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Estimating error bounds for binary subdivision curves/surfaces

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Abstract

We estimate error bounds between binary subdivision curves/surfaces and their control polygons after k-fold subdivision in terms of the maximal differences of the initial control point sequences and constants that depend on the subdivision mask. The bound is independent of the process of subdivision and can be evaluated without recursive subdivision. Our technique is independent of parameterizations therefore it can be easily and efficiently implemented. This is useful and important for pre-computing the error bounds of subdivision curves/surfaces in advance in many engineering applications such as curve/surface intersection, mesh generation, NC machining, surface rendering and so on.

Keywords: Subdivision curve; Subdivision surface; Subdivision depth; Error bound

1. Introduction

Subdivision is an important method for generating smooth curves and surfaces. Efficiency of subdivision algorithms, their flexibility and simplicity have found their way into wide applications in Computer Graphics and Computer Aided Geometric Design (CAGD). A widely used, efficient and intuitive way to specify, represent and reason about curved, surfaces, nonlinear geometry for design and modeling is the control polygon paradigm. For many applications, e.g., rendering, intersection testing or design, this raises the question just how well the control polygon approximates the exact curved and surface geometry. Several researchers give several answers to this question. Nairn et al. [8] show that the maximal distance between a Bézier segment and its control polygon is bounded in terms of the differences of the control

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point sequence and a constant that depends only on the degree of the polynomial. Lutterkort et al. [7] derived a sharp bound on the distance between a spline and its B-spline control polygon. Their bound yields a piecewise linear envelope enclosing the spline and the control polygon. Recently, Karavelas et al. [6] derived sharp bounds for the distance between a planar parametric Bézier curve and parameterizations of its control polygon based on the Greville abscissae. Cheng [2] gave an algorithm to estimate subdivision depths for rational curves and surfaces. The subdivision depth is not estimated for the given curve/surface directly. Their algorithm computes a subdivision depth for the polynomial curve/surface of which the given rational curve/surface is the image under the standard perspective projection. The existing methods for computing the bounds on the approximation of polynomials and splines by their control structures are all based on the parameterizations, so that it is very difficult for them to be generalized to the subdivision surfaces.

In this paper, we estimate error bounds for binary subdivision curves/surfaces in terms of the maximal differences of the initial control point sequence and constants that depend on the subdivision mask. Our technique is independent of parameterizations and therefore it can be easily and efficiently implemented. The paper is organized as follows.

In Section 2 we prove the first main result of the paper about the estimation of error bounds between binary subdivision curves and their control polygon after k-fold subdivision. Then as an application of our result we find error bounds for 4-point interpolatory [4], 6-point interpolatory [9], cubic B-spline [5] and Chaikin's [1] subdivision schemes. In Section 3 we generalize the main result of Section 2 to estimate the error bounds between subdivision surfaces and their control polygons. We end this section by estimating the error bounds for tensor product form of Chaikin's, cubic B-spline, 4-point interpolatory and 6-point interpolatory subdivision schemes. In Section 4 we summarize the results for future research directions.

2. The error bounds for subdivision curves

Let $p_i^k \in \mathbb{R}^N,\ i \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N,\ N \geqslant 2$, where k is a nonnegative integer. A binary subdivision process [3] is defined by

$$\begin{cases} p_{2i}^{k+1} = \sum_{j=0}^{m} a_j p_{i+j}^k, \\ p_{2i+1}^{k+1} = \sum_{j=0}^{m} b_j p_{i+j}^k. \end{cases} \tag{1}$$

Here $m > 0$ and nondegeneracy in the summations is that

$$|a_0| + |b_0| > 0 \quad \text{and} \quad |a_m| + |b_m| > 0.$$

The coefficients $\{a_j\}_{j=0}^m$ and $\{b_j\}_{j=0}^m$ are called subdivision mask. Given initial values $p_i^0 \in \mathbb{R}^N,\ i \in \mathbb{Z}$, then in the limit $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^N$. The sequence of control points $\{p_i^k\}$ is related, in a natural way, with the diadic mesh points $t_i^k = i/2^k,\ i \in \mathbb{Z}$. The process (1) then defines a scheme whereby p_{2i}^{k+1} replaces the value p_i^k at the mesh point $t_{2i}^{k+1} = t_i^k$ and p_{2i+1}^{k+1} is inserted at the new mesh point $t_{2i+1}^{k+1} = (t_i^k + t_{i+1}^k)/2$. A necessary condition for the uniform convergence of the subdivision process (1) on the diadic points for arbitrary initial data, is that

$$\sum_{j=0}^{m} a_j = \sum_{j=0}^{m} b_j = 1. \tag{2}$$

We now establish our first main result to find error bounds between subdivision curves and their control polygons.

