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Best one-sided approximation of polynomials under $\mathbf{L}_1$ norm ☆

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Abstract

In this paper, we develop an analytic solution for the best one-sided approximation of polynomials under $\mathbf{L}_1$ norm, that is, we find two polynomials with lower degree which bound the given polynomial such that the areas between the bounding polynomials and the given polynomial attain minimum. The key ingredient of our technique is a characterization for one-sided approximations based on orthogonal polynomials. This result is applied in the degree reduction of interval polynomial/Bézier curves in Computer Aided Design.

Keywords: One-sided approximation; Orthogonal polynomials; Interval Bézier curves

1. Introduction

Let $\mathbb{R}[x]$ be the polynomial ring of a single variable with coefficients in real field $\mathbb{R}$, and

$$\mathbf{P}_n = \{\, p(x) \in \mathbb{R}[x] \,|\, \text{the degree of } p(x) \text{ is exactly } n\},$$

$$\mathbf{\Pi}_n = \{\, p(x) \in \mathbb{R}[x] \,|\, \text{the degree of } p(x) \text{ is less than or equal to } n\}.$$

In this paper, we consider the following one-sided approximation problem:

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Problem 1. *Given a polynomial* $p(x) \in \mathbf{P}_n$, *we want to find two polynomials* $\bar{q}_1, \bar{q}_2 \in \mathbf{\Pi}_m$ $(m < n)$ *such that*

1. $\bar{q}_1(x) \leqslant p(x) \leqslant \bar{q}_2(x)$ *for* $x \in [-1, 1]$, *and*
2.

$$\|p - \bar{q}_1\|_1 = \min_{\substack{q \in \mathbf{\Pi}_m \\ q(x) \leqslant p(x)}} \|p - q\|_1, \tag{1.1}$$

$$\|p - \bar{q}_2\|_1 = \min_{\substack{q \in \mathbf{\Pi}_m \\ q(x) \geqslant p(x)}} \|p - q\|_1 \tag{1.2}$$

where $\|\cdot\|_1$ *denotes* $\mathbf{L}_1$ *norm*:

$$\|f\|_1 = \int_{-1}^{1} |f(x)| \, \mathrm{d}x. \tag{1.3}$$

$\bar{q}_1(x)$ and $\bar{q}_2(x)$ provide polynomial bounds for a given polynomial, and $\bar{q}_1(x)$ and $\bar{q}_2(x)$ are called *lower bound* and *upper bound*, respectively.

Besides its own importance in approximation theory, the motivation to consider the above problem is that it provides a solution for the degree reduction of interval Bézier curves, the details of which will be discussed in Sections 3 and 4.

The above problem can be reformulated in another form.

Problem 2. *Fix* $l(l < n)$ *real numbers* λ_i, $i = 1, \ldots, l$, *we want to find two polynomials* q_1 *and* q_2 *of degree* n:

$$q_1(x) = x^n + \lambda_l x^{n-1} + \cdots + \lambda_1 x^{n-l} + b_{n-l-1} x^{n-l-1} + \cdots + b_1 x + b_0, \tag{1.4}$$

$$q_2(x) = -x^n - \lambda_l x^{n-1} - \cdots - \lambda_1 x^{n-l} + c_{n-l-1} x^{n-l-1} + \cdots + c_1 x + c_0, \tag{1.5}$$

where $b_0, \ldots, b_{n-l-1}$ *and* $c_0, \ldots, c_{n-l-1}$ *are free parameters*, *such that*

1. $q_i(x) \geqslant 0$, $x \in [-1, 1]$, $i = 1, 2$, *and*
2. $\|q_1\|_1$ *and* $\|q_2\|_1$ *attain minimum values for all the choices of the free parameters under condition* 1.

The connection between Problems 1 and 2 can be related as follows. Suppose

$$p = a_n x^n + \cdots + a_1 x + a_0 \in \mathbf{P}_n.$$

Let $l = n - m - 1$ and $\lambda_i = a_{i+m}/a_n$, $i = 1, \ldots, l$. If we have obtained the solution for Problem 2, then the solution for Problem 1 is

1. If $a_n > 0$, $\bar{q}_1(x) = p(x) - a_n q_1(x)$, $\bar{q}_2(x) = p(x) + a_n q_2(x)$ is the solution of Problem 1.
2. If $a_n < 0$, $\bar{q}_1(x) = p(x) + a_n q_2(x)$, $\bar{q}_2(x) = p(x) - a_n q_1(x)$ is the solution of Problem 1.

For the case of $m=n-1$, the authors developed an analytic solution for the above problem [5]. As far as the authors are aware, no similar results have ever been obtained so far. In this paper, we are going to tackle the problem for the case of $m=n-2$, and we found analytic solution also exists. The key ingredient to solve Problem 1 is a characterization for the one-sided approximation based on Jacobi polynomials.

The organization of the paper is as follows. In the next section, we briefly recall some preliminary knowledge about Jacobi polynomials. For completeness and compassion, in Section 3 we list the existing results obtained in [5] for the case of $m=n-1$. In Section 4, the main results for the case of $m=n-2$ are derived. Finally in Section 5, we provide an example and discuss applications of the one-sided approximation problem in the degree reduction of interval Bézier curves.

