Determination of Free Parameters in Algebraic Surface Blending

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Abstract

In the paper we propose a method to determine parameters that appear in algebraic surface blending. By minimizing the surface energy and adding some point restrictions, we can select the free parameters such that a blending surface with reasonable shape is constructed. The method seems to be extensible for other surface blending problems, although we concentrate on algebraic surface blending .

1. Introduction

Surface blending is a modelling technique which constructs smooth transitions between given surfaces. It has wide applications in many different areas such as mechanical design and manufacturing, computer graphics and animation, etc. Besides keeping the specified order of geometric continuity with the given surfaces, the blending surface must satisfy some physical requirements of the design and its shape should be aesthetically pleasing.

In the past two decades, much literature has focused on the subject and many approaches have been proposed to solve the problem. Depending on the representations of the transitional surfaces, blending methods can be classified into several categories, such as parametric methods, implicit methods and combined methods. In this paper, we only discuss methods for algebraic surface blending.

The classic method of blending two intersecting surfaces works by replacing the intersection curve and its vicinity with part of a canal surface that is generated by a rolling ball and which has G^1 continuous contact to the initial surfaces [16]. Hoffmann and Hopcroft [10, 11] proposed a general approach for blending surfaces by means of the potential method. Li et al [12] suggested to use functional splines to blend algebraic surfaces. Allen, Dutta [1] and Pratt [14] constructed blending surfaces using cyclides and supercyclides. Wu and Zhou [19] applied the method of Gröbner bases to algebraic surface blending. Other blending methods using algebraic surfaces include Liming skill [13], the substituting method [15], the ideal-theory-based method [18] and Wu's method [20]. All of these methods produce algebraic blending surfaces of high degree in general and some of them cannot be easily generalized to blend surfaces with higher-order contact or to blend more than two surfaces.

The main drawback to blend several surfaces using a single algebraic surface is that the blending surface usually has a high algebraic degree, especially for the case where high order of contact with the initial surfaces is required. Algebraic surfaces of high degree are complicated in topology and their shape is hard to control. Furthermore, high degree algebraic surfaces are computationally more expensive in subsequent geometric operations.

An approach to overcome the above drawbacks is to use piecewise algebraic surfaces (PAS for short) instead. The idea of PAS comes from Sederberg [17] and was applied by Bajaj et al, Dahmen and Thamm-Schwar, Hartmann, and Xu et al, in interpolation and free form modelling [2, 7, 21]. In [8], several approaches [4, 3, 5] are summarized to construct piecewise algebraic blending surfaces. The examples in [8] suggest some advantages of using PAS as blending surfaces. First, the piecewise algebraic blending surfaces in general have much lower degree than the blending surfaces generated by other methods. In particular, Gröbner bases' method and syzygy modules' method can produce blending surfaces of lowest possible degree in theory. Second, the methods are general in the sense that they work for any number of given surfaces in arbitrary positions and for any order of geometric continuity. Thirdly, the expressions of the blending surfaces using the syzygy modules' method are relatively simple. However, there is still much work that needs to be done.

This paper is devoted to finding suitable free parameter values in the solutions of the algebraic surface blending problem. The main idea of our parameter-determination technique is to minimize the energy of the blending surface and to add some point restrictions as an auxiliary means.

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The main advantages of our approach are: (a) the method is almost automatic with only a few interactive manipulations; (b) the method is efficient since only linear equations are to be solved; and (c) the method seems to be extensible for other surface blending problems, although we concentrate on algebraic surface blending.

The rest of this paper is organized as follows. In Section 2, we briefly review the algebraic surface blending problem and the syzygy module method. In Section 3, we describe our method to choose suitable free parameter values in the blending surfaces. In Section 4, several examples are provided to illustrate our approach. Conclusions and further research problems are drawn in Section 5.

2. Surface blending with PAS

In this section, we review the surface blending method using piecewise algebraic surfaces. We first recall some basic notations and preliminary knowledge about geometric continuity of algebraic surfaces.

