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Dimensions of spline spaces over general T-meshes*

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Abstract: The spline spaces over general T-meshes and the periodic spline spaces over regular T-meshes were studied. With the method based on B-nets the dimension formulae were proposed as well.

Key words: T-mesh; spline; Bézier ordinate; dimension

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一般T网格上样条空间的维数

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摘要:研究了一般T网格上的样条空间和规则T网格上的周期样条空间,并用基于B网的方法给出了它们的维数公式.

关键词:T网格;样条;Bézier纵标;维数

0 Introduction

T-meshes are formed by a set of horizontal line segments and a set of vertical line segments, where T-junctions are allowed. It could be thought of as a transition between the simple tensor-product meshes and the complicated triangular meshes. Over the T-meshes, we can define many types of spline spaces. Currently, there are basically two types of spline spaces defined over T-meshes: one is from Sederberg *et al*[1], and the other is from Deng *et al*[2].

Ref. [1] invented T-spline. It is a point-based spline over T-meshes, i.e., for every vertex, a blending function is defined. Each of the blending functions comes from some tensor-product spline space. Though this type of spline supports many valuable operations within a consistent framework, some of them, such as, evaluation and local refinement, are not simple. In the T-spline theory, the basis functions are rational which lead to evaluations and are not simple; on the other hand, the local refinement is dependent on the structure of the mesh, and its complexity is uncertain. Furthermore, it is an ongoing problem

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whether T-spline blending functions are always linearly independent[3]. The reason for these problems is mainly that the spline over every cell of the mesh is not a polynomial, but a piecewise polynomial.

Forcing functions to be a tensor-product polynomial on every cell and to satisfy the specified smoothness across common edges, Deng *et al* introduced the spline spaces over T-meshes[2]. They derived the dimension formulae for the spline spaces over regular T-meshes via a method based on B-nets. Here "regular" means that the whole domains are rectangles without holes inside. Comparing with Sederberg's T-spline, the surface modeling based on the spline spaces over T-meshes allows local refinement to be really local. And the evaluation is quick since this type of spline is polynomial, not rational.

However, all these works about spline spaces over T-meshes only take into account the regular T-meshes, and the spline surfaces defined over such T-meshes will have topological equivalent to the disks. In the practice of surface modeling, the boundary of the domain of a complex surface may be an arbitrary polygon, and the domain may also have one or several holes. For example, if we fit a spline surface over T-meshes according to the range data of some mask, then we need the domain T-mesh to have at least three holes inside, one for the mouth, and the other two for two eyes. On the other hand, if we want to model a simple close surface or a cylindrical surface with a close section, then we need periodic splines. Hence in this paper, we consider the spline function spaces over general T-meshes. For the regular T-meshes, periodic splines are discussed as well. With the method based on B-nets proposed by Deng *et al*[2], we derive the dimension formulae of these spline spaces.

1　Spline spaces over T-meshes

In this section, we first review some concepts related to T-meshes and the spline function spaces over T-meshes.

1.1　T-meshes

T-meshes are formed by a set of horizontal line segments and a set of vertical line segments, where T-junctions are allowed. We can also think that the T-mesh is a simply connected domain formed by the union of a set of rectangles whose sides are horizontal or vertical. Fig. 1 illustrates two examples of T-meshes, while in Fig. 2 two examples of non-T-meshes are shown.

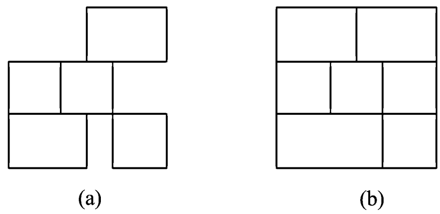

Fig. 1　Two examples of T-meshes

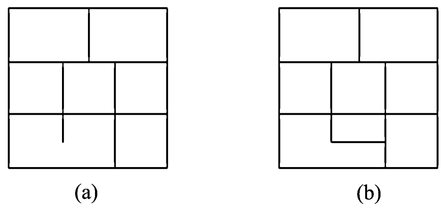

Fig. 2　Two examples of non-T-meshes

Fig. 3 gives a more complicated T-mesh, where three parts with gray fill-in are excluded from the T-mesh. Hence there are three holes in the T-mesh. Here each small rectangle in the T-mesh is called a cell or a facet. The corners of these rectangles are called vertices of the T-mesh. If a vertex is on the boundary, then it is called a boundary vertex. Otherwise, it is called an interior

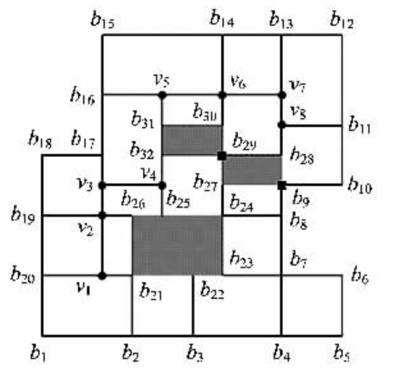

Fig. 3　A T-mesh with notations

vertex. For example, $b_i (i = 1, \cdots, 32)$ in Fig. 3 are boundary vertices, and all the other vertices v_i $(i = 1, \cdots, 8)$ are interior vertices.

