURFACES

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In this article, we use piecewise algebraic surfaces for filling holes. A general algorithm of filling holes applying to any G^k continuity is presented, also, we propose two ways of space partition in our procedure.

1 Introduction

To construct a smooth transition surface for mending holes intersected by given surfaces is called filling holes. Filling holes widely appears in mechanical design, geometrical modeling, computer graphics and animation. Implicit algebraic surfaces play an important role in such area.

Early in 1964[6], G^0 filling holes was drawn out: Coons surface mends four-side hole formed of two pairs of Bézier curves. Filling n-side polygonal holes with smooth parametric surfaces get much research in these years. Gregory et al. gave a tutorial report on bicubic method[8], and some further research was proposed in later years[9].

While, few reseach on filling holes with algebraic surfaces was done. Sederberg[11] used algebraic surfaces to C^0 interpolate points and curves, and Bajaj [1] extended the result to C^1 continuity. In blending corner of table, Bajaj[2] smoothed out the three edges with three quadratic surfaces and formed a 3-side hole, then he construct a quintic surfaces to mend the hole.

However, algebraic surfaces generally have multiple sheets and singularities especially when the degree of the surface is relatively high. Therefore many researchers suggest to use low degree algebraic surfaces. We made our first attempt to tangentially blend three cylinders using piecewise algebraic surface[5]. The results show one of the most important advantages of blending algebraic surfaces with piecewise algebraic surfaces: the degree of the piecewise algebraic blending surface is relatively low compared with a single algebraic blending surface.

In this paper, we use piecewise algebraic surfaces instead of a single surface for filling holes, ought to decrease the degree of blending surfaces. §2 gives some preliminary knowledges about geometric continuity of algebraic surfaces and the geometric continuity conditions for several algebraic surface patches meeting at a common vertex. §3 presents a general algorithm of filling holes.

In §4, two examples are provided.

2 Geometric continuity of algebraic surface patches

Modeling with piecewise algebraic surfaces is based on the theory of geometric continuity. We abbreviate algebraic surface $f(x, y, z) = 0$ (f is a polynomial about x, y, z) to $\mathbf{V}(f)$.

2.1 Geometric continuity of algebraic surfaces

There are many definitions about the geometric continuity of algebraic surfaces, and most of these definitions are equivalent in nature[7]. For two algebraic surfaces, Warren [12] gave a definition of geometric continuity called *rescaling continuity* between the two algebraic surfaces.

Definition 2.1 *Let $\mathbf{V}(f)$ and $\mathbf{V}(g)$ be two algebraic surfaces which intersect transversally at an irreducible algebraic curve C. $\mathbf{V}(f)$ and $\mathbf{V}(g)$ are called meet with G^k rescaling continuity along common curve C if*

(1) $\mathbf{V}(f)$ and $\mathbf{V}(g)$ are smooth along C except at a finite points.

(2) there exist polynomials $a(x, y, z), b(x, y, z)$, where a, b are not identically zero on C, such that af and bg are C^k continuous over C.

The above definition of geometric continuity coincides with geometric intuition. For example, G^1 rescaling continuity is equivalent to tangent continuity; G^2 rescaling continuity is the same as curvature continuity. A general characterization of rescaling continuity is stated in the following theorem [7].

Theorem 2.2 *Let $\mathbf{V}(f)$ and $\mathbf{V}(h)$ be two algebraic surfaces which intersect transversally at an irreducible algebraic curve $C := \mathbf{V}(f) \cap \mathbf{V}(h)$. Then the surfaces $\mathbf{V}(f)$ and $\mathbf{V}(g)$ meet with G^k continuity along common curve C if and only if there are polynomials $\alpha(x, y, z) \neq 0$ and $\beta(x, y, z) = 0$ such that $g = \alpha f + \beta h^{k+1}$.*

In practical applications, $\mathbf{V}(h)$ is often assumed to be a plane. In this case, we have

Corollary 2.3 *Assume algebraic surface $g = 0$ of degree m and algebraic surface $f = 0$ of degree $n\,(n \leq m)$ meet along common algebraic curve in plane π. If there exist polynomials $\alpha(x, y, z)$ of degree $m - n$ and $\beta(x, y, z)$ of degree $m - k - 1$ such that $g = \alpha f + \beta \pi^{k+1}$, then algebraic surfaces $g = 0$ and $f = 0$ meet with G^k continuity along the common curve.*

2.2 *Geometric continuity of algebraic surface patches meeting around a common vertex*