Theorem 1. *Given initial control polygon $p_i^0 = p_i$, $i \in \mathbb{Z}$, let the values p_i^k, $k \geqslant 0$ be defined recursively by subdivision process* (1) *together with necessary condition* (2). *Suppose P^k be the piecewise linear interpolation to the values p_i^k and P^∞ be the limit curve of the process* (1). *If*

$$\sum_{j=0}^{m} |c_j| < 1 \quad \text{and} \quad \sum_{j=0}^{m} |d_j| < 1, \tag{3}$$

where

$$c_j = \sum_{i=0}^{j} (a_i - b_i) \quad \text{and} \quad d_j = a_j - c_j,$$

then error bounds between limit curve and its control polygon after k-fold subdivision are

$$\| P^k - P^\infty \|_\infty \leqslant \gamma\beta \left(\frac{\delta^k}{1-\delta} \right), \tag{4}$$

where

$$\gamma = \max\left\{ \sum_{j=0}^{m-1} |\tilde{a_j}|, \sum_{j=0}^{m-1} |\tilde{b_j}| \right\}, \quad \tilde{a_j} = \sum_{i=j+1}^{m} a_i, \quad \tilde{b_j} = \sum_{i=j+1}^{m} b_i, \quad j \geqslant 1, \quad \tilde{b_0} = \sum_{i=1}^{m} b_i - \frac{1}{2},$$

$$\beta = \max_i \| p_{i+1}^0 - p_i^0 \| \quad \text{and} \quad \delta = \max\left\{ \sum_{j=0}^{m} |c_j|, \sum_{j=0}^{m} |d_j| \right\}.$$

Proof. Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$th mesh, then

$$\| P^{k+1} - P^k \|_\infty \leqslant \max\{\aleph_k^1, \aleph_k^2\}, \tag{5}$$

where

$$\begin{cases} \aleph_k^1 = \max_i \| p_{2i}^{k+1} - p_i^k \|, \\ \aleph_k^2 = \max_i \| p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) \|. \end{cases} \tag{6}$$

From (1) and (2) we obtain

$$p_{2i}^{k+1} - p_i^k = \sum_{j=0}^{m-1} \tilde{a_j} (p_{i+j+1}^k - p_{i+j}^k), \tag{7}$$

where

$$\tilde{a_j} = \sum_{i=j+1}^{m} a_i$$

and

$$p_{2i+1}^{k+1} - \frac{1}{2}(p_i^k + p_{i+1}^k) = \sum_{j=0}^{m-1} \tilde{b_j}(p_{i+j+1}^k - p_{i+j}^k), \tag{8}$$

where

$$\tilde{b_j} = \sum_{i=j+1}^{m} b_i, \quad j \geqslant 1, \qquad \tilde{b_0} = \sum_{i=1}^{m} b_i - \frac{1}{2}.$$

From (1) we have

$$p_{2i+1}^k - p_{2i}^k = \sum_{j=0}^{m} (b_j - a_j) p_{i+j}^{k-1},$$

$$p_{2i+2}^k - p_{2i+1}^k = -b_0 p_i^{k-1} + \sum_{j=1}^{m} (a_{j-1} - b_j) p_{i+j}^{k-1} + a_m p_{i+m+1}^{k-1}.$$

By using (2) and induction on m we can get

$$p_{2i+1}^k - p_{2i}^k = \sum_{j=0}^{m} c_j (p_{i+j+1}^{k-1} - p_{i+j}^{k-1}), \tag{9}$$

where

$$c_j = \sum_{i=0}^{j} (a_i - b_i)$$

and

$$p_{2i+2}^k - p_{2i+1}^k = \sum_{j=0}^{m} d_j (p_{i+j+1}^{k-1} - p_{i+j}^{k-1}), \tag{10}$$

where

$$d_j = \sum_{i=0}^{j} (b_i - a_i) + a_j = a_j - c_j.$$

From (6) to (8) we have

$$\aleph_k^1 \leqslant \left(\sum_{j=0}^{m-1} |\tilde{a_j}| \right) \max_i \| p_{i+1}^k - p_i^k \|, \tag{11}$$

$$\aleph_k^2 \leqslant \left(\sum_{j=0}^{m-1} |\tilde{b_j}| \right) \max_i \; \| p_{i+1}^k - p_i^k \|. \tag{12}$$

Using (9) and (10) recursively gives

$$\max_i \| p_{i+1}^k - p_i^k \| \leqslant \left(\sum_{j=0}^{m} |c_j| \right)^k \max_i \; \| p_{i+1}^0 - p_i^0 \|, \tag{13}$$

$$\max_i \| p_{i+1}^k - p_i^k \| \leqslant \left(\sum_{j=0}^{m} |d_j| \right)^k \max_i \| p_{i+1}^0 - p_i^0 \|. \tag{14}$$