Interior vertices have two types. One is crossing, for example, v_2 in Fig. 3; and the other is T-junctional, for example, v_1 in Fig. 3. We call them crossing vertices and T-vertices respectively. Boundary vertices also have two types. If a boundary vertex is on the boundary of some cell, but not any corner of the cell, then it is called an inner b-vertex; otherwise, it is called a free b-vertex. For example, b_{17} in Fig. 3 is an inner b-vertex, and b_{16} a free b-vertex. Especially, if a free b-vertex is the common corner and the only intersection point of two cells, then it is called a singular b-vertex, such as b_9 and b_{29} in Fig. 3.

The line segment connecting two adjacent vertices is called an edge of the T-mesh. If an edge is on the boundary of the T-mesh, then it is called a boundary edge; otherwise it is called an interior edge. For example, in Fig. 3, $v_1 b_{20}$ and $v_1 v_2$ are interior edges while $b_2 b_3$ and $b_{21} b_{22}$ are boundary edges.

Two cells are called adjacent if they share a common edge as part of their boundaries. If one cell is above or below the other, then they are called adjacent vertically. If one cell is on the left or right of the other, then they are called adjacent horizontally. A cell is called adjacent to a grid line (an edge or composition of several edges) if some boundary line of the cell is part of the grid line. For example, in Fig. 3, $v_3 v_4 v_5 b_{16}$ and $v_6 b_{14} b_{15} b_{16}$ are adjacent vertically, $v_3 v_4 v_5 b_{16}$ and $v_2 b_{17} b_{18} b_{19}$ are adjacent horizontally, and $v_3 v_4 v_5 b_{16}$ is adjacent to $v_1 b_{15}$.

A composite edge (shortly, c-edge) is the longest possible line segment in the T-mesh. The inner vertices (vertices except the end points of the line segment) are all either interior T-vertices or inner b-vertices. For example, in Fig. 3, $v_6 b_{16}$, $v_5 b_{25}$, $v_2 b_{16}$, and $b_{15} b_{16}$ are c-edges, while $v_1 v_3$, $v_2 b_{17}$, $v_2 b_{15}$ and $b_{16} b_{17}$ are not. C-edges have three types. If a c-edge is just a boundary edge, then it is called a boundary c-edge, for example, $b_1 b_2$ in Fig. 3. An interior c-edge is the c-edge which consists of only interior edges, for example, $v_1 v_2$ and $v_6 b_{16}$ in Fig. 3. A mixed c-edge is the c-edge which consists of both interior edges and boundary edges, for example, $v_2 b_{16}$ and $v_5 b_{25}$ in Fig. 3. Mixed c-edges and interior c-edges are also called non-boundary c-edges.

The regular T-mesh considered by Deng *et al*[2] is such a T-mesh that the union of its cells forms a rectangle, see Fig. 1(b). In a regular T-mesh, a c-edge is either a boundary c-edge or an interior c-edge.

1.2 The spline space

Given a T-mesh $\mathscr{T}$, we use $\mathscr{F}$ to denote all the cells in $\mathscr{T}$ and Ω to denote the region occupied by all the cells in $\mathscr{T}$. The spline space over the given T-mesh $\mathscr{T}$ is defined as

$$\mathscr{S}(m,n,\alpha,\beta,\mathscr{T}) := \{s(x,y) \in C^{\alpha,\beta}(\Omega) \mid s(x,y)|_{\phi} \in \mathscr{P}_{mn}, \forall \phi \in \mathscr{F}\},$$

where $\mathscr{P}_{mn}$ is the space of all polynomials with bi-degree (m,n), and $C^{\alpha,\beta}(\Omega)$ is the space consisting of all bivariate functions which are continuous in Ω with order α along the x direction and with order β along the y direction.

It is obvious that the spline spaces are linear spaces and the study as to its dimensions property is an interesting problem. The results are useful for surface modeling with the spline spaces over T-meshes.

2 Review of B-net method

B-net method is used in the proof of the dimension formulae proposed by Deng *et al*[2]. We will also use B-net method to discuss the dimensions of the spline spaces over general T-meshes.

