High Accurate Approximation of Ellipsoid surface patch by bicubic Bézier Polynomials*

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Abstract

In this paper, the approximation of a Ellipsoid surface patch using bicubic Bézier polynomials is considered. The approximation is sixth order accurate. Furthermore the adjacent approximation surface patches have the same tangent plane at their common boundary.
Keywords: Approximation, Ellipsoid, Accurate, Patch.

1 Introduction

Bézier curves and surfaces are widely used in the geometric modelling, but they could not denote circle, sphere and the like exactly. Hence in practice, the requirement of approximation of sphere and the like arises when conic sections or rational curves are not available or are not recommended.

Many authors have worked with the approximation of circle by Bézier polynomials [1, 2, 3], and in [4] we give a perfect approximation for a octant of ellipsoid surface. This paper we consider the approximation of a ellipsoid surface patch by bicubic polynomials. The approximation turns out to have sixth order accuracy, giving a very small error.

In the following, we introduce the error functions

$$\varepsilon_1(t) = \frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} - 1 \tag{1.1}$$

for the case of ellipse and

$$\varepsilon_2(s,t) = \frac{x^2(s,t)}{a^2} + \frac{y^2(s,t)}{b^2} + \frac{z^2(s,t)}{c^2} - 1 \tag{1.2}$$

for the ellipsoid.

2 The approximation of elliptic arc

An ellipse is defined as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a, b \in \mathbb{R}. \tag{2.1}$$

Let the elliptic arc to be approximated be given by its angular width $0 < \beta \leqslant \pi/2$, starting in the point $\binom{0}{b}$ on the positive y-axis, see Fig.1(a).

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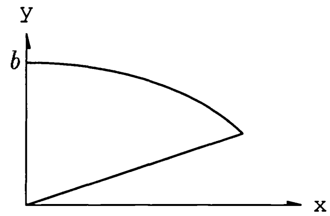
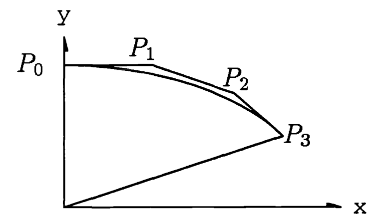


Fig.1(a)the elliptic arc



(b)the approximation curve and its control points

We intend to find a cubic Bézier curve $p(t)$ to approximate this elliptic arc. Here, $p(t)$ is of form

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{i=0}^{3} P_i B_i^3(t) \tag{2.2}$$

with the control points P_i and the Bernstein basic functions $B_i^n(t)$ given by

$$P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}, \quad B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \quad t \in (0,1), \quad i = 0, 1, \ldots, n. \tag{2.3}$$

The Bézier curve $p(t)$ is required that passes through the end points and there has a collinear tangents with the ellipse arc. We rewrite the ellipse equation in a parametric form as

$$f(\theta) = \begin{pmatrix} x(\theta) \\ y(\theta) \end{pmatrix} = \begin{pmatrix} a\sin\theta \\ b\cos\theta \end{pmatrix}, \tag{2.4}$$

where θ is the angle starting from positive y-axis as shown in Fig.1(a). Then the control points p_i (see Fig.1(b)) can be computed as follows.

$$\begin{aligned} &P_0 = \begin{pmatrix} 0 \\ b \end{pmatrix}, \quad P_3 = \begin{pmatrix} a\sin\beta \\ b\cos\beta, \end{pmatrix}, \quad P_1 = \begin{pmatrix} 0 \\ b \end{pmatrix} + h \cdot \frac{df}{d\theta}\Big|_{\theta=0} = \begin{pmatrix} ah \\ b \end{pmatrix}, \\ &P_2 = \begin{pmatrix} a\sin\beta \\ b\cos\beta \end{pmatrix} - h \cdot \frac{df}{d\theta}\Big|_{\theta=\beta} = \begin{pmatrix} a\sin\beta - ah\cos\beta \\ b\cos\beta + bh\sin\beta \end{pmatrix}, \end{aligned} \tag{2.5}$$