In geometric modeling, especially in closed solid modeling, we often need to deal with the problem of smoothly joining several algebraic surface patches meeting around a common vertex V. Let $S = \{S_i\}_{i=1}^m$ be a set of algebraic surface patches which meet at the common vertex V, and C_i be the intersection curve of S_i and S_{i+1} on the boundary, $i = 1, 2, \ldots, m$. Here the subscripts are moduled by m. Obviously, $C_i \cap C_j = V (i \neq j)$. The shape of the collection of the surface patches $\{S_i\}_{i=1}^m$ looks like an umbrella, so we call S an *umbrella*[4]. Umbrella S is called G^k continuous if S_i and S_{i+1} meet with G^k continuity along C_i, $i = 1, 2, \ldots, m$.

We consider defining regions for an umbrella (figure 1) composed of tetrahedrons $T_i = Z_1 Z_2 V_{i-1} V_i$, $i = 1, 2, ..., m$ with common edge $Z_1 Z_2$, where the neighboring tetrahedrons T_i and T_{i+1} share a same face $\pi_i = Z_1 Z_2 V_i$.

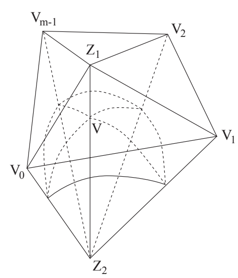

Figure 1. defining region for an umbrella

In each tetrahedron T_i, an algebraic surface patch $g_i(x, y, z) = 0$ of degree n is defined. If we construct g_i in such a way that

$$g_{i+1} = g_i + \alpha_i \pi_i^{k+1}, \quad i = 1, 2, \ldots, m, \tag{1}$$

here, α_i is polynomial with degree $n - k - 1$, then the set of algebraic surface patches $g_i = 0$, $i = 1, 2, \ldots, m$ forms a G^k continuous umbrella.

Note that, the system of equations (1) implies a consistency condition:

$$\sum_{i=1}^{m} \alpha_i \pi_i^{k+1} = 0. \tag{2}$$

Theorem 2.4 *The umbrella defined by (1)is G^k continuous, if and only if the consistency condition (2) has non-zero solutions.*

3 Algorithm for filling holes

Given hole intersected by several algebraic surfaces, the process of constructing a piecewise algebraic surface to blend the given algebraic surfaces mainly consists of two steps:

(1) Space partition, i.e., determine the defining region for the piecewise algebraic blending surface.

(2) Construct the piecewise algebraic surface which blend the given surfaces with G^k continuity.

We will explore these two steps separately.

3.1 Space partition and subdivision

Space partition is to decide the definition region of each algebraic surface patches. It varies with different situations, and makes geometric modeling with piecewise algebraic surfaces quite difficult but flexible.

Let f_i, $i = 1, 2, \ldots, m$ be the given algebraic surfaces, F_i are transversal planes, blending surfaces meet with f_i at $C_i = (f_i = 0) \cap (F_i = 0)$. The neighbouring C_i and C_{i+1} meet at a certain point and all C_i form a hole.

(1) The defining region of the blending surface is bounded by planes F_i, $i = 1, 2 \ldots, r$.

(2) We divide the whole defining region to small blocks T_j, and define an algebraic surface patch g_j in each T_j. Here, all f_i and g_j whose defining region shares a common transversal plane meet at the transitional plane with G^k continuity.

(3) When the parameters are not enough to get a solution, we will subdivide the defining region into smaller blocks. In blending problem transitional curve C_i not meeting, we often subdivide the region to smaller pieces around the same common vertex shown in figure 2. While, in filling holes, the neighbouring C_i meet at a certain point. In order to avoid cutting one curve to two segment and get more free parameters and fewer consistency conditions, we subdivide region at the intersection point of C_i shown as figure 3.

3.2 Construction of the blending surface

After determining the defining region of the blending surface, the construction of the blending surface can be achieved by solving some algebraic equations.

