

双曲面片的高精度多项式逼近

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摘要 用三次 Bézier 曲线逼近双曲线段, 在端点保持 GC^1 插值, 给出单边逼近的误差, 并进行最优插值点的选择, 得到最优的误差估计; 在此基础上, 用双三次 Bézier 多项式逼近单叶和双叶双曲面片, 给出误差估计, 逼近达到六阶精度. 相邻的逼近片之间 GC^1 连续.

关键词 逼近, 双曲面片, 高精度, 双三次多项式

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High Accuracy Approximation of Hyperboloid Surface Patch by Bicubic Bézier Polynomials

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Abstract Bézier curve is required to interpolate the end points of given curve segment with GC^1 continuity. One-sided approximation error is given and the choice of optimal interpolation points is studied. the optimal error estimate is obtained. Based on the above results for curve segments, the approximation of a hyperboloid surface patch using bicubic Bézier polynomials both for one sheet and two sheets is worked out. Its approximation accuracy is of sixth order. Furthermore the adjacent approximation surface patches have the same tangent plane at their common boundaries.

Key words Bézier approximation, hyperboloid, high accuracy, bicubic polynomials

1 引言

在自由型曲线和曲面的表示和设计中, Bézier 曲线和曲面有着广泛的应用, 但它们不能严格地表示圆锥截线和椭球、双曲面等二次曲面. 因而, 当造型系统不能有效地处理有理曲线和曲面时, 出于不同的几何造型系统之间信息的交换的需要, 常常要求用多项式曲线和曲面来逼近圆锥截线以及椭球、双曲面等二次曲面.

许多人致力于研究用 Bézier 多项式逼近圆^[1-3], 在文献[4]中, 我们对 1/8 椭球面给出了很好的逼近, 文献[5]则给出了对椭球曲面片的高精度逼近, 逼近达到六阶精度. 本文首先考虑用三次 Bézier 多项式逼近双曲线段, 再用双三次多项式分别逼近单叶双曲面片与双叶双曲面片, 逼近达到六阶精度.

设双曲线方程为

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad a, b \in \mathbb{R} \quad (1)$$

双曲面方程为

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad a, b, c \in \mathbb{R} \quad (2)$$

或

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1, \quad a, b, c \in \mathbb{R} \quad (3)$$

三次 Bézier 曲线及双三次 Bézier 曲面分别定义为

$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P_0 B_0^3(t) + P_1 B_1^3(t) + P_2 B_2^3(t) + P_3 B_3^3(t) \quad (4)$$

$$P(s, t) = \begin{pmatrix} X(s, t) \\ Y(s, t) \\ Z(s, t) \end{pmatrix} = \sum_{i+j=0}^3 P_{ij} B_i^3(s) B_j^3(t) \quad (5)$$

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其中, P_i, P_{ij} , $i=0, \dots, 3$, $j=0, \dots, 3$ 分别是 Bézier 曲线和曲面的控制点.

$$P_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}, \quad P_{ij} = \begin{pmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{pmatrix},$$

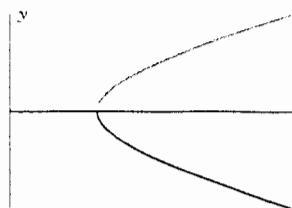
$B_i(t)$ 为 Bernstein 基函数.

引入逼近双曲线的误差函数

$$\epsilon_1(t) = \frac{x(t)^2}{a^2} - \frac{y(t)^2}{b^2} - 1 \quad (6)$$

与逼近单叶和双叶双曲线面的误差函数, 分别为

$$\epsilon_2(s, t) = \frac{X(s, t)^2}{a^2} + \frac{Y(s, t)^2}{b^2} - \frac{Z(s, t)^2}{c^2} - 1 \quad (7)$$



a 给定双曲线弧

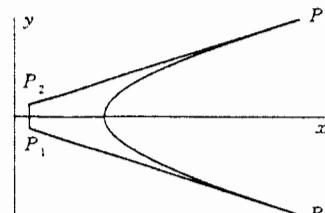
$$\epsilon_2(s, t) = \frac{X(s, t)^2}{a^2} + \frac{Y(s, t)^2}{b^2} - \frac{Z(s, t)^2}{c^2} + 1 \quad (8)$$

2 双曲线的多项式逼近

首先考虑标准双曲线方程(1), 由于对称性, 只需考虑位于第一、四象限的一条曲线, 其参数方程定义为

$$f(\theta) = \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} a \sec \theta \\ b \tan \theta \end{pmatrix}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad (9)$$

若限制弧度宽度为 2α , $0 < \alpha < \frac{\pi}{2}$, 即令 $-\alpha \leq \theta \leq \alpha$, 得到一段有限弧, 如图 1 a 所示.



