

# Morgan-Scott 剖分上样条空间 $S_6^3(\Delta_{ms})$ 的维数<sup>\*</sup>

陈之兵 邓建松 奚梅成 冯玉瑜

(中国科学技术大学数学系)

**摘要** 众所周知,Morgen-Scott 剖分上的样条空间的维数依赖于剖分的几何性质. 本文证明了 D. Diener 提出的猜想对  $r = 3$  是正确的.

**关键词** 计算机辅助几何设计, 多变量样条, M-S 剖分

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## 1 引言

设  $\Delta_{ms}$  是 Morgen-Scott 剖分(图 1),  $V_1, V_2, V_3$  和  $\hat{V}_1, \hat{V}_2, \hat{V}_3$  分别是剖分边界上以及内部的顶点,  $V_i, \hat{V}_i$  位于  $\hat{V}_{i+1}, \hat{V}_{i+2}$  连线的两边,  $i = 1, 2, 3$ . 我们约定下标是模 3 的, 即  $V_4 = V_1$ ,  $V_{-1} = V_2$  等等. 用  $T$  表示  $\Delta\hat{V}_1\hat{V}_2\hat{V}_3$ ,  $T_i, \hat{T}_i$  分别表示  $\Delta V_i\hat{V}_{i+1}\hat{V}_{i+2}$  和  $\Delta\hat{V}_{i-1}V_{i+1}\hat{V}_i$ ,  $i = 1, 2, 3$ . 又设  $(r_i, s_i, t_i)$  是  $V_i$  对于三角形  $T$  的面积坐标, 即  $V_i = r_i\hat{V}_i + s_i\hat{V}_{i+1} + t_i\hat{V}_{i+2}$ ,  $i = 1, 2, 3$ . 显然  $r_i < 0$ . 又因为在同一个内顶点处的边有不同的斜率, 从而  $s_i \neq 0 \neq t_i$ . 在剖分  $\Delta_{ms}$  上定义样条空间

$$S_d^r(\Delta_{ms}) = \{s \in C^r(\Delta_{ms}) : s|_{\text{子三角形}} \in \mathbf{P}_d\}$$

这里  $\mathbf{P}_d$  是维数为  $(d+2)(d+1)/2$  的总次数  $\leq d$  的两变量多项式空间. 空间  $S_d^r(\Delta_{ms})$  的维数不仅依赖于  $r, d$ , 还依赖于剖分的几何性质.

我们回顾一些已知的结果: 在 [1] 中证明了  $\dim S_2^1(\Delta_{ms}) = 6$  或 7, 并指出若剖分是对称的, 那么维数为 7. [2] 中指出即使不对称的情况, 维数也可能为 7. 最近 [3, 4, 5] 从不同的途径, 完满地解决了这一问题, 得到了

$$6 \leq \dim S_2^1(\Delta_{ms}) \leq 7 \text{ 且 } \dim S_2^1(\Delta_{ms}) = 7 \Leftrightarrow V_i\hat{V}_i, i = 1, 2, 3, \text{ 交于一点.}$$

对一般的  $r$ , 在 [4] 中证明了如下的定理:  $\alpha + \sigma \leq \dim S_{2r}^r(\Delta_{ms}) \leq \alpha + \sigma + 1$ , 且只要

$$\begin{aligned} s_1s_2s_3 &= t_1t_2t_3, \quad \text{当 } r \text{ 为奇数} \\ s_1s_2s_3 &= \pm t_1t_2t_3, \quad \text{当 } r \text{ 为偶数} \end{aligned} \tag{1}$$

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不成立, 必有  $\dim S_{2r}^r(\Delta_{ms}) = \alpha + \sigma$ . 这里

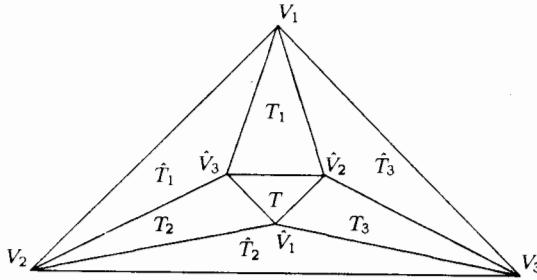


图 1 Morgan-Scott 剖分

Fig.1 Morgan-Scott partition

$V_i\hat{V}_i, i = 1, 2, 3$ , 交于一点.