If

$$\delta = \max \left\{ \sum_{j=0}^{m} |c_j|, \sum_{j=0}^{m} |d_j| \right\}$$

then from (11) to (14) we have

$$\aleph_k^1 \leqslant \delta^k \left(\sum_{j=0}^{m-1} |\tilde{a_j}| \right) \max_i \| p_{i+1}^0 - p_i^0 \|, \tag{15}$$

$$\aleph_k^2 \leqslant \delta^k \left(\sum_{j=0}^{m-1} |\tilde{b_j}| \right) \max_i \; \| p_{i+1}^0 - p_i^0 \|. \tag{16}$$

If

$$\gamma = \max \left\{ \sum_{j=0}^{m-1} |\tilde{a_j}|, \sum_{j=0}^{m-1} |\tilde{b_j}| \right\} \quad \text{and} \quad \beta = \max_i \| p_{i+1}^0 - p_i^0 \|$$

then from (5), (15) and (16)

$$\| P^{k+1} - P^k \|_\infty \leqslant \gamma \beta \delta^k. \tag{17}$$

Using triangle inequality we get

$$\| P^k - P^\infty \|_\infty \leqslant \gamma \beta \left(\frac{\delta^k}{1-\delta} \right).$$

This completes the proof. □

Remark 2. Here we point out that the famous binary subdivision schemes satisfy condition (3). Our claim is supported by the following corollaries.

Corollary 3. *Given* $p_i^0 = p_i$, $i \in \mathbb{Z}$, *let the values* p_i^k, $k \geqslant 0$ *be defined recursively by* 4*-point interpolatory binary subdivision process* [4]. *Suppose* P^k *be the piecewise linear interpolation to the values* p_i^k *and* P^∞ *be the limit curve of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \gamma\beta\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\gamma = \max\left\{1, \frac{1}{2} + |w| + \left|\frac{1}{2} + w\right|\right\}, \quad \delta = \frac{1}{2} + 2|w| \quad \textit{and} \quad \beta = \max_i \|p_{i+1}^0 - p_i^0\|.$$

Proof. A 4-point interpolatory binary subdivision scheme have following subdivision mask

$$(a_0, a_1, a_2, a_3) = (0, 1, 0, 0),$$

$$(b_0, b_1, b_2, b_3) = (-w, \tfrac{1}{2} + w, \tfrac{1}{2} + w, -w).$$

The range of w that guarantees a C^1 continuous limit curve is $0 < w < (-1 + \sqrt{5})/8$. Since above subdivision mask satisfy (3), for this range of w, then by Theorem 1 we have the result. The maximum range of w for which condition (3) satisfied is approximately $-0.2499 \leqslant w \leqslant 0.2499$. □

Corollary 4. *Given* $p_i^0 = p_i$, $i \in \mathbb{Z}$, *let the values* p_i^k, $k \geqslant 0$ *be defined recursively by* 6*-point interpolatory binary subdivision process* [9]. *Suppose* P^k *be the piecewise linear interpolation to the values* p_i^k *and* P^∞ *be the limit curve of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \gamma\beta\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\gamma = \max\{2, \tfrac{1}{2} + |\theta| + |\tfrac{1}{2} - \theta| + |\tfrac{17}{16} + 2\theta| + |\tfrac{1}{16} + 2\theta|\}, \quad \delta = \tfrac{1}{2} + 2|\theta| + 2|\tfrac{1}{16} + 2\theta|$$

and

$$\beta = \max_i \|p_{i+1}^0 - p_i^0\|.$$

Proof. A 6-point interpolatory binary subdivision scheme have following subdivision mask

$$(a_0, a_1, a_2, a_3, a_4, a_5) = (0, 0, 1, 0, 0, 0),$$

$$(b_0, b_1, b_2, b_3, b_4, b_5) = (\theta, -(\tfrac{1}{16} + 3\theta), (\tfrac{9}{16} + 2\theta), (\tfrac{9}{16} + 2\theta), -(\tfrac{1}{16} + 3\theta), \theta).$$

The range of θ that guarantees a continuous curvature of the limit curve is $0 < \theta < 0.02$. Since above subdivision mask satisfy (3), for this range of θ, then by Theorem 1 we have the result. The maximum range of θ for which condition (3) satisfied is approximately $-0.1041 \leqslant \theta \leqslant 0.0624$. □

