Approximate μ-Bases of Rational Curves and Surfaces

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Abstract. The μ-bases of rational curves and surfaces are newly developed tools which play an important role in connecting parametric forms and implicit forms of curves and surfaces. However, exact μ-bases may have high degree with complicated rational coefficients and are often hard to compute (especially for surfaces), and sometimes they are not easy to use in geometric modeling and processing applications. In this paper, we introduce approximate μ-bases for rational curves and surfaces, and present an algorithm to compute approximate μ-bases. The algorithm amounts to solving a generalized eigenvalue problem and some quadratic programming problems with linear constraints. As applications, approximate implicitization and degree reduction of rational curves and surfaces with approximate μ-bases are discussed. Both the parametric equations and the implicit equations of the approximate curves/surfaces are easily obtained by using the approximate μ-bases. As indicated by the examples, the proposed algorithm may be a useful alternative to other methods for approximate implicitization.

Keywords: approximate μ-bases, approximate implicitization.

1 Introduction

The concept of μ-bases was first introduced in [9] to derive a compact representation for the implicit equation of a planar rational curve. The basic idea of μ-bases originates in a method called *moving curves and surfaces* to implicitize rational curves and surfaces [16]. The μ-basis of a planar rational curve of degree n consists of two polynomials $p(x, y; t)$ and $q(x, y; t)$ which are linear in x, y and degree μ and $n - \mu$ in t respectively, where $0 \leqslant \mu \leqslant \lfloor n/2 \rfloor$. The resultant of p and q with respect to t gives the implicit equation of the rational curve. In the generic case, $\mu = \lfloor n/2 \rfloor$, and thus the implicit equation of a rational curve can be expressed as a determinant of size $\lfloor n/2 \rfloor \times \lfloor n/2 \rfloor$, whereas previous resultant-based methods express the implicit equation as either an $n \times n$ determinant or an $2n \times 2n$ determinant. The μ-basis can not only compute the implicit equation of a rational curve, but also recover the parametric equation of the curve conveniently. Thus μ-bases connect the and the parametric form of a curve.

The concept of μ-bases was subsequently generalized to rational ruled surfaces[2,6] and general rational surfaces [7]. Various algorithms to compute the

M.-S. Kim and K. Shimada (Eds.): GMP 2006, LNCS 4077, pp. 175–188, 2006.

μ-bases for rational curves and rational surfaces were developed [3,10,19]. Applications of μ-bases to implicitization, singular point computation and surface reparameterization are explored as well [4,5,8]. Thus μ-bases provide a new tool to study curves and surfaces in geometric modeling.

However, the use of exact μ-bases leads to some problems in applications. First, general μ-bases may have very complicated rational coefficients and/or high degree, and they are therefore hard to use in practice. Second, it is very costly to compute μ-bases, especially for surfaces. Finally, curves and surfaces in CAD systems are usually described by floating point coefficients, and in these situations, exact μ-bases are often unnecessary. To overcome these difficulties, we introduce the concept of *approximate* μ-bases. These bases have low degree and are described by floating point coefficients. They can be found by numerical techniques.

A direct application of approximate μ-bases is the *approximate implicitization* (see [1,11,17,18]) of rational curves and surfaces. As an obvious advantage of the new approach, both a parametric and an implicit representation of the approximating curve or surface are available, and the parametric equation can be easily recovered by evaluating the exterior product of the approximate μ-bases. In addition, the new approach can also be used as a degree reduction technique for rational curves and surfaces. See [12,13,14,15] for more information on this topic.

The organization of the paper is as follows. Section 2 reviews some preliminary results about the μ-bases of rational curves and surfaces. Section 3 introduces approximate μ-bases for rational curves and presents an algorithm to compute them. Applications of approximate μ-bases to approximate implicitization and degree reduction are discussed. In Section 4, we generalize the results of Section 3 to rational surfaces. Finally we conclude this paper.

2 μ-Bases of Rational Curves and Surfaces

Consider a planar rational curve in homogenous form

$$\mathbf{P}(t) = (a(t), b(t), c(t)), \tag{1}$$

where $a(t)$, $b(t)$, $c(t)$ are relatively prime polynomials whose maximum degree equals n. A *moving line* is a family of lines with parameter t,

$$L(x, y; t) := A(t)x + B(t)y + C(t) = 0, \tag{2}$$

where $A(t), B(t), C(t)$ are polynomials. For simplicity, sometimes we write a moving line as $\mathbf{L}(t) := (A(t), B(t), C(t))$. The moving line (2) is said to *follow* the rational curve (1) if

$$\mathbf{L}(t) \cdot \mathbf{P}(t) = A(t)a(t) + B(t)b(t) + C(t)c(t) \equiv 0. \tag{3}$$

A *μ-basis* of a planar rational curve of degree n consists of two independent moving lines $p = p_1x+p_2y+p_3$ and $q = q_1x+q_2y+q_3$ that follow the curve, where the degree in t of p and q sums up to n. Let $\mathbf{p} = (p_1, p_2, p_3)$ and $\mathbf{q} = (q_1, q_2, q_3)$. Then the μ-basis has the following properties [3]:

1. $\mathbf{p} \times \mathbf{q} = \kappa(a, b, c)$ for some non-zero constant κ.
2. For any moving line $\mathbf{l}(t)$, there exist polynomials $h_1(t)$ and $h_2(t)$ such that
$$\mathbf{l}(t) = h_1\mathbf{p} + h_2\mathbf{q}.$$
3. The resultant of p and q with respect to t gives the implicit equation of the rational curve (1).