## 2 平滑条件

我们利用两变量样条的 Bernstein-Bézier 表示. 在剖分  $\Delta_{ms}$  中令

$$\begin{aligned} P_{i,j,k}^{[l]} &= (iV_l + j\hat{V}_{l+1} + k\hat{V}_{l+2})/6, & i+j+k &= 6, l = 1, 2, 3 \\ \hat{P}_{i,j,k}^{[l]} &= (i\hat{V}_{l-1} + jV_{l+1} + kV_l)/6, \end{aligned} \quad (2)$$

分别表示子三角形  $T_l, \hat{T}_l$  内的格点. 相应地, 分别用  $\beta_{i,j,k}^{[l]}, \hat{\beta}_{i,j,k}^{[l]}$  表示  $P_{i,j,k}^{[l]}, \hat{P}_{i,j,k}^{[l]}$  处的 Bézier 纵坐标.

又设  $a_{i,j}, b_{i,j}, c_{i,j}$  是顶点  $V_j$  关于三角形  $T_i$  的面积坐标, 即

$$V_j = a_{i,j}V_i + b_{i,j}\hat{V}_{i+1} + c_{i,j}\hat{V}_{i+2} \quad (3)$$

为表示的方便, 引进移位算子<sup>[6]</sup>  $E_l, l = 1, 2, 3$  定义为

$$\begin{aligned} E_1 f_{i,j,k} &:= f_{i+1,j,k} \\ E_2 f_{i,j,k} &:= f_{i,j+1,k} \\ E_3 f_{i,j,k} &:= f_{i,j,k+1} \end{aligned} \quad (4)$$

那么样条  $s \in S_6^3(\Delta_{ms})$  限制在  $T_l$  上的 B-B 多项式表示为

$$s_l(a, b, c) = \sum_{i+j+k=6} \beta_{i,j,k}^{[l]} \frac{6!}{i!j!k!} a^i b^j c^k \quad (5)$$

这里  $(a, b, c)$  是关于三角形  $T_l$  的面积坐标.

同样  $s$  限制在  $\hat{T}_l$  上的 B-B 多项式表示为

$$\hat{s}_l(\hat{a}, \hat{b}, \hat{c}) = \sum_{i+j+k=6} \hat{\beta}_{i,j,k}^{[l]} \frac{6!}{i!j!k!} \hat{a}^i \hat{b}^j \hat{c}^k \quad (6)$$

这里  $(\hat{a}, \hat{b}, \hat{c})$  是关于三角形  $\hat{T}_l$  上的面积坐标.

我们假定样条  $s$  在三角形  $T$  上为 0, 那么由相邻两个多项式片在公共边上的光滑条件<sup>[7]</sup>知

$$\begin{aligned} \beta_{i,j,k}^{[l]} &= 0, i \leq 3 \\ \hat{\beta}_{i,j,k}^{[l]}, &\quad i \geq 3 \end{aligned} \quad l = 1, 2, 3 \quad (7)$$

$$\alpha = \binom{2r+2}{2}, \sigma = 3 \sum_{j=1}^r (r+1-3j)_+$$

并且作者猜想, 条件(1)对于  $\dim S_{2r}^r(\Delta_{ms}) = \alpha + \sigma + 1$  也是充分的.