Corollary 5. *Given* $p_i^0 = p_i,\ i \in \mathbb{Z}$, *let the values* $p_i^k,\ k \geqslant 0$ *be defined recursively by cubic b-spline subdivision process* [5]. *Suppose* P^k *be the piecewise linear interpolation to the values* p_i^k *and* P^∞ *be the limit curve of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant 2\left(\frac{1}{2}\right)^k \beta,$$

where

$$\beta = \max_i \|p_{i+1}^0 - p_i^0\|.$$

Corollary 6. *Given* $p_i^0 = p_i,\ i \in \mathbb{Z}$, *let the values* $p_i^k,\ k \geqslant 0$ *be defined recursively by Chaikin's subdivision process* [1]. *Suppose* P^k *be the piecewise linear interpolation to the values* p_i^k *and* P^∞ *be the limit curve of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant (\tfrac{1}{2})^{k+1} \beta,$$

where

$$\beta = \max_i \|p_{i+1}^0 - p_i^0\|.$$

3. The error bounds for subdivision surfaces

In this section, first we define basic concepts and settle some notations required for fair reading and better understanding. Then we will present our main result to estimate error bounds for binary subdivision surfaces. We end this section by estimating error bounds for tensor product form of Chaikin's, cubic B-spline, 4-point interpolatory and 6-point interpolatory subdivision schemes.

Definition. Let $p_{i,j}^k \in \mathbb{R}^N,\ i \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N,\ N \geqslant 2$, where k is a nonnegative integer. A tensor product of binary subdivision process (1) is defined by

$$\begin{cases} p_{2i,2j}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_r a_s p_{i+r,j+s}^k, \\ p_{2i,2j+1}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} a_r b_s p_{i+r,j+s}^k, \\ p_{2i+1,2j}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} b_r a_s p_{i+r,j+s}^k, \\ p_{2i+1,2j+1}^{k+1} = \sum_{r=0}^{m}\sum_{s=0}^{m} b_r b_s p_{i+r,j+s}^k, \end{cases} \tag{18}$$

where m is greater than zero. The coefficients $\{a_j\}_{j=0}^m$ and $\{b_j\}_{j=0}^m$ are called subdivision mask. Given initial values $p_{i,j}^0 \in \mathbb{R}^N,\ i,j \in \mathbb{Z}$, then in the limit $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^N$. The sequence of control points $\{p_{i,j}^k\}$ is related, in a natural way, with the diadic mesh points $(i/2^k, j/2^k),\ i,j \in \mathbb{Z}$. The process (18) then defines a scheme whereby $p_{2i,2j}^{k+1}$, $p_{2i+2,2j}^{k+1}$, $p_{2i,2j+2}^{k+1}$

and $p^{k+1}_{2i+2,2j+2}$ replaces the values $p^k_{i,j}$, $p^k_{i+1,j}$, $p^k_{i,j+1}$ and $p^k_{i+1,j+1}$ at the mesh points $(i/2^k, j/2^k)$, $((i+1)/2^k, j/2^k)$, $(i/2^k, (j+1)/2^k)$ and $((i+1)/2^k, (j+1)/2^k)$ respectively. The control points $p^{k+1}_{2i+1,2j}$, $p^{k+1}_{2i,2j+1}$, $p^{k+1}_{2i+1,2j+1}$, $p^{k+1}_{2i+2,2j+1}$ and $p^{k+1}_{2i+1,2j+2}$ are inserted at the new mesh points $((i+1)/(2^{k+1}), j/(2^{k+1}))$, $(i/(2^{k+1}), (j+1)/(2^{k+1}))$, $((i+1)/(2^{k+1}), (j+1)/(2^{k+1}))$, $((i+2)/(2^{k+1}), (j+1)/(2^{k+1}))$ and $((i+1)/(2^{k+1}), (j+2)/(2^{k+1}))$ respectively. A necessary condition for the convergence of the subdivision process (18) for arbitrary initial data, is that

$$\sum_{j=0}^{m} a_j = \sum_{j=0}^{m} b_j = 1. \tag{19}$$

Notations. Here, we settle some basic notations required for fair reading and better understanding. In our analysis we suppose

$$\delta = \max\left\{ \sum_{r=0}^{m}|a_r|\sum_{s=0}^{m}|c_s|, \sum_{r=0}^{m}|a_r|\sum_{s=0}^{m}|d_s|, \sum_{r=0}^{m}|b_r|\sum_{s=0}^{m}|c_s|, \sum_{r=0}^{m}|b_r|\sum_{s=0}^{m}|d_s| \right\} < 1, \tag{20}$$

where

$$c_s = \sum_{t=0}^{s}(a_t - b_t) \quad \text{and} \quad d_s = a_s - c_s,$$

$$\eta_1 = |a_0|\left(\sum_{t=1}^{m}|a_t| + \sum_{s=1}^{m-1}|\tilde{a}_s|\right), \quad \eta_2 = |a_0|\left(\sum_{t=1}^{m}|b_t| + \sum_{s=1}^{m-1}|\tilde{b}_s|\right) + \frac{1}{2},$$

$$\eta_3 = |b_0|\left(\sum_{t=1}^{m}|a_t| + \sum_{s=1}^{m-1}|\tilde{a}_s|\right), \quad \eta_4 = |b_0|\left(\sum_{t=1}^{m}|b_t| + \sum_{s=1}^{m-1}|\tilde{b}_s|\right) + \frac{1}{4}, \tag{21}$$