2. Jacobi orthogonal polynomials

Let $L^2[-1,1]$ be the function space endowed with norm $\|.\|_2$:

$$\|f\|_2^2=\int_{-1}^{1} f^2(x)\,\mathrm{d}x. \tag{2.1}$$

For any two functions $f,g\in L^2[-1,1]$, the *inner product* of f and g with respect to a weight function $w(x)\geqslant 0$ is defined as

$$\langle f,g\rangle=\int_{-1}^{1} w(x)f(x)g(x)\,\mathrm{d}x. \tag{2.2}$$

If $\langle f,g\rangle=0$, we say f and g are *orthogonal* with respect to weight $w(x)$. A set S of polynomials is called an *orthogonal polynomial system* if S forms a basis of $L^2[-1,1]$ and any two elements of S are orthogonal. Each element of S is called an orthogonal polynomial.

Orthogonal polynomials have many important properties. We will list some of these properties in the following lemmas. Before proceeding, we introduce some notations.

Given a polynomial $p\in\mathbf{P}_n$:

$$p=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0, \tag{2.3}$$

let $\deg(p)=n$ denote the *degree* of polynomial p, and $\mathrm{LC}(p)=a_n$ the *leading coefficient*. Define

$$\mathbf{P}_n^+=\{p(x)\in\mathbf{P}_n\,|\,\mathrm{LC}(p)=1\}.$$

Also we define coefficient operator C_i of p:

$$\mathrm{C}_i(p)=a_i,\quad i=0,\ldots,n. \tag{2.4}$$

Lemma 1 ([1]). *Given a weight function $w(x)$, if $p_n(x)\in\mathbf{P}_n^+$, $n=0,1,\ldots$ satisfy*

$$\int_{-1}^{1} w(x)p_n^2(x)\,\mathrm{d}x=\min_{q\in\mathbf{P}_n^+}\int_{-1}^{1} w(x)q^2(x)\,\mathrm{d}x \tag{2.5}$$

then $\{p_n(x)\}$ are the orthogonal polynomials with respect to weight $w(x)$ on $[-1,1]$.

If we choose $w(x)=w^{(\alpha,\beta)}(x)=(1-x)^{\alpha}(1+x)^{\beta},\ \alpha,\beta>-1$, then the corresponding orthogonal polynomials are the famous Jacobi polynomials over $[-1,1]$. Denote the Jacobi polynomial of degree n by $J_n^{(\alpha,\beta)}(x)$, by the orthogonality of Jacobi polynomials we have

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x)J_m^{(\alpha,\beta)}(x)J_n^{(\alpha,\beta)}(x)\,\mathrm{d}x=\delta_{m,n}=\begin{cases}1 & \text{for } n=m,\\ 0 & \text{for } n\neq m.\end{cases} \tag{2.6}$$

Lemma 2 ([1]). *Jacobi polynomial $J_n^{(\alpha,\beta)}(x)$ has the following property*:

1. *Simple Real Zeros*: *All the zeros of $J_n^{(\alpha,\beta)}(x)$ are simple and real, and lie inside of* $(-1,1)$.
2. *Explicit Form*: *There exist constants $C_n^{(\alpha,\beta)}$ such that*

$$C_n^{(\alpha,\beta)}J_n^{(\alpha,\beta)}(x)=\sum_{m=0}^{n}\binom{\alpha+n}{n-m}\binom{\alpha+\beta+n+m}{m}\left(\frac{x-1}{2}\right)^m, \tag{2.7}$$

where the choice of $C_n^{(\alpha,\beta)}$ makes $\|J_n^{(\alpha,\beta)}\|_2=1$.

Let

$$K_n^{(\alpha,\beta)}(x)=\frac{C_n^{(\alpha,\beta)}J_n^{(\alpha,\beta)}(x)}{\mathrm{LC}(C_n^{(\alpha,\beta)}J_n^{(\alpha,\beta)}(x))}, \tag{2.8}$$

which is monic. We will use these monic polynomials to construct our analytic solutions for the one-sided approximation.

From (2.7), we have

$$\mathbf{C}_{n-1}(K_n^{(\alpha,\beta)}(x))=\frac{\mathbf{C}_{n-1}(C_n^{(\alpha,\beta)}J_n^{(\alpha,\beta)}(x))}{\mathrm{LC}(C_n^{(\alpha,\beta)}J_n^{(\alpha,\beta)}(x))}=\frac{(\alpha-\beta)n}{\alpha+\beta+2n}. \tag{2.9}$$

Lemma 3. *Suppose $K_n^{(\alpha,\beta)}(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$, and*

$$I_n^{(\alpha,\beta)}(b_{n-1},\ldots,b_0)=\int_{-1}^{1} w^{(\alpha,\beta)}(x)(x^n+b_{n-1}x^{n-1}+\cdots+b_0)^2\,\mathrm{d}x.$$

then $\{a_i\}_{i=0}^{n-1}$ is the unique solution of the following linear system of equations:

$$\int_{-1}^{1} x^i w^{(\alpha,\beta)}(x)(x^n+b_{n-1}x^{n-1}+\cdots+b_0)\,\mathrm{d}x=0,\quad i=0,1,\ldots,n-1. \tag{2.10}$$

where $\{b_i\}_{i=0}^{n-1}$ are unknowns.