2.1. Notation and preliminaries

Let $\mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \cdots, x_n]$ denote the set of polynomials in $x_1, \cdots, x_n$ with real coefficients and $\mathbb{R}[\mathbf{x}]^m$ denote the set of m-dimensional row vectors with entries in the polynomial ring $\mathbb{R}[\mathbf{x}]$. Given a set of polynomials $f_1, \cdots, f_s \in \mathbb{R}[\mathbf{x}]$, the set of common zeros of $f_1, \cdots, f_s$ is called a *variety*, denoted by $V(f_1, \cdots, f_s)$, i.e.,

$$\{(a_1, \cdots, a_n) \in \mathbb{R}^n : f_i(a_1, \cdots, a_n) = 0, 1 \leq i \leq s\}.$$

Especially, algebraic surfaces and algebraic curves in $\mathbb{R}^3$ are varieties. A variety V is said to be *reducible* if it can be expressed as the union of two proper subvarieties of V. Otherwise, it is said to be *irreducible*.

Next we come to the syzygy module. Let $\mathbf{F} = (\mathbf{F}_1, \cdots, \mathbf{F}_s)$ be an s-tuple, where $\mathbf{F}_i \in \mathbb{R}[\mathbf{x}]^m$, $i = 1, \cdots, s$. The set

$$\mathbf{\Phi} := \left\{ (h_1, \cdots, h_s)^T \in \mathbb{R}[\mathbf{x}]^s : \sum_{i=1}^{s} h_i \mathbf{F}_i = \mathbf{0} \right\}$$

is a *module*, called a *syzygy module* of $(\mathbf{F}_1, \cdots \mathbf{F}_s)$, and is denoted by $\mathrm{Syz}(\mathbf{F}_1, \cdots, \mathbf{F}_s)$, or simply by $\mathrm{Syz}(\mathbf{F})$.

Let $\mathbf{G}_i$, $i = 1, \cdots, k$ be a set of elements in $\mathbf{\Phi}$. If for any $\mathbf{G} \in \mathbf{\Phi}$, there are polynomials $a_i \in \mathbb{R}[\mathbf{x}]$, $i = 1, \cdots, k$ such that

$$\mathbf{G} = a_1 \mathbf{G}_1 + \cdots + a_k \mathbf{G}_k,$$

then $\mathbf{G}_i$, $i = 1, \cdots, k$ is called a *generating set* of $\mathbf{\Phi}$. By Hilbert's Basis Theorem, any module over a polynomial ring has a finite generating set. Given the set of polynomials $\mathbf{F}$, a generating set of $\mathrm{Syz}(\mathbf{F})$ can be computed using the Gröbner bases technique [6].

In the following, we will strict our discussion over polynomial ring $\mathbb{R}[x, y, z]$.

2.2. Geometric continuity of algebraic surfaces

Geometric continuity provides an important characterization for the smoothness of geometric entities. One often used definition of geometric continuity between algebraic surfaces is called rescaling continuity [9].

Definition 2.1 *Let $V(f)$ and $V(g)$ be two algebraic surfaces which intersect transversally at an irreducible algebraic curve C, where $f, g \in \mathbb{R}[x, y, z]$. We say that $V(f)$ and $V(g)$ meet with G^k rescaling continuity along the common curve C if*

- *$V(f)$ and $V(g)$ are smooth along C except at a finite number of points;*
- *there exist two polynomials $a, b \in \mathbb{R}[x, y, z]$, which are not identically zero over C, such that af and bg are C^k continuous on C.*

A general characterization of G^k continuity for two algebraic surfaces on their common boundary is stated in the following theorem [9]:

Theorem 2.2 *Let $V(f)$ and $V(h)$ be two algebraic surfaces which intersect transversally at an irreducible algebraic curve $C := V(f) \cap V(h)$. The surfaces $V(f)$ and $V(g)$ meet with G^k continuity along the common curve C if and only if there are polynomials $\alpha(x, y, z) \neq 0$ and $\beta(x, y, z)$ such that $g = \alpha f + \beta h^{k+1}$.*

In practical applications, $V(h)$ is often assumed to be a plane. In this case, we have the following result.

Proposition 2.3 *Assume that an algebraic surface $V(g)$ of degree n and an algebraic surface $V(f)$ of degree $m (n \geq m)$ meet along a common algebraic curve in a plane $V(\pi)$. If there exist polynomials $\alpha(x, y, z)$ of degree $n - m$ and $\beta(x, y, z)$ of degree $n - k - 1$ such that $g = \alpha f + \beta \pi^{k+1}$, then algebraic surfaces $V(g)$ and $V(f)$ meet with G^k continuity along the common curve.*

2.3. Algebraic surface blending

The algebraic surface blending problem can be formulated as follows.