b 逼近曲线及其控制点

图 1 双曲线弧的逼近

考虑 GC^1 连续的插值, 令 Bezier 曲线 $P(t)$ 经过双曲线段的端点, 且在端点与双曲线有相同的切线方向, 那么控制点 P_i (如图 1 b 所示) 可以计算出来,

$$P_1 = \begin{pmatrix} a \sec \alpha \\ -b \tan \alpha \end{pmatrix},$$

$$P_2 = P_1 + h \cdot \left. \frac{df}{d\theta} \right|_{\theta=-\alpha} = \begin{pmatrix} a \sec \alpha (1 - h \tan \alpha) \\ b(-\tan \alpha + h \sec^2 \alpha) \end{pmatrix},$$

$$P_3 = P_2 - h \cdot \left. \frac{df}{d\theta} \right|_{\theta=\alpha} = \begin{pmatrix} a \sec \alpha (1 - h \tan \alpha) \\ b(\tan \alpha - h \sec^2 \alpha) \end{pmatrix},$$

$$P_4 = \begin{pmatrix} a \sec \alpha \\ b \tan \alpha \end{pmatrix}.$$

其中 h 是一个待定的正常数. 基于以上控制点, 我们得到多项式逼近曲线的参数表示,

$$\begin{cases} x(t) = a \sec \alpha [B_0^3(t) + (1 - h \tan \alpha)(B_1^3(t) + \\ B_2^3(t)) + B_3^3(t)] \\ y(t) = b[-\tan \alpha (B_0^3(t) + B_1^3(t) - B_2^3(t) - \\ B_3^3(t)) + h \sec^2 \alpha (B_1^3(t) - B_2^3(t))] \end{cases} \quad (10)$$

于是得到误差函数 $\epsilon_1(t)$ 为

$$\begin{aligned} \epsilon_1(t) &= \frac{x^2(t)}{a^2} - \frac{y^2(t)}{b^2} - 1 \\ &= (-9h^2 \sec^2 \alpha - 12h \tan \alpha \sec^2 \alpha + 12 \tan^2 \alpha) \\ &\quad (t^2(1-t)^4 + t^4(1-t)^2) + \\ &\quad [18 \sec^2 \alpha (\sec^2 \alpha + \tan^2 \alpha)h^2 - \\ &\quad 72h \tan \alpha \sec^2 \alpha + 40 \tan^2 \alpha]t^3(1-t)^3 \end{aligned} \quad (11)$$

显然 $\epsilon_1(t) = \epsilon_1(1-t)$, 由明显的对称性, 设 $\epsilon_1(1/2) = 0$, 可以

$$解出 h = \frac{4}{3} \frac{1-\cos \alpha}{\tan \alpha} = \frac{4}{3} \tan \frac{\alpha}{2} \cos \alpha.$$

将 h 代入式(11), 可得

$$\epsilon_1(t) = -4 \tan^2 \alpha \tan^4 \frac{\alpha}{2} t^2 (1-t)^2 (2t-1)^2 \quad (12)$$

由于 $\max_{t \in [0, 1]} t^2(1-t)^2(2t-1)^2 = 1/108$, 于是得到

$$\|\epsilon_1(t)\|_\infty = \frac{4}{27} \frac{\sin^6 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}} \cdot \sec^2 \alpha \quad (13)$$

从式(13)可知, 我们的逼近是单边逼近. 表 1 给出了当 α 取不同值时的误差函数的极大模.

表 1 单边逼近的误差

$\alpha =$	$\frac{\pi}{3}$	$\frac{2\pi}{9}$	$\frac{\pi}{6}$	$\frac{\pi}{9}$	$\frac{\pi}{12}$
$\ \epsilon_1(t)\ _\infty \approx$	1.235×10^{-2}	4.576×10^{-4}	6.364×10^{-5}	4.743×10^{-6}	7.988×10^{-7}

由于上述逼近是单边逼近, 且逼近曲线总在双曲线的左边(因为 $\epsilon_1(t) \leq 0$), 这就意味着存在另一种绝对误差更小的逼近. 我们考虑在上述 GC^1 连续的假设下重新选择插值点. 令 $\epsilon_1(r) = 0$, 则同时有 $\epsilon_1(1-r) = 0$. 因此可设 $0 < r < 1/2$, 将误差函数 $\epsilon_1(t)$ 改写为 $\epsilon_1(t) = t^2(1-t)^2 p_2(t)$.

其中, $p_2(t) = (-9h^2 \sec^2 \alpha - 12h \tan \alpha \sec^2 \alpha + 12 \tan^2 \alpha) [t^2(1-t)^2] + [18 \sec^2 \alpha (\sec^2 \alpha + \tan^2 \alpha)h^2 - 72h \tan \alpha \sec^2 \alpha + 40 \tan^2 \alpha]t(1-t)$.