The concept of μ-bases can be generalized to rational surfaces. Let

$$\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)), \tag{4}$$

be a rational surface in homogeneous form, where a, b, c, d are bivariate polynomials in s and t, and $\gcd(a,\ b,\ c,\ d) = 1$. We assume that the rational surface (4) is given by a proper parameterization. A *moving plane* is defined by

$$L(x, y, z; s, t) := A(s,t)x + B(s,t)y + C(s,t)z + D(s,t) = 0,$$

where A, B, C, D are polynomials in s and t. Sometimes we use $\mathbf{L}(s,t) := (A(s,t), B(s,t), C(s,t), D(s,t))$ to denote the moving plane. The moving plane $\mathbf{L}(s,t)$ is said to *follow* the rational surface (4) if and only if

$$\mathbf{L}(s,t) \cdot \mathbf{P}(s,t) = aA + bB + cC + dD \equiv 0. \tag{5}$$

A *μ-basis* of the rational surface (4) consists of three moving planes $\mathbf{p}, \mathbf{q}, \mathbf{r}$ following (4) such that

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa\mathbf{P}(s,t) \tag{6}$$

for some nonzero constant κ. Here $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ is the *exterior product* of $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$,

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \left(\begin{vmatrix} p_2 & p_3 & p_4 \\ q_2 & q_3 & q_4 \\ r_2 & r_3 & r_4 \end{vmatrix}, -\begin{vmatrix} p_1 & p_3 & p_4 \\ q_1 & q_3 & q_4 \\ r_1 & r_3 & r_4 \end{vmatrix}, \begin{vmatrix} p_1 & p_2 & p_4 \\ q_1 & q_2 & q_4 \\ r_1 & r_2 & r_4 \end{vmatrix}, -\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \right). \tag{7}$$

Furthermore, $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are said to form a *minimal μ-basis* of the rational surface (4) if $\mathbf{p}, \mathbf{q}, \mathbf{r}$ have minimal degree. Unlike curves, for surfaces many possible notions of minimal degree exist. One notion that works well for tensor product surfaces is the following.

1. among all triples $\mathbf{p}, \mathbf{q}, \mathbf{r}$ satisfying (6), $\deg_t(\mathbf{p}) + \deg_t(\mathbf{q}) + \deg_t(\mathbf{r})$ is minimal, and
2. among all triples $\mathbf{p}, \mathbf{q}, \mathbf{r}$ satisfying (6) and the previous condition, $\deg_s(\mathbf{p}) + \deg_s(\mathbf{q}) + \deg_s(\mathbf{r})$ is minimal.

Here, $\deg_t(\mathbf{p}) = \max_{1 \leqslant i \leqslant 4}(\deg_t(p_i))$ when $\mathbf{p} = (p_1, p_2, p_3, p_4)$, and $\deg_t(\mathbf{q})$, $\deg_t(\mathbf{r})$, $\deg_s(\mathbf{p})$, $\deg_s(\mathbf{q})$, $\deg_s(\mathbf{r})$ are defined similarly. Sometimes we refer to the three polynomials

$$p = \mathbf{p} \cdot \mathbf{X}, \quad q = \mathbf{q} \cdot \mathbf{X}, \quad r = \mathbf{r} \cdot \mathbf{X}, \qquad \mathbf{X} = (x, y, z, 1),$$

as the μ-basis of the rational surface (4). As observed in [7], a μ-basis forms a basis for the set of all the moving planes following $\mathbf{P}(s,t)$.

3 Approximate μ-Bases of Rational Curves

In this section, we introduce the novel concept of approximate μ-bases for rational curves and present an algorithm to compute them. The applications to approximate implicitization and to degree reduction of rational curves are also discussed.

3.1 Approximate μ-Bases

For the given rational curve $\mathbf{P}(t)$ defined in (1), if a moving line satisfies

$$A(t)a(t) + B(t)b(t) + C(t)c(t) \approx 0, \tag{8}$$

then we call the moving line $A(t)x + B(t)y + C(t) = 0$ an *approximate moving line* of $\mathbf{P}(t)$. Here "$\approx$" means that the left hand side of the equation (8) is approximately zero with respect to some criteria which will be specified later.

An *approximate μ-basis* of the rational curve $\mathbf{P}(t)$ consists of two approximate moving lines $\mathbf{p}(t)$ and $\mathbf{q}(t)$ such that $\mathbf{p}(t) \times \mathbf{q}(t)$ is a good approximation to $\mathbf{P}(t)$. It is obvious that a different choice for the approximation criteria will lead to a different specification for the approximate μ-basis. In the next subsection, we will provide more details of the criteria in order to facilitate the computation of approximate μ-bases.

3.2 Computation

We describe the computation of the first and of the second approximate moving line.

Computing the first line. The moving line is written in Bézier form,

$$\mathbf{p}(t) = \sum_{i=0}^{\mu} \mathbf{p}_i B_i^{\mu}(t) = (p_1(t), p_2(t), p_3(t)),$$

where $\mathbf{p}_i = (p_{i1}, p_{i2}, p_{i3})$, $p_i(t) = \sum_{j=0}^{\mu} p_{ji} B_j^{\mu}(t)$, and $0 < \mu \leqslant \lfloor n/2 \rfloor$. In order to deal with condition (8), we introduce the following optimization problem:

$$\int_0^1 (\mathbf{P}(t) \cdot \mathbf{p}(t))^2 \mathrm{d}t = \int_0^1 (a(t)p_1(t) + b(t)p_2(t) + c(t)p_3(t))^2 \mathrm{d}t \to \min\,. \tag{9}$$

Furthermore, we normalize the approximate moving line by imposing

$$\int_0^1 (p_1(t)^2 + p_2(t)^2) \mathrm{d}t = 1. \tag{10}$$

In order to find the first approximate moving line, we minimize (9) subject to (10). Let

$$a(t)p_1(t) + b(t)p_2(t) + c(t)p_3(t) = \mathbf{g}(t) \cdot \mathbf{x},$$

where $\mathbf{g}(t)$ is a vector of dimension $3(\mu + 1)$ with the components

$$g_i(t) = a(t)B_i^\mu(t), \quad g_{\mu+1+i}(t) = b(t)B_i^\mu(t), \quad g_{2\mu+2+i}(t) = c(t)B_i^\mu(t),$$