where

$$\tilde{a_s} = \sum_{t=s+1}^{m} a_t \quad \text{and} \quad \tilde{b_s} = \sum_{t=s+1}^{m} b_t,$$

$$\tau_1 = \sum_{t=1}^{m}|a_t| + \sum_{r=0}^{m}|a_r|\sum_{s=1}^{m-1}|\tilde{a}_s|, \quad \tau_2 = \sum_{t=1}^{m}|a_t| + \sum_{r=0}^{m}|b_r|\sum_{s=1}^{m-1}|\tilde{a}_s|,$$

$$\tau_3 = \sum_{t=1}^{m}|b_t| + \sum_{r=0}^{m}|a_r|\sum_{s=1}^{m-1}|\tilde{b}_s| + \frac{1}{2}, \quad \tau_4 = \sum_{t=1}^{m}|b_t| + \sum_{r=0}^{m}|b_r|\sum_{s=1}^{m-1}|\tilde{b}_s| + \frac{1}{2}, \tag{22}$$

and

$$\xi_1 = \sum_{t=1}^{m}|a_t|\left(\sum_{t=1}^{m}|a_t| + \sum_{s=1}^{m-1}|\tilde{a}_s|\right), \quad \xi_2 = \sum_{t=1}^{m}|a_t|\left(\sum_{t=1}^{m}|b_t| + \sum_{s=1}^{m-1}|\tilde{b}_s|\right),$$

$$\xi_3 = \sum_{t=1}^{m} |b_t| \left(\sum_{t=1}^{m} |a_t| + \sum_{s=1}^{m-1} |\tilde{a_s}| \right), \quad \xi_4 = \sum_{t=1}^{m} |b_t| \left(\sum_{t=1}^{m} |b_t| + \sum_{s=1}^{m-1} |\tilde{b_s}| \right) + \frac{1}{4}. \tag{23}$$

We introduce forward difference operators $\{\Delta^k_{i,j,t}\}$, $t = 1, 2, 3$, along the mesh directions defined as

$$\begin{cases} \Delta^k_{i,j,1} = p^k_{i+1,j} - p^k_{i,j}, \\ \Delta^k_{i,j,2} = p^k_{i,j+1} - p^k_{i,j}, \\ \Delta^k_{i,j,3} = p^k_{i+1,j+1} - p^k_{i,j+1}. \end{cases} \tag{24}$$

We now present our main result to estimate error bounds for subdivision surfaces.

Theorem 7. *Given initial control polygon $p^0_{i,j} = p_{i,j}$, $i, j \in \mathbb{Z}$, let the values $p^k_{i,j}$, $k \geqslant 0$ be defined recursively by subdivision process* (18) *together with* (19). *Suppose P^k be the piecewise linear interpolation to the values $p^k_{i,j}$ and P^∞ be the limit surface of the subdivision process* (18). *If* (20) *hold then error bounds between limit surface and its control polygon after k-fold subdivision is*

$$\|P^k - P^\infty\|_\infty \leqslant \{\eta\beta_1 + \tau\beta_2 + \xi\beta_3\} \left(\frac{\delta^k}{1-\delta} \right),$$

where

$$\eta = \max\{\eta_1, \eta_2, \eta_3, \eta_4\}, \quad \tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\},$$

$$\xi = \max\{\xi_1, \xi_2, \xi_3, \xi_4\}, \quad \eta_t, \tau_t, \xi_t, \ t = 1, \ldots, 4,$$

are defined in (21)–(23),

$$\beta_t = \max_{i,j} \|\Delta^0_{i,j,t}\|, \quad \{\Delta^0_{i,j,t}\}, \ t = 1, 2, 3,$$

are defined in (24).

Proof. Let $\|.\|_\infty$ denote the uniform norm. Since the maximum difference between P^{k+1} and P^k is attained at a point on the $(k+1)$th mesh, then

$$\|P^{k+1} - P^k\|_\infty \leqslant \max\{M^1_k, M^2_k, M^3_k, M^4_k\}, \tag{25}$$

where

$$\begin{cases} M^1_k = \max\limits_{i,j} \|p^{k+1}_{2i,2j} - p^k_{i,j}\|, \\ M^2_k = \max\limits_{i,j} \left\| p^{k+1}_{2i+1,2j} - \frac{1}{2}(p^k_{i,j} + p^k_{i+1,j}) \right\|, \\ M^3_k = \max\limits_{i,j} \left\| p^{k+1}_{2i,2j+1} - \frac{1}{2}(p^k_{i,j} + p^k_{i,j+1}) \right\|, \\ M^4_k = \max\limits_{i,j} \left\| p^{k+1}_{2i+1,2j+1} - \frac{1}{4}(p^k_{i,j} + p^k_{i+1,j} + p^k_{i,j+1} + p^k_{i+1,j+1}) \right\|. \end{cases} \tag{26}$$