Problem 2.4 *Given m initial algebraic surfaces $V(f_i)$ and additional m auxiliary surfaces $V(h_i)$, where $f_i, h_i \in \mathbb{R}[x, y, z], i = 1, 2, \cdots, m$. Suppose that $V(f_i)$ and $V(h_i)$ intersect transversally at a curve $C_i = V(f_i, h_i)$, $i = 1, 2, \cdots, m$. The problem is to find a (piecewise) algebraic surface $V(f)$ such that $V(f)$ meets $V(f_i)$ with G^k continuity along the curve C_i, $i = 1, 2, \cdots, m$.*

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The general approach for constructing piecewise algebraic blending surfaces is summarized as follows.

1. According to the given initial surfaces and transversal surfaces, determine the defining region for the PAS. Some heuristic rules should be applied in this step.
2. Form a system of algebraic equations from the geometric continuity conditions for each pair of adjacent surface patches.
3. Solve the system of algebraic equations to obtain piecewise algebraic surfaces with free parameters.
4. Determine these free parameters to control the shape of the blending surface.

We use a simple example to illustrate the above approach. For the details about the method, the reader is referred to [8].

Example 1 Consider the G^1 blending of two coaxial cylinders with different radii :

$$\begin{cases} f_1 : y^2 + z^2 - 1 = 0, & x \geq 1 \\ f_2 : y^2 + z^2 - 4 = 0, & x \leq -1 \end{cases}$$

The auxiliary planes are given by

$$F_1 : x - 1 = 0 \quad \text{and} \quad F_2 : x + 1 = 0.$$

For simplicity, we use a single surface patch to blend the two cylinders. According to Theorem 2.2 and Proposition 2.3, the blending surface $V(f)$ has to satisfy

$$f = \alpha_1 f_1 + \beta_1 F_1^2 = \alpha_2 f_2 + \beta_2 F_2^2$$

with $\alpha_i, \beta_i \in \mathbb{R}[x, y, z], i = 1, 2$. It follows that

$$\alpha_1 f_1 - \alpha_2 f_2 + \beta_1 F_1^2 - \beta_2 F_2^2 = 0.$$

This polynomial equation (with $\alpha_1, \alpha_2, \beta_1, \beta_2$ being unknowns) can be solved by computing the Gröbner basis of the syzygy module $\mathrm{Syz}(f_1, -f_2, F_1^2, -F_2^2)$. With a symbolic computation software such as Maple, the Gröbner basis with graded lex order is computed as

$$\begin{cases} g_1 = 4\,y^2 + 4\,z^2 - 16 + (6 - 3\,x)(x + 1)^2 \\ g_2 = -4\,x\,(y^2 + z^2 - 4) - 3\,(x + 1)^2 \\ g_3 = (y^2 + z^2 - 1)(x + 1)^2 \\ g_4 = (y^2 + z^2 - 1)(y^2 + z^2 - 4) \end{cases}$$

Thus the blending surface is $f = \sum_{i=1}^{4} a_i g_i$, where $a_i \in \mathbb{R}[x, y, z], i = 1, \cdots, 4$ are free polynomials.

One advantage of Gröbner basis method is that it can obtain blending surfaces of lowest possible degree. In this example, the lowest degree blending surfaces are given by $f = a_1 g_1 + a_2 g_2 = 0$, where a_1 and a_2 are constants.

There are still problems need further investigation in surface blending with PASs, i.e., problems in Step 1 and Step 4. In this paper, we will propose a method to solve the problem in the last step, i.e., determining the free parameters in the surface blending problem.

3. Determination of free parameters

After obtaining all the blending surfaces of lowest degree by the syzygy module method, we need to choose suitable free parameters to adjust the shape of the final blending surface. In [4], the authors control the shape of the blending surface by expressing each surface patch into B-B form and adjust the free parameters according to the the Bézier ordinates. This is a very sophisticated approach and is totally non-automatic. In [8], the authors adjust the free parameters by interpolating or least-square approximating a set of specified points. However, this approach does not produce a reasonable shape in certain circumstances. Thus how to determine the free parameters of the blending surfaces and how to avoid multiple sheets of the algebraic surface patches is still an unsolved problem.

