Available online at www.sciencedirect.com

SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 194 (2006) 267–283

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Dimensions of spline spaces over T-meshes

Jiansong Deng*, Falai Chen, Yuyu Feng

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026, PR China

Received 16 March 2005; received in revised form 22 July 2005

Abstract

A T-mesh is basically a rectangular grid that allows T-junctions. In this paper, we propose a method based on Bézier nets to calculate the dimension of a spline function space over a T-mesh. When the order of the smoothness is less than half of the degree of the spline functions, a dimension formula is derived which involves only the topological quantities of the T-mesh. The construction of basis functions is briefly discussed. Furthermore, the dimension formulae for T-meshes after mesh operations, such as edge insertion and mesh merging, are also obtained.

Keywords: Spline; Bézier ordinate; Dimension; T-mesh

1. Introduction

T-meshes are formed by a set of horizontal line segments and a set of vertical line segments, where T-junctions are allowed. See Fig. 1 for examples. The traditional tensor-product B-spline functions, which are a basic tool in the design of free-form surfaces, are defined over special T-meshes, where no T-junctions appear. The tensor-product B-spline surfaces have the drawback that arises from the mathematical properties of the tensor-product B-spline basis functions. Two global knot vectors, which are shared by all the basis functions, do not allow local modification of the domain partition. Thus, if we want to construct a surface which is flat in the most part of the domain, but sharp in a small region, we have to use more control points not only in the sharp region, but also in the regions propagating from the sharp region along horizontal and vertical directions to maintain the tensor-product mesh structure. The

* Corresponding author. Tel.: 0086 551 3601009; fax: 0086 551 3601005.
E-mail address: dengjs@ustc.edu.cn (J. Deng).

doi:10.1016/j.cam.2005.07.009

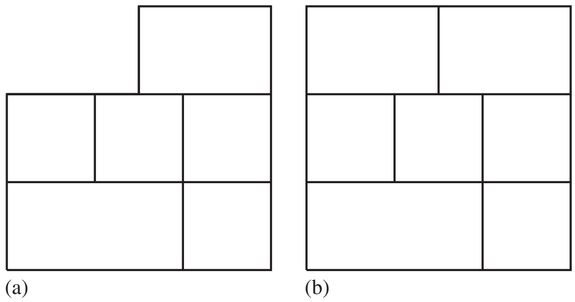


Fig. 1. Examples of T-meshes.

superfluous control points are a big burden to modeling systems. In [6], Sederberg et al. explained the troubles caused by these superfluous control points in detail.

To overcome this limitation, we need the local refinement of B-spline surfaces, i.e., to insert a single control point without propagating an entire row or column of control points. In 1988, the hierarchical B-splines were introduced by Forsey and Bartels [4], and two concepts were defined: local refinement using an efficient representation and multi-resolution editing. In principle, the hierarchical B-spline surfaces are the accumulation of tensor-product surfaces with different resolutions and domains. Weller and Hagen [8] in 1995 discussed the tensor-product splines with knot segments. In fact, they defined a spline space over a more general T-mesh, where crossing, T-junctional, and L-junctional vertices are allowed. But its dimensions are estimated and its basis functions are given over the mesh induced by some semi-regular basis functions. In 2003, Sederberg et al. [6] invented the notion of T-splines. A T-spline is a point-based spline, i.e., for every vertex, a basis function of the spline space is defined. Each of the basis functions comes from some tensor-product spline space. Though this type of splines supports many valuable operations within a consistent framework, but some of them, say, local refinement, are not simple. In the T-spline theory, the local refinement is dependent on the structure of the mesh, and its complexity is uncertain [5]. The reason leading to this problem is that the spline function over every cell of the mesh is not a polynomial, but a piecewise polynomial. On the other hand, since the basis functions do not form a partition of unity, T-splines are rational, which leads to complicated computations in subsequent geometric operations. Sederberg et al. [5] put forward the problem on how to construct polynomial T-splines.

In this paper, we introduce polynomial spline functions over T-meshes. We force the spline function on every cell to be a tensor-product polynomial, and to achieve the specified smoothness across the common edges. To distinguish this type of splines from T-splines, we call them *splines over T-meshes*. Splines over T-meshes have several advantages over T-splines. First, local refinement becomes very simple. Second, the splines are piecewise polynomials, instead of rational functions. Thirdly, there is a hierarchical structure in such kind of splines. We expect them to play an important role in adaptive surface approximation and modeling. To build up the theory of splines over T-meshes, two fundamental problems have to be solved. The first problem is to determine the dimensions of spline function spaces over T-meshes, which is the main task of current paper. The second problem is the construction of basis functions of splines over T-meshes. We will explore this problem in a future paper.

The organization of the rest part of the paper is as follows. We give the definitions and notations of T-meshes and the spline spaces over T-meshes in Section 2. In Section 3, a method based on Bézier nets (simply called *B-nets*) is proposed to compute the dimension of spline function spaces over T-meshes.

When the order of the smoothness is less than half of the degree of the polynomials in the spline space, a dimension formula is derived in Section 4 which involves only the topological quantities of the T-mesh. The construction of the basis functions of the spline space over a T-mesh is also briefly discussed. In Section 5, the dimension formulae for T-meshes after mesh operations, such as edge insertion and mesh merging are obtained. Finally, in Section 6, we conclude the paper with some future research problems.

2. Spline spaces over T-meshes

In this section, we first present some concepts in a T-mesh, and then introduce spline function spaces over T-meshes.

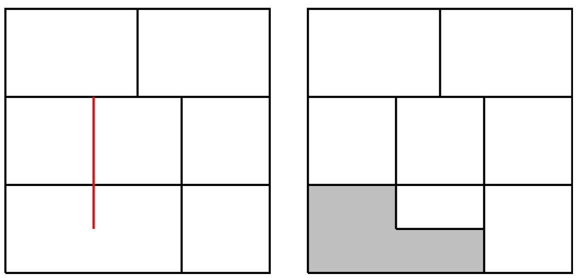
2.1. T-meshes

A *T-mesh* is basically a rectangular grid that allows T-junctions [6]. We assume that the end points of each grid line in the T-mesh must be on two other grid lines, and each *cell* or *facet* in the grid must be a rectangle. Fig. 1 illustrates two examples of T-meshes, while in Fig. 2 two examples of non-T-meshes are shown.