From (18) and (19) we get

$$p^{k+1}_{2i,2j} - p^k_{i,j} = \sum_{r=0}^{m} a_r \left(\sum_{s=0}^{m} a_s (p^k_{i+r,j+s} - p^k_{i,j}) \right). \tag{27}$$

Since

$$\begin{aligned}
&\sum_{s=0}^{m} a_s (p^k_{i+r,j+s} - p^k_{i,j}) \\
&\quad = a_0(p^k_{i+r,j} - p^k_{i,j}) + a_1(p^k_{i+r,j+1} - p^k_{i,j}) \\
&\qquad + a_2(p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) \\
&\qquad + a_3(p^k_{i+r,j+3} - p^k_{i+r,j+2} + p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}) + \dots \\
&\qquad + a_m(p^k_{i+r,j+m} - \dots + p^k_{i+r,j+3} - p^k_{i+r,j+2} + p^k_{i+r,j+2} - p^k_{i+r,j+1} + p^k_{i+r,j+1} - p^k_{i,j}).
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{s=0}^{m} a_s (p^k_{i+r,j+s} - p^k_{i,j}) = a_0(p^k_{i+r,j} - p^k_{i,j}) &+ \sum_{t=1}^{m} a_t (p^k_{i+r,j+1} - p^k_{i,j}) \\
&+ \sum_{s=1}^{m-1} \tilde{a}_s (p^k_{i+r,j+s+1} - p^k_{i+r,j+s}),
\end{aligned}$$

where

$$\tilde{a_s} = \sum_{t=s+1}^{m} a_t.$$

Taking sum on both side of above equation we get

$$\begin{aligned}
&\sum_{r=0}^{m} a_r \left(\sum_{s=0}^{m} a_s (p^k_{i+r,j+s} - p^k_{i,j}) \right) \\
&\quad = a_0 \sum_{r=0}^{m} a_r (p^k_{i+r,j} - p^k_{i,j}) + \sum_{t=1}^{m} a_t \left(\sum_{r=0}^{m} a_r (p^k_{i+r,j+1} - p^k_{i,j}) \right) \\
&\qquad + \sum_{r=0}^{m} a_r \left(\sum_{s=1}^{m-1} \tilde{a}_s (p^k_{i+r,j+s+1} - p^k_{i+r,j+s}) \right).
\end{aligned} \tag{28}$$

Since

$$\begin{aligned}\sum_{r=0}^{m} a_r(p^k_{i+r,j} - p^k_{i,j})\\ &= a_1(p^k_{i+1,j} - p^k_{i,j}) + a_2(p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j})\\ &\quad + a_3(p^k_{i+3,j} - p^k_{i+2,j} + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}) + \cdots\\ &\quad + a_m(p^k_{i+m,j} - \cdots + p^k_{i+3,j} - p^k_{i+2,j} + p^k_{i+2,j} - p^k_{i+1,j} + p^k_{i+1,j} - p^k_{i,j}).\end{aligned}$$

Hence

$$\sum_{r=0}^{m} a_r(p^k_{i+r,j} - p^k_{i,j}) = \sum_{t=1}^{m} a_t(p^k_{i+1,j} - p^k_{i,j}) + \sum_{s=1}^{m-1} \tilde{a}_s(p^k_{i+s+1,j} - p^k_{i+s,j}).$$

Similarly

$$\begin{aligned}\sum_{r=0}^{m} a_r(p^k_{i+r,j+1} - p^k_{i,j}) &= a_0(p^k_{i,j+1} - p^k_{i,j}) + \sum_{t=1}^{m} a_t(p^k_{i+1,j+1} - p^k_{i,j})\\ &\quad + \sum_{s=1}^{m-1} \tilde{a}_s(p^k_{i+s+1,j+1} - p^k_{i+s,j+1}).\end{aligned}$$

Substituting these sum into (28) and then from (27) we have

$$\begin{aligned}p^{k+1}_{2i,2j} - p^k_{i,j} &= \left(a_0 \sum_{t=1}^{m} a_t\right)(p^k_{i+1,j} - p^k_{i,j})\\ &\quad + \left(\sum_{t=1}^{m} a_t\right)^2 (p^k_{i+1,j+1} - p^k_{i,j+1}) + \sum_{t=1}^{m} a_t(p^k_{i,j+1} - p^k_{i,j})\\ &\quad + a_0 \sum_{s=1}^{m-1} \tilde{a}_s(p^k_{i+s+1,j} - p^k_{i+s,j})\\ &\quad + \left(\sum_{t=1}^{m} a_t\right) \sum_{s=1}^{m-1} \tilde{a}_s(p^k_{i+s+1,j+1} - p^k_{i+s,j+1})\\ &\quad + \sum_{r=0}^{m} a_r \left(\sum_{s=1}^{m-1} \tilde{a}_s(p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right).\end{aligned} \tag{29}$$