In this section, we present a technique to choose the free parameters in the algebraic blending surfaces with only a few interactions. The method is based on minimizing the blending surface energy and adding some point restrictions as an auxiliary means. The examples in the next section seem to suggest that the proposed technique produces reasonable result in most circumstances.

3.1. Minimizing surface energy

A (piecewise) algebraic surface is defined as the zero set of an implicit function f. The shape of the algebraic surface is totally determined by the implicit function. One way to get a reasonable shape of the algebraic surface is to minimize the smooth energy of the surface which is a functional of the corresponding implicit function.

In our application – constructing (piecewise) algebraic blending surfaces, we aim that the blending surface will not have multiple sheets and that the shape of the surface is aesthetically pleasing. Hence in this paper we choose the thin plate energy model as follows, which minimizes the surface area.

$$E(f) = \frac{1}{2} \int_{\Omega} \|\nabla^2 f\|^2_{Frobenius}\, dxdydz \qquad (1)$$

For example 1, assume that the blending surface is expressed as $f = \sum_{i=1}^{4} a_i g_i$, where $a_4 = 1 - \sum_{i=1}^{3} a_i$. The thin plate energy of f can be written as

$$E(f) = \frac{1}{2}\,\mathbf{x}^T \mathbf{H} \mathbf{x} + \mathbf{b}^T \mathbf{x}$$

Here

$$\mathbf{x} = (a_1, a_2, a_3)^T$$

$$\mathbf{H} = \begin{pmatrix} 234112/9 & 218368/9 & 213248/9 \\ 218368/9 & 391552/9 & 192128/9 \\ 213248/9 & 192128/9 & 470656/15 \end{pmatrix}$$

$$\mathbf{b} = \frac{1}{9}\,(-218368, -270592, -253184)^T,$$

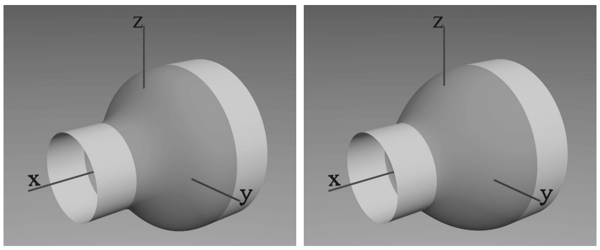
and the integral domain is $\Omega = \{\, |x| \le 1, |y| \le 2, |z| \le 2\}$.

Solving the equation $\nabla E(f) = 0$ gives

$$\mathbf{x} = \frac{1}{17070338}\,(-146624, 6476809, 11008845)^T$$

So we get the blending surface (see Figure 1.a).

Using the same method we can also get the lowest degree blending surface (see Figure 1.b).



a. Quartic blending surface b. Cubic blending surface

Figure 1. Blending surfaces of example 1

3.2. Point constraints

In some situations, the blending surface based merely on minimizing the surface energy may have an undesirable shape. In this case, we need to add some point restrictions as an auxiliary means.

An algebraic surface $V(f)$ divides $\mathbb{R}^3$ into three parts: the surface itself $f = 0$, the interior of the surface $f < 0$ and the exterior of the surface $f > 0$. Instead of requiring that the surface passes through (or approximates) a point set, we only impose the constraint that a point set is inside (or outside) the surface. Thus we need to solve the following optimization problem

$$\begin{aligned} \textbf{Min} \quad & E(f) \\ s.t \quad & f(\mathbf{p}_i) \le 0, \quad i = 1, \cdots, r \\ & f(\mathbf{q}_j) \ge 0, \quad j = 1, \cdots, s \end{aligned}$$

The point sets $\{\, \mathbf{p}_i\}$ and $\{\, \mathbf{q}_j\}$ can be chosen interactively.

4. Examples and results

In this section, we provide several examples .