A grid point in a T-mesh is also called a *vertex* of the T-mesh. If a vertex is on the boundary grid line of the T-mesh, then is called a *boundary vertex*. Otherwise, it is called an *interior vertex*. For example, $b_i, i = 1, \ldots, 10$, in Fig. 3 are boundary vertices, while all the other vertices $v_i, i = 1, \ldots, 5$, are *interior vertices*. Interior vertices have two types. One is *crossing*, for example, v_2 in Fig. 3; and the other one is *T-junctional*, for example, v_1 in Fig. 3. We call them *crossing vertices* and *T-vertices*, respectively.

The line segment connecting two adjacent vertices on a grid line is called an *edge* of the T-mesh. If an edge is on the boundary of the T-mesh, then it is called a *boundary edge*; otherwise it is called an *interior edge*. For example, in Fig. 3, $b_{10}v_1$ and v_1v_2 are interior edges while b_1b_2 is a boundary edge. Two cells are called *adjacent* if they share a common edge as part of their boundaries. If one cell is above (below) the other, then they are called *adjacent vertically*. If one cell is on the left (right) of the other, then they are called *adjacent horizontally*. A cell is called *adjacent* to a grid line (an edge or composition of several edges) if some boundary line of the cell is part of the grid line.

A *composite edge* (shortly, *c-edge*) is a line segment which consists of several interior edges. It is the longest possible line segment, such that the inner vertices (vertices except the end points of the line



(a) (b)

Fig. 2. Examples of non-T-meshes.

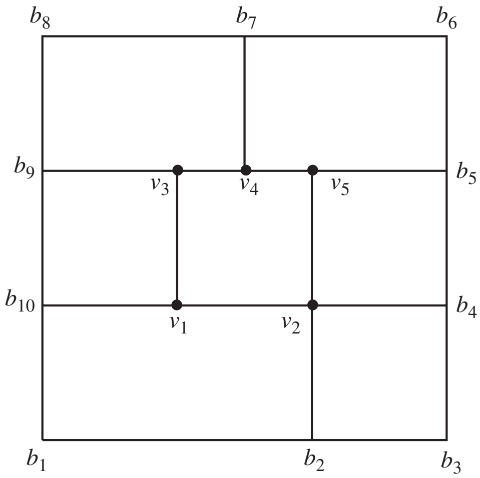


Fig. 3. A T-mesh with notations.

segment) of which are all T-vertices. For example, in Fig. 3, v_2b_{10}, b_5b_9, and v_2v_5 are c-edges, while v_1b_{10} and v_1b_4 are not. We especially define every boundary edge to be a c-edge. The concept of c-edges is a key to the proof of the dimension formula in Section 4.

For simplicity, in this paper, we consider only T-meshes whose boundary grid lines form a rectangle, see Fig. 1(b). We call this type of T-meshes *regular T-meshes*.

2.2. Spline spaces over T-meshes

Given a T-mesh $\mathscr{T}$, we use $\mathscr{F}$ to denote all the cells in $\mathscr{T}$ and Ω to denote the region occupied by all the cells in $\mathscr{T}$. Define

$$\mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) := \{s(x, y) \in C^{\alpha,\beta}(\Omega) | s(x, y)|_{\phi} \in \mathbb{P}_{mn} \text{ for any } \phi \in \mathscr{F}\}, \tag{1}$$

where $\mathbb{P}_{mn}$ is the space of all the polynomials with bi-degree (m, n), and $C^{\alpha,\beta}(\Omega)$ is the space consisting of all the bivariate functions which are continuous in Ω with order α along x direction and with order β along y direction. It is obvious that $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ is a linear space. We call it *the spline space over the given T-mesh* $\mathscr{T}$.

In the following section, we will propose a method to calculate the dimensions of the spline spaces over T-meshes. When $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, a dimension formula which involves only the topological quantities of the T-meshes is derived in Section 4.

3. The B-net method

In the theory of multi-variate splines, in order to calculate the dimension of a spline space, we first need to transfer the smoothness conditions into algebraic forms [7]. There are many approaches to address this problem. Among them, the B-net method [3] is a dominant one. We will apply this method to calculate the dimensions of the spline spaces over T-meshes.

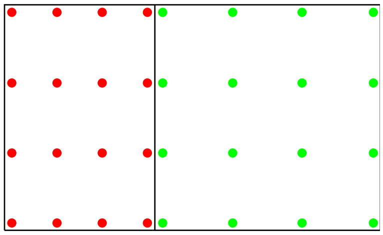


Fig. 4. The Bézier ordinates of two bi-cubic polynomials.

3.1. Review of the B-net method

Let $\pi_1(x, y)$ and $\pi_2(x, y)$ be two polynomials with bi-degree (m, n), defined over two adjacent domains $[x_0, x_1] \times [y_0, y_1]$ and $[x_1, x_2] \times [y_0, y_1]$, respectively. They can be expressed in the *Bernstein–Bézier* forms:

$$\pi_1(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} b^1_{j,k} B^m_j \left(\frac{x - x_0}{x_1 - x_0} \right) B^n_k \left(\frac{y - y_0}{y_1 - y_0} \right),$$

$$\pi_2(x, y) = \sum_{j=0}^{m} \sum_{k=0}^{n} b^2_{j,k} B^m_j \left(\frac{x - x_1}{x_2 - x_1} \right) B^n_k \left(\frac{y - y_0}{y_1 - y_0} \right), \tag{2}$$

where $B^m_j(t)$ and $B^n_k(t)$ are the *Bernstein polynomials*. $\{b^1_{j,k}\}$ and $\{b^2_{j,k}\}$ are called the *Bézier ordinates* of $\pi_1(x, y)$ and $\pi_2(x, y)$, respectively.