将其写为关于 h 的多项式 $p_2(t) = -4 \tan^2 \alpha (2t+1)(2t-3) + 12h \sec^2 \alpha \tan \alpha (4t^2 - 4t - 1) - 9h^2 \sec^4 \alpha (4t^2 - 4t +$

$\cos^2 \alpha$.

由 $\epsilon_1(r) = 0$, 可得 $p_2(r) = 0$, 于是解出

$$h = \frac{\sin 2\alpha [4r^2 - 4r - 1 + \sqrt{\sin^2 \alpha (4r^2 - 4r) + 1 + 3\cos^2 \alpha}]}{3(4r^2 - 4r + \cos^2 \alpha)} \quad (14)$$

由于 r 是 $p_2(t) = 0$ 的根, 则误差函数可写作

$$\epsilon_1(t) = C(r)t^2(1-t)^2(t-r)(t-1+r) \quad (15)$$

其中, $C(r)$ 是 $p_2(t)$ 中 t^2 的系数, 把它写为关于 h 的多项式, 由式(14)知, h 是 r 的函数, 所以将其记为关于 r 的函数 $C(r) = -[16\tan^2 \alpha - 48h\sec^2 \alpha \tan \alpha + 36h^2\sec^4 \alpha] = -[6h\sec^2 \alpha - 4\tan \alpha]^2$.

解方程 $\epsilon_1'(t) = C(r) \cdot t(1-t)(1-2t)(3t^2 - 3t - 2r^2 + 2r) = 0$, 得到误差函数的极值点 $t_1 = 0$, $t_2 = 1$, $t_3 = 1/2$, $t_{4,5} = \frac{1}{2} \pm \frac{1}{6}\sqrt{24r^2 - 24r + 9}$. 于是得到误差函数的极值 $\epsilon_1(t_1) = \epsilon_1(t_2) = 0$, $\epsilon_1(t_3) = -\frac{1}{64}C(r)(2r-1)^2$, $\epsilon_1(t_4) = \epsilon_1(t_5) =$

$$\epsilon_1(t_4) = 0, \epsilon_1(t_5) = -\frac{1}{64}C(r)(2r-1)^2, \epsilon_1(t_4) = \epsilon_1(t_5) =$$

$$\frac{4}{27}C(r)r^3(1-r)^3.$$

为使误差函数模最小, 令上下振幅相等, 即 $|\epsilon_1(t_3)| = |\epsilon_1(t_4)|$, 于是有

$$\frac{1}{64}(1-2r)^2 = \frac{4}{27}r^3(1-r)^3 \quad (16)$$

令 $T = \frac{8}{3}r(1-r)$, 那么,

$$(1-2r)^2 = 1 - \frac{3}{2}T \quad (17)$$

将其代入式(16), 得到 $T^3 + 3T - 2 = 0$, 解得 $T = \sqrt[3]{\sqrt{2} + 1} - \sqrt[3]{\sqrt{2} - 1}$. 将 T 代入(17), 得到

$$r = \frac{1}{2} \left(1 - \sqrt{1 + \frac{3}{2} \left(\sqrt[3]{\sqrt{2} - 1} - \sqrt[3]{\sqrt{2} + 1} \right)} \right) \approx 0.3373 \quad (18)$$

从式(14)~(16)及式(18), 可以求得 $\|\epsilon_1(t)\|_\infty$. 表 2 所示为 $\epsilon_1(r) = 0$ 时的 $\|\epsilon_1(t)\|_\infty$.

表 2 双边逼近的误差

$\alpha =$	$\frac{\pi}{3}$	$\frac{2\pi}{9}$	$\frac{\pi}{6}$	$\frac{\pi}{9}$	$\frac{\pi}{12}$
$\ \epsilon_1(t)\ _\infty \approx$	8.527×10^{-3}	3.242×10^{-4}	4.529×10^{-5}	3.384×10^{-4}	5.704×10^{-5}

比较表 1 和表 2 中的结果可以看出, 表 2 中的误差是表 1 中相应误差的 70% 左右, 但后者的逼近不是单边逼近.