在 [3], [4] 中从不同的途径对  $r = 1, 2$  证明了上述猜想. 本文的目的是用 B 网方法证明  $r = 3$  时猜想也是正确的, 即证明了

$$\dim S_6^3(\Delta_{ms}) = 32 \Leftrightarrow s_1 s_2 s_3 = t_1 t_2 t_3, \text{ 即}$$

余下的光滑条件为

$$\begin{aligned}\hat{\beta}_{i,j,k}^{[l]} &= (a_{l,l+1}E_1 + b_{l,l+1}E_2 + c_{l,l+1}E_3)^j \beta_{k,0,i}^{[l]} \\ \hat{\beta}_{i,j,k}^{[l]} &= (a_{l+1,l}E_1 + b_{l+1,l}E_2 + c_{l+1,l}E_3)^k \beta_{j,i,0}^{[l+1]}\end{aligned}\quad (8)$$

这里  $i \leq 2, i+j+k = 6, 1 \leq j, k \leq 3, l = 1, 2, 3$ . 由于  $\hat{\beta}_{i,j,k}^{[l]}$  由两个方程决定, 从而得到如下齐次方程组:

$$(a_{l,l+1}E_1 + b_{l,l+1}E_2 + c_{l,l+1}E_3)^j \beta_{k,0,i}^{[l]} = (a_{l+1,l}E_1 + b_{l+1,l}E_2 + c_{l+1,l}E_3)^k \beta_{j,i,0}^{[l+1]} \quad (9)$$

这里  $i \leq 2, 1 \leq j, k \leq 3, i+j+k = 6, l = 1, 2, 3$ .

对给定的  $l = 1, 2, 3$ , 考虑(9)中的来自纵标  $\hat{\beta}_{2,3,1}^{[l]}, \hat{\beta}_{2,2,2}^{[l]}, \hat{\beta}_{2,1,3}^{[l]}$  的三个方程, 注意到(7)式, 得到齐次方程组:

$$\begin{pmatrix} a_{l+1,l}^3 & -a_{l,l+1} \\ a_{l+1,l}^2 & -a_{l,l+1}^2 \\ a_{l+1,l} & -a_{l,l+1}^3 \end{pmatrix} \begin{pmatrix} \beta_{4,2,0}^{[l+1]} \\ \beta_{4,0,2}^{[l]} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (10)$$

从(3)式可证得

$$1 - a_{l+1,l}a_{l,l+1} \neq 0 \quad (11)$$

因而必有

$$\beta_{4,2,0}^{[l+1]} = \beta_{4,0,2}^{[l]} = 0, l = 1, 2, 3 \quad (12)$$

现考虑(9)中余下的光滑条件. 对每一  $l$  而言, 这样的格点有三个:

$$\hat{\beta}_{0,3,3}^{[l]}, \hat{\beta}_{1,2,3}^{[l]}, \hat{\beta}_{1,3,2}^{[l]}$$

对纵标  $\hat{\beta}_{0,3,3}^{[l]}$ , 由光滑条件

$$\begin{aligned}\hat{\beta}_{0,3,3}^{[l]} &= (a_{l,l+1}E_1 + b_{l,l+1}E_2 + c_{l,l+1}E_3)^3 \beta_{3,0,0}^{[l]} \\ &= a_{l,l+1}^3 \beta_{6,0,0}^{[l]} + 3a_{l,l+1}^2 c_{l,l+1} \beta_{5,0,1}^{[l]} + 3a_{l,l+1}^2 b_{l,l+1} \beta_{5,1,0}^{[l]} + 6a_{l,l+1} b_{l,l+1} c_{l,l+1} \beta_{4,1,1}^{[l]}\end{aligned}$$

及

$$\begin{aligned}\hat{\beta}_{0,3,3}^{[l]} &= (a_{l+1,l}E_1 + b_{l+1,l}E_2 + c_{l+1,l}E_3)^3 \beta_{3,0,0}^{[l+1]} \\ &= a_{l+1,l}^3 \beta_{6,0,0}^{[l+1]} + 3a_{l+1,l}^2 c_{l+1,l} \beta_{5,0,1}^{[l+1]} + 3a_{l+1,l}^2 b_{l+1,l} \beta_{5,1,0}^{[l+1]} + 6a_{l+1,l} b_{l+1,l} c_{l+1,l} \beta_{4,1,1}^{[l+1]}\end{aligned}$$