Let $\pi_1(x,y)$ and $\pi_2(x,y)$ be two polynomials with bi-degree (m,n), defined over two adjacent domains

$$[x_0, x_1] \times [y_0, y_1]$$

and

$$[x_1, x_2] \times [y_0, y_1],$$

respectively. They can be expressed in Bernstein-

Bézier forms:

$$\pi_1(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}b_{j,k}^{1}B_j^m\Big(\frac{x-x_0}{x_1-x_0}\Big)B_k^n\Big(\frac{y-y_0}{y_1-y_0}\Big),$$

$$\pi_2(x,y)=\sum_{j=0}^{m}\sum_{k=0}^{n}b_{j,k}^{2}B_j^m\Big(\frac{x-x_1}{x_2-x_1}\Big)B_k^n\Big(\frac{y-y_0}{y_1-y_0}\Big),$$

where $B_j^m(t)$ and $B_k^n(t)$ are Bernstein polynomials. $\{b_{j,k}^1\}$ and $\{b_{j,k}^2\}$ are called the Bézier ordinates of $\pi_1(x,y)$ and $\pi_2(x,y)$ respectively. It is well known that $\pi_1(x,y)$ and $\pi_2(x,y)$ are r times differentiable across their common boundary if and only if [3]

$$\frac{1}{(x_1-x_0)^i}\Delta^{i,0}\,b_{m-i,j}^{1}=\frac{1}{(x_2-x_1)^i}\Delta^{i,0}\,b_{0,j}^{2},$$

$$j=0,\cdots,n,\ i=0,\cdots,r.$$

Here the difference operators are defined by

$$\Delta^{i,0}\,b_{j,k}=\Delta^{i-1,0}\,b_{j+1,k}-\Delta^{i-1,0}\,b_{j,k}$$

with $\Delta^{0,0}b_{j,k}=b_{j,k}$.

The geometric meaning of the above conditions is illustrated in Fig. 4. The Bézier ordinates of two bicubic Bézier functions are shown in solid and circle, respectively. Suppose $\pi_1(x,y)$ and $\pi_2(x,y)$ are $C^k(k=0,1,2,3)$ continuous along their common boundary, then the 0th to the kth columns of the circle ordinates are determined by the $(3-k)$th to the 3rd columns of the solid ordinates.

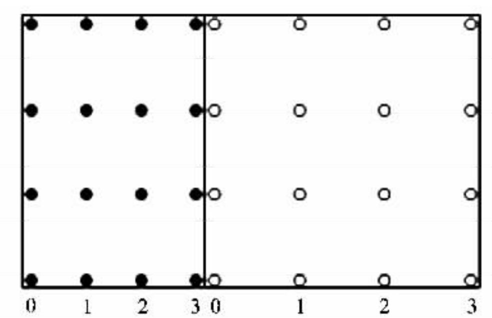

Fig. 4 Bézier ordinates of two bicubic polynomials

If we define a polynomial of bi-degree $(k,3)$ $(k=1,2,3)$ with the $(3-k)$th to the 3rd columns of the solid ordinates as its Bézier ordinates, and similarly define a polynomial of bi-degree $(k,3)$ with the 0th to the kth columns of the circle ordinates as its Bézier ordinates, then these two polynomials are the same. Especially, for the third order continuity conditions, the two polynomials $\pi_1(x,y)$ and $\pi_2(x,y)$ are identical.

Similar to the cases of regular T-meshes in Lemma 3.1 of Ref. [2], for general T-meshes, we have the following lemma, which will play an important role in the proof of the dimension formulae of the spline spaces over general T-meshes.

Lemma 2.1 Given a T-mesh $\mathscr{T}$ and a spline space $\mathscr{S}(m,n,\alpha,\beta,\mathscr{T})$ defined over $\mathscr{T}$, consider a horizontal non-boundary c-edge which consists of ℓ interior edges and has $\ell+1$ cells adjacent to it. Then the $\beta+1$ rows of the Bézier ordinates near the c-edge in each of these cells will define an identical polynomial of bi-degree $(m,\ \beta)$ (see Fig. 5). Similarly, consider a vertical non-boundary c-edge. The $\alpha+1$ columns of the Bézier ordinates near the c-edge in every adjacent cell will define an identical polynomial of bi-degree $(\alpha,\ n)$.

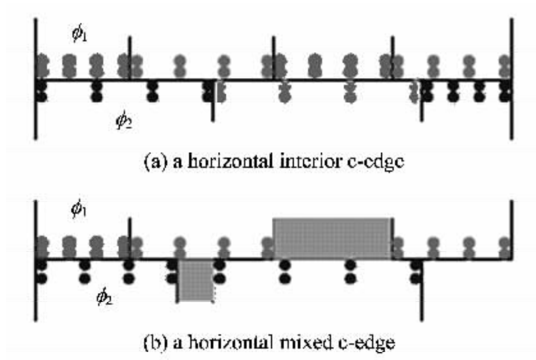

Fig. 5 Bézier ordinates in every cell near a non-boundary c-edge ($m=3,\beta=1$)

Proof We just prove the case for horizontal non-boundary c-edge. Suppose ϕ_1 and ϕ_2 are the two leftmost adjacent cells near the c-edge with ϕ_2 beneath ϕ_1. According to the previous analysis, we know that $\beta+1$ rows of the Bézier ordinates near the c-edge in ϕ_1 and ϕ_2 will define the same polynomial with bi-degree $(m,\ \beta)$. If $\ell>1$, then we select the leftmost cell in the rest of the adjacent cells of the c-edge. The new cell will be adjacent vertically to ϕ_1 or ϕ_2. Therefore $\beta+1$ rows of the Bézier ordinates near the c-edge in the new cell will define the same polynomial. By this fashion, we can run through all the cells, and thus all the $\beta+1$ rows of the Bézier ordinates near the c-edge define an identical polynomial. □

3 Dimension formula of spline spaces over general T-meshes

In this section, we will derive a dimension formula for the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ over a general T-mesh $\mathcal{T}$ when $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$.