Similarly from (18) and (19) we obtain

$$\begin{aligned} p^{k+1}_{2i+1,2j} - \frac{1}{2}(p^k_{i,j} + p^k_{i+1,j}) = {} & \left(a_0 \sum_{t=1}^{m} b_t - \frac{1}{2}\right)(p^k_{i+1,j} - p^k_{i,j}) \\ & + \left(\sum_{t=1}^{m} a_t \sum_{t=1}^{m} b_t\right)(p^k_{i+1,j+1} - p^k_{i,j+1}) + \sum_{t=1}^{m} a_t (p^k_{i,j+1} - p^k_{i,j}) \\ & + a_0 \sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+s+1,j} - p^k_{i+s,j}) \\ & + \left(\sum_{t=1}^{m} a_t\right) \sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ & + \sum_{r=0}^{m} b_r \left(\sum_{s=1}^{m-1} \tilde{a}_s (p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right), \end{aligned} \tag{30}$$

$$\begin{aligned} & p^{k+1}_{2i,2j+1} - \frac{1}{2}(p^k_{i,j} + p^k_{i,j+1}) \\ & \quad = \left(b_0 \sum_{t=1}^{m} a_t\right)(p^k_{i+1,j} - p^k_{i,j}) + \left(\sum_{t=1}^{m} a_t \sum_{t=1}^{m} b_t\right)(p^k_{i+1,j+1} - p^k_{i,j+1}) \\ & \qquad + \left(\sum_{t=1}^{m} b_t - \frac{1}{2}\right)(p^k_{i,j+1} - p^k_{i,j}) + b_0 \sum_{s=1}^{m-1} \tilde{a}_s (p^k_{i+s+1,j} - p^k_{i+s,j}) \\ & \qquad + \left(\sum_{t=1}^{m} b_t\right) \sum_{s=1}^{m-1} \tilde{a}_s (p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ & \qquad + \sum_{r=0}^{m} a_r \left(\sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right), \end{aligned} \tag{31}$$

$$\begin{aligned} & p^{k+1}_{2i+1,2j+1} - \frac{1}{4}(p^k_{i,j} + p^k_{i+1,j} + p^k_{i,j+1} + p^k_{i+1,j+1}) \\ & \quad = \left(b_0 \sum_{t=1}^{m} b_t - \frac{1}{4}\right)(p^k_{i+1,j} - p^k_{i,j}) + \left(\sum_{t=1}^{m} b_t \sum_{t=1}^{m} b_t - \frac{1}{4}\right)(p^k_{i+1,j+1} - p^k_{i,j+1}) \\ & \qquad + \left(\sum_{t=1}^{m} b_t - \frac{1}{2}\right)(p^k_{i,j+1} - p^k_{i,j}) + \left(\sum_{t=1}^{m} b_t\right) \sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+s+1,j+1} - p^k_{i+s,j+1}) \\ & \qquad + b_0 \sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+s+1,j} - p^k_{i+s,j}) + \sum_{r=0}^{m} b_r \left(\sum_{s=1}^{m-1} \tilde{b}_s (p^k_{i+r,j+s+1} - p^k_{i+r,j+s})\right), \end{aligned} \tag{32}$$

From (24), (26) and (29)–(32) we have

$$M_k^1 \leqslant |a_0| \left(\sum_{t=1}^{m} |a_t| + \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,1}^k\| + \left(\sum_{t=1}^{m} |a_t| + \sum_{r=0}^{m} |a_r| \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,2}^k\| + \sum_{t=1}^{m} |a_t| \left(\sum_{t=1}^{m} |a_t| + \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,3}^k\|, \tag{33}$$

$$M_k^2 \leqslant \left(|a_0| \sum_{t=1}^{m} |b_t| + |a_0| \sum_{s=1}^{m-1} |\tilde{b}_s| + \frac{1}{2} \right) \max_{i,j} \|\Delta_{i,j,1}^k\| + \left(\sum_{t=1}^{m} |a_t| + \sum_{r=0}^{m} |b_r| \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,2}^k\| + \sum_{t=1}^{m} |a_t| \left(\sum_{t=1}^{m} |b_t| + \sum_{s=1}^{m-1} |\tilde{b}_s| \right) \max_{i,j} \|\Delta_{i,j,3}^k\|, \tag{34}$$