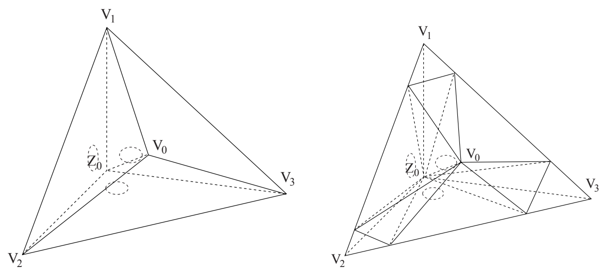

Figure 2. Subdividing for blending region

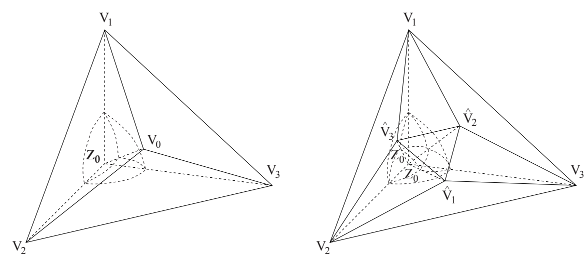

Figure 3. Subdividing for filling holes region

Suppose we are given r algebraic surfaces $f_i = 0$, $i = 1, 2, \ldots, r$. The blending surface consists of s surface patches $g_j = 0$ (or equivalently, the defining region consists of s tetrahedrons). We assume the surface patches g_j have the same degree n $(n \geq deg(f_i))$. Since the blending surface meet each $f_i = 0$ at some curve (which is the intersection of $f_i = 0$ with some plane F_i) with G^k continuity, there exists some g_j such that

$$g_j = \gamma_i f_i + \beta_i F^{k+1}$$

where γ_i and β_i are polynomials, and $deg(\gamma_i) \leq deg(g_j) - deg(f_i)$ and $deg(\beta_i) \leq deg(g_j) - k - 1$.

On the other hand, at each place where the partition has the structure of an umbrella, a consistency condition like (2) must be held. If there are t umbrella structures in the defining region, then there are t corresponding consistency conditions.

Thus we get a system of algebraic equations with polynomials α_i, β_i and γ_i being unknowns. When we know the degrees of α_i, β_i and γ_i in advance, the above algebraic equations can be converted to a system of linear equations with the coefficients of α_i, β_i and γ_i being unknowns. After solving the system of the linear equations, the piecewise algebraic blending surface can be constructed.

It should be noted that the resulting blending surface heavily relies on the space partition. For the given algebraic surfaces f_i, different space partition can result in different solutions and different number of solutions.

4 Examples

In this section, we will present two specific examples to demonstrate the process of constructing piecewise algebraic surfaces to fill given hole. In the first example, we use 3 patches of quartic algebraic surfaces to G^1 mend the hole; and in the second one, we construct a piecewise cubic algebraic surfaces (7 patches) to fill the same hole.

4.1 Hole in Bajaj's blending corner of table

Consider a corner of three faces that sonsist of the first quadrants of xy, yz, zx planes[2]. The three edges are smoothed out by three quadratic surfaces: a cone $f_1 = 10yz - 25z^2 + 40z - y^2 + 10xy - 8y - 25x^2 + 40x - 16 = 0$, another cone $f_2 = 4z^2 + 4xz - 12z + 4y^2 + 4xy - 12y + x^2 - 6x + 9 = 0$, and a circular cylinder $f_3 = (x-1)^2 + (y-1)^2 - 1 = 0$. These three surfaces, respectively intersected with $F_1 = y - 1 = 0$, $F_2 = x - 1 = 0$, $F_3 = z - 1 = 0$, produces three circles:

$$\begin{aligned} C_1 &= \left\{ \left(\tfrac{1+2t+t^2}{1+t^2}, 1, \tfrac{2}{1+t^2} \right) \right\}, \\ C_2 &= \left\{ \left(1, \tfrac{1+2t+t^2}{1+t^2}, \tfrac{2}{1+t^2} \right) \right\}, \\ C_3 &= \left\{ \left(\tfrac{1+2t+t^2}{1+t^2}, 21+t^2, 1 \right) \right\}. \end{aligned} \tag{3}$$

Now, we look for a surface $g(x, y, z) = 0$ which smoothly interpolates the three curves and fills the hole generated by the three curves. Also, g respectively meets with $f_i = 0$ at the three curves satisfying G^1 continuity. The algebraic degree of blending surfaces must be at least five [2].

However, quintic surfaces have more multiple sheets and singularities than lower degree algebraic surfaces. Sederberg[11] showed that algebraic surfaces of degree as low as three might be adequate for many free form modelling applications.

We use piecewise algebraic surfaces instead of a single surface in blending in order to decrease the degree of blending surfaces.

4.2 Filling hole with piecewise quartic surfaces

In the blending region $x \leqslant 1, y \leqslant 1, z \leqslant 1$, we divide the region into three blocks meeting at a common vertex, it is to say that, we want to fill the hole with three surface patches meeting at a common vertex.