It is well known that $\pi_1(x, y)$ and $\pi_2(x, y)$ are r times differentiable across their common boundary, if and only if [3]

$$\frac{1}{(x_1 - x_0)^i} \Delta^{i,0} b^1_{m-i,j} = \frac{1}{(x_2 - x_1)^i} \Delta^{i,0} b^2_{0,j}, \quad j = 0, \ldots, n, \quad i = 0, \ldots, r. \tag{3}$$

Here the difference operators are defined by

$$\Delta^{i,0} b_{j,k} = \Delta^{i-1,0} b_{j+1,k} - \Delta^{i-1,0} b_{j,k} \tag{4}$$

with $\Delta^{0,0} b_{j,k} = b_{j,k}$.

The geometric meaning of the above conditions are illustrated in Fig. 4. The Bézier ordinates of two bicubic Bézier functions are shown in red and green, respectively. Suppose $\pi_1(x, y)$ and $\pi_2(x, y)$ are C^0 continuous along their common boundary, then the 3rd column of the red ordinates should be coincide with the 0th column of the green ordinates. Hence the whole function defined over $[x_0, x_2] \times [y_0, y_1]$ has $(3 + 1)^2 + 3 \times (3 + 1) = 28$ free coefficients.

Now we assume the two polynomials $\pi_1(x, y)$ and $\pi_2(x, y)$ are C^1 continuous across their common boundary, then the 1st column of the green ordinates are determined by the 2nd and 3rd columns of the red ordinates. In this case, the whole function has 24 free coefficients. If we define a polynomial of bi-degree $(1, 3)$ with the 2nd and 3rd columns of the red ordinates as its Bézier ordinates, and similarly define a

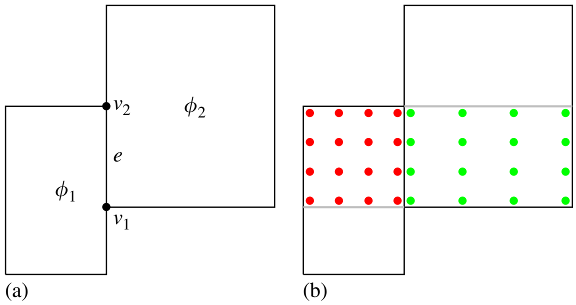


Fig. 5. Two horizontal adjacent cells.

polynomial of bi-degree (1, 3) with the 0th and 1st columns of the green ordinates as its Bézier ordinates, then these two polynomials are the same.

Similarly, for the second order smoothness conditions, the 2nd column of the green ordinates are determined by the 1st, 2nd and 3rd columns of the red ordinates, and the whole function has 20 free coefficients. For the third order smoothness conditions, the two polynomials $\pi_1(x, y)$ and $\pi_2(x, y)$ are identical.

3.2. Applications in T-meshes

Given a T-mesh $\mathscr{T}$, Let $\bar{\mathscr{S}}(m, n, \mathscr{T}) = \mathscr{S}(m, n, -1, -1, \mathscr{T})$ be the piecewise polynomial function space without continuity constraints between adjacent cells. Then it is easy to verify that

$$\dim \bar{\mathscr{S}}(m, n, \mathscr{T}) = F(m + 1)(n + 1),$$

where F is the number of the cells in $\mathscr{T}$. Any function in $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ can be thought of as a function in $\bar{\mathscr{S}}(m, n, \mathscr{T})$ that satisfies additional smoothness conditions.

Now we consider smoothness conditions. Select an arbitrary function $s \in \bar{\mathscr{S}}(m, n, \mathscr{T})$. Let ϕ_1 and ϕ_2 be two horizontal adjacent cells in the T-mesh $\mathscr{T}$ as shown in Fig. 5(a). Assume ϕ_1 and ϕ_2 are defined over $[x_0, x_1] \times [y_0, y_1]$ and $[x_1, x_2] \times [y_2, y_3]$, respectively. Suppose $s|_{\phi_k} = s_k$, and the Bézier ordinates of s_k over ϕ_k are b^k_{ij}, $i = 0, \ldots, m$, $j = 0, \ldots, n$, $k = 1, 2$. Then, according to the subdivision algorithm for tensor-product polynomials [3], we compute the Bézier ordinates for polynomial s_k restricted over $[x_k, x_{k+1}] \times [\underline{y}, \bar{y}]$, $k = 1, 2$, where $\bar{y} = \min(y_1, y_3)$ and $\underline{y} = \max(y_0, y_2)$ (see Fig. 5(b)). Assume the new Bézier ordinates are $\bar{b}^k_{ij}$, $i = 0, \ldots, m$, $j = 0, \ldots, n$, $k = 1, 2$. According to Eq. (3), we obtain the smoothness conditions satisfied by these two sets of the Bézier ordinates. These conditions lead to the smoothness conditions on the original two sets of Bézier ordinates b^k_{ij}, $i = 0, \ldots, m$, $j = 0, \ldots, n$, $k = 1, 2$.

The similar smoothness conditions can be derived for any two vertical adjacent cells in $\mathscr{T}$. All the smooth conditions can be collected into a homogeneous linear system

$$Ac = 0,$$

where A is a matrix and c is the vector with all the Bézier ordinates collected in some order. Since the number of elements in c is $F(m + 1)(n + 1)$, it follows that

$$\dim \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) = F(m + 1)(n + 1) - \operatorname{rank} A. \quad (5)$$

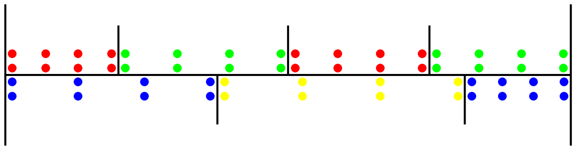


Fig. 6. Bézier ordinates in every cell near a c-edge ($m=3$, $\beta=1$).

To facilitate the computation of rank A, we apply the concept of determining sets introduced by Alfeld and Schumaker [1,2] in the following subsection.