Proof. By Lemma 1, $\{a_i\}_{i=0}^{n-1}$ is the unique solution of the following linear system of equations:

$$\begin{cases}\dfrac{\partial}{\partial b_{n-1}}I_n^{(\alpha,\beta)}(b_{n-1},\ldots,b_0)=0,\\ \cdots\\ \dfrac{\partial}{\partial b_0}I_n^{(\alpha,\beta)}(b_{n-1},\ldots,b_0)=0.\end{cases} \tag{2.11}$$

It is easy to check (2.11) and (2.10) are equivalent. □

Note that it is not a good idea to obtain the representations of $\{a_i\}_{i=0}^{n-1}$ by solving (2.10), for the coefficient matrix are ill-conditioned. We mention them just for proving some results in later theorems.

3. The case of $m = n - 1$

For completeness and comparison, here we restate the results obtained by Lou et al. [5] for the case of $m = n - 1$. Notice that in [5], all the discussions happened on $[0,1]$. For consistency, here we make some transformations and give the results on $[-1,1]$.

Theorem 4. *Let*

$$\mathbf{H}_n^1 = \{p(x) \in \mathbf{P}_n \mid \mathrm{LC}(p) = 1,\ p(x) \geqslant 0,\ x \in [-1,1]\}.$$

If $p_n^* \in \mathbf{H}_n^1$ *satisfies*

$$\|p_n^*\|_1 = \min_{p_n \in \mathbf{H}_n^1} \|p_n\|_1 \tag{3.1}$$

then

$$p_n^*(x) = \begin{cases} [K_k^{(0,0)}(x)]^2, & \text{if } n = 2k, \\ (1+x)[K_k^{(0,1)}(x)]^2, & \text{if } n = 2k+1. \end{cases} \tag{3.2}$$

where $K_k^{(\alpha,\beta)}$ *is defined in* (2.8).

Theorem 5. *Let*

$$\mathbf{G}_n^1 = \{q(x) \in \mathbf{P}_n \mid \mathrm{LC}(q) = -1,\ q(x) \geqslant 0,\ x \in [-1,1]\}.$$

If $q_n^* \in \mathbf{G}_n^1$ *satisfies*

$$\|q_n^*\|_1 = \min_{q_n \in \mathbf{G}_n^1} \|q_n\|_1, \tag{3.3}$$

then

$$q_n^*(x) = \begin{cases} (1-x)[K_k^{(1,0)}(x)]^2, & n = 2k+1, \\ (1-x^2)[K_k^{(1,1)}(x)]^2, & n = 2k+2. \end{cases} \tag{3.4}$$

where $K_k^{(\alpha,\beta)}$ *is defined in* (2.8).

4. Main results

Now we come to the case of $m = n - 2$ (i.e. $l = 1$) for Problem 2. Given a number $\lambda \in \mathbb{R}$, we let

$$\mathbf{H}_n^2 = \{p(x) \in \mathbf{P}_n \mid \mathrm{LC}(p) = 1,\ \mathbf{C}_{n-1}(p) = \lambda,\ p(x) \geqslant 0,\ x \in [-1,1]\},$$

$$\mathbf{G}_n^2 = \{q(x) \in \mathbf{P}_n \mid \mathrm{LC}(q) = -1,\ \mathbf{C}_{n-1}(p) = -\lambda,\ q(x) \geqslant 0,\ x \in [-1,1]\}.$$

Lemma 6. *If q_1 and q_2 are the solutions of Problem* 2 *with $l=1$, then both q_1 and q_2 have n real roots, and at least $n-1$ roots lie inside* $[-1,1]$.

Proof. For $\mathbf{H}_n^2$ and $\mathbf{G}_n^2$ are convex close sets, the existence of the solutions for Problem 2 is obvious. Hence

$$\|q_1\|_1 = \min_{q\in\mathbf{H}_n^2} \|q\|_1, \tag{4.1}$$

$$\|q_2\|_1 = \min_{q\in\mathbf{G}_n^2} \|q\|_1. \tag{4.2}$$

We will prove the lemma for $q_1 \in \mathbf{H}_n^2$. the proof for $q_2 \in \mathbf{G}_n^2$ is similar.

(1) Any root of equation $q_1(x)=0$ is real.
Assume that equation $q_1(x)=0$ has a complex root. Since the equation with real coefficients always has a pair of conjugate complex roots, $q_1(x)$ has the following form:

$$q_1(x) = (x^2+ax+b)f(x),$$

where $a,b\in\mathbb{R}$, and $a^2<4b$, $f(x)\in\mathbb{P}_{n-2}$, and for all $x\in[0,1]$, $f(x)\geqslant 0$. Then

$$q_1(x) > \left(x^2+ax+\frac{a^2}{4}\right)f(x)$$

except for finite points. But $(x^2+ax+a^2/4)f(x)\in\mathbb{H}_n^2$, and $\|q_1\|_1 > \|(x^2+ax+a^2/4)f(x)\|_1$, this contradicts (4.1).