得

$$\begin{aligned}a_{l,l+1}^3 \beta_{6,0,0}^{[l]} + 3a_{l,l+1}^2 c_{l,l+1} \beta_{5,0,1}^{[l]} + 3a_{l,l+1}^2 b_{l,l+1} \beta_{5,1,0}^{[l]} + 6a_{l,l+1} b_{l,l+1} c_{l,l+1} \beta_{4,1,1}^{[l]} \\ = a_{l+1,l}^3 \beta_{6,0,0}^{[l+1]} + 3a_{l+1,l}^2 c_{l+1,l} \beta_{5,0,1}^{[l+1]} + 3a_{l+1,l}^2 b_{l+1,l} \beta_{5,1,0}^{[l+1]} + 6a_{l+1,l} b_{l+1,l} c_{l+1,l} \beta_{4,1,1}^{[l+1]}\end{aligned}$$

(13)

类似地, 考虑纵标  $\hat{\beta}_{1,2,3}^{[l]}, \hat{\beta}_{1,3,2}^{[l]}$ , 可以得到

$$a_{l,l+1}^2 \beta_{5,0,1}^{[l]} + 2a_{l,l+1} b_{l,l+1} \beta_{4,1,1}^{[l]} = a_{l+1,l}^3 \beta_{5,1,0}^{[l+1]} + 3a_{l+1,l}^2 c_{l+1,l} \beta_{4,1,1}^{[l+1]}, l = 1, 2, 3 \quad (14)$$

$$a_{l,l+1}^3 \beta_{5,0,1}^{[l]} + 3a_{l,l+1}^2 b_{l,l+1} \beta_{4,1,1}^{[l]} = a_{l+1,l}^2 \beta_{5,1,0}^{[l+1]} + 2a_{l+1,l} c_{l+1,l} \beta_{4,1,1}^{[l+1]}, l = 1, 2, 3 \quad (15)$$

令

$$\Delta_l := 1 - a_{l,l+1} a_{l+1,l}, l = 1, 2, 3 \quad (16)$$

(14)式减去(15)式乘以  $a_{l+1,l}$  得

$$a_{l,l+1}^2 \Delta_l \beta_{5,0,1}^{[l]} = -a_{l,l+1} b_{l,l+1} (2 - 3a_{l,l+1} a_{l+1,l}) \beta_{4,1,1}^{[l]} + a_{l+1,l}^2 c_{l+1,l} \beta_{4,1,1}^{[l+1]}, l = 1, 2, 3 \quad (17)$$

(14)式乘以  $a_{l,l+1}$  减去(15)式得

$$a_{l+1,l}^2 \Delta_l \beta_{5,1,0}^{[l+1]} = -a_{l+1,l} c_{l+1,l} (2 - 3 a_{l,l+1} a_{l+1,l}) \beta_{4,1,1}^{[l+1]} + a_{l,l+1}^2 b_{l,l+1} \beta_{4,1,1}^{[l]}, l = 1, 2, 3 \quad (18)$$

于是,(13),(17),(18)构成了由九个方程,关于十二个未知数:

$$\beta := (\beta_{4,1,1}^{[1]}, \beta_{4,1,1}^{[2]}, \beta_{4,1,1}^{[3]}, \beta_{5,0,1}^{[1]}, \beta_{5,0,1}^{[2]}, \beta_{5,0,1}^{[3]}, \beta_{5,1,0}^{[1]}, \beta_{5,1,0}^{[2]}, \beta_{5,1,0}^{[3]}, \beta_{6,0,0}^{[1]}, \beta_{6,0,0}^{[2]}, \beta_{6,0,0}^{[3]})^T$$