3.3. Determining sets

Suppose that $\mathscr{T}$ is an arbitrary T-mesh, and the cells of $\mathscr{T}$ are denoted by $\phi_1, \ldots, \phi_\ell$, where $\phi_k = [x_0^k, x_1^k] \times [y_0^k, y_1^k]$. For any $s \in \mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$, let s_k denote the polynomial $s|_{\phi_k}$. Assume the Bézier ordinates of s_k over ϕ_k are b_{ij}^k, $i = 0, \ldots, m$, $j = 0, \ldots, n$. Each of the Bézier ordinate b_{ij}^k is associated with a *domain point*

$$P_{ij}^k = \left(\frac{(m-i)x_0^k + i x_1^k}{m}, \frac{(n-j)y_0^k + j y_1^k}{n} \right).$$

Let $\mathscr{B}(m, n, \mathscr{T})$ denote the set of domain points. For $t = P_{ij}^k \in \mathscr{B}(m, n, \mathscr{T})$, let λ_t denote the linear functional on $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ defined by $\lambda_t s = b_{ij}^k$. A set of domain points $\mathscr{P} \subset \mathscr{B}(m, n, \mathscr{T})$ is called a *determining set* [2] for the spline space $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ if for $s \in \mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$,

$$\lambda_t s = 0, \quad \forall t \in \mathscr{P} \Longrightarrow s = 0.$$

If $\mathscr{P}$ is a determining set for $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$, then it follows that dim $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) \leqslant \#\mathscr{P}$, where $\#\mathscr{P}$ denotes the cardinality of the set $\mathscr{P}$. If $\mathscr{Q}$ is a determining set, and each of its nontrivial subsets is not a determining set, then it is called a *minimal determining set*. It follows from basic linear algebra that the number of points in any minimal determining set equals the dimension of the spline space. The Bézier ordinates corresponding to the points in a minimal determining set can be chosen arbitrarily and they uniquely determine a spline s in $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$. For the details about determining sets, the reader is referred to [2]. Though the results in [2] were obtained for spline spaces over triangular meshes, similar conclusions hold for the spline spaces over T-meshes.

The following lemma states a key fact which will be used in the proof of the dimension formula in Section 4.

Lemma 3.1. *Given a T-mesh $\mathscr{T}$ and the spline space $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ defined over $\mathscr{T}$, consider an interior horizontal c-edge which consists of ℓ edges and has $\ell + 1$ cells adjacent to it. Then the $\beta + 1$ rows of the Bézier ordinates near the c-edge in each of these cells define an identical polynomial of bi-degree (m, β) (see Fig. 6). Similarly, consider an interior vertical c-edge. The $\alpha + 1$ columns of the Bézier ordinates near the c-edge in every adjacent cell define an identical polynomial of bi-degree (α, n).*

Proof. We just prove the case for horizontal c-edges. Suppose ϕ_1 and ϕ_2 are the two leftmost adjacent cells near the c-edge with ϕ_2 beneath ϕ_1. By the similar analysis as in Subsection 3.1, the $\beta + 1$ rows of

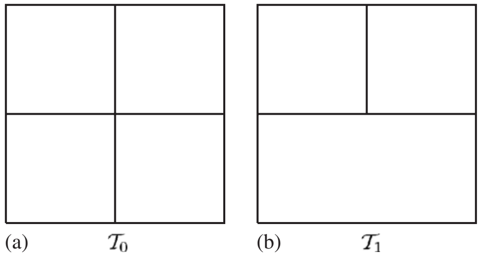


Fig. 7. The nontrivial simplest T-meshes.

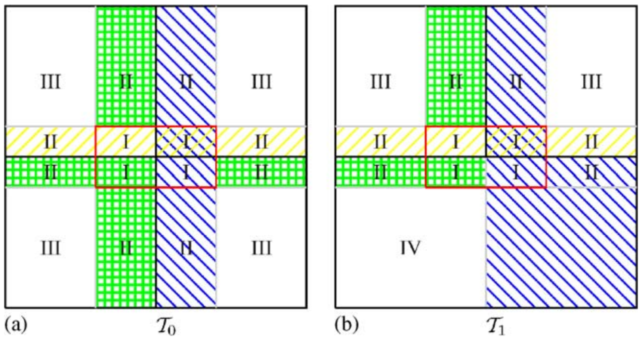


Fig. 8. The constraints on the B-nets.

the Bézier ordinates near the c-edge in ϕ_1 and ϕ_2 define the same polynomial with bi-degree (m, β). If $\ell > 1$, then we select the leftmost cell in the rest of the adjacent cells near the c-edge. The new cell will be adjacent vertically to either ϕ_1 or ϕ_2. Therefore, the $\beta + 1$ rows of the Bézier ordinates near the c-edge in the new cell defines the same polynomial. By this fashion, we can run through all the cells adjacent to the c-edge, and thus all the $\beta + 1$ rows of the Bézier ordinates near the c-edge define an identical polynomial. □

Now we illustrate some examples to show how to use the B-net method to calculate the dimensions of spline spaces.

Example 1. Suppose we are given a simple T-mesh $\mathscr{T}_0$ as shown in Fig. 7(a). The dimension of $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T}_0)$ can be calculated as follows. Consider the constraints imposed on the Bézier ordinates over the four cells. As shown in Fig. 8, for each cell, the part labeled with III does not have any constraints on it, the size of which is $(m - \alpha)(n - \beta)$, that is, we have $(m - \alpha)(n - \beta)$ free Bézier ordinates for each part labeled with III. The domain points corresponding to these free Bézier ordinates are collected into the determining set.

For any adjacent parts labeled with II, if one part is fixed, another part is also determined by the smoothness conditions. Hence without loss of generality, we can put the $2(m-\alpha)(\beta+1)+2(\alpha+1)(n-\beta)$ domain points in the parts II with green crosshatches into the determining set.

For the parts labeled with I in the red frame, it is a little more tricky. Let us fix one part with label I, say the one with green crosshatches. With the $(\alpha + 1)(\beta + 1)$ Bézier ordinates, we can define a polynomial

$p_{11}(x,y)$ of bi-degree (α,β). Similarly, we can define two polynomials $p_{12}(x,y)$ and $p_{21}(x,y)$ of bi-degree (α,β), respectively, using the Bézier ordinates in the region with just the yellow lines or the blue lines in the red frame. By the smoothness conditions, it is obviously that $p_{11}(x,y)=p_{12}(x,y)=p_{21}(x,y)$.