Example 2 Consider the G^1 blending of two coaxial elliptic cylinders with the same major axis and minor axis :

$$\begin{cases} f_1 : 2\,y^2 + z^2 = 2, & x \ge 1 \\ f_2 : y^2 + 2\,z^2 = 2, & x \le -1 \end{cases}$$

The auxiliary planes are given by

$$F_1 : x - 1 = 0 \quad \text{and} \quad F_2 : x + 1 = 0$$

Using the syzygy module method, we obtain the blending surface $f = \sum_{i=1}^{4} a_i g_i$, where

$$\begin{cases} g_1 = -8\,x\,(y^2 + 1/2\,z^2 - 1) + (3\,z^2 - 2)(x-1)^2 \\ g_2 = 16\,x\,(y^2 + 1/2\,z^2 - 1) + (3\,y^2 - 2)(x-1)^2 \\ g_3 = (y^2 + 2\,z^2 - 2)(y^2 + 1/2\,z^2 - 1) \\ g_4 = (2\,x^2 + 4\,x + 2)(y^2 + 1/2\,z^2 - 1) \end{cases}$$

Assume that a_i, $i = 1, \cdots, 4$ are constants and $a_4 = 1 - \sum_{i=1}^{3} a_i$. Similarly writing $E(f)$ of the integral domain $\Omega = \{\, |x| \le 1, |y| \le 3/2, |z| \le 3/2\}$ into matrix form and solving the equation $\nabla E(f) = 0$, we get

$$\mathbf{x} = \left(t, 2\,t - \frac{36207}{81682}, \frac{4634}{40841}\right)^T$$

where t is a free parameter. The blending surface corresponding to $t = 1$ is shown in Figure 2.

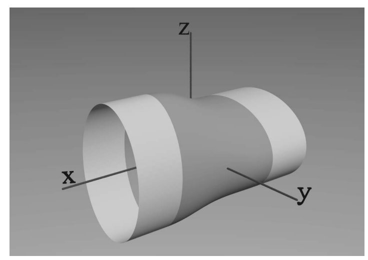


Figure 2. Blending surface of example 2

Example 3 Consider the G^1 blending of two cylinders whose axes are perpendicular to each other :

$$\begin{cases} f_1 : y^2 + z^2 - 1 = 0, & x \ge 2 \\ f_2 : x^2 + z^2 - 1 = 0, & y \ge 2 \end{cases}$$

and the corresponding auxiliary planes are

$$F_1 : x - 2 = 0 \quad \text{and} \quad F_2 : y - 2 = 0$$

Using the syzygy module method, we obtain five generators for the blending surface:

$$\begin{cases} g_1 = (x + y - 4)(y^2 + z^2 - 1) + (x + y)(x-2)^2 \\ g_2 = (z^2 - 4\,y + 11)(y^2 + z^2 - 1) + (z^2 - 4\,x - 5)(x-2)^2 \\ g_3 = (y^2 - 4\,y + 4)(x-2)^2 \\ g_4 = (y^2 - 4\,y + 4)(y^2 + z^2 - 1) + (y^2 - 4\,y + 4)(x-2)^2 \\ g_5 = (x^2 + z^2 - 1)(x-2)^2 \end{cases}$$

Notice that using g_1, one can get the lowest degree blending surface (see Figure 4.a). But its shape is fixed.

Unfortunately, the blending surface $f = \sum_{i=1}^{5} a_i g_i = 0$ obtained by minimizing the thin plate energy (with integral domain $\Omega = \{\, |x| \le 2, |y| \le 2, |z| \le 2\}$) is not closed and has multiple sheets (see Figure 3). To get a proper blending

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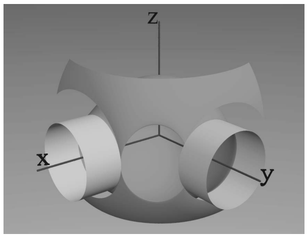
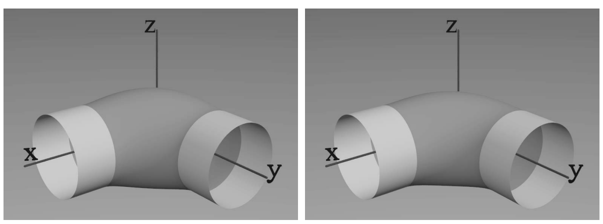