We select points $V_1 = (1,1,-2)$, $V_2 = (1,-2,1)$, $V_3 = (-2,1,1)$, $V_0 = (0,0,0)$, $Z_0 = (1,1,1)$, and divided transversal planes: $\pi_1 = x - y = 0, \pi_2 = x - z = 0, \pi_3 = y - z = 0$. $\{g_i\}_{i=1}^3$ is defined in each tetrahedron $Z_0V_0V_{i-1}V_i$. g_i meet with f_i at F_i with G^1 continuity, also, g_i fit G^1 continuity, we get

$$\begin{cases} \sum_{i=1}^3 \alpha_i \pi_i^2 = 0, \\ \gamma_1 f_1 + \beta_1 F_1^2 + \alpha_1 \pi_1^2 = \gamma_2 f_2 + \beta_2 F_2^2, \\ \gamma_1 f_1 + \beta_1 F_1^2 - \alpha_3 \pi_3^2 = \gamma_3 f_3 + \beta_3 F_3^2, \end{cases} \tag{4}$$

Here, $\alpha_i, \beta_i, \gamma_i$ are polynomial of degree $n-2$.

First, we try to find piecewise cubic surfaces in the three tetrahedrons partition: $\alpha_i, \beta_i, \gamma_i$ are linear polynomials. But the coefficients of linear functions are not enough to get a solution.

Then we consider quartic surfaces — $\alpha_i, \beta_i, \gamma_i$ are quadratic polynomials. We convert equations system (4) to a system of linear equations of the coefficients of monomials. The equations system contains 105 euqations and 90 unknown coefficients, and the rank of this 105×90 system is 77, the dimension of solution space is 13. As in figure 4 shown is a solution:

$$\begin{cases} g_1 = \gamma_1 f_1 + \beta_1 F_1^2, \\ g_2 = \gamma_2 f_2 + \beta_2 F_2^2, \\ g_3 = \gamma_3 f_3 + \beta_3 F_3^2, \end{cases}$$

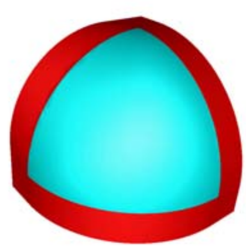

Figure 4. Four quartic surfaces patches

Here,

$$\begin{cases}\gamma_1 = 25x^2 - 182y^2 - 21xy - 25xz + 10yz - 60x + 300y - 16z, \\ \beta_1 = -5730x^2 + 63y^2 - 4920z^2 + \frac{4129}{3}xy \\ \qquad +3272xz - 493yz + \frac{29027}{3}x + 1579y + 10060z - 9537, \\ \gamma_2 = \frac{1295}{2}x^2 - \frac{395}{2}y^2 + \frac{1565}{4}xy + \frac{625}{4}xz - \frac{125}{2}yz - \frac{6745}{4}x + \frac{915}{2}y + 100z, \\ \beta_2 = -\frac{8773}{6}x^2 - \frac{9705}{2}y^2 - 4360z^2 - \frac{10797}{4}xy - \frac{6309}{4}xz - \frac{2785}{2}yz \\ \qquad +\frac{80855}{12}x + \frac{23471}{2}y + 9416z - 9537, \\ \gamma_3 = -625x^2 - 790y^2 + 4780z^2 + 775xy + 625xz \\ \qquad -250yz + 1250x + 2620y - 9160z, \\ \beta_3 = -5405x^2 - 6090y^2 + z^2 + 1806xy + \frac{1375}{3}xz - 249yz \\ \qquad +\frac{28157}{3}x + 13233y + 402z - 9537. \end{cases}$$

4.3 Filling hole with piecewise cubic surfaces

In this subsection, We ought to find cubic surfaces to fill the same hole. As in §3.1 described, we use space partition as figure 3 (right) shown. $V_1 = (1,1,-2)$, $V_2 = (1,-2,1)$, $V_3 = (-2,1,1)$, $\widehat{V}_1 = (\frac{1}{4}, -\frac{1}{8}, -\frac{1}{8})$, $\widehat{V}_2 = (-\frac{1}{8}, \frac{1}{4}, -\frac{1}{8})$, $\widehat{V}_3 = (-\frac{1}{8}, -\frac{1}{8}, \frac{1}{4})$, $Z_0 = (1,1,1)$. Also,

$$\begin{cases}\pi_1 = -2x + 3y - 1 = 0, & \pi_2 = 3x - 2y - 1 = 0, \\ \pi_3 = 3x - 2z - 1 = 0, & \pi_4 = -2x + 3z - 1 = 0, \\ \pi_5 = -2y + 3z - 1 = 0, & \pi_6 = 3y - 2z - 1 = 0, \\ \pi_7 = 3x + 3y - 5z - 1 = 0, & \pi_8 = 3x - 5y + 3z - 1 = 0, \\ \pi_9 = -5x + 3y + 3z - 1 = 0. \end{cases} \tag{5}$$