$i = 0, 1, \ldots, \mu$, and $\mathbf{x} = (p_{0,1}, \cdots, p_{\mu,1}, p_{0,2}, \cdots, p_{\mu,2}, p_{0,3}, \cdots, p_{\mu,3})$ is a vector which consists of all the unknown coefficients of $\mathbf{p}(t)$. The objective function (9) can be rewritten as

$$\int_0^1 (a(t)p_1(t) + b(t)p_2(t) + c(t)p_3(t))^2 \mathrm{d}t = \int_0^1 \mathbf{x} \cdot \mathbf{g}(t)^T \mathbf{g}(t) \cdot \mathbf{x}^T \mathrm{d}t = \mathbf{x}\mathbf{M}\mathbf{x}^T,$$

where $\mathbf{M}$ is a positive semi-definite $3(\mu+1) \times 3(\mu+1)$ matrix. Similarly, the normalization condition is rewritten as

$$\int_0^1 (p_1(t)^2 + p_2(t)^2) \mathrm{d}t = \mathbf{x}\mathbf{N}\mathbf{x}^T,$$

where $\mathbf{N} = \mathrm{diag}(\mathbf{D}, \mathbf{D}, 0)$ is a positive semi-definite $3(\mu+1) \times 3(\mu+1)$ matrix, and $\mathbf{D} = (d_{ij})$ is a positive definite $\mu+1 \times \mu+1$ matrix. The components of the matrices are

$$m_{ij} = \int_0^1 g_i(t)g_j(t)\mathrm{d}t \quad \text{and} \quad d_{ij} = \int_0^1 B_{i-1}^\mu(t)B_{j-1}^\mu(t)\mathrm{d}t$$

the optimization problem can be rewritten as

$$\mathbf{x}\mathbf{M}\mathbf{x}^T \to \min \quad \text{subject to} \quad \mathbf{x}\mathbf{N}\mathbf{x}^T = 1. \tag{11}$$

If $\det \mathbf{M} = 0$, then there exists $\bar{\mathbf{x}} = (\bar{p}_{0,1}, \cdots, \bar{p}_{\mu,1}, \bar{p}_{0,2}, \cdots, \bar{p}_{\mu,2}, \bar{p}_{0,3}, \cdots, \bar{p}_{\mu,3})$ such that

$$\int_0^1 (a(t)\bar{p}_1(t) + b(t)\bar{p}_2(t) + c(t)\bar{p}_3(t))^2 \mathrm{d}t = \bar{\mathbf{x}}\mathbf{M}\bar{\mathbf{x}}^T = 0.$$

In this case, $a(t)\bar{p}_1(t) + b(t)\bar{p}_2(t) + c(t)\bar{p}_3(t) \equiv 0$, which means $\mathbf{p}(t)$ is an *exact* moving line. Otherwise, if $\det \mathbf{M} \neq 0$, i.e., if $\mathbf{M}$ is positive definite, then there do not exist exact moving lines of degree μ. The solution of (11) then defines an *approximate* moving line.

The problem (11) can be solved by using Lagrangian multipliers. A short computation leads to the equations

$$\det(\mathbf{M} - \lambda\mathbf{N}) = 0, \quad (\mathbf{M} - \lambda\mathbf{N})\mathbf{x}^T = 0, \tag{12}$$

and $\mathbf{x}\mathbf{M}\mathbf{x}^T = \lambda$. Therefore computing an approximate moving line $\mathbf{p}(t)$ is equivalent to solving the generalized eigenvalue problem (12).

The determinant $\det(\mathbf{M} - \lambda\mathbf{N})$ is a polynomial of degree $\gamma = 2(\mu+1)$ in λ. Suppose the zeros of $\det(\mathbf{M} - \lambda\mathbf{N})$ are $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_\gamma$, which are the generalized eigenvalues, and the corresponding eigenvectors are $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_\gamma$. Since $\mathbf{x}\mathbf{M}\mathbf{x}^T = \lambda$, the optimal solution is given by $\mathbf{x} = \mathbf{x}_1$. Thus we get one element $\mathbf{p}(t) = (\mathbf{x}_{11} \cdot \mathbf{t}, \mathbf{x}_{12} \cdot \mathbf{t}, \mathbf{x}_{13} \cdot \mathbf{t})$ of the approximate μ-basis. Here $\mathbf{x}_i = (\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3})$, $\mathbf{x}_{ij}$ is a vector of dimension $\mu+1$, $i = 1, 2, \ldots, \gamma$, $j = 1, 2, 3$, and $\mathbf{t} = (B_0^\mu(t), B_1^\mu(t), \ldots, B_\mu^\mu(t))$.

Computing the second line. An obvious choice for the second element $\mathbf{q}(t)$ of the approximate μ-basis is $\mathbf{q}(t) = (\mathbf{x}_{21} \cdot \mathbf{t}, \mathbf{x}_{22} \cdot \mathbf{t}, \mathbf{x}_{23} \cdot \mathbf{t})$. However, such a choice may have some limitations. First, the degree of $\mathbf{q}(t)$ must be the same as $\mathbf{p}(t)$. Second, it may happen that the curve $\mathbf{p}(t) \times \mathbf{q}(t)$ is not defined at some parameter values in $[0, 1]$, i.e., there exists $t_0 \in [0, 1]$ such that the third component of $\mathbf{p}(t_0) \times \mathbf{q}(t_0)$ is zero. Third, $\mathbf{p}(t) \times \mathbf{q}(t)$ may not be a good approximation of the given curve $\mathbf{P}(t)$. In this section, we develop other techniques to find the second element $\mathbf{q}(t)$ of the approximate μ-basis. We assume that $\deg(\mathbf{q}) = \bar{\mu} \geqslant \mu$.