Figure 3. Surface with multiple sheets

surface, we add point constraint as: $f(-1/4, -1/4, 0) \leq 0$. Solve the constrained minimization problem, we get:

$$(a_1, a_2, a_3, a_4, a_5)^T = \begin{pmatrix} 0.3387104474 \\ -0.2071533438 \\ -0.04550660490 \\ -0.2473380977 \end{pmatrix}$$

The corresponding blending surface is shown in Figure 4.b.



a. Cubic blending surface b. Quartic blending surface

Figure 4. Blending surfaces of example 3

Example 4 Given three cylinders as follows :

$$\begin{cases} f_1 = y^2 + z^2 - r_1^2 = 0, & x > h_1 > r_1 > 0 \\ f_2 = z^2 + x^2 - r_2^2 = 0, & y > h_2 > r_2 > 0 \\ f_3 = x^2 + y^2 - r_3^2 = 0, & z > h_3 > r_3 > 0 \end{cases}$$

We want to seek a G^k continuous PAS $f = 0$ in the region $\{(x, y, z) : |x| \leq h_1, |y| \leq h_2, |z| \leq h_3\}$ which meets $f_1 = 0$, $f_2 = 0$ and $f_3 = 0$ at

$$F_1 : x - h_1 = 0; \quad F_2 : y - h_2 = 0; \quad F_3 : z - h_3 = 0$$

with G^k continuity respectively.

For this particular problem, the defining region of the blending surface can be determined as follows [4]. Choose three surface patches $V(G_1)$, $V(G_3)$, and $V(G_5)$ to meet the three cylinders with G^k continuity, respectively. Now we need other three surface patches $V(G_2)$, $V(G_4)$, and $V(G_6)$ which serve as the transitional surfaces between $V(G_1)$, $V(G_3)$, and $V(G_5)$. Thus in total we need six patches to compose the blending surface. Since the transversal planes $V(F_i)(i = 1, 2, 3)$ of the three cylinders intersect at one common point, the defining region of the blending surface can be defined as the composition of six tetrahedrons as shown in Figure 5.

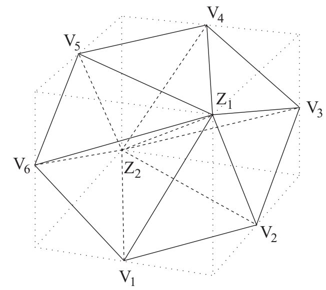


Figure 5. Partition of the defining region

The vertices of the defining region are as follows: $Z_1 = (h_1, h_2, h_3)$, $Z_2 = (-h_1, -h_2, -h_3)$, $V_1 = (h_1, 0, -h_3)$, $V_2 = (0, h_2, -h_3)$, $V_3 = (-h_1, h_2, 0)$, $V_4 = (-h_1, 0, h_3)$, $V_5 = (0, -h_2, h_3)$, $V_6 = (h_1, -h_2, 0)$. Let $\pi_i = 0$ be the plane passing through Z_1, Z_2 and V_i for $i = 1, \cdots, 6$ and $T_i = Z_1 Z_2 V_{i-1} V_i, i = 1, \cdots, 6$. Then

$$\begin{aligned} \pi_4 = \pi_1 : &\quad h_2 h_3 x - 2h_1 h_3 y + h_1 h_2 z = 0 \\ \pi_5 = \pi_2 : &\quad -2h_2 h_3 x + h_1 h_3 y + h_1 h_2 z = 0 \\ \pi_6 = \pi_3 : &\quad -h_2 h_3 x - h_1 h_3 y + 2h_1 h_2 z = 0 \end{aligned}$$

Assume the surface patch in the tetrahedron T_i is defined by $G_i = 0$ and that G_i satisfies $G_{i+1} = G_i + \alpha_i \pi_i^{k+1}, \quad i = 1, \cdots, 6$, where α_i are polynomials in x, y, z. By the consistency condition, α_i satisfies $\sum_{i=1}^{6} \alpha_i \pi_i^{k+1} = 0$. On the other hand, $V(G_1)$, $V(G_3)$ and $V(G_5)$ meet $V(f_1)$, $V(f_2)$ and $V(f_3)$ with G^k continuity respectively, so