where h is a positive constant to be determined later. Then we have:

$$\begin{cases} x(t) & = a[hB_1^3(t) + (\sin\beta - h\cos\beta)B_2^3(t) + \sin\beta B_3^3(t)] \\ y(t) & = b[B_0^3(t) + B_1^3(t) + (\cos\beta + h\sin\beta)B_2^3(t) + \cos\beta B_3^3(t)] \end{cases}$$

and the error function $\varepsilon_1(t)$ is of the form

$$\begin{aligned} \varepsilon_1(t) &= [hB_1^3(t) + (\sin\beta - h\cos\beta)B_2^3(t) + \sin\beta B_3^3(t)]^2 \\ &+ [B_0^3(t) + B_1^3(t) + (\cos\beta + h\sin\beta)B_2^3(t) + \cos\beta B_3^3(t)]^2 - 1 \\ &= f_1^2(t,\beta,h) + f_2^2(t,\beta,h) - 1. \end{aligned} \tag{2.6}$$

here,

$$\begin{aligned} f_1(t,\beta,h) &= hB_1^3(t) + (\sin\beta - h\cos\beta)B_2^3(t) + \sin\beta B_3^3(t) \\ f_2(t,\beta,h) &= B_0^3(t) + B_1^3(t) + (\cos\beta + h\sin\beta)B_2^3(t) + \cos\beta B_3^3(t) \end{aligned} \tag{2.7}$$

Hence, $\varepsilon_1(t)$ defined in (1.1) is a polynomial of degree 6. The error function can be written in the form

$$\varepsilon_1(t) = \sum_{i=0}^{6} b_i B_i^6(t). \tag{2.8}$$

Using the method as same as in [4], the Bézier coefficients in (2.8) can be decided:

$$
\begin{aligned}
&b_0 = b_1 = b_5 = b_6 = 0, \quad b_2 = b_4 = \frac{1}{5}[3h^2 + 2h\sin\beta - 2(1-\cos\beta)] \\
&b_3 = \frac{1}{10}[-9h^2\cos\beta + 18h\sin\beta - 10(1-\cos\beta)].
\end{aligned} \tag{2.9}
$$

So

$$
\varepsilon_1(t) = b_2 B_2^6(t) + b_3 B_3^6(t) + b_4 B_4^6(t). \tag{2.10}
$$

In order to determine the free parameter h, for obvious symmetry, we let $\varepsilon_1(\frac{1}{2}) = 0$, it requires $2b_3 + 3b_2 = 0$, thus h can be determined, and

$$
h = \frac{4}{3}\tan\frac{\beta}{4}.
$$

Then we have

$$
\varepsilon_1(t) = 15b_2 \cdot t^2(1-t)^2(2t-1)^2 \tag{2.11}
$$

here

$$
\begin{aligned}
15b_2 &= 9h^2 + 6h\sin\beta - 6(1-\cos\beta) \\
&= 16\sin^2\frac{\beta}{4}\left(\frac{1}{\cos\frac{\beta}{4}} - \cos\frac{\beta}{4}\right)^2 = 16\frac{\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}}.
\end{aligned} \tag{2.12}
$$

Hence

$$
\varepsilon_1(t) = 16\frac{\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}} t^2(1-t)^2(2t-1)^2. \tag{2.13}
$$

Because of

$$
\max_{t\in[0,1]} t^2(1-t)^2(2t-1)^2 = 1/108,
$$

finally we have:

Theorem 1 *The Bézier curve* $p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ *obtained by above control points, when* $h = \frac{4}{3}\tan\frac{\beta}{4}$, *it interpolates the arc at* $\begin{pmatrix} 0 \\ b \end{pmatrix}$, $\begin{pmatrix} a\sin(\beta/2) \\ b\cos(\beta/2) \end{pmatrix}$ *and* $\begin{pmatrix} a\sin\beta \\ b\cos\beta \end{pmatrix}$ *and never enters inside the ellipse. The error is given by*

$$
\|\varepsilon_1(t)\|_\infty = \max_{t\in[0,1]}\left\{\frac{x^2(t)}{a^2} + \frac{y^2(t)}{b^2} - 1\right\} = \frac{4}{27}\frac{\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}}.
$$

This result is the same as that in [2] or [3] for the circle case ($a = b = 1$).
At last of this section, we note that from (2.11), (2.12), the approximation is one-sided.