3 双曲面片的多项式逼近

我们首先考虑单叶双曲面式(2), 它的参数方程为

$$f(\theta, \varphi) = \begin{bmatrix} a \sec \beta \cos \varphi \\ b \sec \beta \sin \varphi \\ c \tan \beta \end{bmatrix}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad 0 \leq \varphi \leq 2\pi \quad (19)$$

考虑如图 2 a 所示的曲面片, 其中 $-\beta \leq \theta \leq \beta < \pi/2$, $0 \leq \varphi \leq \alpha \leq \pi/2$.



a 单叶双曲面片 d 双叶双曲面片

图 2 待逼近的双曲面片

我们用含两个参数的双三次 Bézier 曲面来逼近这一曲面片. 由方程(5), 只要求出各控制点的坐标, 就可以得到逼近双曲面的多项式曲面片. 仍然要求 $P(s, t)$ 在顶点 $(\sec \beta, 0, \tan \beta)^T$, $(\sec \beta, 0, -\tan \beta)^T$, $(\sec \beta \cos \alpha, \sec \beta \sin \alpha, 0)^T$, $(-\sec \beta \cos \alpha, -\sec \beta \sin \alpha, 0)^T$ 与 $(\sec \beta \cos \alpha, \sec \beta \sin \alpha, \tan \beta)^T$ 插值于该双曲面片, 且在这些点与双曲面片有相同的切平面.

有了上述插值要求, 容易算出

$$P_{00} = \begin{bmatrix} \sec \beta \\ 0 \\ \tan \beta \end{bmatrix}, \quad P_{10} = \begin{bmatrix} \sec \beta \\ 0 \\ -\tan \beta \end{bmatrix},$$

$$P_{01} = \begin{bmatrix} \sec \beta \cos \alpha \\ b \sec \beta \sin \alpha \\ \tan \beta \end{bmatrix}, \quad P_{11} = \begin{bmatrix} \sec \beta \cos \alpha \\ b \sec \beta \sin \alpha \\ -\tan \beta \end{bmatrix}.$$

$$\text{根据以上假设, 可令 } \frac{\partial P(s, t)}{\partial s} \Big|_{(0,0)} = \sigma_1 \frac{\partial f}{\partial \theta} \Big|_{(\beta,0)}, \quad \frac{\partial P(s, t)}{\partial s} \Big|_{(1,0)} = \sigma_1 \frac{\partial f}{\partial \theta} \Big|_{(-\beta,0)}, \quad \text{于是有}$$

$$P_{10} = \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) \\ 0 \\ c(\tan \beta - h_1 \sec^2 \beta) \end{bmatrix},$$

$$P_{20} = \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) \\ 0 \\ -c(\tan \beta - h_1 \sec^2 \beta) \end{bmatrix}.$$

其中, $h_1 = -\frac{1}{3}\sigma_1$. 同样可设

$$\frac{\partial P(s, t)}{\partial t} \Big|_{(0,0)} = \sigma_2 \frac{\partial f}{\partial \varphi} \Big|_{(\beta,0)}, \quad \frac{\partial P(s, t)}{\partial t} \Big|_{(0,1)} = \sigma_2 \frac{\partial f}{\partial \varphi} \Big|_{(\beta,\alpha)},$$

$$\frac{\partial P(s, t)}{\partial t} \Big|_{(1,0)} = \sigma_2 \frac{\partial f}{\partial \varphi} \Big|_{(-\beta,0)}, \quad \frac{\partial P(s, t)}{\partial t} \Big|_{(1,1)} = \sigma_2 \frac{\partial f}{\partial \varphi} \Big|_{(-\beta,\alpha)},$$

$$\frac{\partial P(s, t)}{\partial s} \Big|_{(0,1)} = \sigma_1 \frac{\partial f}{\partial \theta} \Big|_{(\beta,\alpha)}, \quad \frac{\partial P(s, t)}{\partial s} \Big|_{(1,1)} = \sigma_1 \frac{\partial f}{\partial \theta} \Big|_{(-\beta,\alpha)}.$$

令 $h_2 = \frac{1}{3}\sigma_2$, 立即得到下面的控制点

$$P_{01} = P_{00} + h_2 \frac{\partial f}{\partial \varphi} \Big|_{(\beta,0)} = \begin{bmatrix} \sec \beta \\ bh_2 \sec \beta \\ \tan \beta \end{bmatrix},$$