Now we determine the Bézier ordinates in the 4th part labeled with I. There are two ways to determine them, one is from $p_{12}(x,y)$, and the other is from $p_{21}(x,y)$. A bit of analysis shows that either way gives the same polynomial as $p_{12}(x,y)$ or $p_{21}(x,y)$. Thus we get an identical polynomial in the four parts labeled with I. Hence we can put the $(\alpha+1)(\beta+1)$ domain points with green crosshatches in part I into the determining set.

From above analysis, the number of domain points in the determining set is

$$4(m-\alpha)(n-\beta)+2(m-\alpha)(\beta+1)+2(\alpha+1)(n-\beta)+(\alpha+1)(\beta+1),$$

which is exactly the dimension of spline space $\mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_0)$, since it is obvious that the determining set is minimal.

Example 2. Consider the T-mesh $\mathscr{T}_1$ given in Fig. 7(b). In order to simulate the situation in the mesh $\mathscr{T}_0$, we subdivide the bottom cell into two sub-cells along the vertical line at the T-junction. Now the difference between Examples 1 and 2 is that the polynomials in the two sub-cells in $\mathscr{T}_1$ should be identical. A similar analysis as in Example 1 shows that there is no over-determination in the T-junction. Thus the dimension of the spline space $\mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_1)$ can be calculated as

$$\begin{aligned}&\dim\ \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_1)\\&\quad=(m+1)(n-\beta)+2(m-\alpha)(n-\beta)+(m-\alpha)(\beta+1)+(\alpha+1)(n-\beta)+(\alpha+1)(\beta+1)\\&\quad=3(m+1)(n+1)-(\alpha+1)(n+1)-2(m+1)(\beta+1)+(\alpha+1)(\beta+1).\end{aligned}$$

4. The dimension formula

In this section, we will derive a dimension formula for the spline space $\mathscr{S}(m,n,\alpha,\beta,\mathscr{T})$ over a given T-mesh $\mathscr{T}$ when $m\geqslant 2\alpha+1$ and $n\geqslant 2\beta+1$. The formula, in the form of weighted Euler formula, depends only on the topological quantities of $\mathscr{T}$. For simplicity of derivation, we assume $\mathscr{T}$ is regular, i.e., Ω is a rectangle. This assumption does not limit the validity of the method for general cases.

4.1. Some notations for a T-mesh

Before we derive the dimension formula, we introduce some notations for a T-mesh as shown in Table 1.

Lemma 4.1. *Given a T-mesh with the notations in Table 1, it follows that*

1.

$$\sum_{i=1}^{\tilde{E}_h}\lambda_i=E_h,\quad \sum_{i=1}^{\tilde{E}_v}\mu_i=E_v. \tag{6}$$

Table 1
Notations for a T-mesh

E_h^b	number of horizontal boundary edges (=c-edges)
E_v^b	number of vertical boundary edges (=c-edges)
E_h	number of horizontal interior edges
E_v	number of vertical interior edges
$\tilde{E}_h$	number of horizontal interior c-edges
$\tilde{E}_v$	number of vertical interior c-edges
λ_i	number of edges on the ith horizontal interior c-edge, $i = 1, \ldots, \tilde{E}_h$
μ_i	number of edges on the ith vertical interior c-edge, $i = 1, \ldots, \tilde{E}_v$
V^+	number of interior crossing vertices
$V^\perp$	number of interior T-vertices
V	number of interior vertices ($V = V^\perp + V^+$)
F	number of cells in the mesh

2.

$$2F - E_h^b - \tilde{E}_h = E_h, \quad 2F - E_v^b - \tilde{E}_v = E_v. \tag{7}$$

3.

$$E - \tilde{E}_h - \tilde{E}_v = V^\perp. \tag{8}$$

Proof.

1. Eqs. (6) obviously hold according to the definitions of λ_i and μ_i.
2. Since every cell has two horizontal boundary lines, each of which is part of some horizontal c-edge, and every interior horizontal c-edge has $\lambda_i + 1$ adjacent cells, by the contribution of boundary edges, it follows that

$$2F = E_h^b + \sum_{i=1}^{\tilde{E}_h} (\lambda_i + 1).$$

By Eqs. (6), we have

$$2F - E_h^b - \tilde{E}_h = E_h.$$

Similarly,

$$2F - E_v^b - \tilde{E}_v = E_v.$$

3. Observe that

$$E - \tilde{E}_h - \tilde{E}_v = (E_h - \tilde{E}_h) + (E_v - \tilde{E}_v).$$

Since $E_h - \tilde{E}_h$ and $E_v - \tilde{E}_v$ represent the numbers of all the T-vertices on all the horizontal and vertical c-edges, respectively, it follows that

$$E - \tilde{E}_h - \tilde{E}_v = V^\perp. \qquad \square$$

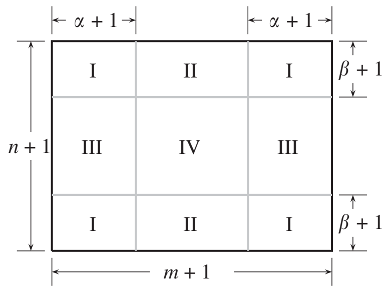


Fig. 9. A cell in T-mesh.

4.2. Dimension formula

Now we are ready to prove the dimension formula for the spline space $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ when $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta + 1$.

Theorem 4.2. *Given a regular T-mesh and a corresponding spline space $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$, and suppose $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$. Then*

$$\begin{aligned}\dim \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) = {} & F(m + 1)(n + 1) - E_h(m + 1)(\beta + 1) \\ & - E_v(\alpha + 1)(n + 1) + V(\alpha + 1)(\beta + 1),\end{aligned} \tag{9}$$

where F, E_h, E_v and V are defined in Table 1.

Proof. For any cell in the given T-mesh, since $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, we can divide the Bézier ordinates in the cell into nine parts as shown in Fig. 9. The Bézier ordinates in part IV are free since no constraints are imposed on them. Hence we can put all of the domain points corresponding to these Bézier ordinates into the determining set. The number of these Bézier ordinates is

$$\begin{aligned}d_1 &= F[(m + 1) - 2(\alpha + 1)][(n + 1) - 2(\beta + 1)] \\ &= F(m + 1)(n + 1) - 2F(\alpha + 1)(n + 1) - 2F(m + 1)(\beta + 1) + 4F(\alpha + 1)(\beta + 1).\end{aligned} \tag{10}$$

In the following, we will count how many free Bézier ordinates there are in parts I, II and III.