(2) At least $n-1$ roots of equation $q_1(x)=0$ lie inside of $[-1,1]$.
Assume there exist two roots $r_1\leqslant r_2$ of $q_1(x)=0$ which are outside of $[-1,1]$. Let

$$q_1(x) = (x-r_1)\,(x-r_2)f(x),$$

where $f(x)\in\mathbb{P}_{n-2}$. We consider three cases:

1. $r_1\leqslant r_2<-1$. In this case, $f(x)\geqslant 0$, $x\in[-1,1]$, so

$$q_1(x) > (x+1)\,(x-(r_1+r_2+1))f(x)\geqslant 0$$

except for finite points. Since for $j=n,n-1$,

$$\mathbf{C}_j((x+1)\,(x-(r_1+r_2+1))f(x)) = \mathbf{C}_j((x-r_1)\,(x-r_2)f(x)),$$

$(x+1)\,(x-(r_1+r_2+1))f(x)\in\mathbf{H}_n^2$. But $\|q_1\|_1 > \|(x+1)\,(x-(r_1+r_2+1))f(x)\|_1$, this contradicts (4.1).

2. $1<r_1\leqslant r_2$. Similar to case 1.
3. $r_1<-1$, $r_2>1$. In this case, $f(x)\leqslant 0$. Let

$$\tilde{q}_1(x) = \begin{cases} (x-1)\,(x-(r_1+r_2-1))f(x), & \text{if } r_1+r_2\leqslant 0,\\ (x+1)\,(x-(r_1+r_2+1))f(x), & \text{if } r_1+r_2>0, \end{cases}$$

it is easy to verify that $\tilde{q}_1\in\mathbf{H}_n^2$, and $\|q_1\|_1 > \|\tilde{q}_1\|_1$, so this also contradicts (4.1). □

From Lemma 6, we know that there must exist $n-1$ roots r_i, $i=1,2,\dots,n-1$ of $q_1(x)=0$ (and $q_2(x)=0$) lying inside of $[-1,1]$. The other root r_n can lie in $(-\infty,-1)$, $[-1,1]$ or $(1,+\infty)$. We need to discuss the location of r_n in these three cases.

Lemma 7. 1. *When $n=2k$ is even, then*

(a) *if* $r_n<-1$, $q_1(x)=(x-r_n)\,(1+x)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}+1-\lambda$,
(b) *if* $-1\leqslant r_n\leqslant 1$, $q_1(x)=(x^k+\mu_{k-1}x^{k-1}+\cdots+\mu_0)^2$, *where* $\lambda=2\mu_{k-1}$,
(c) *if* $r_n>1$, $q_1(x)=-(x-r_n)\,(1-x)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}-1-\lambda$.

2. *When $n=2k+1$ is odd, then*

(a) *if* $r_n<-1$, $q_1(x)=(x-r_n)\,(x^k+\mu_{k-1}x^{k-1}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-1}-\lambda$,
(b) *if* $-1\leqslant r_n\leqslant 1$, $q_1(x)=(1+x)\,(x^k+\mu_{k-1}x^{k-1}+\cdots+\mu_0)^2$, *where* $\lambda=2\mu_{k-1}+1$,
(c) *if* $r_n>1$, $q_1(x)=-(x-r_n)\,(1-x)(1+x)(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}-\lambda$.

Proof. We only prove the case of n is even and $r_n<-1$, and the others' proofs are similar. Suppose $n=2k$, and $n-1$ roots of $q_1(x)=0$ lie inside of $[-1,1]$ and the root r_n is less than -1. When $x\to-\infty$, $q_1(x)\to+\infty$, so $x=-1$ must be one of the roots of $q_1(x)=0$. Since $q_1(x)$ is a nonnegative polynomial over $[-1,1]$, it must have the form $q_1(x)=(x-r_n)\,(x+1)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, where $\mu_{k-2},\dots,\mu_0$ are real numbers. By $\mathbf{C}_{n-1}(q_1)=\lambda$, we have $r_n=2\mu_{k-2}+1-\lambda$. □

Similarly, for $q_2\in\mathbf{G}_n^2$ we have

Lemma 8. 1. *When $n=2k$ is even, then*

(a) *if* $r_n<-1$, $q_2(x)=(x-r_n)\,(1-x)(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}-1-\lambda$,
(b) *if* $-1\leqslant r_n\leqslant 1$, $q_2(x)=(1-x)(1+x)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $\lambda=2\mu_{k-2}$,
(c) *if* $r_n>1$, $q_2(x)=-(x-r_n)\,(1+x)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}+1-\lambda$.

2. *When $n=2k+1$ is odd, then*

(a) *if* $r_n<-1$, $q_2(x)=(x-r_n)\,(1-x)\,(1+x)\,(x^{k-1}+\mu_{k-2}x^{k-2}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-2}-\lambda$,
(b) *if* $-1\leqslant r_n\leqslant 1$, $q_2(x)=(1-x)\,(x^k+\mu_{k-1}x^{k-1}+\cdots+\mu_0)^2$, *where* $\lambda=2\mu_{k-1}-1$,
(c) *if* $r_n>1$, $q_2(x)=-(x-r_n)\,(x^k+\mu_{k-1}x^{k-1}+\cdots+\mu_0)^2$, *where* $r_n=2\mu_{k-1}-\lambda$.

Proof. Similar to Lemma 7. □

When $r_n\notin[-1,1]$, from Lemmas 7 and 8, we know q_i, $i=1,2$ have the following forms:

$$q_i(x)=c(x-2\mu_{l-1}+\gamma)w^{(\alpha,\beta)}(x)\,(x^l+\mu_{l-1}x^{l-1}+\cdots+\mu_0)^2, \tag{4.3}$$

where $c=1$ or -1, α and β equal 0 or 1, $l=k$ or $k-1$, and $\gamma=\lambda$, $\lambda-1$ or $\lambda+1$.