$\{g_i\}_{i=1}^7$ is defined in each tetrahedron. Then we get equations system:

$$\begin{cases}\alpha_5\pi_5^2 + \alpha_9\pi_9^2 - \alpha_8\pi_8^2 - \alpha_4\pi_4^2 = 0, \\ \alpha_1\pi_1^2 + \alpha_7\pi_7^2 - \alpha_9\pi_9^2 - \alpha_6\pi_6^2 = 0, \\ \alpha_3\pi_3^2 + \alpha_8\pi_8^2 - \alpha_7\pi_7^2 - \alpha_2\pi_2^2 = 0, \\ \gamma_1 f_1 + \beta_1 F_1^2 + \alpha_1\pi_1^2 = \gamma_2 f_2 + \beta_2 F_2^2 + \alpha_2\pi_2^2, \\ \gamma_2 f_2 + \beta_2 F_2^2 + \alpha_3\pi_3^2 = \gamma_3 f_3 + \beta_3 F_3^2 + \alpha_4\pi_4^2. \end{cases} \tag{6}$$

Here, α_i, β_i, γ_i are linear polynomials.

We also convert this equations system to a system of linear equations of the coefficients of monomials. The equations system contains 100 euqations and 60 unknown coefficients, and the rank of this 100×60 system is 55, the dimension of solution space is 5. As in figure 5 shown is a solution:

$$\begin{cases} g_1 = \gamma_1 f_1 + \beta_1 F_1^2, & g_2 = \gamma_2 f_2 + \beta_2 F_2^2, \\ g_3 = \gamma_3 f_3 + \beta_3 F_3^2, & g_4 = g_1 + \alpha_1 \pi_1^2, \\ g_5 = g_2 + \alpha_3 \pi_3^2, & g_6 = g_3 + \alpha_5 \pi_5^2, \\ g_7 = g_4 + \alpha_7 \pi_7^2. \end{cases}$$

Here

$$\begin{cases} \gamma_1 = -\frac{5}{16} y - \frac{5}{16} z + \frac{15}{16}, \\ \beta_1 = \frac{40103}{648} x + \frac{193}{48} y - \frac{77539}{1296} z - \frac{1033}{432}, \\ \gamma_2 = \frac{1}{20} x + \frac{7}{100} y + \frac{1}{20} z - \frac{11}{50}, \\ \beta_2 = -\frac{11}{15} x + \frac{1070909}{16200} y - \frac{209251}{3240} z + \frac{3323}{1350}, \\ \gamma_3 = -\frac{5}{4} x - \frac{5}{4} y - \frac{5}{4} z + 5, \\ \beta_3 = x + y + z + 1, \\ \alpha_1 = -\frac{58219}{5400} x - \frac{140317}{16200} y + \frac{19459}{1620} z + \frac{6193}{900}, \\ \alpha_3 = -\frac{3}{10} x - \frac{61219}{5400} y + \frac{7447}{1800} z + \frac{4691}{675}, \\ \alpha_5 = -\frac{13711}{1620} x - \frac{37}{60} y + \frac{2599}{1620} z + \frac{187}{27}, \\ \alpha_7 = -\frac{28399}{21600} x - \frac{30289}{21600} y + \frac{95}{72} z + \frac{13609}{10800}, \end{cases}$$

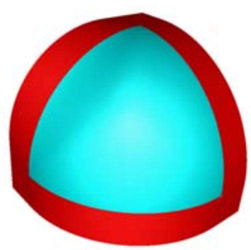

Figure 5. Seven cubic surfaces patches

5 Conclusion

In this paper, we use piecewise algebraic surfaces instead of one algebraic surface to fill holes smoothly. It seems that the piecewise blending surfaces have relatively low degree and have more free parameters to be used for controling the shape of blending surface. The result shows that the piecewise algebraic surface have potential applications in modeling a variaty of geometric objects.

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