Let $\mathbf{y}$ be the vector consisting of the coefficients of $\mathbf{q}(t)$. In order to define a reasonable curve from $\mathbf{p} \times \mathbf{q} := \bar{\mathbf{P}}(t) := (\bar{a}, \bar{b}, \bar{c})$, $\mathbf{q}$ must satisfy $p_1(t)q_2(t) - q_2(t)p_1(t) \neq 0$ for all $t \in [0, 1]$. On the other hand, we expect that $\bar{\mathbf{P}}(t)$ is a good approximation of $\mathbf{P}(t)$, i.e., $\bar{a}/a \approx \bar{b}/b \approx \bar{c}/c$. Hence, we minimize

$$\int_0^1 (a\bar{b} - \bar{a}b)^2 + (b\bar{c} - \bar{b}c)^2 + (c\bar{a} - \bar{c}a)^2 \mathrm{d}t = \mathbf{y}\bar{\mathbf{M}}\mathbf{y}^T.$$

Summing up, we need to solve the optimization problem

$$\mathbf{y}\bar{\mathbf{M}}\mathbf{y}^T \to \min \quad \text{subject to} \quad \mathbf{y}\mathbf{N}\mathbf{y}^T = 1 \text{ and } p_1q_2 - p_2q_1 \neq 0, \quad t \in [0, 1]. \tag{13}$$

We write $p_1q_2 - p_2q_1$ in Bernstein-Bézier form. Suppose that its Bézier coefficient vector is $\mathbf{L}\mathbf{y}^T$, where $\mathbf{L}$ is a $(\mu + \bar{\mu} + 1) \times (3\bar{\mu} + 3)$ matrix. Then the constraint $p_1q_2 - p_2q_1 \neq 0$ can be replaced by the sufficient linear conditions $\mathbf{L}\mathbf{y}^T \leqslant -\mathbf{E}$, where $\mathbf{E} = (e_1, e_2, \cdots, e_{3\bar{\mu}+3})^T$, e_i is a small positive number, $i = 1, \ldots, 3(\mu+1)$. Thus instead of solving (13), we will solve

$$\mathbf{y}\bar{\mathbf{M}}\mathbf{y}^T \to \min \quad \text{subject to} \quad \mathbf{y}\mathbf{N}\mathbf{y}^T = 1 \quad \text{and} \quad \mathbf{L}\mathbf{y}^T \leqslant -\mathbf{E}. \tag{14}$$

In order to simplify this problem, we first solve a series of simpler optimization problems,

$$\mathbf{y}\bar{\mathbf{M}}\mathbf{y}^T \to \min \quad \text{subject to} \quad y_i = 1 \quad \text{and} \quad \mathbf{L}\mathbf{y}^T \leqslant -\mathbf{1}, \tag{15}$$

where $i = 1, \ldots, 3\bar{\mu} + 3$, and $\mathbf{1} = (1, 1, \cdots, 1)$ is a column vector of dimension $3\bar{\mu} + 3$. If $\{\mathbf{y} | y_i = 1, \ \mathbf{L}\mathbf{y}^T \leqslant -\mathbf{1}\} \neq \emptyset$, then there exists a solution for the corresponding problem.

Suppose we obtain m solutions $\mathbf{y}_1, \ldots, \mathbf{y}_m$, $m \leqslant 3\bar{\mu} + 3$. Then the coefficient vector of $\mathbf{q}(t)$ is defined as an affine combination

$$\mathbf{y} = \sum_{i=1}^m \alpha_i \mathbf{y}_i, \quad \text{where} \quad \alpha_i \in [0, 1] \quad \text{and} \quad \sum_{i=1}^m \alpha_i = 1.$$

In the following, we propose a technique to determine the optimal coefficients. In order to find them, we maximize the angle between the two moving lines $p(t) = 0$ and $q(t) = 0$. Since the normals of the two lines are $(p_2, -p_1)$ and $(q_2, -q_1)$ respectively, we will minimize $\int_0^1 (p_1q_1 + p_2q_2)^2 \mathrm{d}t$. This leads to the following optimization problem:

$$\alpha\tilde{\mathbf{M}}\alpha^T \to \min \quad \text{subject to} \quad \sum_{i=1}^m \alpha_i = 1 \quad \text{and} \quad 0 \leqslant \alpha_i \leqslant 1, i = 1, \ldots, m. \tag{16}$$

Consequently, in order to find the second moving line, we need solve at most $3\bar{\mu}+4$ quadratic programming problems with linear constraints.

Remark 1. If $\bar{\mu}=\mu$, we can set

$$\mathbf{q}(t)=\alpha_2\mathbf{x}_2+\ldots+\alpha_l\mathbf{x}_l, \tag{17}$$

where $x_2,\ldots,\mathbf{x}_l$ are generalized eigenvectors defined in (12), and α_i, $i=2,\ldots,l$ are the coefficients. Here we choose $l\geqslant 2$ such that $\lambda_l\leqslant 2\sqrt{\lambda_1}$. The coefficients can be computed by solving a quadratic programming problem.