3 The approximation of ellipsoid patch

The main purpose of this paper is to generalize the above method to surface case. An ellipsoid is defined by

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a, b, c \in \mathbb{R}. \tag{3.1}
$$

Let the parametric representation of the ellipsoid surface be

$$f(\theta,\varphi)=\begin{pmatrix} a\sin\theta\cos\varphi \\ b\sin\theta\sin\varphi \\ c\cos\theta \end{pmatrix},\quad 0\leqslant\theta\leqslant\pi,\ 0\leqslant\varphi\leqslant 2\pi. \tag{3.2}$$

Initially, we consider a patch as shown in Fig.2(a) on the ellipsoid with $0\leqslant\theta\leqslant\beta\leqslant\pi/2$ and $0\leqslant\varphi\leqslant\alpha\leqslant\pi/2$.

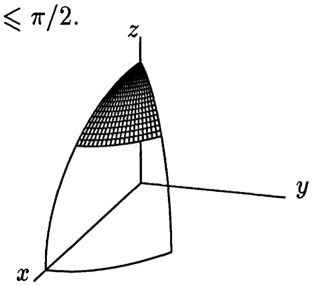


Fig.2(a)the ellipsoid patch $0\leqslant\theta\leqslant\beta\leqslant\frac{\pi}{2}$

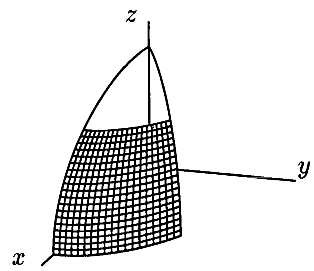


Fig.2(b)the ellipsoid patch $0\leqslant\beta\leqslant\theta\leqslant\frac{\pi}{2}$

We intend to approximate the ellipsoid surface patch using bicubic Bézier surface with two parameters. Let

$$p(s,t)=\sum_{i=0}^{3}\sum_{j=0}^{3}P_{ij}B_i^3(s)B_j^3(t). \tag{3.3}$$

In order to determine the control points P_{ij}, it is required that $P(s,t)$ interpolates the patch at points $(0,0,c)^T$, $(a\sin\beta,0,c\cos\beta)^T$, $(a\sin\beta\cos\alpha, b\sin\beta\sin\alpha, c\cos\beta)^T$, and has the same tangent plane with ellipsoid at these three points, but in the increasing direction of θ and φ, the tangent at these points may have different length.

By requirements of interpolation, it is easy to compute that

$$P_{00}=P_{01}=P_{02}=P_{03}=\begin{pmatrix}0\\0\\c\end{pmatrix},\ P_{30}=\begin{pmatrix}a\sin\beta\\0\\c\cos\beta\end{pmatrix},\ P_{33}=\begin{pmatrix}a\sin\beta\cos\alpha\\b\sin\beta\sin\alpha\\c\cos\beta\end{pmatrix}.$$

From $\frac{\partial P(s,t)}{\partial s}\Big|_{(0,0)}=\sigma_1\frac{\partial f}{\partial\theta}\Big|_{(0,0)}$ and $\frac{\partial P(s,t)}{\partial s}\Big|_{(1,0)}=\sigma_1\frac{\partial f}{\partial\theta}\Big|_{(\beta,0)}$, we have

$$P_{10}=\begin{pmatrix}ah_1\\0\\c\end{pmatrix},\quad P_{20}=\begin{pmatrix}a\sin\beta-ah_1\cos\beta\\0\\c\cos\beta+ch_1\sin\beta\end{pmatrix}.$$