Lemma 9. *The system of equations*

$$\frac{\partial}{\partial v_i}\int_{-1}^{1}[c(x-2v_{l-1}+\gamma)w^{(\alpha,\beta)}(x)\,(x^l+v_{l-1}x^{l-1}+\cdots+v_0)^2]\,\mathrm{d}x=0,\quad i=0,\dots,l-1 \tag{4.4}$$

has the same solutions $(v_0,\dots,v_{l-1})$ *as the system of equations*

$$\frac{\partial}{\partial v_i}\int_{-1}^{1}[w^{(\alpha,\beta)}(x)\ (x^l+v_{l-1}x^{l-1}+\cdots+v_0)^2]\,\mathrm{d}x=0,\quad i=0,\dots,l-1. \tag{4.5}$$

Proof. It is obvious that we can omit the factor c in (4.4) for $c\neq 0$. From

$$\frac{\partial}{\partial v_i}\int_{-1}^{1}[(x-2v_{l-1}+\gamma)w^{(\alpha,\beta)}(x)\ (x^l+v_{l-1}x^{l-1}+\cdots+v_0)^2]\,\mathrm{d}x=0,\quad i=l-1,\dots,0,$$

we have

$$\int_{-1}^{1} f_l(x)g_l(x)\,\mathrm{d}x=0, \tag{4.6}$$

$$\int_{-1}^{1} f_l(x)\ (x-2v_{l-1}+\gamma)x^j\,\mathrm{d}x=0,\quad j=l-2,\dots,0, \tag{4.7}$$

where $f_l(x)=w^{(\alpha,\beta)}(x)\ (x^l+v_{l-1}x^{l-1}+\cdots+v_0)$, $g_l(x)=3v_{l-1}x^{l-1}+(v_{l-2}-\gamma)x^{l-2}+v_{l-3}x^{l-3}+\cdots+v_0$. From (4.7),

$$\int_{-1}^{1} x^{j+1}f_l(x)\,\mathrm{d}x=(2v_{l-1}-\gamma)\int_{-1}^{1} x^j f_l(x)\,\mathrm{d}x,\quad j=l-2,\dots,0. \tag{4.8}$$

Substituting the above equations into (4.6), we get

$$g_l(2v_{l-1}-\gamma)\int_{-1}^{1} f_l(x)\,\mathrm{d}x=0. \tag{4.9}$$

Because

$$g_l(2v_{l-1}-\gamma)=(\bar{f}_l(x)-(x^l-2v_{l-1}x^{l-1}+\gamma x^{l-2}))|_{x=2v_{l-1}-\gamma}=\bar{f}_l(2v_{l-1}-\gamma),$$

where $\bar{f}_l(x)=f_l(x)/w^{(\alpha,\beta)}(x)=x^l+v_{l-1}x^{l-1}+\cdots+v_0$, and all roots of $\bar{f}_l(x)$ lie inside $[-1,1]$ and $2v_{l-1}-\gamma\notin[-1,1]$, therefore $g_l(2v_{l-1}-\gamma)=\bar{f}_l(2v_{l-1}-\gamma)\neq 0$. So (4.9) is equivalent to

$$\int_{-1}^{1} w^{(\alpha,\beta)}(x)\ (x^l+v_{l-1}x^{l-1}+\cdots+v_0)\,\mathrm{d}x=0. \tag{4.10}$$

Substituting the above equation into (4.8), we obtain

$$\int_{-1}^{1} x^j w^{(\alpha,\beta)}(x)\ (x^l+v_{l-1}x^{l-1}+\cdots+v_0)\,\mathrm{d}x=0,\quad j=l-1,\dots,0. \tag{4.11}$$

This completes the proof of the theorem. □

Theorem 10. *In the general forms* (4.3) *of* q_i, $i=1,2$, $(x^l+\mu_{l-1}x^{l-1}+\cdots+\mu_0)=K_l^{(\alpha,\beta)}(x)$.

Proof. The statement follows immediately from Theorem 3 and Lemma 9. □

When $r_n \in [-1,1]$, from Lemmas 7 and 8, q_i, $i=1,2$, have the following form:

$$q_i(x) = w^{(\alpha,\beta)}(x)\ (x^l + v_{l-1}(x) + v_{l-2}x^{l-2} + \cdots + v_0)^2, \tag{4.12}$$

where α and β equal 0 or 1, $l=k$ or $k-1$, and $v_{l-1} = \lambda/2$, $(\lambda-1)/2$ or $(\lambda+1)/2$. To get the final forms of q_1 and q_2, we will turn to Jacobi polynomials to determine the polynomial $x^l + v_{l-1}(x) + v_{l-2}x^{l-2} + \cdots + v_0$ appearing in (4.12).