3.1 Some notations for a T-mesh

Before we derive the dimension formula, we introduce some notations for a T-mesh as shown in Tab. 1.

Tab. 1 Notations for a T-mesh

notations	description
E_h^b	number of horizontal boundary edges
E_v^b	number of vertical boundary edges
E_h	number of horizontal interior edges
E_v	number of vertical interior edges
$\widetilde{E}_h^b$	number of horizontal boundary c-edges
$\widetilde{E}_v^b$	number of vertical boundary c-edges
$\widetilde{E}_h$	number of horizontal non-boundary c-edges
$\widetilde{E}_v$	number of vertical non-boundary c-edges
λ_j	number of interior edges on the jth horizontal non-boundary c-edge, $j=1,\cdots,\widetilde{E}_h$
μ_j	number of interior edges on the jth vertical non-boundary c-edge, $j=1,\cdots,\widetilde{E}_v$
E	number of interior edges
V^+	number of interior crossing vertices
$V^\perp$	number of interior T-vertices
V	number of interior vertices
V_f^b	number of free b-vertices
V_s^b	number of singular b-vertices
V^b	number of boundary vertices
F	number of cells in the mesh

Similar to Lemma 4.1 for a regular T-mesh in Ref. [2], for a general T-mesh, we have the following topological equations.

Lemma 3.1 Given a T-mesh with the notations in Tab. 1, then

(Ⅰ)

$$\sum_{j=1}^{\widetilde{E}_h}\lambda_j = E_h, \sum_{j=1}^{\widetilde{E}_v}\mu_j = E_v. \quad (1)$$

(Ⅱ)

$$2F-\widetilde{E}_h^b-\widetilde{E}_h = E_h, 2F-\widetilde{E}_v^b-\widetilde{E}_v = E_v. \quad (2)$$

(Ⅲ)

$$E-\widetilde{E}_h-\widetilde{E}_v = V^\perp. \quad (3)$$

(Ⅳ)

$$V^b+V_s^b = E_h^b+E_v^b,\ V_f^b+V_s^b = \widetilde{E}_h^b+\widetilde{E}_v^b. \quad (4)$$

Proof

(Ⅰ) Eq. (1) holds obviously according to the definition of λ_i and μ_i.

(Ⅱ) Since every cell has two horizontal boundary lines, each of which is part of some horizontal c-edge, and every horizontal non-boundary c-edge has λ_i+1 adjacent cells, by the contribution of boundary c-edges, it follows that

$$2F = \widetilde{E}_h^b + \sum_{j=1}^{\widetilde{E}_h}(\lambda_j+1).$$

By Eq. (1), we have

$$2F-\widetilde{E}_h^b-\widetilde{E}_h = E_h.$$

Similarly,

$$2F-\widetilde{E}_v^b-\widetilde{E}_v = E_v.$$

(Ⅲ) Observe that

$$E-\widetilde{E}_h-\widetilde{E}_v = (E_h-\widetilde{E}_h)+(E_v-\widetilde{E}_v).$$

Since $E_h-\widetilde{E}_h$ and $E_v-\widetilde{E}_v$ represent the numbers of all interior T-vertices on all horizontal and vertical c-edges, respectively, it follows that

$$E-\widetilde{E}_h-\widetilde{E}_v = V^\perp.$$

(Ⅳ) If there are no singular b-vertices in the T-mesh, according to the simple connectivity, it follows that

$$V^b = E_h^b+E_v^b.$$

Note that we can give a little perturbation to each singular b-vertex to isolate the two cells intersected at it. Then we have

$$V^b+V_s^b = E_h^b+E_v^b.$$

It is easy to know that the number of inner b-vertices is equal to the number of boundary edges but not boundary c-edges. Hence we have

$$V_f^b+V_s^b = \widetilde{E}_h^b+\widetilde{E}_v^b.$$

□

3.2 The dimension formula

Now we are ready to prove the dimension formula for the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ when $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$.

Theorem 3.2 Given a general T-mesh and a corresponding spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, then

$$\dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) = F(m+1)(n+1)+V(\alpha+1)(\beta+1)- V_s^b(\alpha+\beta+2)-E_h(m+1)(\beta+1)- E_v(\alpha+1)(n+1),$$

where F is the number of cells in $\mathcal{T}$, E_h and E_v are the number of horizontal interior edges and the number of vertical interior edges respectively, V and V_s^b are the number of interior vertices and the number of singular b-vertices respectively.