Suppose $h_1=\sigma_1/3, h_2=\sigma_2/3$. Due to the assumptions, we also have

$$\frac{\partial P(s,t)}{\partial t}\Big|_{(1,0)}=\sigma_2\frac{\partial f}{\partial\varphi}\Big|_{(\beta,0)},\quad \frac{\partial P(s,t)}{\partial t}\Big|_{(1,1)}=\sigma_2\frac{\partial f}{\partial\varphi}\Big|_{(\beta,\alpha)}$$
$$\frac{\partial P(s,t)}{\partial s}\Big|_{(0,1)}=\sigma_1\frac{\partial f}{\partial\theta}\Big|_{(0,\alpha)},\quad \frac{\partial P(s,t)}{\partial s}\Big|_{(1,1)}=\sigma_1\frac{\partial f}{\partial\theta}\Big|_{(\beta,\alpha)}$$

and the other control points can be decided:

$$P_{31}=P_{30}+h_2\frac{\partial f}{\partial\varphi}\Big|_{(\beta,0)}=\begin{pmatrix}a\sin\beta\\bh_2\sin\beta\\c\cos\beta\end{pmatrix},\quad P_{13}=P_{03}+h_1\frac{\partial f}{\partial\theta}\Big|_{(0,\alpha)}=\begin{pmatrix}ah_1\cos\alpha\\bh_1\sin\alpha\\c\end{pmatrix},$$

$$P_{32}=P_{33}-h_2\frac{\partial f}{\partial\varphi}\Big|_{(\beta,\alpha)}=\begin{pmatrix}a\sin\beta\cos\alpha+ah_2\sin\beta\sin\alpha\\b\sin\beta\sin\alpha-bh_2\sin\beta\cos\alpha\\c\cos\beta\end{pmatrix},$$

$$P_{23} = P_{33} - h_1 \frac{\partial f}{\partial \theta}\Big|_{(\beta,\alpha)} = \begin{pmatrix} a\sin\beta\cos\alpha - ah_1\cos\beta\cos\alpha \\ b\sin\beta\sin\alpha - bh_1\cos\beta\sin\alpha \\ c\cos\beta + ch_1\sin\beta \end{pmatrix}.$$

Now, only middle control points $P_{11}, P_{12}, P_{21}, P_{22}$ are left to be determined by using two-order mixed derivatives. From

$$\frac{\partial^2 p(s,t)}{\partial s \partial t}\Big|_{(0,0)} = \sigma_1\sigma_2 \frac{\partial^2 f}{\partial\theta\partial\varphi}\Big|_{(0,0)},$$

and

$$\frac{\partial^2 p(s,t)}{\partial s \partial t}\Big|_{(0,0)} = 9(P_{11} - P_{10} - P_{01} + P_{00}), \quad \sigma_1\sigma_2 \frac{\partial^2 f}{\partial\theta\partial\varphi}\Big|_{(0,0)} = \sigma_1\sigma_2 \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix},$$

we get

$$P_{11} = P_{10} + h_1 h_2 \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} = \begin{pmatrix} ah_1 \\ bh_1h_2 \\ c \end{pmatrix}.$$

By the same argument, from

$$\frac{\partial^2 P(s,t)}{\partial s \partial t}\Big|_{(0,1)} = \sigma_1\sigma_2 \frac{\partial^2 f}{\partial\theta\partial\varphi}\Big|_{(0,\alpha)}, \quad \frac{\partial^2 P(s,t)}{\partial s \partial t}\Big|_{(1,0)} = \sigma_1\sigma_2 \frac{\partial^2 f}{\partial\theta\partial\varphi}\Big|_{(\beta,0)}$$
$$\frac{\partial^2 P(s,t)}{\partial s \partial t}\Big|_{(1,1)} = \sigma_1\sigma_2 \frac{\partial^2 f}{\partial\theta\partial\varphi}\Big|_{(\beta,\alpha)}$$