$$\begin{aligned}
 P_{01} &= P_{03} - h_2 \frac{\partial f}{\partial \varphi} \Big|_{(\beta, \alpha)} = \begin{cases} \sec \beta \cos \alpha + ah_2 \sec \beta \sin \alpha \\ b \sec \beta \sin \alpha - bh_2 \sec \beta \cos \alpha \\ -ctan \beta \end{cases}, \\
 P_{31} &= P_{30} + h_2 \frac{\partial f}{\partial \varphi} \Big|_{(-\beta, \alpha)} = \begin{cases} \sec \beta \\ bh_2 \sec \beta \\ -ctan \beta \end{cases}, \\
 P_{32} &= P_{33} - h_2 \frac{\partial f}{\partial \varphi} \Big|_{(-\beta, \alpha)} = \begin{cases} \sec \beta \cos \alpha + ah_2 \sec \beta \sin \alpha \\ b \sec \beta \sin \alpha - bh_2 \sec \beta \cos \alpha \\ -ctan \beta \end{cases}, \\
 P_{23} &= P_{33} + h_1 \frac{\partial f}{\partial \theta} \Big|_{(-\beta, \alpha)} = \begin{cases} \sec \beta \cos \alpha - ah_1 \sec \beta \tan \beta \cos \alpha \\ b \sec \beta \sin \alpha - bh_1 \sec \beta \tan \beta \sin \alpha \\ -ctan \beta + ch_1 \sec^2 \beta \end{cases}, \\
 P_{13} &= P_{03} - h_1 \frac{\partial f}{\partial \theta} \Big|_{(\beta, \alpha)} = \begin{cases} \sec \beta \cos \alpha - ah_1 \sec \beta \tan \beta \cos \alpha \\ b \sec \beta \sin \alpha - bh_1 \sec \beta \tan \beta \sin \alpha \\ ctan \beta - ch_1 \sec^2 \beta \end{cases}.
 \end{aligned}$$

现在, 只剩下控制点 $P_{11}, P_{12}, P_{21}, P_{22}$ 的坐标还没有确定, 我们可以要求在插值点的二阶混合偏导数满足

$$\frac{\partial^2 P(s, t)}{\partial s \partial t} \Big|_{(0,0)} = \sigma_1 \sigma_2 \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{(\beta, \alpha)}, \text{ 而 } \frac{\partial^2 P(s, t)}{\partial t \partial \theta} \Big|_{(0,0)} = 9(P_{11} -$$

$$P_{10} - P_{01} + P_{00}), \quad \sigma_1 \sigma_2 \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{(\beta, \alpha)} = \sigma_1 \sigma_2 \begin{bmatrix} 0 \\ b \sec \beta \tan \beta \\ 0 \end{bmatrix}, \text{ 则}$$

$$P_{11} = \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) \\ bh_2 \sec \beta (1 - h_1 \tan \beta) \\ c (\tan \beta - h_1 \sec^2 \beta) \end{bmatrix}.$$

同样, 设

$$\begin{aligned}
 \frac{\partial^2 P(s, t)}{\partial s \partial t} \Big|_{(1,1)} &= \sigma_1 \sigma_2 \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{(\beta, \alpha)}, \\
 \frac{\partial^2 P(s, t)}{\partial s \partial t} \Big|_{(1,0)} &= \sigma_1 \sigma_2 \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{(-\beta, \alpha)}, \\
 \frac{\partial^2 P(s, t)}{\partial s \partial t} \Big|_{(0,1)} &= \sigma_1 \sigma_2 \frac{\partial^2 f}{\partial \theta \partial \varphi} \Big|_{(-\beta, \alpha)};
 \end{aligned}$$

于是, 剩下的控制点的坐标都被确定

$$\begin{aligned}
 P_{12} &= \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) (\cos \alpha + h_2 \sin \alpha) \\ b \sec \beta (1 - h_1 \tan \beta) (\sin \alpha - h_2 \cos \alpha) \\ c (\tan \beta - h_1 \sec^2 \beta) \end{bmatrix}, \\
 P_{21} &= \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) \\ bh_2 \sec \beta (1 - h_1 \tan \beta) \\ -c (\tan \beta - h_1 \sec^2 \beta) \end{bmatrix}, \\
 P_{22} &= \begin{bmatrix} \sec \beta (1 - h_1 \tan \beta) (\cos \alpha + h_2 \sin \alpha) \\ b \sec \beta (1 - h_1 \tan \beta) (\sin \alpha - h_2 \cos \alpha) \\ -c (\tan \beta - h_1 \sec^2 \beta) \end{bmatrix}.
 \end{aligned}$$

于是, $P(s, t) = \begin{pmatrix} X(s, t) \\ Y(s, t) \\ Z(s, t) \end{pmatrix}$ 的各坐标为

$$X(s, t) = \sum_{i,j=0}^3 X_{ij} B_i^3(s) B_j^3(t) = a f_1(s, \beta, h_1) g_1(t, \alpha, h_2),$$

$$Y(s, t) = \sum_{i,j=0}^3 Y_{ij} B_i^3(s) B_j^3(t) = b f_1(s, \beta, h_1) g_2(t, \alpha, h_2),$$