Theorem 11. *Suppose*

$$x^l + v_{l-1}x^{l-1} = \sum_{j=0}^{l} b_j J_j^{(\alpha,\beta)}(x),$$

$$v_{l-2}x^{l-2} + \cdots + v_0 = \sum_{j=0}^{l-2} c_j J_j^{(\alpha,\beta)}(x),$$

where v_j, $j=0,1,\ldots,l-1$ are coefficients appearing in the representation (4.12) *of $q_1(x)$ or $q_2(x)$. Then $c_j = -b_j = -\int_{-1}^{1} w^{(\alpha,\beta)}(x)\ (x^l + v_{l-1}x^{l-1})J_j^{(\alpha,\beta)}(x)\,\mathrm{d}x$, $j=0,\ldots,l-2$.*

Proof. From the orthogonality of Jacobi polynomials $J_j^{(\alpha,\beta)}(x)$, we have

$$\int_{-1}^{1} q_i(x)\,\mathrm{d}x = \int_{-1}^{1} w^{(\alpha,\beta)}(x)\left(\sum_{j=0}^{l} b_j J_j^{(\alpha,\beta)}(x) + \sum_{j=0}^{l-2} c_j J_j^{(\alpha,\beta)}(x)\right)^2 \mathrm{d}x$$

$$= \left\|\sum_{j=0}^{l} b_j J_j^{(\alpha,\beta)}(x)\right\|_2^2 + \left\|\sum_{j=0}^{l-2} c_j J_j^{(\alpha,\beta)}(x)\right\|_2^2 + 2\left\langle \sum_{j=0}^{l} b_j J_j^{(\alpha,\beta)}(x), \sum_{j=0}^{l-2} c_j J_j^{(\alpha,\beta)}(x)\right\rangle$$

$$= \sum_{j=0}^{l} b_j^2 + \sum_{j=0}^{l-2} c_j^2 + 2\sum_{j=0}^{l-2} b_j c_j = \sum_{j=0}^{l-2} (b_j + c_j)^2 + b_l^2 + b_{l-1}^2,$$

The above integral achieves minimum value if and only if $c_j = -b_j$, $j=0,1,\ldots,l-2$. □

The final forms of $q_1(x)$ and $q_2(x)$ relating with parameter λ are given by the following two theorems.

Theorem 12. 1. *When $n=2k$ is even, then*

(a) *if* $\lambda > n/(n-1)$, $q_1(x) = (x + \lambda - 1/(n-1))\ (1+x)[K_{k-1}^{(0,1)}(x)]^2$,
(b) *if* $\lambda < -n/(n-1)$, $q_1(x) = -(x + \lambda + 1/(n-1))\ (1-x)[K_{k-1}^{(1,0)}(x)]^2$.

(c) *if* $-n/(n-1) \leqslant \lambda \leqslant n/(n-1)$, $q_1(x) = (x^k + (\lambda/2)x^{k-1} + \sum_{j=0}^{k-2} c_j J_j^{(0,0)}(x))^2$, *where*

$$c_j = -\int_{-1}^{1} \left(x^k + \frac{\lambda}{2}x^{k-1}\right) J_j^{(0,0)}(x)\,\mathrm{d}x, \quad j = 0,\dots,k-2.$$

2. *When* $n = 2k+1$ *is odd, then*

(a) *if* $\lambda > 1$, $q_1(x) = (x+\lambda)\,[K_k^{(0,0)}(x)]^2$,
(b) *if* $\lambda < -1$, $q_1(x) = -(x+\lambda)\,(1-x)\,(1+x)\,[K_{k-1}^{(1,1)}(x)]^2$,
(c) *if* $-1 \leqslant \lambda \leqslant 1$, $q_1(x) = (1+x)\,(x^k + ((\lambda-1)/2)x^{k-1} + \sum_{j=0}^{k-2} c_j J_j^{(0,1)}(x))^2$, where

$$c_j = -\int_{-1}^{1} (1+x)\left(x^k + \frac{\lambda-1}{2}x^{k-1}\right) J_j^{(0,1)}(x)\,\mathrm{d}x, \quad j = 0,\dots,k-2.$$

Proof. We just prove the case of 1(a). Suppose $n = 2k$ and $r_n < -1$. From Lemma 7, one has

$$q_1(x) = (x - r_n)\,(1+x)\,(x^{k-1} + \mu_{k-2}x^{k-2} + \cdots + \mu_0)^2,$$

where $r_n = 2\mu_{k-2} + 1 - \lambda$. so

$$q_1(x) = (x - r_n)\,(1+x)\,[K_{k-1}^{(0,1)}(x)]^2$$

by Theorem 10. Thus

$$\mu_{k-2} = \frac{1-n/2}{n-1} \quad \text{and} \quad r_n = \frac{1}{n-1} - \lambda$$

by using Eq. (2.9). Note that

$$r_n < -1 \quad \Leftrightarrow \quad \lambda > \frac{n}{n-1},$$

therefore case 1(a) holds. For the other cases, the proofs are similar. □

Theorem 13. 1. *When* $n = 2k$ *is even, then*

(a) *if* $\lambda > (n-2)/(n-1)$, $q_2(x) = (x + \lambda + 1/(n-1))\,(1-x)\,[K_{k-1}^{(1,0)}(x)]^2$,
(b) *if* $\lambda < -(n-2)/(n-1)$, $q_2(x) = -(x + \lambda - 1/(n-1))\,(1+x)\,[K_{k-1}^{(0,1)}(x)]^2$,
(c) *if* $-(n-2)/(n-1) \leqslant \lambda \leqslant (n-2)/(n-1)$, $q_2(x) = (1-x)\,(1+x)\,(x^{k-1} + (\lambda/2)x^{k-2} + \sum_{j=0}^{k-3} c_j J_j^{(1,1)}(x))^2$,
where

$$c_j = -\int_{-1}^{1} (1-x)\,(1+x)\left(x^{k-1} + \frac{\lambda}{2}x^{k-2}\right) J_j^{(1,1)}(x)\,\mathrm{d}x, \quad j = 0,\dots,k-3.$$