we have

$$P_{12} = \begin{pmatrix} ah_1\cos\alpha + ah_1h_2\sin\alpha \\ bh_1\sin\alpha - bh_1h_2\cos\alpha \\ c \end{pmatrix} \qquad P_{21} = \begin{pmatrix} a\sin\beta - ah_1\cos\beta \\ bh_2\sin\beta - bh_1h_2\cos\beta \\ c\cos\beta + ch_1\sin\beta \end{pmatrix}$$
$$P_{22} = \begin{pmatrix} a\sin\beta\cos\alpha - ah_1\cos\beta\cos\alpha + ah_2\sin\beta\sin\alpha - ah_1h_2\cos\beta\sin\alpha \\ b\sin\beta\sin\alpha - bh_1\cos\beta\sin\alpha - bh_2\sin\beta\cos\alpha + bh_1h_2\cos\beta\cos\alpha \\ c\cos\beta + ch_1\sin\beta \end{pmatrix}$$

Let $P_{ij} = \begin{pmatrix} X_{ij} \\ Y_{ij} \\ Z_{ij} \end{pmatrix}$, $P(s,t) = \begin{pmatrix} X(s,t) \\ Y(s,t) \\ Z(s,t) \end{pmatrix}$, then from (3.3), we have

$$X(s,t) = \sum_{i,j=0}^{3} X_{ij} B_i^3(s) B_j^3(t) = af_1(s,\beta,h_1) f_2(t,\alpha,h_2) \tag{3.4}$$

here,

$$f_1(s,\beta,h_1) = h_1 B_1^3(s) + (\sin\beta - h_1\cos\beta) B_2^3(s) + \sin\beta B_3^3(s),$$
$$f_2(t,\alpha,h_2) = B_0^3(t) + B_1^3(t) + (\cos\alpha + h_2\sin\alpha) B_2^3(t) + \cos\alpha B_3^3(t).$$

we also have:

$$Y(s,t) = \sum_{i,j=0}^{3} Y_{ij} B_i^3(s) B_j^3(t) = bf_1(s,\beta,h_1) f_1(t,\alpha,h_2),$$
$$f_1(t,\alpha,h_2) = h_2 B_1^3(t) + (\sin\alpha - h_2\cos\alpha) B_2^3(t) + \sin B_3^3(t),$$
$$Z(s,t) = \sum_{i,j=0}^{3} Z_{ij} B_i^3(s) B_j^3(t) = cf_2(s,\beta,h_1), \tag{3.5}$$

$$f_2(s,\beta,h_1) = B_0^3(s) + B_1^3(s) + (\cos\beta + h_1 \sin\beta) B_2^3(s) + \cos\beta B_3^3(s).$$

Substituting above X(s,t),Y(s,t),Z(s,t) into (1.2) gives

$$\begin{aligned}\varepsilon_2(s,t) &= \frac{X^2(s,t)}{a^2} + \frac{Y^2(s,t)}{b^2} + \frac{Z^2(s,t)}{c^2} - 1 \\ &= f_1^2(s,\beta,h_1)[f_2^2(t,\alpha,h_2) + f_1^2(t,\alpha,h_2) - 1] + f_2^2(s,\beta,h_1) + f_1^2(s,\beta,h_1) - 1\end{aligned} \tag{3.6}$$

It is easy to find out that $f_2^2(t,\alpha,h_2) + f_1^2(t,\alpha,h_2) - 1$ and $f_2^2(s,\beta,h_1) + f_1^2(s,\beta,h_1) - 1$ are just the error interpolating elliptic arc introduced in (2.6) and (2.7). When $h_2 = \frac{4}{3}\tan\frac{\alpha}{4}$ and $h_1 = \frac{4}{3}\tan\frac{\beta}{4}$, we have

$$\varepsilon_2(s,t) \leqslant \frac{4}{27}\frac{\sin^6\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}} f_1^2(s,\beta,h_1) + \frac{4}{27}\frac{\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}}.$$

Combine the above discussion,we state the following main theorem.