$$Z(s, t) = \sum_{i,j=0}^3 Z_{ij} B_i^3(s) B_j^3(t) = c f_2(s, \beta, h_1).$$

其中,

$$\begin{aligned}
 f_1(s, \beta, h_1) &= \sec \beta [B_0^3(s) + (1 - h_1 \tan \beta) B_1^3(s) + \\
 &\quad (1 - h_1 \tan \beta) B_2^3(s) + B_3^3(s)] \\
 g_1(t, \alpha, h_2) &= B_0^3(t) + B_1^3(t) + (\cos \alpha + h_2 \sin \alpha) B_2^3(t) + \\
 &\quad \cos \alpha B_3^3(t) \\
 f_2(s, \beta, h_1) &= \tan \beta B_0^3(s) + (\tan \beta - h_1 \sec^2 \beta) (B_1^3(s) - \\
 &\quad B_2^3(s)) - \tan \beta B_3^3(s) \\
 g_2(t, \alpha, h_2) &= h_2 B_1^3(t) + (\sin \alpha - h_2 \cos \alpha) B_2^3(t) + \\
 &\quad \sin \alpha B_3^3(t)
 \end{aligned} \tag{20}$$

将以上的 $X(s, t), Y(s, t), Z(s, t)$ 代入式(7), 得到插值误差函数的表达式为

$$\begin{aligned}
 \epsilon_2(s, t) &= \frac{X^2(s, t)}{a^2} + \frac{Y^2(s, t)}{b^2} - \frac{Z^2(s, t)}{c^2} - 1 \\
 &= f_1^2(s, \beta, h_1) [g_1^2(t, \alpha, h_2) + g_2^2(t, \alpha, h_2) - 1] + \\
 &\quad f_1^2(s, \beta, h_1) - f_2^2(s, \beta, h_1) - 1.
 \end{aligned}$$

容易看出, $g_1^2(t, \alpha, h_2) + g_2^2(t, \alpha, h_2) - 1$ 正是文献[5]中逼近宽为 α 的椭圆弧时的误差函数. 利用文献[5]中结果或直接计算可知, 当 $h_2 = \frac{4}{3} \tan \frac{\alpha}{4}$ 时,

$$g_1^2 + g_2^2 - 1 = 16 \frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} \cdot t^2 (1-t)^2 (2t-1)^2 \tag{21}$$

而 $f_1^2(s, \beta, h_1) - f_2^2(s, \beta, h_1) - 1$ 正是式(10). (11). 所以当 $h_1 = \frac{4}{3} \tan \frac{\beta}{2} \cos \beta, h_2 = \frac{4}{3} \tan \frac{\alpha}{4}$ 时, 由式(21), (13)得到

$$\epsilon_2(s, t) \leq \frac{4}{27} \frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} f_1^2(s, \beta, h_1) + \frac{4}{27} \frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} \sec^2 \beta.$$

注意到 $|1 - h_1 \tan \beta| = \frac{1}{3} |4 \cos \beta - 1| \leq 1$, 导致 $|f_1(s, \beta, h_1)| \leq \sec \beta [B_0^3(s) + B_1^3(s) + B_2^3(s) + B_3^3(s)] = \sec \beta$.

综上所述, 我们有以下定理.

定理 1. 设双三次 Bézier 曲面的控制点 P_{ij} 由上述方法定出, 若取

$$h_1 = \frac{4}{3} \tan \frac{\beta}{2} \cos \beta, \quad h_2 = \frac{4}{3} \tan \frac{\alpha}{4},$$

则 Bézier 曲面

$$P(s, t) = \sum_{i,j=0}^3 P_{ij} B_i^3(s) B_j^3(t), \quad s, t \in [0, 1]$$

插值于双曲面式(19)上, 如图 2 a 所示, 曲面片 ($-\beta \leq \theta \leq \beta < \frac{\pi}{2}, 0 \leq \varphi \leq \alpha \leq \frac{\pi}{2}$) 的顶点 $P_{00}, P_{03}, P_{30}, P_{33}$, 且在这些点与曲面片有相同的切平面, 逼近误差 $\epsilon_2(s, t)$ 满足

$$\begin{aligned}
 \|\epsilon_2(s, t)\|_\infty &= \max_{s, t \in [0, 1]} \left[\frac{X^2(s, t)}{a^2} + \frac{Y^2(s, t)}{b^2} - \frac{Z^2(s, t)}{c^2} - 1 \right] \\
 &\leq \frac{4}{27} \left[\frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} + \frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} \right] \sec^2 \beta.
 \end{aligned}$$

表 3 所示为当 $\beta = \frac{\alpha}{2}$ 时, 逼近误差 $\epsilon_2(s, t)$ 的最大值.