2. *When* $n = 2k+1$ *is odd, then*

(a) *if* $\lambda > 1$, $q_2(x) = (x+\lambda)\,(1-x)\,(1+x)\,[K_{k-1}^{(1,1)}(x)]^2$,
(b) *if* $\lambda < -1$, $q_2(x) = -(x+\lambda)\,[K_k^{(0,0)}(x)]^2$,

(c) *if* $-1 \leqslant \lambda \leqslant 1$, $q_2(x) = (1-x)(x^k + ((\lambda+1)/2)x^{k-1} + \sum_{j=0}^{k-2} c_j J_j^{(1,0)}(x))^2$, *where*

$$c_j = -\int_{-1}^{1} (1-x)\left(x^k + \frac{\lambda+1}{2}x^{k-1}\right) J_j^{(1,0)}(x)\,\mathrm{d}x, \quad j = 0,\ldots,k-2.$$

Proof. Similar to the proof of Theorem 12. □

We end this section with an example which involves the most complex cases in Theorems 12 and 13.

Example. Given polynomial $p(x) = -3x^4 + 3x^3 - x + 1$, we are going to find two quadratic polynomial bounds for $p(x)$. In this case, we have $n = 4$, $m = 2$, $l = 1$ and $\lambda = -1$.

By Theorem 12, when $-\frac{4}{3} < \lambda < \frac{4}{3}$, $q_1(x)$ has the following form:

$$q_1(x) = \left(x^2 - \frac{1}{2}x + c_0 J_0^{(0,0)}(x)\right)^2.$$

Since

$$J_0^{(0,0)}(x) = \frac{1}{\sqrt{2}}, \quad c_0 = -\int_{-1}^{1}\left(x^2 - \frac{1}{2}x\right) J_0^{(0,0)}(x)\,\mathrm{d}x = -\frac{\sqrt{2}}{3},$$

we have

$$q_1(x) = \left(x^2 - \frac{1}{2}x - \frac{1}{3}\right)^2.$$

On the other hand, by Theorem 13, $q_2(x)$ has the form

$$q_2(x) = -\left(x - \frac{4}{3}\right)(1+x)[K_1^{(0,1)}(x)]^2 = -\left(x - \frac{4}{3}\right)(1+x)\left(x - \frac{1}{3}\right)^2.$$

Thus

$$\bar{q}_1(x) = p(x) + (-3)q_2(x) = -3x^2 + \frac{14}{9}x + \frac{5}{9},$$

$$\bar{q}_2(x) = p(x) - (-3)q_1(x) = -\frac{5}{4}x^2 + \frac{4}{3}$$

are the polynomial lower bound and upper bound of $p(x)$ respectively. Fig. 1 show the approximation results.

5. Applications

As an application of the results for one-sided approximation, we provide an algorithm for the degree reduction of an interval Bézier polynomial/curve which is a hot spot in Computer Aided Design and Geometric Modeling.

Interval Bézier curves are new representation forms for parametric curves, and were first introduced into Geometric Modeling community by Sederberg and Farouki [7]. The main advantage to use interval representations of curves is that such representation forms can embody a complete description

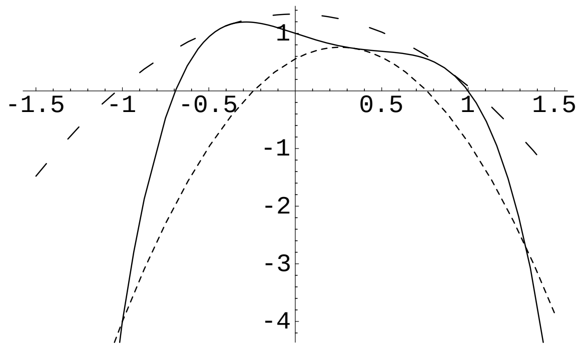

Fig. 1. Top: $\bar{q}_2$, middle: p, and bottom: $\bar{q}_1$.

about coefficient errors along with the curve, and they are convenient for tolerance analyses in geometric modeling. In the following, we will briefly describe the definition of an interval Bézier polynomial. For the details about interval polynomials/curves, the reader is referred to [7,4].

An interval polynomial is a polynomial whose coefficients are intervals:

$$[p](t) = \sum_{k=0}^{n} [a_k, b_k] B_k^n(t), \quad -1 \leqslant t \leqslant 1, \tag{5.1}$$

where

$$B_k^n(t) = \frac{1}{2^n} \binom{n}{k} (1-t)^{n-k}(1+t)^k$$

are Bernstein basis functions over $[-1, 1]$. Interval operations are defined by

$$[a,b] + [c,d] = [a+c, b+d]$$

$$[a,b] - [c,d] = [a-c, b-d]$$

$$[a,b] \times [c,d] = [\min(ac,ad,bc,bd), \max(ac,ad,bc,bd)]$$

$$[a,b]/[c,d] = [a,b] \times [1/d, 1/c].$$

Polynomials

$$p_{\min}(t) = \sum_{k=0}^{n} a_k B_k^n(t) \quad \text{and} \quad p_{\max}(t) = \sum_{k=0}^{n} b_k B_k^n(t) \tag{5.2}$$

are called *lower bound* and *upper bound* of $[p](t)$, respectively. From the positivity of Bernstein basis functions, it is easy to see

$$p_{\min}(t) \leqslant p_{\max}(t), \quad t \in [-1, 1].$$

Thus

$$[p](t) = [p_{\min}(t), p_{\max}(t)]$$

and the graph of an interval polynomial is the area bounded by $p_{\min}(t)$ and $p_{\max}(t)$.