First, we consider the boundary edges (c-edges). There are E_h^b and E_v^b horizontal boundary edges and vertical boundary edges, respectively. In Fig. 10, we use green crosshatch lines to represent the free Bézier ordinates.

It is easy to show that the number is

$$\begin{aligned}d_2 &= E_h^b(m - \alpha)(\beta + 1) + E_v^b(\alpha + 1)(n - \beta) \\ &= E_h^b(m + 1)(\beta + 1) + E_v^b(\alpha + 1)(n + 1) - (E_h^b + E_v^b)(\alpha + 1)(\beta + 1).\end{aligned} \tag{11}$$

The domain points corresponding to this part of the Bézier ordinates can be added into the determining set.

Next, we consider the interior horizontal c-edges (Fig. 11). Referring to Table 1, there are $\tilde{E}_h$ interior horizontal c-edges in the given T-mesh, and there are λ_i edges on the ith c-edge, i.e., $\lambda_i + 1$ cells are adjacent to the ith c-edge. These cells queue in two horizontal rows. Consider the cell in the down-left

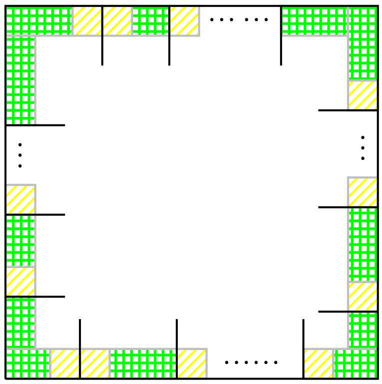


Fig. 10. Free Bézier ordinates along the boundary of a regular T-mesh.

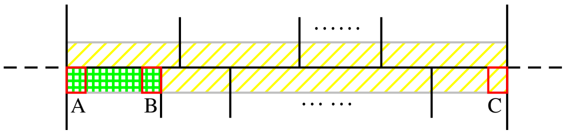


Fig. 11. An interior horizontal c-edge.

corner. If we fix its top $\beta+1$ rows of the Bézier ordinates ($(m+1)(\beta+1)$ Bézier ordinates in total), according to Lemma 3.1, all the $\beta+1$ rows of the Bézier ordinates in each cell adjacent to the c-edge will be fixed as well. But this does not mean that we have $(m+1)(\beta+1)$ free Bézier ordinates coming from this c-edge, since, if we consider this horizontal c-edge within the original T-mesh, two end parts with the size of $(\alpha+1)(\beta+1)$ could be possibly determined cyclically. Hence we have just $[(m+1)-2(\alpha+1)](\beta+1)$ affirmative free Bézier ordinates, and put the corresponding domain points into the determining set. The total number is

$$\begin{aligned} d_3 &= \tilde{E}_h[(m+1)-2(\alpha+1)](\beta+1) \\ &= \tilde{E}_h(m+1)(\beta+1) - 2\tilde{E}_h(\alpha+1)(\beta+1). \end{aligned} \tag{12}$$

Similarly, we can get the following number of affirmative free Bézier ordinates from interior vertical c-edges (the corresponding domain points are added into the determining set):

$$\begin{aligned} d_4 &= \tilde{E}_v(\alpha+1)[(n+1)-2(\beta+1)] \\ &= \tilde{E}_v(\alpha+1)(n+1) - 2\tilde{E}_v(\alpha+1)(\beta+1). \end{aligned} \tag{13}$$

Now only the Bézier ordinates around every interior vertices in parts labeled with I have to be determined. There are two types of interior vertices, one is crossing, and the other one is T-junctional. Based on the fixed Bézier ordinates in parts II, III and IV as shown in Fig. 9, in the following we will show that the Bézier ordinates around the interior T-vertices will be determined by the Bézier ordinates around the interior crossing vertices and the boundary vertices.

Select an interior T-vertex, and a binary tree structure can be build in the following fashion. The root is the T-vertex, and its two child nodes are the two end vertices of the c-edge where the root lies (Note: a

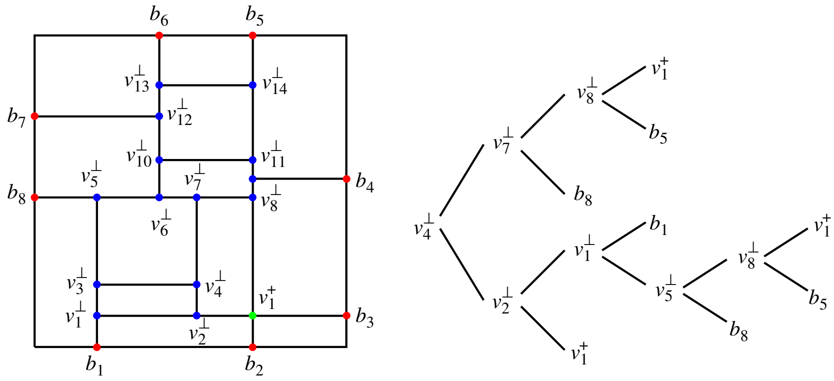


Fig. 12. A T-junctional vertex and its binary tree structure.

T-vertex must lie on some c-edge). If the node is a crossing vertex or a boundary vertex, it has no child. Otherwise, the node is a T-vertex. If the node has appeared in some ancestor node, then it has no child; else, the node has two children which are the two end vertices of the c-edge where the node lies. By repeating this procedure, we get a binary tree, where the leaf nodes are crossing vertices or boundary vertices. See Fig. 12 for an example, where $v_4^{\perp}$ is a T-vertex.

For an interior horizontal c-edge, referring to Fig. 11 and according to the analysis for calculating d_3, if the top $\beta + 1$ rows of the Bézier ordinates in the down-left corner (with green crosshatches) are fixed, then all the Bézier ordinates labeled with yellow lines in every cell adjacent to the c-edge will be fixed as well. In the part with green crosshatches, the Bézier ordinates in the red frames A and B are in part I, and need to be determined. By the subdivision algorithm of tensor product surfaces [3], we can prove that the determination of the Bézier ordinates in the frames A and B is equivalent to the determination of the Bézier ordinates in the frames A and C. The number of Bézier ordinates in each red frame is $(\alpha+1)(\beta+1)$.