表 3 $\|\epsilon_2(s,t)\|_\infty$

$\alpha =$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{8}$
$\ \epsilon_2(s,t)\ _\infty \leqslant$	2.181×10^{-3}	1.28×10^{-4}	1.99×10^{-5}	1.60×10^{-6}	2.76×10^{-7}

下面考虑双叶双曲面片, 如图 2 b 所示.

$$f(\theta, \varphi) = \begin{bmatrix} \tan \theta \cos \varphi \\ \tan \theta \sin \varphi \\ \csc \theta \end{bmatrix},$$

$$-\beta \leq \theta \leq \beta < \frac{\pi}{2}, 0 \leq \varphi \leq \alpha \leq \frac{\pi}{2},$$

的多项式逼近. 仍设逼近多项式曲面 $P(s,t)$ 由式(5)定义, 并要求 $P(s,t)$ 插值于双叶双曲面片的顶点 $P_{00}, P_{03}, P_{30}, P_{33}$, 且在这些点与曲面片有相同的切平面. 用与计算单叶双曲面片的控制点类似的算法, 得到控制点 $P_{ij}, i,j=0,1,2,3$ 如下:

$$P_{00} = \begin{bmatrix} \tan \beta \\ 0 \\ \csc \beta \end{bmatrix}, P_{01} = \begin{bmatrix} \tan \beta \\ bh_2 \tan \beta \\ \csc \beta \end{bmatrix},$$

$$P_{02} = \begin{bmatrix} \tan \beta (\cos \alpha + h_2 \sin \alpha) \\ b \tan \beta (\sin \alpha - h_2 \cos \alpha) \\ \csc \beta \end{bmatrix}, P_{03} = \begin{bmatrix} \tan \beta \cos \alpha \\ b \tan \beta \sin \alpha \\ \csc \beta \end{bmatrix}.$$

$$P_{10} = \begin{bmatrix} a(\tan \beta - h_1 \sec^2 \beta) \\ 0 \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} a(\tan \beta - h_1 \sec^2 \beta) \\ bh_2(\tan \beta - h_1 \sec^2 \beta) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} a(\tan \beta - h_1 \sec^2 \beta)(\cos \alpha + h_2 \sin \alpha) \\ b(\tan \beta - h_1 \sec^2 \beta)(\sin \alpha - h_2 \cos \alpha) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{13} = \begin{bmatrix} a \cos \alpha (\tan \beta - h_1 \sec^2 \beta) \\ b \sin \alpha (\tan \beta - h_1 \sec^2 \beta) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{20} = \begin{bmatrix} -a(\tan \beta - h_1 \sec^2 \beta) \\ 0 \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{21} = \begin{bmatrix} -a(\tan \beta - h_1 \sec^2 \beta) \\ -bh_2(\tan \beta - h_1 \sec^2 \beta) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{22} = \begin{bmatrix} -a(\tan \beta - h_1 \sec^2 \beta)(\cos \alpha + h_2 \sin \alpha) \\ -b(\tan \beta - h_1 \sec^2 \beta)(\sin \alpha - h_2 \cos \alpha) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{23} = \begin{bmatrix} -a \cos \alpha (\tan \beta - h_1 \sec^2 \beta) \\ -b \sin \alpha (\tan \beta - h_1 \sec^2 \beta) \\ \csc \beta (1 - h_1 \tan \beta) \end{bmatrix},$$

$$P_{30} = \begin{bmatrix} -\tan \beta \\ 0 \\ \csc \beta \end{bmatrix}, P_{31} = \begin{bmatrix} -\tan \beta \\ -bh_2 \tan \beta \\ \csc \beta \end{bmatrix},$$

$$P_{32} = \begin{bmatrix} -\tan \beta (\cos \alpha + h_2 \sin \alpha) \\ -b \tan \beta (\sin \alpha - h_2 \cos \alpha) \\ \csc \beta \end{bmatrix},$$

$$P_{33} = \begin{bmatrix} -\tan \beta \cos \alpha \\ -b \tan \beta \sin \alpha \\ \csc \beta \end{bmatrix}.$$

由以上控制点产生的双三次 Bézier 曲面片为

$$P(s,t) = \begin{bmatrix} X(s,t) \\ Y(s,t) \\ Z(s,t) \end{bmatrix} = \begin{bmatrix} af_1(s, \beta, h_1)g_1(t, \alpha, h_2) \\ bf_2(s, \beta, h_1)g_2(t, \alpha, h_2) \\ cf_1(s, \beta, h_1) \end{bmatrix}.$$

其中 f_1, g_1, f_2, g_2 为由式(20)所定义的函数, 于是由式(8)所定义的逼近误差

$$\epsilon_2^*(s,t) = f_1^2(g_1^2 + g_2^2 - 1) - (f_1^2 - f_2^2 - 1) \quad (22)$$