Example 1. Given a rational curve $\mathbf{P}(t)=(a(t),b(t),c(t))$ of degree 12:

$$\begin{aligned}
a(t)&=654t^{12}-5904t^{11}+20592t^{10}-38720t^9+63360t^8-126720t^7+177408t^6\\
&\quad-101376t^5+24576t-4096,\\
b(t)&=-173t^{12}+4752t^{11}-50688t^{10}+264000t^9-760320t^8+1241856t^7\\
&\quad-1005312t^6+760320t^4-675840t^3+270336t^2-49152t+4096,\\
c(t)&=189t^{12}+660t^{11}-14916t^{10}+45760t^9+47520t^8-570240t^7+1478400t^6\\
&\quad-2027520t^5+1647360t^4-788480t^3+202752t^2-24576t+8192.
\end{aligned}$$

Set $\mu=2,\bar{\mu}=3$. With the method presented in the previous sub-subsection, the approximate μ-bases are computed as

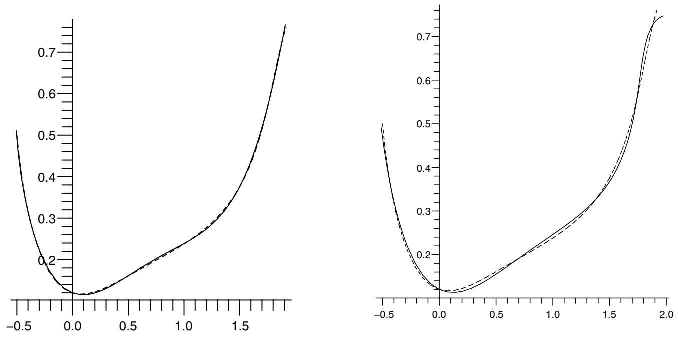
$$\begin{aligned}
\mathbf{p}=(&-0.05102032592B_0^2(t)-0.4807954605B_1^2(t)-1.038288624B_2^2(t),\\
&-0.06065488069B_0^2(t)-0.1692187275B_1^2(t)-1.694077190B_2^2(t),\\
&0.005200267248B_0^2(t)-0.2832738286B_1^2(t)+3.279553214B_2^2(t)),\\
\mathbf{q}=(&-0.04159716413B_0^3(t)+0.9016040373B_1^3(t)+1.379605454B_2^3(t)\\
&-1.727525622B_3^2(t),\ 0.5204348143B_0^3(t)+0.6001839987B_1^3(t)\\
&+1.997824400B_2^3(t)-0.1722884058B_3^3(t),\ -0.2875610577B_0^3(t)\\
&+1.146012829B_1^3(t)-4.681463727B_2^3(t)+3.431341524B_3^3(t)).
\end{aligned}$$

As a comparison, the exact μ-basis computed by the algorithm in [3] consists of two moving lines of degree six, and the coefficients in the μ-basis are integers with approximately forty digits.

3.3 Applications

We present two applications of approximate μ-bases of rational curves to degree reduction and to approximate implicitization, respectively.

Degree reduction. Based on the approximate μ-basis, a degree reduced rational curve $\bar{\mathbf{P}}(t)$ can be obtained directly from the exterior product of $\mathbf{p}(t)$ and $\mathbf{q}(t)$. Assume the error between the original curve and the degree reduced curve is measured by



a. Approximate μ-basis b. Eck's method

Fig. 1. Degree reduction without constraints

$$e(\mathbf{P}, \bar{\mathbf{P}}) := \sqrt{\int_0^1 \left(\left(\frac{a(t)}{c(t)} - \frac{\tilde{a}(t)}{\tilde{c}(t)} \right)^2 + \left(\frac{b(t)}{c(t)} - \frac{\tilde{b}(t)}{\tilde{c}(t)} \right)^2 \right) \mathrm{d}t}.$$

For the curve in Example 1, the approximation error is 0.00332. Figure 1.a illustrates the approximation result, where the original curve is dashed, and the degree reduced curve is solid.

As a comparison, if we use Eck's method [13] to reduce the same degree of the curve in Example 1, the degree reduction error is 0.0114. See Figure 1.b for an illustration.

In some cases, boundary conditions [13] are required. In order to satisfy them, we require that $\mathbf{p}(t)$ respects the conditions

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}(\mathbf{P}(t) \cdot \mathbf{p}(t))|_{t=0} = 0, \quad \frac{\mathrm{d}^i}{\mathrm{d}t^i}(\mathbf{P}(t) \cdot \mathbf{p}(t))|_{t=1} = 0, \quad i = 0, 1, \ldots, k. \tag{18}$$

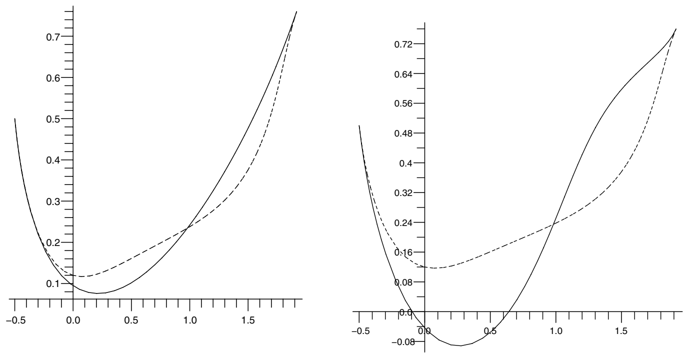
The conditions (18) can be written in matrix form $\mathbf{Q}\mathbf{x}^T = 0$, where $\mathbf{Q}$ is a matrix of order $2(k+1) \times 3(\mu+1)$. Hence $\mathbf{p}(t)$ is the solution of the following optimization problem:

$$\mathbf{x}\mathbf{M}\mathbf{x}^T \to \min \quad \text{subject to} \quad \mathbf{x}\mathbf{N}\mathbf{x}^T = 1 \quad \text{and} \quad \mathbf{Q}\mathbf{x}^T = 0. \tag{19}$$

In order to find $\mathbf{q}(t)$, we add $\mathbf{Q}\mathbf{x}^T = 0$ to (15).