Proof For any cell in the given T-mesh, since $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, we can divide the Bézier ordinates in the cell into nine parts as shown in Fig. 6.

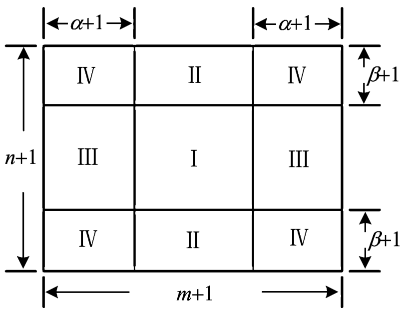

Fig. 6 A cell in T-mesh

The Bézier ordinates in Part I are free since no constraints are imposed on them. Hence we have obtained the first component in the dimension formula:

$$d_1 = F[(m+1)-2(\alpha+1)][(n+1)-2(\beta+1)] = F(m+1)(n+1)-2F(\alpha+1)(n+1)- 2F(m+1)(\beta+1)+4F(\alpha+1)(\beta+1).$$

In the following we will consider how many free Bézier ordinates there exist in parts Ⅱ, Ⅲ and Ⅳ.

Firstly, we consider the horizontal boundary c-edges. There are $\widetilde{E}_h^b$ horizontal boundary c-edges. It is easy to know that the number of free Bézier ordinates in Parts Ⅱ near all the horizontal boundary c-edges is

$$d_2^b = \widetilde{E}_h^b[(m+1)-2(\alpha+1)](\beta+1) = \widetilde{E}_h^b(m+1)(\beta+1)-2\widetilde{E}_h^b(\alpha+1)(\beta+1).$$

Secondly, we consider the horizontal non-boundary c-edges as shown in Fig. 7. Referring to Tab. 1, there are $\widetilde{E}_h$ horizontal non-boundary c-edges in the given T-mesh, and there are λ_j interior edges on the jth c-edge, i. e., λ_j+1 cells are adjacent to the jth non-boundary c-edge. These cells queue in two horizontal rows. Consider the cell in the down-left corner. If we fix its top at $\beta+1$ rows of the Bézier ordinates ($(m+1)(\beta+1)$ Bézier ordinates in total), according to Lemma 2.1, all the $\beta+1$ rows of the Bézier ordinates in each cell adjacent to the c-edge will be fixed as well. But this does not mean that we have $(m+1)(\beta+1)$ free Bézier ordinates coming from this c-edge, since, if we consider this horizontal non-boundary c-edge within the original T-mesh, two end parts with the size of $(\alpha+1)(\beta+1)$ could be possibly determined cyclically. Hence we have just $[(m+1)-2(\alpha+1)](\beta+1)$ affirmative free Bézier ordinates. The total number is

$$d_2^c = \widetilde{E}_h[(m+1)-2(\alpha+1)](\beta+1) = \widetilde{E}_h(m+1)(\beta+1)-2\widetilde{E}_h(\alpha+1)(\beta+1).$$

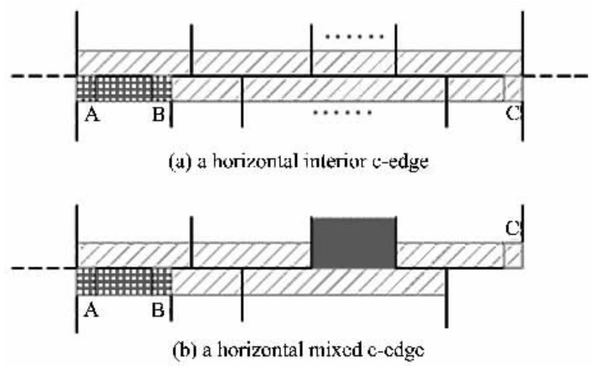

Fig. 7 Horizontal non-boundary c-edges

Therefore, the total number of free Bézier ordinates in Parts Ⅱ is

$$d_2 = d_2^b + d_2^c = (\widetilde{E}_h+\widetilde{E}_h^b)[(m+1)-2(\alpha+1)](\beta+1) = (\widetilde{E}_h+\widetilde{E}_h^b)(m+1)(\beta+1)- 2(\widetilde{E}_h+\widetilde{E}_h^b)(\alpha+1)(\beta+1).$$

Similarly, we can get the following number of free Bézier ordinates in Parts Ⅲ near all the vertical c-edges:

$$d_3 = (\widetilde{E}_v+\widetilde{E}_v^b)(\alpha+1)[(n+1)-2(\beta+1)] = (\widetilde{E}_v+\widetilde{E}_v^b)(\alpha+1)(n+1)- 2(\widetilde{E}_v+\widetilde{E}_v^b)(\alpha+1)(\beta+1).$$

Till now, only the Bézier ordinates around every vertices in parts labeled with Ⅳ have to be determined. Following the analysis procedure[2], we know that the Bézier ordinates around interior T-vertices and inner b-vertices will be determined by the Bézier ordinates around interior crossing vertices and free b-vertices. So we only have to consider free b-vertices and interior crossing vertices.