Theorem 2 *suppose the control points P_{ij} chosen as above and $h_1 = \frac{4}{3}\tan\frac{\beta}{4}$, $h_2 = \frac{4}{3}\tan\frac{\alpha}{4}$, then, the Bézier surface*

$$P(s,t) = \sum_{i,j=0}^{3} P_{ij} B_i^3(s) B_j^3(t), \quad s,t \in [0,1]$$

interpolates the ellipsoid surface patch Fig2(a) at points P_{00}, P_{30}, P_{33} and have the same tangent plane with ellipsoid at these points. The interpolating error $\varepsilon_2(s,t)$ satisfies

$$\begin{aligned}||\varepsilon_2(s,t)||_\infty &= \max_{s,t\in[0,1]}\left[\frac{X^2(s,t)}{a^2} + \frac{Y^2(s,t)}{b^2} + \frac{Z(s,t)^2}{c^2} - 1\right] \\ &\leqslant \frac{4}{27}\left[\frac{\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}} + \frac{\sin^6\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}\right].\end{aligned} \tag{3.7}$$

Proof: If h_1, h_2 are chosen as above, since (2.6) and (2.13), (3.6) is equal to

$$\varepsilon_2(s,t) = f_1^2(s,\beta,h_1)\frac{16\sin^6\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}} t^2(1-t)^2(2t-1)^2 + \frac{16\sin^6\frac{\beta}{4}}{\cos^2\frac{\beta}{4}} s^2(1-s)^2(2s-1)^2$$

In order to prove theorem, we only need to prove

$$|f_1(s,\beta,h_1)| \leqslant 1.$$

Because of

$$\sin\beta - h_1\cos\beta = \sin\beta - \frac{4}{3}\tan\frac{\beta}{4}\cos\beta \geqslant 0,$$

and $0 \leqslant h_1 < 1$, it is obvious that

$$|f_1(s,\beta,h_1)| \leqslant B_1^3(s) + B_2^3(s) + B_3^3(s) \leqslant 1.$$

The theorem is proved.

Considering another patch(see Fig.2(b)) the algorithm is similar to the previous one. The calculation shows that the control points are

$$P_{00} = \begin{pmatrix} a\sin\beta \\ 0 \\ c\cos\beta \end{pmatrix}, \quad P_{01} = \begin{pmatrix} a\sin\beta \\ bh_2\sin\beta \\ c\cos\beta \end{pmatrix}, \quad P_{02} = \begin{pmatrix} a\sin\beta\cos\alpha + ah_2\sin\beta\sin\alpha \\ b\sin\beta\sin\alpha - bh_2\sin\beta\cos\alpha \\ c\cos\beta \end{pmatrix},$$

$$P_{03}=\begin{pmatrix} a\sin\beta\cos\alpha\\ b\sin\beta\sin\alpha\\ c\cos\beta \end{pmatrix}, P_{10}=\begin{pmatrix} a\sin\beta+ah_1\cos\beta\\ 0\\ c\cos\beta-ch_1\sin\beta \end{pmatrix}, P_{11}=\begin{pmatrix} a\sin\beta+ah_1\cos\beta\\ bh_2\sin\beta+bh_1h_2\cos\beta\\ c\cos\beta-ch_1\sin\beta \end{pmatrix},$$

$$P_{12}=\begin{pmatrix} (a\sin\beta+ah_1\cos\beta)(\cos\alpha+h_2\sin\alpha)\\ (b\sin\alpha-bh_2\cos\alpha)(\sin\beta+h_1\cos\beta)\\ c\cos\beta-ch_1\sin\beta \end{pmatrix}, \quad P_{13}=\begin{pmatrix} a\sin\beta\cos\alpha+ah_1\cos\beta\cos\alpha\\ b\sin\beta\sin\alpha+bh_1\cos\beta\sin\alpha\\ c\cos\beta-ch_1\sin\beta \end{pmatrix},$$