Example 2. We continue the previous example. If we impose C^1 end-points interpolation conditions, and — in order to simplify the computation — set $\bar{\mu} = \mu = 2$, then

$$\begin{aligned}\mathbf{p} = (&0.7771853455(1-t)^2 + 2.355233260t(1-t) + 0.8057279534t^2,\\ &- 0.6976122684(1-t)^2 - 0.7195012836t(1-t) + 0.4112876045t^2,\\ &0.7373988081(1-t)^2 - 2.767665960t(1-t) - 1.856287881t^2),\end{aligned}$$



a. Approximate μ-bases b. Eck's method

Fig. 2. Degree reduction with C^1 constraints

$$\begin{aligned}\mathbf{q} = (&1.524480118(1-t)^2 + 1.879443954t(1-t) - 0.5076826073t^2,\\ &- 0.6343217809(1-t)^2 + 1.998870979t(1-t) + 0.5295267213t^2,\\ &1.079400949(1-t)^2 - 5.200881704t(1-t) + 0.5705104408t^2).\end{aligned}$$

The error is 0.0525. If we use Eck's method to obtain an end-point C^1 interpolation reduction, the approximate error is 0.156. Figure 2.a and Figure 2.b depict the degree-reduced curves.

A more detailed comparison with other techniques for degree reduction may be a subject for further research. Unlike most existing techniques, our method can handle rational curves, and it generates a truly rational curve.

Approximate implicitization. By computing the resultant of $p = \mathbf{p}(t) \cdot (x, y, 1)$ and $q = \mathbf{q}(t) \cdot (x, y, 1)$ with respect to t, we obtain the approximate implicit equation of the original curve. Note that the curve defined by the implicit equation has — at the same time — a rational parameterization.

Example 3. An approximate implicit equation of the curve in Example 1 is

$$\begin{aligned}F(x,y) := {}& 0.3790735866\,x^5 - 0.1877970243\,x^4y - 0.7592764650\,x^3y^2\\ &+ 0.038491110\,x^2y^3 + 0.2978093541\,xy^4 - 0.3360449642\,y^5\\ &- 0.0124850348\,x^4 + 1.716117584\,x^3y + 1.20600026\,x^2y^2\\ &- 0.740052063\,xy^3 + 3.030997755\,y^4 - 1.929933707\,x^3\\ &- 4.54990339\,x^2y + 0.903009110\,xy^2 - 9.592329447\,y^3\\ &+ 3.245601207\,x^2 - 0.76223425\,xy + 13.59527146\,y^2\\ &- 0.40069231\,x - 7.860226927\,y + 0.769956394 = 0.\end{aligned}$$

4 Approximate μ-Bases of Rational Surfaces

We generalize the results for approximate μ-bases of rational curves to rational surfaces. Since the discussions are similar to those for rational curves, we just outline the main results.

4.1 Definition and Computation

Consider a rational parametric surface of bi-degree (m, n) in homogeneous form,

$$\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)) := \sum_{i=0}^{m}\sum_{j=0}^{n} \mathbf{P}_{ij}\omega_{ij} B_i^m(s) B_j^n(t), \quad (20)$$

where $\mathbf{P}_{ij} = (x_{ij}, y_{ij}, z_{ij}, 1)$ and ω_{ij}, $i = 0, 1, \ldots, m$, $j = 0, 1, \ldots, n$ are control points and their corresponding weights respectively. An *approximate moving plane* of $\mathbf{P}(s,t)$ is a moving plane $A(s,t)x + B(s,t)y + C(s,t)z + D(s,t) = 0$ which minimizes

$$\int_0^1\int_0^1 (A(s,t)a(s,t) + B(s,t)b(s,t) + C(s,t)c(s,t) + D(s,t)d(s,t))\,\mathrm{d}s\,\mathrm{d}t \quad (21)$$

subject to the normalization condition

$$\int_0^1\int_0^1 (A(s,t)^2 + B(s,t)^2 + C(s,t)^2 + D(s,t)^2)\,\mathrm{d}s\,\mathrm{d}t = 1. \quad (22)$$

An *approximate μ-basis* of $\mathbf{P}(s,t)$ consists of three approximate moving planes

$$\begin{aligned}\mathbf{p}(s,t) &= (p_1(s,t), p_2(s,t), p_3(s,t), p_4(s,t)),\\ \mathbf{q}(s,t) &= (q_1(s,t), q_2(s,t), q_3(s,t), q_4(s,t)),\\ \mathbf{r}(s,t) &= (r_1(s,t), r_2(s,t), r_3(s,t), r_4(s,t)),\end{aligned}$$

such that $[\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)] \neq 0$ approximates $\mathbf{P}(s,t)$ with respect to some criteria.

We represent the three moving planes in Bernstein-Bézier form,

$$\begin{pmatrix}\mathbf{p}(s,t)\\ \mathbf{q}(s,t)\\ \mathbf{r}(s,t)\end{pmatrix} = \sum_{i=0}^{m_0}\sum_{j=0}^{n_0}\begin{pmatrix}\mathbf{p}_{ij}\\ \mathbf{q}_{ij}\\ \mathbf{r}_{ij}\end{pmatrix} B_i^m(s)\, B_j^n(t) \quad (23)$$

with control points $\mathbf{p}_{ij} = (p_{ij1}, p_{ij2}, p_{ij3}, p_{ij4})$, etc. While each of the three moving planes could have different degrees, we choose all of them to be equal to m_0, n_0.

