Polynomial Interpolant Approximation with Geometric Continuity is Much Better than Hermite Interpolation *

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July 5, 2001

Abstract

General speaking, Bézier curve interpolation with geometric continuity has more free parameter than corresponding Hermite interpolation, therefore with the same interpolation polynomial degree, the approximation error of the former is smaller. But sometimes there is an unexpected big difference between them. In this paper, as approximating a quarter unit circle, we show that the error $\varepsilon_3^*(t)$ of cubic Hermite interpolation is of 400 times the error $\varepsilon_3(t)$ for cubic Bézier curve interpolation with geometric continuity, i.e., $\|\varepsilon_3^*(t)\|_\infty \geqslant 400\cdot\|\varepsilon_3(t)\|_\infty$. Furthermore, for the error $\varepsilon_4^*(t,c)$ of Hermite interpolation of degree 4 with a free parameter c, we still have $\min\limits_{c\in\mathbb{R}} \|\varepsilon_4^*(t,c)\|_\infty \geqslant 6.3\cdot\|\varepsilon_3(t)\|_\infty$.

Keywords Approximation, Bézier curves, Geometric continuity

1 Introduction

Polynomial curves and surfaces are simplest and play an essential role in the representation and design of free form curves and surfaces. For a given planar parametric curve $F(t) = (x(t), y(t))^T$, $0 \leqslant t \leqslant 1$, sometimes it is necessary to approximate this curve by polynomial curve in the geometric modeling application, due to the need of communication between different CAD systems.

The common method of approximation to a given curve is interpolation at end point of the given curve with geometric continuity. For example, we need to find a cubic polynomial curve $p(t)$ in Bézier form,

$$p(t) = \sum_{i=0}^{3} P_i B_i^3(t), \tag{1.1}$$

*Supported by NKBRSF on Mathematic Mechanics (grant G1998030600), the Research Fund for the Doctoral Program of Higher Education.

such that

$$\begin{aligned} p(0) &= F(0), \quad p'(0) = \sigma_0 F'(0), \\ p(1) &= F(1), \quad p'(1) = \sigma_1 F'(1), \end{aligned} \tag{1.2}$$

where P_i, $i = 0, 1, 2, 3$ are control points, and

$$B_i^m(t) = \binom{m}{i} t^i (1-t)^{m-i}, \quad i = 0, \ldots, m \tag{1.3}$$

Bernstein basis functions, as well as σ_0 and σ_1 any real numbers.

Another choice of interpolation conditions is require polynomial curve $H(t)$ interpolates $F(t)$ and its derivatives at end points $F(0)$ and $F(1)$. Anyway we always hope the error between the approximation curve $p(t)$ or $H(t)$ and the given curve $F(t)$ is small, the smaller the better.

It is obvious that the approximation with geometric continuity is better, since it has two free parameters σ_0 and σ_1, when $\sigma_0 = \sigma_1 = 1$, we have $p(t) = H(t)$.

In this paper, for the case $\sigma_0 = \sigma_1 = \sigma$, let $\varepsilon_3^*(t)$, $\varepsilon_3(t)$ be the errors of cubic Hermite interpolation and cubic Bézier curve interpolation with geometric continuity to a quarter unit circle respectively. We will prove that $\|\varepsilon_3^*(t)\|_\infty \geqslant 400 \cdot \|\varepsilon_3(t)\|_\infty$. Furthermore, if $\varepsilon_4^*(t, c)$ denotes the error of Hermite interpolation of degree 4 with a free parameter c, we still have $\min\limits_{c \in \mathbb{R}} \|\varepsilon_4^*(t, c)\|_\infty \geqslant 6.3 \|\varepsilon_3(t)\|_\infty$.

2 Comparison between two kinds of interpolant approximation

Let

$$F(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2} \tag{2.1}$$

be a quarter unit circle, we use a cubic polynomial curve in Bézier form interpolating and having the same tangent direction with the circle at end point to approximate it. Let the Bézier curve be

$$p(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{i=0}^{3} P_i B_i^3(t), \quad t \in [0, 1], \tag{2.2}$$

where P_i and $B_i^3(t)$, $i = 0, 1, 2, 3$ are control points and Berntein basis functions respectively. Because $p(t)$ interpolates and has the same tangent direction at end points, it is easy to compute

$$P_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad P_1 = \begin{pmatrix} 1 \\ \sigma/3 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \sigma/3 \\ 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.3}$$

where $\sigma \in \mathbb{R}$ is an arbitrary constant. Then $p(t)$ satisfies

$$\begin{aligned} p(0) &= F(0), \\ p(1) &= F(\pi/2) \end{aligned} \quad \text{and} \quad \begin{aligned} p'(0) &= \sigma F'(0), \\ p'(1) &= \sigma F'(\pi/2). \end{aligned} \tag{2.4}$$

From (2.2) and (2.3), we have

$$\begin{aligned} x(t) &= B_0^3(t) + B_1^3(t) + \tfrac{1}{3}\sigma B_2^3(t), \\ y(t) &= \tfrac{1}{3}\sigma B_1^3(t) + B_2^3(t) + B_3^3(t). \end{aligned} \tag{2.5}$$