的方程组

$$M\beta = 0 \quad (19)$$

其中

$$M = (A_1, A_2) \quad (20)$$

$$A_1 = \begin{pmatrix} a_{12}b_{12}(2 - 3a_{12}a_{21}) & -a_{21}^2c_{21} & 0 & a_{12}^2\Delta_1 & 0 \\ a_{13}c_{13}(2 - 3a_{13}a_{31}) & 0 & -a_{31}^2b_{31} & 0 & a_{13}^2\Delta_3 \\ 0 & a_{23}b_{23}(2 - 3a_{23}a_{32}) & -a_{32}^2c_{32} & 0 & 0 \\ -a_{12}^2b_{12} & a_{21}c_{21}(2 - 3a_{21}a_{12}) & 0 & 0 & 0 \\ -a_{13}^2c_{13} & 0 & a_{31}b_{31}(2 - 3a_{31}a_{13}) & 0 & 0 \\ 0 & -a_{23}^2b_{23} & a_{32}c_{32}(2 - 3a_{32}a_{23}) & 0 & 0 \\ 6a_{12}b_{12}c_{12} & -6a_{21}b_{21}c_{21} & 0 & 3a_{12}^2c_{12} & 3a_{12}^2b_{12} \\ 0 & 6a_{23}b_{23}c_{23} & -6a_{32}b_{32}c_{32} & 0 & 0 \\ -6a_{13}b_{13}c_{13} & 0 & 6a_{31}b_{31}c_{31} & -3a_{13}^2c_{13} & -3a_{13}^2b_{13} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{23}^2\Delta_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_{21}^2\Delta_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{31}^2\Delta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{32}^2\Delta_2 & 0 & 0 & 0 \\ -3a_{21}^2c_{21} & -3a_{21}^2b_{21} & 0 & 0 & a_{12}^3 & -a_{21}^3 & 0 \\ 3a_{23}^2c_{23} & 3a_{23}^2b_{23} & -3a_{32}^2c_{32} & -3a_{32}^2b_{32} & 0 & a_{23}^3 & -a_{32}^3 \\ 0 & 0 & 3a_{31}^2c_{31} & 3a_{31}^2b_{31} & -a_{13}^3 & 0 & a_{31}^3 \end{pmatrix}$$

### 3 主要定理的证明

由第二节的光滑条件,我们得出了线性方程组(19),从而有

$$\dim S_6^3(\Delta_{ms}) = \binom{6+2}{2} + 12 - \text{rank } M = 40 - \text{rank } M \quad (21)$$

为此只需化简  $M$  并求其秩. 首先推导出面积坐标  $a_{i,j}, b_{i,j}, c_{i,j}, i, j = 1, 2, 3, i \neq j$  与  $r_i, s_i, t_i, i = 1, 2, 3$  之间的关系. 由(16),(17)可得

$$\begin{aligned}
 V_1 &= r_1 \hat{V}_1 + s_1 \hat{V}_2 + t_1 \hat{V}_3 = (r_1 \quad s_1 \quad t_1) \begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{pmatrix} \\
 &= (a_{21} \quad b_{21} \quad c_{21}) \begin{pmatrix} V_2 \\ \hat{V}_3 \\ \hat{V}_1 \end{pmatrix} = (c_{21} \quad a_{21} \quad b_{21}) \begin{pmatrix} \hat{V}_1 \\ V_2 \\ \hat{V}_3 \end{pmatrix} \\
 &= (c_{21} \quad a_{21} \quad b_{21}) \begin{pmatrix} 1 & 0 & 0 \\ t_2 & r_2 & s_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{pmatrix} \\
 &= (a_{31} \quad b_{31} \quad c_{31}) \begin{pmatrix} V_3 \\ \hat{V}_1 \\ \hat{V}_2 \end{pmatrix} = (b_{31} \quad c_{31} \quad a_{31}) \begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ V_3 \end{pmatrix} \\
 &= (b_{31} \quad c_{31} \quad a_{31}) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_3 & t_3 & r_3 \end{pmatrix} \begin{pmatrix} \hat{V}_1 \\ \hat{V}_2 \\ \hat{V}_3 \end{pmatrix}
 \end{aligned}$$