当取 $h_1 = \frac{4}{3} \tan \frac{\beta}{2} \cos \beta$, $h_2 = \frac{4}{3} \tan \frac{\alpha}{4}$ 时, 由式(21)以及式(10)~(12)知,

$$\|g_1^2 + g_2^2 - 1\| \leq \frac{4}{27} \frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} \quad (23)$$

以及

$$\|f_1^2 - f_2^2 - 1\| \leq \frac{4}{27} \frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} \cdot \sec^2 \beta \quad (24)$$

注意到

$$\begin{aligned} \|\tan \beta - h_1 \sec^2 \beta\| &= \|\tan \beta - \frac{4}{3} \tan \frac{\beta}{2} \sec \beta\| \\ &= \frac{1}{3} |\tan \beta| \|1 - 2 \tan^2 \frac{\beta}{2}\| \leq \frac{1}{3} |\tan \beta|. \end{aligned}$$

从而得到

$$\begin{aligned} \|f_2(s, \beta, h_1)\| &\leq \tan \beta [B_0^3(s) + B_1^3(s) + B_2^3(s) + B_3^3(s)] \\ &= \tan \beta < \sec \beta \end{aligned} \quad (25)$$

从式(22)~(25)得到

$$\|\epsilon_2^*(s,t)\|_\infty \leq \frac{4}{27} \left[\frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} \tan^2 \beta + \frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} \sec^2 \beta \right].$$

这就证明了定理 2.

定理 2. 由上述控制点产生的 Bézier 曲面

$$P(s,t) = \begin{bmatrix} af_1(s, \beta, h_1)g_1(t, \alpha, h_2) \\ bf_2(s, \beta, h_1)g_2(t, \alpha, h_2) \\ cf_1(s, \beta, h_1) \end{bmatrix}$$

在诸顶点 $P_{00}, P_{03}, P_{30}, P_{33}$ 插值于上述双叶双曲面片, 且在这些点与曲面片有相同的切平面, 误差函数 $\epsilon_2^*(s,t)$ 满足

$$\|\epsilon_2^*(s,t)\|_\infty \leq \frac{4}{27} \left[\frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} \tan^2 \beta + \frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} \sec^2 \beta \right] \leq$$

$$\frac{4}{27} \left[\frac{\sin^6 \frac{\alpha}{4}}{\cos^2 \frac{\alpha}{4}} + \frac{\sin^6 \frac{\beta}{2}}{\cos^2 \frac{\beta}{2}} \right] \sec^2 \beta.$$

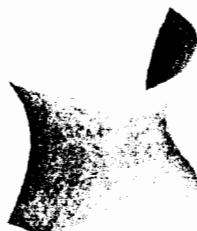
表4所示为当 $\beta = \frac{\alpha}{2}$ 时,误差函数 $\epsilon_2^*(s,t)$ 对不同的 α 时的最大值。

表4 $\|\epsilon_2^*(s,t)\|_\infty$

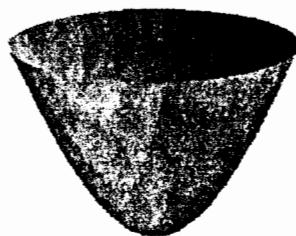
$\alpha =$	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{8}$
$\ \epsilon_2(s,t)\ _\infty \leq$	1.64×10^{-3}	7.96×10^{-5}	1.15×10^{-5}	8.53×10^{-7}	1.44×10^{-7}

从表3和表4易见,逼近达到六阶精度,并且,相邻的逼近片之间达到 GC^1 连续,图3 a, 3 b 显示出相邻两片逼近曲

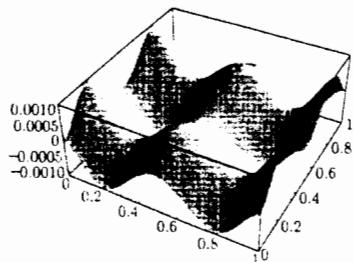
面片的 GC^1 拼接的结果,误差函数的图像如图3 c, 3 d 所示。



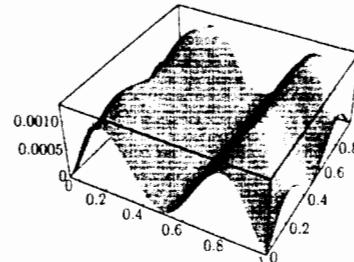
a 两片单叶双曲面片的逼近曲面片的拼接



b 两片双叶双曲面片的逼近曲面片的拼接



c $\epsilon_2^*(s,t)$, $\beta = \frac{\alpha}{2}$, $\alpha = \pi/2$



d $\epsilon_2^*(s,t)$, $\beta = \frac{\alpha}{2}$, $\alpha = \pi/8$

图3 双曲面片的 GC^1 逼近与误差

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