For every non-singular free b-vertex or interior crossing vertex, we have $(\alpha+1)(\beta+1)$ free Bézier ordinates. For every singular b-vertex, we have $2(\alpha+1)(\beta+1)-(\alpha+1)-(\beta+1)$ free Bézier ordinates. Totally we have

$$d_4=(V_f^b-V_s^b+V^+)(\alpha+1)(\beta+1)+ V_s^b[2(\alpha+1)(\beta+1)-(\alpha+1)-(\beta+1)]= (V_f^b+V_s^b+V^+)(\alpha+1)(\beta+1)-V_s^b(\alpha+\beta+2)$$

free Bézier ordinates.

Now the dimension of the spline space is the sum of $d_i(i=1,2,3,4)$:

$$\begin{aligned}&\dim\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})=\sum_{i=1}^{4}d_i=\\&F(m+1)(n+1)-(2F-\widetilde{E}_v^b-\widetilde{E}_v)(\alpha+1)(n+1)-\\&(2F-\widetilde{E}_h^b-\widetilde{E}_h)(m+1)(\beta+1)+\\&(4F-2\widetilde{E}_h^b-2\widetilde{E}_v^b-2\widetilde{E}_h-2\widetilde{E}_v+V_f^b+V_s^b+V^+)\cdot\\&(\alpha+1)(\beta+1)-V_s^b(\alpha+\beta+2)=\\&F(m+1)(n+1)-E_v(\alpha+1)(n+1)-\\&E_h(m+1)(\beta+1)+V(\alpha+1)(\beta+1)-V_s^b(\alpha+\beta+2),\end{aligned}$$

since

$$\begin{aligned}&4F-2\widetilde{E}_h^b-2\widetilde{E}_v^b-2\widetilde{E}_h-2\widetilde{E}_v+V_f^b+V_s^b+V^+=\\&(2F-\widetilde{E}_h^b-\widetilde{E}_h)+(2F-\widetilde{E}_v^b-\widetilde{E}_v)+\\&(V_f^b+V_s^b-\widetilde{E}_h^b-\widetilde{E}_v^b)-\widetilde{E}_h-\widetilde{E}_v+V^+=\\&E_h+E_v-\widetilde{E}_h-\widetilde{E}_v+V^+=V^{\perp}+V^+=V.\end{aligned}$$

This completes the proof of the theorem. □

Now we illustrate some examples to show how to use Theorem 3.2 to calculate the dimensions of spline spaces.

Example 3.3 Suppose we are given a general T-mesh $\mathcal{T}_1$ as shown in Fig. 8. In $\mathcal{T}_1, F=5, E_h=2, E_v=2$, and $V=V_s^b=0$, then the dimension of the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}_1)$ is

$$\dim\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}_1)= 5(m+1)(n+1)-2(m+1)(\beta+1)- 2(\alpha+1)(n+1).$$

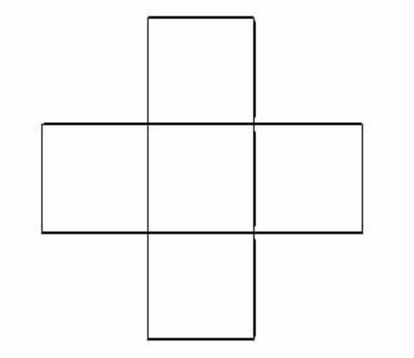

Fig. 8 A general T-mesh $\mathcal{T}_1$

Example 3.4 Suppose we are given a general T-mesh $\mathcal{T}_2$ as shown in Fig. 9. In $\mathcal{T}_2, F=8, E_h=8, E_v=8, V=0$, and $V_s^b=1$, then the dimension of the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}_2)$ is

$$\dim\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}_2)= 8(m+1)(n+1)-(\alpha+\beta+2)- 8(m+1)(\beta+1)-8(\alpha+1)(n+1).$$

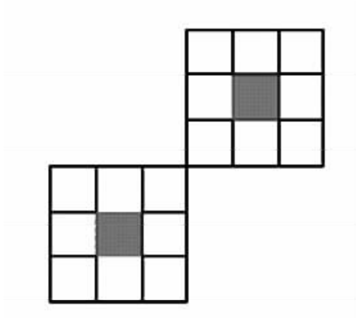

Fig. 9 A general T-mesh $\mathcal{T}_2$

4 Periodic spline spaces over regular T-meshes

In many applications, it is necessary to approximate a function that is known to be periodic. As it is usually desirable to work with periodic approximation functions in such cases, we devote this section to the study of periodic spline spaces over regular T-meshes.