Similar to the curve case, the approximate moving planes can be obtained by solving the generalized eigenvalue problems

$$\det(\mathbf{M} - \lambda\mathbf{N}) = 0, \quad (\mathbf{M} - \lambda\mathbf{N})\mathbf{x}^T = 0, \quad (24)$$

where both $\mathbf{M}$ and $\mathbf{N}$ are semi-positive definitive matrices of order $4(m_0+1)(n_0+1)$. It follows that $\det(\mathbf{M} - \lambda\mathbf{N})$ is a polynomial of degree $\gamma = 3(m_0+1)(n_0+1)$

in λ. Assume the zeros of $\det(\mathbf{M} - \lambda\mathbf{N})$ are $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_\gamma$, and their corresponding generalized eigenvectors are $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_\gamma$. Then we can take $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ to be coefficients of $\mathbf{p}(s,t)$, $\mathbf{q}(s,t)$, and $\mathbf{r}(s,t)$, respectively. But if we expect $[\mathbf{p}(s,t), \mathbf{q}(s,t), \mathbf{r}(s,t)]$ to represent a rational surface patch over $[0,1]^2$, then for any $(s,t) \in [0,1]^2$,

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \neq 0. \qquad (25)$$

must hold. In order to satisfy this condition, we only select $\mathbf{y}_1$ and $\mathbf{y}_2$ as the coefficients of $\mathbf{p}$ and $\mathbf{q}$ respectively. The coefficient vector $\mathbf{z}$ of the element $\mathbf{r}$ is set to be the linear combination of $\mathbf{y}_3, \ldots, \mathbf{y}_l$ for some $l < \gamma$: $\mathbf{z} = \sum_{i=3}^{l} \alpha_i \mathbf{y}_i$. The coefficients will be determined by requiring (25) holds and the angles between $\mathbf{r}$ and $\mathbf{p}$ (and $\mathbf{q}$) are not too small. Then $\mathbf{r}$ is the solution of the following problem

$$\mathbf{z}\bar{\mathbf{M}}\mathbf{z}^T \to \min \quad \text{subject to} \quad \mathbf{z}\mathbf{N}\mathbf{z}^T = 1 \quad \text{and} \quad \mathbf{L}\mathbf{z}^T \leqslant -\mathbf{E}. \qquad (26)$$

where $\mathbf{L}$ is a matrix of size $(3m_0+1)(3n_0+1) \times 4(m_0+1)(n_0+1)$. This problem can be solved in a similar way as in the curve case.

4.2 Examples and Applications

We provides two examples to illustrate some applications of approximate μ-bases of rational surfaces — approximate implicitization and degree reduction.

Example 4. Given a bicubic surface defined by:

$$\begin{aligned}
a(s,t) = {} & \frac{1}{4}(3s^3t^3 - 6s^2t^3 + 3st^3 - 9s^3t^2 + 18s^2t^2 - 9st^2 \\
& + 9s^3t - 18s^2t + 9st - 3s^3 + 6s^2 + 9s), \\
b(s,t) = {} & -3s^3t^3 + 3s^3t^2 + 3s^2t^3 - 3s^2t^2 + 3t, \\
c(s,t) = {} & \frac{1}{2}(3s^3t^3 - 6s^3t^2 - 6s^2t^3 + 9s^2t^2 + 3st^3 + 3s^3 + 2t^3 - 6s^2 - 6t^2 + 4), \\
d(s,t) = {} & \frac{1}{5}(-s^3t^3 + 3s^3t^2 + 3s^2t^3 - 3s^3t - 9s^2t^2 - 3st^3 + s^3 + 9s^2t + 9st^2 \\
& + t^3 - 3s^2 - 9st - 3t^2 + 3s + 3t + 4).
\end{aligned}$$

A linear approximate μ-basis can be computed as

$$\begin{aligned}
\mathbf{p} = (& -0.3179763603 + 0.01129769996s - 0.005706462242t, \\
& -0.05643886419 - 0.02026545412s + 0.00006577262458t, \\
& 0.03369450553 - 0.01391833112s - 0.001238259243t, \\
& -0.08609427199 + 1.0s + 0.2529721651t), \\
\mathbf{q} = (& -0.08463698360 - 0.03287295861s + 0.00661036891t, \\
& 0.4542005463 + 0.01115597009s - 0.02382379534t, \\
& -0.03488009368 + 0.00195597407s - 0.00519198472t, \\
& 0.04412801971 + 0.3068345570s - 1.340134559t),
\end{aligned}$$

$$\begin{aligned}\mathbf{r} = (&-10.22862360 + 1.851570346s + 17.56377500t,\\ &15.86139849 - 21.35392833s + 0.2563277660t,\\ &5.474325695 + 2.113435284s - 0.650356134t,\\ &-14.90107308 + 36.65715509s - 33.69498870t).\end{aligned}$$

A cubic rational parametric surface can be obtained from $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$. The approximation error between the original surface and the new surface is 0.0264.

By eliminating s, t from $\mathbf{p} \cdot (x, y, z, 1) = \mathbf{q} \cdot (x, y, z, 1) = \mathbf{r} \cdot (x, y, z, 1) = 0$, one obtains an approximate cubic implicit equation for the given surface,

$$\begin{aligned}F(x, y, z) := {}& 0.2063266181\, x^3 - 0.2187304282\, x^2y - 0.04778543831\, x^2z\\ &+ 0.3449682259\, xy^2 + 0.1115861554\, xzy - 0.01039875210\, xz^2\\ &+ 0.03791855132\, y^3 + 0.004111063445\, zy^2 - 0.005055353827\, z^2y\\ &+ 0.001142730024\, z^3 - 1.317908681\, x^2 + 1.286241307\, xy - 0.1681655\, xz\\ &- 1.252094004\, y^2 - 0.3605440520\, zy + 0.1401044281\, z^2 - 2.990107224\, x\\ &- 3.033340596\, y - 8.359734692\, z + 19.68286052 = 0.\end{aligned}$$

Note that the exact implicit degree of the surface is 18.