类似可得其它来自  $V_2$  和  $V_3$  的关系,从而易得

$$\left. \begin{aligned}
 a_{21} &= s_1/r_2, \quad b_{21} = t_1 - s_1 s_2/r_2, \quad c_{21} = r_1 - s_1 t_2/r_2, \\
 a_{31} &= t_1/r_3, \quad b_{31} = r_1 - s_3 t_1/r_3, \quad c_{31} = s_1 - t_1 t_3/r_3, \\
 a_{12} &= t_2/r_1, \quad b_{12} = r_2 - s_1 t_2/r_1, \quad c_{12} = s_2 - t_1 t_2/r_1, \\
 a_{32} &= s_2/r_3, \quad b_{32} = t_2 - s_2 s_3/r_3, \quad c_{32} = r_2 - s_2 t_3/r_3, \\
 a_{13} &= s_3/r_1, \quad b_{13} = t_3 - s_1 s_3/r_1, \quad c_{13} = r_3 - s_3 t_1/r_1, \\
 a_{23} &= t_3/r_2, \quad b_{23} = r_3 - s_2 t_3/r_2, \quad c_{23} = s_3 - t_2 t_3/r_2.
 \end{aligned} \right\} \quad (22)$$

由(16) $\Delta_i, i = 1, 2, 3$  的表达式和(22)可以得到

$$c_{21} = r_1 \Delta_1, \quad b_{31} = r_1 \Delta_3, \quad b_{12} = r_2 \Delta_1, \quad c_{32} = r_2 \Delta_2, \quad c_{13} = r_3 \Delta_3, \quad b_{23} = r_3 \Delta_2. \quad (23)$$

将表达式(23)代入(20)可以发现,前六行分别含有公因子  $\Delta_1, \Delta_3, \Delta_2, \Delta_1, \Delta_3, \Delta_2$ , 利用行变换将这些公因子消去,然后依次做下述列变换(这里括号中的数字表示矩阵  $M$  的列):

$$\begin{aligned}
 (4) \times \frac{1}{a_{12}^2}, \quad (5) \times \frac{1}{a_{13}^2}, \quad (6) \times \frac{1}{a_{23}^2} \\
 (7) \times \frac{1}{a_{21}^2}, \quad (8) \times \frac{1}{a_{31}^2}, \quad (9) \times \frac{1}{a_{32}^2} \\
 (1) - (4) \times 2r_2 \Delta_1 a_{12}, \quad (1) - (5) \times 2r_3 \Delta_3 a_{13} \\
 (2) - (6) \times 2r_3 \Delta_2 a_{23}, \quad (2) - (7) \times 2r_1 \Delta_1 a_{21} \\
 (3) - (8) \times 2r_1 \Delta_3 a_{31}, \quad (3) - (9) \times 2r_2 \Delta_2 a_{32}
 \end{aligned}$$