$$P_{20}=\begin{pmatrix} a\\ 0\\ ch_1 \end{pmatrix}, \quad P_{21}=\begin{pmatrix} a\\ bh_2\\ ch_1 \end{pmatrix}, \quad P_{22}=\begin{pmatrix} a\cos\alpha+ah_2\sin\alpha\\ b\sin\alpha-bh_2\cos\alpha\\ ch_1 \end{pmatrix}, \quad P_{23}=\begin{pmatrix} a\cos\alpha\\ b\sin\alpha\\ ch_1 \end{pmatrix},$$

$$P_{30}=\begin{pmatrix} a\\ 0\\ 0 \end{pmatrix}, \quad P_{31}=\begin{pmatrix} a\\ bh_2\\ 0 \end{pmatrix}, \quad P_{32}=\begin{pmatrix} a\cos\alpha+ah_2\sin\alpha\\ b\sin\alpha-bh_2\cos\alpha\\ 0 \end{pmatrix}, \quad P_{33}=\begin{pmatrix} a\cos\alpha\\ b\sin\alpha\\ 0 \end{pmatrix}.$$

Based on control points, the bicubic Bézier surface can be written as:

$$P(s,t)=\begin{pmatrix} X(s,t)\\ Y(s,t)\\ Z(s,t) \end{pmatrix}=\begin{pmatrix} aq_1(s,\beta,h_1)f_2(t,\alpha,h_2)\\ bq_1(s,\beta,h_1)f_1(t,\alpha,h_2)\\ cq_2(s,\beta,h_1) \end{pmatrix}.$$

here, f_1 andf_2 are the same functions as in (2.7), but

$$\begin{aligned} q_1(s,\beta,h_1)&=\sin\beta B_0^3(s)+(\sin\beta+h_1\cos\beta)B_1^3(s)+B_2^3(s)+B_3^3(s)\\ &=f_2(1-s,\frac{\pi}{2}-\beta,h_1)\\ q_2(s,\beta,h_1)&=\cos\beta B_0^3(s)+(\cos\beta-h_1\sin\beta)B_1^3(s)+h_1B_2^3(s)\\ &=f_1(1-s,\frac{\pi}{2}-\beta,h_1), \end{aligned}$$

so the interpolating error $\varepsilon_2^*(s,t)$ is equal to:

$$\begin{aligned} \varepsilon_2^*(s,t)&=q_1^2(f_2^2+f_1^2-1)+q_1^2+q_2^2-1\\ &=f_2^2(1-s,\frac{\pi}{2}-\beta,h_1)[f_2^2(t,\alpha,h_2)+f_1^2(t,\alpha,h_2)-1]\\ &+f_2^2(1-s,\frac{\pi}{2}-\beta,h_1)+f_1^2(1-s,\frac{\pi}{2}-\beta,h_1)-1 \end{aligned} \tag{3.8}$$

Let $h_1=\frac{4}{3}\tan(\frac{\pi}{8}-\frac{\beta}{4}), \quad h_2=\frac{4}{3}\tan(\frac{\alpha}{4})$, we have the following

Theorem 3 *The Bézier surface*

$$P(s,t)=\begin{pmatrix} aq_1(s,\beta,h_1)f_2(t,\alpha,h_2)\\ bq_1(s,\beta,h_1)f_1(t,\alpha,h_2)\\ cq_2(s,\beta,h_1) \end{pmatrix}$$

based on the above assumptions interpolates the second ellipsoid patch shown in Fig.2(b) at points $P_{00}, P_{03}, P_{30}, P_{33}$, and have the same tangent plane with ellipsoid at these points. And the error function $\varepsilon_2^*(s,t)$ satisfies

$$||\varepsilon_2^*(s,t)||_\infty \leqslant \frac{4}{27}\left[\frac{\sin^6\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}+\frac{\sin^6(\frac{\pi}{8}-\frac{\beta}{4})}{\cos^2(\frac{\pi}{8}-\frac{\beta}{4})}\right].$$