$$G_i = \gamma_i f_i + \beta_i F_i^{k+1}, \quad i = 1, 3, 5$$

where β_i, γ_i are polynomials in x, y, z. Thus we obtain a system of equations

$$\sum_{i=1}^{6} \alpha_i \pi_i^{k+1} = 0$$

$$\gamma_1 f_1 + \beta_1 F_1^{k+1} + \alpha_1 \pi_1^{k+1} + \alpha_2 \pi_2^{k+1} = \gamma_2 f_2 + \beta_2 F_2^{k+1}$$

$$\gamma_1 f_1 + \beta_1 F_1^{k+1} - \alpha_6 \pi_6^{k+1} - \alpha_5 \pi_5^{k+1} = \gamma_3 f_3 + \beta_3 F_3^{k+1}$$

Taking $k = 1$, $h_1 = h_2 = h_3 = \frac{3}{5}$ and $r_1 = r_2 = r_3 = \frac{1}{5}$, we can obtain three generators with the lowest degree by solving the above system of equations.

$$(\alpha_1^j, \cdots, \alpha_6^j, \beta_1^j, \cdots, \beta_3^j, \gamma_1^j, \cdots, \gamma_3^j)^T, \ j = 1, 2, 3$$

the corresponding expressions for the PAS $f^j = 0$ are:

$$G_i^j = \begin{cases} \gamma_{(i+1)/2}^j f_1 + \beta_{(i+1)/2}^j F_{(i+1)/2}^2, \ i = 1, 3, 5 \\ G_{i-1}^j + \alpha_{i-1}^j \pi_{i-1}^2, \ i = 2, 4, 6 \end{cases}$$

Assume that the blending surface is $f = \sum_{j=1}^{3} a_j f^j$, and let $a_3 = 1 - \sum_{i=1}^{2} a_i$. We need to choose a_1 and

a_2. Without loss of generality, we apply our method on $G_1 = \sum_{i=1}^{3} a_i G_1^i$. Define the thin plate energy $E(G_1)$ in $\Omega = \{\frac{1}{4} \leq x \leq \frac{3}{5}, -\frac{2}{5} \leq y \leq \frac{2}{5}, -\frac{2}{5} \leq z \leq \frac{2}{5}\}$. Add different point restriction of $G_1(0.5, 0.5, 0.5) \leq 0$ and $G_1(-0.2, -0.2, -0.2) \leq 0, G_1(-0.1, -0.1, -0.1) \leq 0$ or $G_1(0, 0, 0) \leq 0$. After solving these constrained minimization problems, we get three different cubic PASs (see Figure 6).

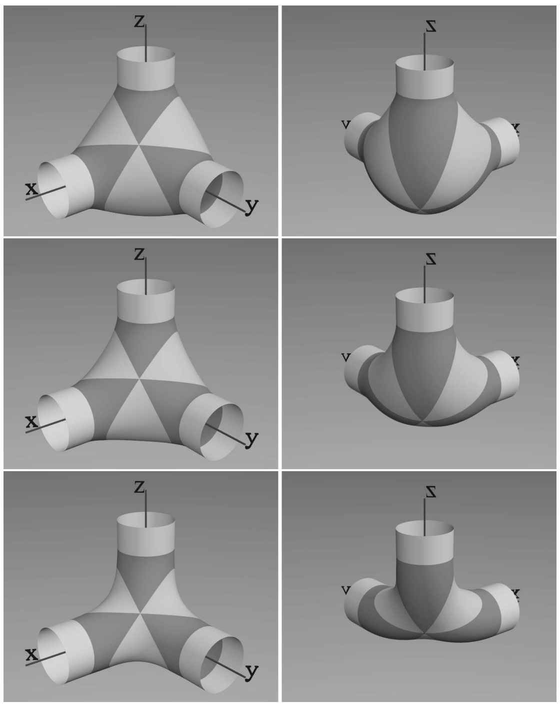


Figure 6. Three different PASs of example 4

5. Conclusion

In this paper, we proposed a method to determine the free parameters which are common in algebraic surface blending (Especially when PAS is used). Mainly depending on minimizing surface energy and adding some point restrictions, we can select those parameters which provide a smooth blending surface of the problem. The method seems to be extensible to many surface blending problems, although we concentrate on algebraic surfaces.

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