可将  $M$  化简为

$$M' = (A'_1, A'_2) \quad (24)$$

$$A'_1 = \begin{pmatrix} r_2 a_{12}^2 a_{21} & r_1 a_{21}^2 & 0 & 1 & 0 \\ r_3 a_{13}^2 a_{31} & 0 & r_1 a_{31}^2 & 0 & 1 \\ 0 & r_3 a_{23}^2 a_{32} & r_2 a_{32}^2 & 0 & 0 \\ r_2 a_{12}^2 & r_1 a_{21}^2 a_{12} & 0 & 0 & 0 \\ r_3 a_{13}^2 & 0 & r_1 a_{31}^2 a_{13} & 0 & 0 \\ 0 & r_3 a_{23}^2 & r_2 a_{32}^2 a_{23} & 0 & 0 \\ \frac{2r_2 r_3 t_2^2 \Delta_1 \Delta_3 a_{13}}{s_3^2} & -\frac{2r_1 r_3 s_1^2 \Delta_1 \Delta_2 a_{23}}{t_3^2} & 0 & c_{12} & \frac{r_2 t_2^2 \Delta_1}{s_3^2} \\ 0 & \frac{2r_1 r_3 t_3^2 \Delta_1 \Delta_2 a_{21}}{s_1^2} & -\frac{2r_1 r_2 s_2^2 \Delta_2 \Delta_3 a_{31}}{t_1^2} & 0 & 0 \\ -\frac{2r_2 r_3 s_3^2 \Delta_1 \Delta_3 a_{12}}{t_2^2} & 0 & \frac{2r_1 r_2 t_1^2 \Delta_2 \Delta_3 a_{32}}{s_3^2} & -\frac{r_3 s_3^2 \Delta_3}{t_2^2} & -b_{13} \end{pmatrix}$$

$$A'_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{r_1 s_1^2 \Delta_1}{t_3^2} & -b_{21} & 0 & 0 & t_2^3 & -s_1^3 & 0 \\ c_{23} & \frac{r_3 t_3^2 \Delta_2}{s_1^2} & -\frac{r_2 s_2^2 \Delta_2}{t_1^2} & -b_{32} & 0 & t_3^3 & -s_2^3 \\ 0 & 0 & c_{31} & \frac{r_1 t_1^2 \Delta_3}{s_2^2} & -s_3^3 & 0 & t_1^3 \end{pmatrix}$$

注意到

$$\det \begin{pmatrix} t_2^3 & -s_1^3 & 0 \\ 0 & t_3^3 & -s_2^3 \\ -s_3^3 & 0 & t_1^3 \end{pmatrix} = t_1^3 t_2^3 t_3^3 - s_1^3 s_2^3 s_3^3 \quad (25)$$

显然当  $t_1 t_2 t_3 \neq s_1 s_2 s_3$  时

$$\text{rank } M = \text{rank } M' = 9$$

从而,由(21)式得

$$\dim S_6^3(\Delta_{ms}) = 40 - 9 = 31, t_1 t_2 t_3 \neq s_1 s_2 s_3 \quad (26)$$

下面考察  $t_1 t_2 t_3 = s_1 s_2 s_3$  的情形,对  $M'$  做如下的行变换:

$$(7) \text{ 行 } \times s_3^3 s_2^3 + (8) \text{ 行 } \times t_2^3 t_1^3 + (9) \text{ 行 } \times t_2^3 s_2^3$$

如此之后,再利用(22)消去前三列中每一列元素的公因子,可得

$$\text{rank } M' = 2 + \text{rank } M'' \quad (27)$$

其中

$$M'' = \begin{pmatrix} s_1 t_2^2 & r_1 s_1^2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ t_1 s_3^2 & 0 & r_1 t_1^2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & s_2 t_3^2 & r_2 s_2^2 & 0 & 0 & 1 & 0 & 0 & 0 \\ r_2 t_2^2 & t_2 s_1^2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ r_3 s_3^2 & 0 & s_3 t_1^2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & r_3 t_3^2 & t_3 s_2^2 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & *_1 & *_2 & *_3 & *_4 & *_5 & *_6 \end{pmatrix}_{7 \times 9} \quad (28)$$

这里

$$\begin{aligned} *_1 &= s_2^3 s_3^2 (s_2 s_3 - r_3 t_2) \\ *_2 &= t_2^2 s_2^3 (s_3 r_2 - t_2 t_3) \\ *_3 &= t_1^2 t_2^2 s_3 (t_1 t_2 - r_1 s_2) \\ *_4 &= s_1^2 s_2^2 t_1 (r_3 t_2 - s_2 s_3) \\ *_5 &= s_2^2 t_2^3 (s_1 s_2 - t_1 r_2) \\ *_6 &= t_1^2 t_2^3 (r_1 s_2 - t_1 t_2) \end{aligned}$$