Proof: From (3.8), (2.6), (2.13), we have

$$\varepsilon_2^*(s,t)=q_1^2(s,\beta,h_1)\frac{16\sin^6\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}t^2(1-t)^2(2t-1)^2+\frac{16\sin^6(\frac{\pi}{8}-\frac{\beta}{4})}{\cos^2(\frac{\pi}{8}-\frac{\beta}{4})}s^2(1-s)^2(2s-1)^2.$$

Then, for the theorem, we only need to prove $|q_1| \leqslant 1$, that is $|\sin\beta + h_1\cos\beta| \leqslant 1$. Obviously,

$$\frac{4}{3}\cos(\frac{\pi}{4} - \frac{\beta}{2}) \leqslant 1 + \cos(\frac{\pi}{4} - \frac{\beta}{2}) = 2\cos^2(\frac{\pi}{8} - \frac{\beta}{4})$$

thus there is

$$\frac{4}{3}\frac{\sin(\frac{\pi}{8} - \frac{\beta}{4})}{\cos(\frac{\pi}{8} - \frac{\beta}{4})}\cos(\frac{\pi}{4} - \frac{\beta}{2}) \leqslant 2\sin(\frac{\pi}{8} - \frac{\beta}{4})\cos(\frac{\pi}{8} - \frac{\beta}{4}) = \sin(\frac{\pi}{4} - \frac{\beta}{2})$$

so, $h_1\sin(\frac{\pi}{2} - \beta) \leqslant 2\sin^2(\frac{\pi}{4} - \frac{\beta}{2}) = 1 - \sin\beta$. Then what we need is proved.

From theorem2 and theorem3, we know that the sixth order approximation error is obtained. The following table shows the error $\varepsilon_2(s,t)$ for different α, β. For simplicity, here take $\alpha = \beta$:

β	$\beta = \frac{\pi}{2}$	$\beta = \frac{\pi}{4}$	$\beta = \frac{\pi}{8}$	$\beta = \frac{\pi}{16}$
$\|\|\varepsilon_2(s,t)\|\|_\infty$	$1.09027 \cdot 10^{-3}$	$1.69822 \cdot 10^{-5}$	$2.65296 \cdot 10^{-7}$	$4.14520 \cdot 10^{-9}$

Furthermore, two adjacent approximation surface patches are not only continuous but also have the same tangent plane at common boundary. The effect of approximation is shown in Fig.3(a),(b), and we show the pictures for error functions in Fig.3(c),(d).

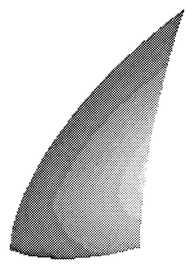


Fig.3(a)the combination of two approximation patches

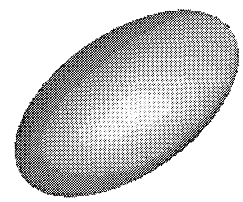


Fig.3(b)the approximation surface of the whole ellipsoid using eight pieces approximation patches.

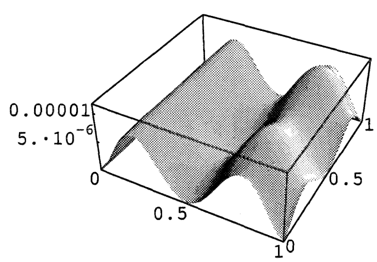


Fig.3(c)the error function $\varepsilon_2(s,t), \alpha = \beta = \frac{\pi}{4}$

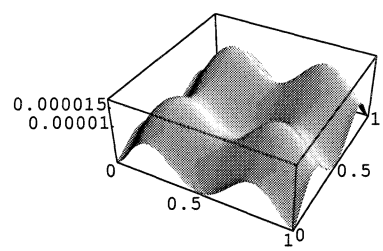


Fig.3(d)the error function $\varepsilon_2^*(s,t), \alpha = \beta = \frac{\pi}{4}$

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