Given a regular T-mesh $\mathcal{T}$, and $\Omega=[x_0,x_1]\times[y_0,y_1]$, we define

$$\begin{aligned}&\overset{o}{\mathcal{S}}_h(m,n,\alpha,\beta,\mathcal{T}):=\\&\{s(x,y)\mid s\in\mathcal{S}(m,n,\alpha,\beta,\mathcal{T}),s^{(j)}(x_0,y)=\\&s^{(j)}(x_1,y),y_0\leqslant y\leqslant y_1,\ j=0,1,\cdots,\alpha\}.\end{aligned}$$

We call $\mathring{\mathcal{S}}_h$ the horizontally periodic spline space over the regular T-mesh $\mathscr{T}$. Similarly, we can define the vertically periodic spline space $\mathring{\mathcal{S}}_v(m,n,\alpha,\beta,\mathscr{T})$ and the bi-directional periodic spline space $\mathring{\mathcal{S}}(m,n,\alpha,\beta,\mathscr{T})$ over the regular T-mesh $\mathscr{T}$.

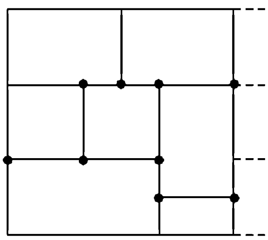

Fig. 10 Horizontally periodic spline

As illustrated in Fig. 10, for a horizontally periodic spline, the Bézier ordinates in Part Ⅲ and Part Ⅳ (referring to Fig. 6) near the vertical boundary edges (c-edges) are interrelated. When counting the number of free Bézier ordinates, we can imagine bending the T-mesh and gluing the two vertical boundary grid lines together. Then the boundary vertices on the vertical boundary grid lines (except the four corner vertices) can be considered as interior vertices, the four corner vertices can be considered as two boundary vertices, vertical boundary edges become vertical interior edges, and new "crossing vertices" and new "vertical c-edges" along the vertical boundary lines can be introduced. Hence we have, with the notation in the proof of Theorem 3.2,

$$d_1 = F[(m+1)-2(\alpha+1)][(n+1)-2(\beta+1)];$$
$$d_2 = (\widetilde{E}_h + E_h^b)[(m+1)-2(\alpha+1)](\beta+1);$$
$$d_3 = (\widetilde{E}_v + \Delta\widetilde{E}_v)(\alpha+1)[(n+1)-2(\beta+1)];$$
$$d_4 = (V^+ + V^b + \Delta V^+ - \Delta V^b)(\alpha+1)(\beta+1).$$

Here $F, \widetilde{E}_h, \widetilde{E}_v, E_h^b, V^+$ and V^b are defined in Tab. 1, $\Delta\widetilde{E}_v$ is the number of increased vertical interior c-edges which come from the vertical boundary edges, ΔV^+ is the number of increased crossing vertices, and ΔV^b is the number of reduced boundary vertices. It is easy to know that $\Delta\widetilde{E}_v = \Delta V^+ + 1, \Delta V^b = E_v^b$, and ΔV^+ is equal to the number of pairs of the boundary vertices with the same y-coordinates on the vertical boundary grid lines (except the four corner vertices).

Thus the dimension of the spline space $\mathring{\mathcal{S}}_h(m,n,\alpha,\beta,\mathscr{T})$ is

$$\begin{aligned}
&\dim \mathring{\mathcal{S}}_h(m,n,\alpha,\beta,\mathscr{T}) = \sum_{i=1}^{4} d_i = \\
&F(m+1)(n+1) - \\
&(2F - E_h^b - \widetilde{E}_h)(m+1)(\beta+1) - \\
&(2F - \Delta\widetilde{E}_v - \widetilde{E}_v)(\alpha+1)(n+1) + \\
&(4F - 2E_h^b - 2\widetilde{E}_h - 2\Delta\widetilde{E}_v - 2\widetilde{E}_v + V^+ + \\
&V^b + \Delta V^+ - \Delta V^b)(\alpha+1)(\beta+1) = \\
&F(m+1)(n+1) - E_h(m+1)(\beta+1) - \\
&(E_v + \Delta E_v)(\alpha+1)(n+1) + \\
&(V + \Delta V)(\alpha+1)(\beta+1) = \\
&\dim \mathcal{S}(m,n,\alpha,\beta,\mathscr{T}) - \Delta E_v(\alpha+1)(n+1) + \\
&\Delta V(\alpha+1)(\beta+1),
\end{aligned}$$

where $\Delta E_v = E_v^b - \Delta V^+ - 1$ is the number of increased vertical interior edges, and $\Delta V = E_v^b - \Delta V^+ - 2 = \Delta E_v - 1$ is the number of increased interior vertices. For example, in Fig. 10, $F = 7$, $E_h = 7, E_v = 5, E_v^b = 6, V = 6$, and $\Delta V^+ = 1$, then, $\Delta E_v = 6 - 1 - 1 = 4, \Delta V = 4 - 1 = 3$.

As a summary, for the horizontally periodic spline space $\mathring{\mathcal{S}}_h(m,n,\alpha,\beta,\mathscr{T})$, we have the following theorem.