In order to estimate the error between the quarter unit circle and Bézier curve (2.2), we introduce the error function in radias

$$\varepsilon_3(t) = x^2(t) + y^2(t) - 1. \tag{2.6}$$

Since $\varepsilon_3(t)$ is a polynomial of degree six, it can be written in Bézier form

$$\varepsilon_3(t) = \sum_{i=0}^{6} b_i B_i^6(t). \tag{2.7}$$

Using (2.4), it is easy to get $b_0 = b_6 = b_1 = b_5 = 0$. Since symmetry, it requires that $\varepsilon_3(t) = \varepsilon_3(1-t)$, hence $b_2 = b_4$. A simple calculation shows that

$$\varepsilon_3''(0) = 30(b_2 - 2b_1 + b_0) = 30b_2, \tag{2.8}$$

but from (2.5) and (2.6), we have

$$\varepsilon_3''(0) = 2\sigma^2 + 4(\sigma - 3),$$

hence,

$$b_2 = \frac{1}{15}(\sigma^2 + 2\sigma - 6). \tag{2.9}$$

In order to determine the arbitrary constant σ, we require point $(x(1/2), y(1/2))^T$ is on the circle, then σ can be decided as

$$\sigma = 4(\sqrt{2} - 1). \tag{2.10}$$

Using $\varepsilon_3(1/2) = 0$, we have the relation

$$3b_2 + 2b_3 = 0. \tag{2.11}$$

So far, we have get all b_i, $i = 0, 1, \ldots, 6$. Therefore we have

$$\varepsilon_3(t) = 2(17 - 12\sqrt{2})t^2(1-t)^2(1-2t)^2. \tag{2.12}$$

It is not difficult to prove

$$\|\varepsilon_3(t)\|_\infty = \frac{1}{54}(17 - 12\sqrt{2}) \approx 5.45 \times 10^{-4}. \tag{2.13}$$

The above result is the same as in [2] and [3].

If we require that the cubic Bézier approximation curve is an Hermite interpolation of the quarter unit circle, that is the Bézier curve

$$H_3(t) = \begin{pmatrix} x^*(t) \\ y^*(t) \end{pmatrix} \tag{2.14}$$

satisfies

$$\begin{array}{l} H_3(0) = F(0), \\ H_3(1) = F(\pi/2) \end{array} \quad \text{and} \quad \begin{array}{l} H_3'(0) = F'(0), \\ H_3'(1) = F'(\pi/2). \end{array} \tag{2.15}$$

We also introduce error function as

$$\varepsilon_3^*(t) = {x^*}^2(t) + {y^*}^2(t) - 1. \tag{2.16}$$

The similar calcualtion shows that

$$\begin{array}{l} x^*(t) = B_0^3(t) + B_1^3(t) + \frac{1}{3}B_2^3(t), \\ y^*(t) = \frac{1}{3}B_1^3(t) + B_2^3(t) + B_3^3(t) \end{array} \tag{2.17}$$

and

$$\varepsilon_3^*(t) = -t^2(1-t)^2(3+2t-2t^2), \tag{2.18}$$

hence

$$\|\varepsilon_3^*(t)\|_\infty = \frac{7}{32}. \tag{2.19}$$

From (2.13) and (2.19), we get

$$\frac{\|\varepsilon_3^*(t)\|_\infty}{\|\varepsilon_3(t)\|_\infty} = \frac{189}{16(17-12\sqrt{2})} \approx 401.277. \tag{2.20}$$

It is a surprise that the error for Hermite interpolation is about 400 times the interpolant error with geometric continuity. Of course, a reasonable explanation is that there is a free parameter for geometric continuous interpolation. In the following section, we consider Hermite interpolation with a free parameter, using Bézier curve of degree 4, and compare both the errors, we will see what will happen in this case.

3 Further discussions

Let Bézier curve of degree 4

$$H_4(t) = \begin{pmatrix} x_4(t) \\ y_4(t) \end{pmatrix} = \sum_{i=0}^{4} Q_i B_i^4(t) \tag{3.1}$$

interpolates the quarter unit circle and has the same derivatives at end points, that is,

$$\begin{array}{l} H_4(0) = F(0), \\ H_4(1) = F(\pi/2) \end{array} \quad \text{and} \quad \begin{array}{l} H_4'(0) = F'(0), \\ H_4'(1) = F'(\pi/2). \end{array} \tag{3.2}$$

A simple calculation shows that

$$\begin{aligned} x_4(t) &= B_0^4(t) + B_1^4(t) + \tfrac{1}{4}B_2^4(t) + c_1 B_2^4(t), \\ y_4(t) &= \tfrac{1}{4}B_1^4(t) + B_3^4(t) + B_4^4(t) + c_2 B_2^4(t). \end{aligned} \tag{3.3}$$

Since symmetry, we take $c_1 = c_2$, using degree elevation, we can express $H_4(t)$ as

$$H_4(t) = \begin{pmatrix} x_4(t) \\ y_4(t) \end{pmatrix} = \begin{pmatrix} x^*(t) + cB_2^4(t) \\ y^*(t) + cB_2^4(t) \end{pmatrix}, \tag{3.4}$$

where $x^*(t)$ and $y^*(t)$ are defined in (2.17), and c is an arbitrary real number. Let

$$\varepsilon_4^*(t,c) = x_4^2(t) + y_4^2(t) - 1. \tag{3.5}$$

We have the following theorem.