直接验证可知

$$\begin{aligned} s_1 t_2^2 *_1 + t_1 s_3^2 *_2 + r_2 t_2^2 *_4 + r_3 s_3^2 *_5 &= 0 \\ r_1 s_1^2 *_1 + s_2 t_3^2 *_3 + t_2 s_1^2 *_4 + r_3 t_3^2 *_6 &= 0 \\ r_1 t_1^2 *_2 + r_2 s_2^2 *_3 + s_3 t_1^2 *_5 + t_3 s_2^2 *_6 &= 0 \end{aligned}$$

因而

$$\text{rank } M'' = 6 \quad (29)$$

从而由(27)

$$\text{rank } M = \text{rank } M' = 2 + 6 = 8 \quad (30)$$

因而由(21)式得

$$\dim S_6^3(\Delta_{ms}) = 40 - 8 = 32, \quad t_1 t_2 t_3 = s_1 s_2 s_3 \quad (31)$$

综合本节的讨论,有下面的定理:

**定理1** 设  $\Delta_{ms}$  为 Morgan-Scott 三角剖分,则

$$31 \leq \dim S_6^3(\Delta_{ms}) \leq 32 \quad (32)$$

且

$$\dim S_6^3(\Delta_{ms}) = 32 \Leftrightarrow t_1 t_2 t_3 = s_1 s_2 s_3 \quad (33)$$

$t_1 t_2 t_3 = s_1 s_2 s_3$  的几何意义是三线  $V_i, V_i, i = 1, 2, 3$ , 交于一点.

最后要指出的是对于  $d > 2r$  的情况,注意到[8,9]中对任意三角剖分上样条空间维数的下界,容易证明([3,4])对 Morgan-Scott 剖分,样条空间  $S_d'(\Delta_{ms})$  的维数即为这一下界.

## 参 考 文 献

- [1] Morgen J, Scott R. The dimension of piecewise polynomials, manuscript., 1977
- [2] Chou Y S, Su L Y, Wang Renhong. The dimension of bivariate spline over triangulations. Inte. Ser. Nummer. Math. 75, Basel, Birkhauser-Verlag., 1985, 71 – 83
- [3] Chen Zhibin, Feng Yuyu, Kozak J. The Bolossom approach to the dimension of the bivariate spline space. IMUSTC., preprint 96 – 001
- [4] Diener D. Instability in the dimension of spaces of bivariate piecewise polynomials of degree  $2r$  and smoothness order  $r$ . SIAM J. Numer. Anal., 1990, 27(2):543 – 551
- [5] Shi Xiquan. The singularity of Morgen-Scott triangulation, Comput. Aided Geom. Design., 1991 (8):201 – 206
- [6] Chang Gengzhe , Davis P J . The convexity of Bernstein polynomials over triangle. J. Appro. Theory., 1984(40):11 – 28
- [7] Bohm W, Farin G, Kahmann J. A survey of curve and surface method in CAGD, Comput. Aided Geom. Design., 1984,(1):1 – 60
- [8] Schumaker L L. Lower bounds for the dimension of the spaces of piecewise polynomials in two variables, in Multivariate Approximation Theory. W. Schempp and K. Zeller ( eds.), Besel, Birkhäuser-Verlag., 1979, 396 – 412
- [9] Schumaker L L. Bounds on the dimension of spaces of multivariate piecewise polynomials. Rocky Mountain J. Math., 1984, (14): 251 – 264

## On the Dimension of Spline Space $S_6^3(\Delta_{\text{ms}})$ for Morgen-Scott Partition

Chen Zhibin, Deng Jiansong, Xi Meicheng, Feng Yuyu

(Dept. of Mathematics of USTC)

**Abstract** As well known, the dimension of spline space for Morgen-Scott partition depends on the geometry of the partition. It is proved in this paper that the conjecture proposed by D. Dinner is correct for  $r = 3$ .

**Key words** computer aided geometric design, multivariate splines, Morgan-Scott partition