Therefore we have proved that if the Bézier ordinates in part I around the two end vertices of a c-edge are fixed, then the Bézier ordinates in part I around every T-vertex on the c-edge are determined correspondingly. Consequently, by traversing the binary tree of the selecting T-vertex, it follows that if the Bézier ordinates around all the leaf vertices (not including T-vertices on the leaves) are fixed, then the Bézier ordinates around the root vertex will be determined correspondingly, that is, we only have to consider boundary vertices (which has been counted in d_2) and crossing vertices. If a leaf is a T-vertex, then we can assume it has been determined since it has appeared in some ancestor node according to the construction of the binary tree.

For every crossing vertex, which are the meeting points of four c-edges, we have $(\alpha + 1)(\beta + 1)$ free Bézier ordinates. Totally we have

$$d_5 = V^{+}(\alpha + 1)(\beta + 1) \tag{14}$$

free Bézier ordinates. The domain points corresponding to this part of Bézier ordinates will be also added into the determining set.

Now the determining set consists of five parts, and the total number is the sum of d_i, $i = 1, 2, 3, 4, 5$. It is easy to verify part by part that if we remove any one Bézier ordinate from the determining set, then it will not be a determining set. Hence we obtain a minimal determining set. Therefore the dimension of the spline space is

$$\begin{aligned}\dim \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) \\ &= \sum_{i=1}^{5} d_i = F(m+1)(n+1) - (2F - E_v^b - \tilde{E}_v)(\alpha+1)(n+1) \\ &\quad - (2F - E_h^b - \tilde{E}_h)(m+1)(\beta+1) + (4F - E_h^b - E_v^b - 2\tilde{E}_h - 2\tilde{E}_v + V^+)(\alpha+1)(\beta+1) \\ &= F(m+1)(n+1) - E_v(\alpha+1)(n+1) - E_h(m+1)(\beta+1) + V(\alpha+1)(\beta+1),\end{aligned}$$

since

$$\begin{aligned}4F - E_h^b - E_v^b - 2\tilde{E}_h - 2\tilde{E}_v + V^+ &= (2F - E_h^b - \tilde{E}_h) + (2F - E_v^b - \tilde{E}_v) - \tilde{E}_h - \tilde{E}_v + V^+ \\ &= E_h + E_v - \tilde{E}_h - \tilde{E}_v + V^+ = V^\perp + V^+ = V.\end{aligned}$$

This completes the proof of the theorem. □

For the special case where $m = n$ and $\alpha = \beta$, we have

Corollary 4.3. *Suppose* $m \geqslant 2\alpha + 1$, *then*

$$\dim \mathscr{S}(m, m, \alpha, \alpha, \mathscr{T}) = (m+1)^2 + (F-1)(m+1)(m-\alpha) - V(m-\alpha)(\alpha+1). \quad (15)$$

Proof. The difference between Eqs. (9) and (15) is

$$(m+1)(\alpha+1)(F - E + V - 1).$$

By Euler's formula, it follows that $F - E + V - 1 = 0$. Hence Eq. (15) holds. □

Remark. When $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, the dimension formula of $\mathscr{S}(m, n, \alpha, \beta, \mathscr{T})$ is a linear combination of F, E_h, E_v and V, where the combinational coefficients depend only on m, n, α, and β. Hence we can think of the dimension formula as a weighted Euler formula. If we remove the constraints $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, the dimension formula may fail to hold.

4.3. Basis functions

The construction of basis functions for the spline spaces over T-meshes is another fundamental problem. A set of basis functions with some "good" properties, such as nonnegativity, unity partition and compact support, will facilitate the applications of splines over T-meshes.

According to the proof of Theorem 4.2, we have an approach to construct a basis for the spline space over a T-mesh. In fact, we have in total $\sum_{i=1}^{5} d_i$ free Bézier ordinates in the proof of Theorem 4.2, and for each of them, we choose its value to be one and all the other free ordinates to be zero. The rest of the Bézier ordinates can be determined by the smoothness conditions. In this way, we obtain a set of spline functions which form a basis for the spline space. Generally, these basis functions form a partition of unity. Unfortunately, their supports are dependent on the structure of the T-mesh, and may not be compact. Furthermore, the basis functions may not be nonnegative. Recently, we have made some progress in the construction of basis functions which posses the three good properties. We will discuss its detailed construction process and its applications in geometric modeling in a forthcoming paper.

5. Operations

The spline space over a T-mesh changes after some operation on the T-mesh. In this section, we discuss the dimension of the spline space after edge insertion and mesh merging.

5.1. Edge insertion

Shape refinement is an important operation in geometric modeling. For tensor-product B-spline surfaces, this operation is achieved by knot insertion. For example, if one wants to add more details in some part of a B-spline surface, one just has to insert more knots to create more control points near the region of interest. For the splines over T-meshes, a similar operation can be achieved by edge insertion. Here edge insertion is to insert an edge in the T-mesh such that a cell is subdivided into two cells. Edge insertion increases the number of cells in the T-mesh by one, i.e., $\mathrm{d}F = 1$. But the increments of the numbers of edges and vertices depend on the geometric configuration of the T-mesh. Let $\mathrm{d}V$, $\mathrm{d}E_h$ and $\mathrm{d}E_v$ represent the increments of the numbers of interior vertices, interior horizontal edges and interior vertical edges respectively, and $\mathscr{T}_1$ the new mesh after edge insertion. Then

$$\mathscr{S}(m, n, \alpha, \beta, \mathscr{T}_1) \supset \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}),$$

and

$$\dim \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}_1) = \dim \mathscr{S}(m, n, \alpha, \beta, \mathscr{T}) + (m+1)(n+1) - \mathrm{d}E_h \cdot (m+1)(\beta+1) - \mathrm{d}E_v \cdot (\alpha+1)(n+1) + \mathrm{d}V \cdot (\alpha+1)(\beta+1),$$

where $m \geqslant 2\alpha+1$, $n \geqslant 2\beta+1$. In Fig. 13, we illustrate some examples of edge insertion. In these examples, the end vertices of the new edge are new to the T-mesh. If an end vertex is identical to a vertex in $\mathscr{T}$, then it will not be counted in $\mathrm{d}V$.