Theorem 1

$$\min_{c\in\mathbb{R}} \max_{t\in[0,1]} |\varepsilon_4^*(t,c)| \geqslant 3.47 \times 10^{-3}.$$

Proof A straightforward calculation shows that

$$\varepsilon_4^*(t,c) = t^2(1-t)^2[72c^2t^2(1-t)^2 + 12c(1+t-t^2) - (3+2t-2t^2)]. \tag{3.6}$$

Let $T = 4t(1-t)$, then $0 \leqslant T \leqslant 1$, and

$$\varepsilon_4^*(t,c) = \frac{1}{32}T^2[9c^2T^2 + (6c-1)T + 6(4c-1)] =: \frac{1}{32}f(T,c). \tag{3.7}$$

In order to find $\min_{c\in\mathbb{R}} \max_{T\in[0,1]} |f(T,c)|$, let

$$g(c) := \max_{T\in[0,1]} |f(T,c)|. \tag{3.8}$$

When $c \geqslant 1/4$,

$$g(c) \geqslant |f(1,c)| = 9c^2 + 30c - 7 \geqslant \frac{17}{16}; \tag{3.9}$$

When $-6 \leqslant c \leqslant 1/5$, we consider

$$f(1/2,c) = \frac{1}{16}(9c^2 + 108c - 26),$$

$f(1/2,c) = 0$ has two roots:

$$\begin{aligned} c_1 &= \tfrac{1}{3}(5\sqrt{14} - 18) \approx 0.2361, \\ c_2 &= -\tfrac{1}{3}(5\sqrt{14} + 18) \approx -12.2361, \end{aligned} \tag{3.10}$$

and

$$f'(1/2,-6) = 0. \tag{3.11}$$

Hence,

$$g(c) \geqslant \left|f(1/2,c)\right| \geqslant |f(1/2,1/5)| > 1/4. \tag{3.12}$$

For the case $c < -6$, since $f(1,c)$ is monotone decrease, and $f(1,c) > 0$, we have

$$g(c) \geqslant f(1,c) \geqslant f(1,-6) = 137. \tag{3.13}$$

We now only need to consider the case $1/5 < c < 1/4$. We note that

$$\begin{aligned} &\frac{d^2}{dT^2} f(0,c) = 12(4c-1) < 0, \\ &f(0 \pm \varepsilon, c) < 0, \\ &f(1,c) > 0, \end{aligned} \tag{3.14}$$

and

$$\frac{df(T,c)}{dT} = 3T(12c^2T^2 + (6c-1)T + 4(4c-1)), \tag{3.15}$$

it has three real roots:

$$\begin{aligned} T_1 &= 0, \\ T_2(c) &= \frac{1-6c-\sqrt{1-12c+228c^2-768c^3}}{24c^2}, \\ T_3(c) &= \frac{1-6c+\sqrt{1-12c+228c^2-768c^3}}{24c^2} =: d(c) \end{aligned} \tag{3.16}$$

and

$$d(c) \in (0,1). \tag{3.17}$$

Hence, we have

$$g(c) = \max(-f(d(c),c), f(1,c)), \tag{3.18}$$

the equation

$$f(1,c) + f(d(c),c) = 0 \tag{3.19}$$

has a unique solution $c^* \in (1/5, 1/4)$. This solution can be solved, we get

$$c^* \in (0.22222502, 0.22222503), \tag{3.20}$$

and

$$\min_{c\in\mathbb{R}} g(c) \approx 0.111206. \tag{3.21}$$

This gives that

$$\min_{c\in\mathbb{R}} \max_{t\in[0,1]} |\varepsilon_4^*(t,c)| \approx \frac{0.111206}{32} \geqslant 3.47 \times 10^3.$$

The theorem is proved.

From the above theorem and (2.13), it follows:

Corollary

$$\min_{c\in\mathbb{R}} \max_{t\in[0,1]} |\varepsilon_4^*(t,c)| \geqslant 6.3 \cdot \max_{t\in[0,1]} |\varepsilon_3(t)|.$$

It should be pointed out that the approximation (1.2) with $\varepsilon_3(1/2) = 0$ is a one-sided (or over-estimated) approximation. If we select optimal σ (see [1]), then we have $\|\varepsilon_3(t)\|_\infty \leqslant 3.92 \times 10^{-4}$. In this case, we have

$$\min_{c\in\mathbb{R}} \max_{t\in[0,1]} |\varepsilon_4^*(t,c)| \geqslant 8.75 \cdot \max_{t\in[0,1]} |\varepsilon_3(t)|.$$

The above example shows that polynomial interpolation with geometric continuity is much better than the corresponding Hermite interpolation.

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