In [6], Rokne discussed the problem of how to reduce the degree of an interval polynomial, that is, how to use (the graph of) an interval polynomial to bound (the graph of) a higher degree interval polynomial such that the bound is as tight as possible. In [5,2,3], we developed much better algorithms to solve the problem. Unfortunately, these algorithms can only reduce the degree of the given interval polynomial by 1 at a time. To reduce the degree of an interval polynomial to a much lower degree, we have to recursively apply the algorithms. However, the results obtained in this paper can reduce the degree of an interval polynomial by 2 at a time. Since the reduction is optimal under certain sense, the final bounds will be much tighter than that by previous approaches.

Given an interval polynomial $[p](t)$, to reduce the degree of $[p](t)$, we only have to find an upper polynomial bound $q_{\max}(t)$ (with lower degree) for polynomial $p_{\max}(t)$ and a lower polynomial bound $q_{\min}(t)$ (with lower degree) for polynomial $p_{\min}(t)$, then $[q](t) = [q_{\min}(t), q_{\max}(t)]$ bounds $[p](t)$. The solution to Problem 1 provides such a bounding interval polynomial $[q](t)$ and it is optimal under $\mathbf{L}_1$ norm. In the following, we only give an example to demonstrate the algorithm.

Let

$$[p](t) = \left[-\frac{1}{3}, 0\right] B_0^7(t) + \left[\frac{5}{2}, 3\right] B_1^7(t) + \left[\frac{7}{2}, 4\right] B_2^7(t) + \left[\frac{1}{3}, \frac{1}{2}\right] B_3^7(t)$$
$$+ \left[-\frac{10}{3}, -3\right] B_4^7(t) + \left[\frac{8}{3}, 3\right] B_5^7(t) + \left[\frac{1}{2}, 1\right] B_6^7(t) + \left[\frac{1}{4}, \frac{3}{4}\right] B_7^7(t),$$

we want to find a quintic interval polynomial which bound $[p](t)$, and the upper bound and lower bound is optimal under $\mathbf{L}_1$ norm. For this purpose, we first compute the polynomial q_1 corresponding to $p_{\min}$ and q_2 corresponding to $p_{\max}$:

$$q_1(t) = \frac{1197}{1075}t^2 + \frac{9}{25}t^3 - \frac{798}{215}t^4 - \frac{6}{5}t^5 + \frac{133}{43}t^6 + t^7,$$
$$q_2(t) = \frac{21}{155} + \frac{1}{25}t - \frac{231}{155}t^2 - \frac{11}{25}t^3 + \frac{147}{31}t^4 + \frac{7}{5}t^5 - \frac{105}{31}t^6 - t^7.$$

so the quintic interval polynomial's lower bound and upper bound are

$$g_1(t) = p_{\min} - \mathrm{LC}(p_{\min})q_1(t)$$
$$= \frac{1}{7680}(2725 - 14805t + 29526t^2 + 22666t^3 - 36295t^4 - 6825t^5),$$

$$g_2(t) = p_{\max} + \mathrm{LC}(p_{\max})q_2(t)$$
$$= \frac{1}{160}(130 - 307t + 630t^2 + 472t^3 - 700t^4 - 105t^5)$$

respectively.

Fig. 2 shows the original interval polynomial and the degree reduced interval polynomial. In the figure, we use solid lines to represent the boundary of the original interval polynomial, and dashed lines to represent the boundary of the degree reduced interval polynomial. From the figure we see that the degree reduced interval polynomial tightly bounds the original interval polynomial.

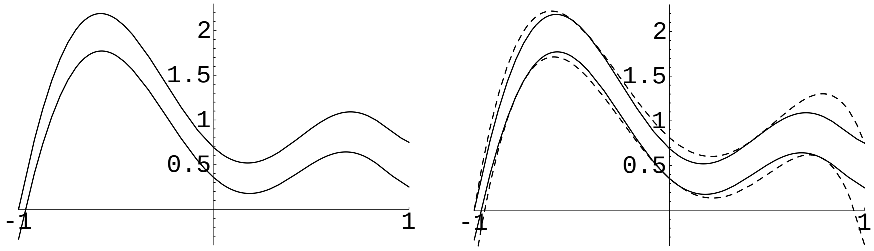

Fig. 2. Degree reduction of interval polynomial.

6. Conclusions

In this paper, we solved a special kind of one-sided approximation problem for the case of $m = n - 2$. The result is applied in the degree reduction of interval polynomials/curves in Computer Aided Design. If $n - m$ is greater than 2, it will involve much more complicated processing. We will discuss the problem in another paper.

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