It is easy to prove that if the new edge is horizontal, then

$$\mathrm{d}F = \mathrm{d}E_h = 1, \quad 0 \leqslant \mathrm{d}V = \mathrm{d}E_v \leqslant 2,$$

and, if the new edge is vertical, then

$$\mathrm{d}F = \mathrm{d}E_v = 1, \quad 0 \leqslant \mathrm{d}V = \mathrm{d}E_h \leqslant 2.$$

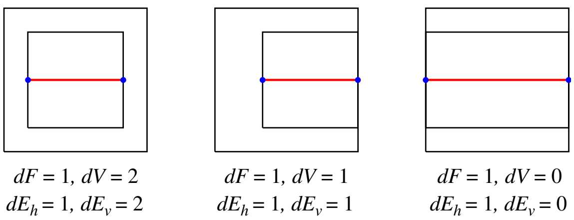


Fig. 13. Examples of edge insertion. (a) $\mathrm{d}F = 1$, $\mathrm{d}V = 2$, $\mathrm{d}E_h = 1$, $\mathrm{d}E_v = 2$, (b) $\mathrm{d}F = 1$, $\mathrm{d}V = 1$, $\mathrm{d}E_h = 1$, $\mathrm{d}E_v = 1$, (c) $\mathrm{d}F = 1$, $\mathrm{d}V = 0$, $\mathrm{d}E_h = 1$, $\mathrm{d}E_v = 0$.

5.2. Merging two T-meshes

Given two T-meshes $\mathscr{T}_1$ and $\mathscr{T}_2$, and suppose they have a common boundary segment and the interiors of both meshes have no intersection. Then we can merge these two T-meshes into one by taking their common segment as a set of interior edges in the new mesh. Denote the new mesh as $\mathscr{T}_1 \cup \mathscr{T}_2$. Suppose there are V_1^b and V_2^b boundary vertices (excluding corner vertices) in $\mathscr{T}_1$ and $\mathscr{T}_2$ along the common boundary segment respectively. Among them, there are V^I duplicate vertices. Therefore, merging these two meshes leads to $V_1^b + V_2^b - V^I + 1$ new interior edges and $V_1^b + V_2^b - V^I$ new interior vertices. Suppose $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$. If the common boundary segment is horizontal, we have the following dimension formula:

$$\begin{aligned}\dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}\cup\mathscr{T}_2) = {} & \dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_1) + \dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_2)\\ & - (V_1^b + V_2^b - V^I + 1)(m+1)(\beta+1)\\ & + (V_1^b + V_2^b - V^I)(\alpha+1)(\beta+1).\end{aligned}$$

Otherwise if the common segment is vertical, then the dimension formula becomes

$$\begin{aligned}\dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}\cup\mathscr{T}_2) = {} & \dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_1) + \dim \mathscr{S}(m,n,\alpha,\beta,\mathscr{T}_2)\\ & - (V_1^b + V_2^b - V^I + 1)(\alpha+1)(n+1)\\ & + (V_1^b + V_2^b - V^I)(\alpha+1)(\beta+1).\end{aligned}$$

6. Conclusions and future work

In this paper, we introduced the splines over T-meshes, which are a generalization of the T-splines invented by Sederberg et al. [5,6]. Based on the B-net method, a scheme to compute the dimensions of spline spaces over T-meshes is presented, and in the case that $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, a dimension formula is derived.

The expectation to introduce the new splines is that they cannot only inherit the attractive properties of the T-splines but also provide more simplicity and flexibility in surface approximation and modeling.

The paper leaves several open problems for further research.

1. The ultimate goal to introduce splines over T-meshes is to apply them in geometric modeling, for example, in surface approximation and interpolation. There are many problems which need to be solved, such as construction of a set of basis functions with good properties, geometric operations and properties of the splines over T-meshes, etc. We will explore these problems in detail in future papers.
2. Currently the dimension formula is obtained under the constraints that $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$. A general dimension formula without these constraints is still unavailable. In fact, we even do not know whether the dimension relies on the geometry of the T-mesh or not.
3. We only considered regular T-meshes in this paper. In the practice of geometric modeling, the types of T-meshes which are useful for specified problems should be explored, and the dimension formula of spline spaces over general T-meshes should be discussed.

Acknowledgements

The authors are support by the Outstanding Youth Grant of NSF of China (No. 60225002), a National Key Basic Research Project of China (2004CB318000), NSF of China (10201030 and 60473132), the TRAPOYT in Higher Education Institute of MOE of China, and SRF for ROCS, SEM.

References

[1] P. Alfeld, Bivariate spline spaces and minimal determining sets, J. Comput. Appl. Math. 119 (2000) 13–27.
[2] P. Alfeld, L.L. Schumaker, The dimensions of bivariate spline spaces of smoothness r for degree $d \geqslant 4r + 1$, Constr. Approx. 3 (1987) 189–197.
[3] G. Farin, Curves and Surfaces for CAGD—A Practical Guide, fifth ed., Morgan Kaufmann Publishers, Los Altos, CA, 2002.
[4] D. Forsey, R.H. Bartels, Hierarchical B-spline refinement, Comput. Graphics 22 (4) (1988) 205–212.
[5] T.W. Sederberg, D.L. Cardon, G.T. Finnigan, N.S. North, J. Zheng, T. Lyche, T-spline simplification and local refinement, ACM Trans. Graphics 23 (3) (2004) 276–283.
[6] T.W. Sederberg, J. Zheng, A. Bakenov, A. Nasri, T-splines and T-NURCCs, ACM Trans. Graphics 22 (3) (2003) 161–172.
[7] R.-H. Wang, Multivariate Spline Functions and Their Applications, Kluwer Academic Publishers, Dordrecht, 2001.
[8] F. Weller, H. Hagen, Tensor-product spline spaces with knot segments, in: M. Dæhlen, T. Lyche, L.L. Schumaker (Eds.), Mathematical Methods for Curves and Surfaces, Vanderbilt University Press, Nashville, TN, 1995, pp. 563–572.