Example 5. We consider a given surface of bi-degree $(5, 5)$,

$$\begin{aligned}a = {}& -25/2\, s^5t^5 + 50\, s^5t^4 + 50\, s^4t^5 - 75\, s^5t^3 - 200\, s^4t^4 - 75\, s^3t^5 + 50\, s^5t^2\\ &+ 300\, s^4t^3 + 300\, s^3t^4 + 50\, s^2t^5 - 25/2\, s^5t - 200\, s^4t^2 - 450\, s^3t^3 - 200\, s^2t^4\\ &- 25/2\, st^5 + 50\, s^4t + 300\, s^3t^2 + 300\, s^2t^3 + 50\, st^4 - 75\, s^3t - 200\, s^2t^2 - 75\, st^3\\ &+ 50\, s^2t + 50\, st^2 - 25/2\, st + 5\, s,\\ b = {}& 25/2\, s^5t^5 - 50\, s^5t^4 - 50\, s^4t^5 + 75\, s^5t^3 + 200\, s^4t^4 + 75\, s^3t^5 - 50\, s^5t^2 - 300\, s^4t^3\\ &- 300\, s^3t^4 - 50\, s^2t^5 + 25/2\, s^5t + 200\, s^4t^2 + 450\, s^3t^3 + 200\, s^2t^4 + 25/2\, st^5\\ &- 50\, s^4t - 300\, s^3t^2 - 300\, s^2t^3 - 50\, st^4 + 75\, s^3t + 200\, s^2t^2 + 75\, st^3\\ &- 50\, s^2t - 50\, st^2 + \frac{25}{2}\, st + 5\, t,\\ c = {}& 50\, s^5t^5 - 100\, s^5t^4 - 150\, s^4t^5 + 50\, s^5t^3 + 300\, s^4t^4 + 150\, s^3t^5 - 150\, s^4t^3\\ &- 300\, s^3t^4 - 50\, s^2t^5 + 150\, s^3t^3 + 100\, s^2t^4 - 50\, s^2t^3 - 5\, s^4 - 5\, t^4 + 10\, s^3\\ &+ 10\, t^3 - 10\, s^2 - 10\, t^2 + 5\, s + 5\, t,\\ d = {}& -1/6\, s^5t^5 + 5/6\, s^5t^4 + 5/6\, s^4t^5 - 5/3\, s^5t^3 - 25/6\, s^4t^4 - 5/3\, s^3t^5 + 5/3\, s^5t^2\\ &+ 25/3\, s^4t^3 + 25/3\, s^3t^4 + 5/3\, s^2t^5 - 5/6\, s^5t - 25/3\, s^4t^2 - 50/3\, s^3t^3\\ &- 25/3\, s^2t^4 - 5/6\, st^5 + 1/6\, s^5 + 25/6\, s^4t + 50/3\, s^3t^2 + 50/3\, s^2t^3 + 25/6\, st^4\\ &+ 1/6\, t^5 - 5/6\, s^4 - 25/3\, s^3t - 50/3\, s^2t^2 - 25/3\, st^3 - 5/6\, t^4 + 5/3\, s^3\\ &+ 25/3\, s^2t + 25/3\, st^2 + 5/3\, t^3 - 5/3\, s^2 - 25/6\, st - 5/3\, t^2 + 5/6\, s + 5/6\, t + 5/6.\end{aligned}$$

An approximate μ-basis of bi-degree (1,1) is computed as

$$\begin{aligned}\mathbf{p} = (&0.2085266555 - 0.007372293503\, s - 0.1428724898\, t + 0.005537995913\, st,\\ &0.1808522216 + 0.2050867177\, s - 0.009074759098\, t + 0.007914352425\, st,\end{aligned}$$

$$
\begin{aligned}
& -0.01878293690+0.01746191813\, s+0.01597674812\, t-0.01869048639\, st, \\
& -0.006408507364-1.0\, s-0.8531656420\, t-0.3816705478\, st), \\
\mathbf{q}= & (-0.08053392989-0.05754492900\, s+0.005132654152\, t+0.006652036180\, st, \\
& 0.08191584666-0.01108549693\, s-0.05549170597\, t+0.003943449557\, st, \\
& -0.1025013651+0.03199060344\, s+0.02845654330\, t-0.03187909295\, st, \\
& 0.02672843678+0.6709819071\, s-0.1475467029\, t-0.002951798803\, st), \\
\mathbf{r}= & (0.1162420910-0.2753649567\, s+0.09208042082\, t-0.07226311792\, st, \\
& -0.02104547875+0.2915682775\, s-0.3705920517\, t+0.1046918855\, st, \\
& -0.4890160153+0.1394880492\, s-0.1022402645\, t+0.03867595945\, st, \\
& 0.1307117232+0.6920411693\, s+1.964792505\, t-2.063871335\, st).
\end{aligned}
$$

The bicubic parametric surface $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ can serve as a degree-reduced surface. The approximation error is 0.0436. An approximate implicit equation of degree six can also be obtained by eliminating s, t from p, q, r. Note that the exact algebraic degree of the surface is 50.

5 Conclusion and Future Work

In this paper, approximate μ-bases of rational curves and surfaces are studied. Algorithms are provided to compute the approximate μ-bases, which amount to solve generalized eigenvalue problems and some quadratic programming problems. Applications of approximate μ-bases in degree reduction and approximate implicitization are explored. Examples seem to suggest that the techniques presented in this paper are competitive with other known methods, but this should be studied further.

In order to compute the approximate μ-bases, quadratic programming problems have to be solved. In the future, we will discuss how to define and compute approximate μ-bases in a more general and efficient approach. Other applications of approximate μ-bases will be explored as well.

Acknowledgments

The authors have been supported by the Outstanding Youth Grant of NSF of China (No.60225002), NSF of China (No. 60533060 and 60473132), a National Key Basic Research Project of China (No. 2004CB318000), and by the Special Research Programme SFB F013 “Numerical and Symbolic Scientific Computing” at Linz, Austria, which has been established by the Austrian Science Fund (FWF).

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