Theorem 4.1 Given a regular T-mesh and a corresponding horizontally periodic spline space $\mathring{\mathcal{S}}_h(m,n,\alpha,\beta,\mathscr{T})$, suppose $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, then

$$\begin{aligned}
&\dim \mathring{\mathcal{S}}_h(m,n,\alpha,\beta,\mathscr{T}) = \\
&\dim \mathcal{S}(m,n,\alpha,\beta,\mathscr{T}) - \Delta E_v(\alpha+1)(n+1) + \\
&\Delta V(\alpha+1)(\beta+1),
\end{aligned}$$

where $\Delta E_v = E_v^b - \Delta V_v^+ - 1, \Delta V = \Delta E_v - 1$; E_v^b is the number of vertical boundary edges, and ΔV_v^+ is the number of pairs of the boundary vertices with the same y-coordinates on the vertical boundary grid lines (except the four corner vertices).

Similarly, for the vertically periodic spline space $\mathring{\mathcal{S}}_v(m,n,\alpha,\beta,\mathscr{T})$, we have

Theorem 4.2 Given a regular T-mesh and a corresponding vertically periodic spline space

$\overset{o}{\mathcal{S}}_v(m,n,\alpha,\beta,\mathcal{T})$, suppose $m\geqslant 2\alpha+1$ and $n\geqslant 2\beta+1$, then

$$\dim \overset{o}{\mathcal{S}}_v(m,n,\alpha,\beta,\mathcal{T}) = \dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) - \Delta E_h(m+1)(\beta+1) + \Delta V(\alpha+1)(\beta+1),$$

where $\Delta E_h = E_h^b - \Delta V_h^+ - 1, \Delta V = \Delta E_h - 1$; E_h^b is the number of horizontal boundary edges, and ΔV_h^+ is the number of pairs of the boundary vertices with the same x-coordinates on the horizontal boundary grid lines (except the four corner vertices).

For the bi-directional periodic spline space $\overset{o}{\mathcal{S}}(m,n,\alpha,\beta,\mathcal{T})$, we have the following theorem.

Theorem 4.3 Given a regular T-mesh and a corresponding bi-directional periodic spline space $\overset{o}{\mathcal{S}}(m,n,\alpha,\beta,\mathcal{T})$, suppose $m\geqslant 2\alpha+1$ and $n\geqslant 2\beta+1$, then

$$\dim \overset{o}{\mathcal{S}}(m,n,\alpha,\beta,\mathcal{T}) = \dim \mathcal{S}(m,n,\alpha,\beta,\mathcal{T}) + \Delta V(\alpha+1)(\beta+1) - \Delta E_h(m+1)(\beta+1) - \Delta E_v(\alpha+1)(n+1),$$

where $\Delta E_h = E_h^b - \Delta V_h^+ - 1, \Delta E_v = E_v^b - \Delta V_v^+ - 1$, $\Delta V = \Delta E_h + \Delta E_v - 1$. Here, E_h^b and E_v^b are the number of horizontal boundary edges and the number of vertical boundary edges respectively, ΔV_h^+ is the number of pairs of the boundary vertices with the same x-coordinates on the horizontal boundary grid lines (except the four corner vertices), and ΔV_v^+ is the number of pairs of the boundary vertices with the same y-coordinates on the vertical boundary grid lines (except the four corner vertices).

5 Conclusion and future work

This paper presents the dimension formula for the spline space $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ over a general T-mesh $\mathcal{T}$ when $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$. The periodic spline spaces defined over regular T-meshes are discussed as well. These dimension formulae will be useful in the surface modeling with these splines, by then we can design some "good" basis functions. We will address the basis function construction of the spline spaces over T-meshes in future paper.

When $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, the dimension formula of $\mathcal{S}(m,n,\alpha,\beta,\mathcal{T})$ is a linear combination of F, E_h, E_v and V, where the combinational coefficients depend only on m, n, α, and β. Hence we can think of the dimension formula as a weighted Euler formula. If we remove the constraints $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$, the Bézier ordinates in every cell cannot be divided into four parts and all the Bézier ordinates will influence each other. The dimension formula may fail to hold. A general dimension formula without these constraints is still unavailable. In fact, we even do not know whether the dimension relies on the geometry of the T-mesh or not. In future research, we will investigate these problems.

References

[1] Sederberg T W, Zheng J, Bakenov A, et al. T-splines and T-NURCCs[J]. ACM Transactions on Graphics, 2003,22(3):161-172.

[2] DENG Jian-song, CHEN Fa-lai , FENG Yu-yu. Dimensions of spline spaces over T-meshes[J]. Journal of Computational and Applied Mathematics, 2006,194(2):267-283.

[3] Sederberg T W, Cardon D L, Finnigan G T, et al. T-spline simplification and local refinement[J]. ACM Transactions on Graphics, 2004, 23(3): 276-283.

[4] Farin G. Curves and Surfaces for CAGD : A Practical Guide[M]. 5th ed.. San Francisco: Morgan Kaufmann Publishers, 2002.