$$M_k^3 \leqslant |b_0| \left(\sum_{t=1}^{m} |a_t| + \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,1}^k\| + \left(\sum_{t=1}^{m} |b_t| + \sum_{r=0}^{m} |a_r| \sum_{s=1}^{m-1} |\tilde{b}_s| + \frac{1}{2} \right) \max_{i,j} \|\Delta_{i,j,2}^k\| + \sum_{t=1}^{m} |b_t| \left(\sum_{t=1}^{m} |a_t| + \sum_{s=1}^{m-1} |\tilde{a}_s| \right) \max_{i,j} \|\Delta_{i,j,3}^k\|, \tag{35}$$

$$M_k^4 \leqslant \left(|b_0| \sum_{t=1}^{m} |b_t| + |b_0| \sum_{s=1}^{m-1} |\tilde{b}_s| + \frac{1}{4} \right) \max_{i,j} \|\Delta_{i,j,1}^k\| + \left(\sum_{t=1}^{m} |b_t| + \sum_{r=0}^{m} |b_r| \sum_{s=1}^{m-1} |\tilde{b}_s| + \frac{1}{2} \right) \max_{i,j} \|\Delta_{i,j,2}^k\| + \left(\sum_{t=1}^{m} |b_t| \sum_{t=1}^{m} |b_t| + \sum_{t=1}^{m} |b_t| \sum_{s=1}^{m-1} |\tilde{b}_s| + \frac{1}{4} \right) \max_{i,j} \|\Delta_{i,j,3}^k\|. \tag{36}$$

If $\eta = \max\{\eta_1, \eta_2, \eta_3, \eta_4\}$, $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$ and $\xi = \max\{\xi_1, \xi_2, \xi_3, \xi_4\}$, where $\eta_t, \tau_t, \xi_t, t = 1, \ldots, 4$ are defined in (21)–(23) then from (25) and (33)–(36) we have

$$\|P^{k+1} - P^k\|_\infty \leqslant \eta\gamma_1 + \tau\gamma_2 + \xi\gamma_3, \tag{37}$$

where

$$\gamma_t = \max_{i,j} \|\Delta_{i,j,t}^k\|, \quad t = 1, 2, 3.$$

From (18), (19) and using similar approach as we did for (9) and (10) we obtain

$$\begin{aligned} p^k_{2i+1,2j} - p^k_{2i,2j} &= \sum_{s=0}^{m} a_s \left(\sum_{r=0}^{m} (b_r - a_r) p^{k-1}_{i+r,j+s} \right) \\ &= \sum_{s=0}^{m} a_s \left(\sum_{r=0}^{m} c_r (p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}) \right), \end{aligned} \tag{38}$$

where

$$c_r = \sum_{t=0}^{r} (a_t - b_t),$$

$$p^k_{2i+2,2j} - p^k_{2i+1,2j} = \sum_{s=0}^{m} a_s \left(\sum_{r=0}^{m} d_r (p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}) \right), \tag{39}$$

where $d_r = a_r - c_r$,

$$p^k_{2i+1,2j+1} - p^k_{2i,2j+1} = \sum_{s=0}^{m} b_s \left(\sum_{r=0}^{m} c_r (p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}) \right), \tag{40}$$

$$p^k_{2i+2,2j+1} - p^k_{2i+1,2j+1} = \sum_{s=0}^{m} b_s \left(\sum_{r=0}^{m} d_r (p^{k-1}_{i+r+1,j+s} - p^{k-1}_{i+r,j+s}) \right), \tag{41}$$

$$p^k_{2i+1,2j+2} - p^k_{2i,2j+2} = \sum_{s=0}^{m} a_s \left(\sum_{r=0}^{m} c_r (p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1}) \right), \tag{42}$$

$$p^k_{2i+2,2j+2} - p^k_{2i+1,2j+2} = \sum_{s=0}^{m} a_s \left(\sum_{r=0}^{m} d_r (p^{k-1}_{i+r+1,j+s+1} - p^{k-1}_{i+r,j+s+1}) \right), \tag{43}$$

$$p^k_{2i,2j+1} - p^k_{2i,2j} = \sum_{r=0}^{m} a_r \left(\sum_{s=0}^{m} c_s (p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}) \right), \tag{44}$$

$$p^k_{2i,2j+2} - p^k_{2i,2j+1} = \sum_{r=0}^{m} a_r \left(\sum_{s=0}^{m} d_s (p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}) \right), \tag{45}$$

$$p^k_{2i+1,2j+1} - p^k_{2i+1,2j} = \sum_{r=0}^{m} b_r \left(\sum_{s=0}^{m} c_s (p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}) \right), \tag{46}$$

$$p^k_{2i+1,2j+2} - p^k_{2i+1,2j+1} = \sum_{r=0}^{m} b_r \left(\sum_{s=0}^{m} d_s (p^{k-1}_{i+r,j+s+1} - p^{k-1}_{i+r,j+s}) \right). \tag{47}$$

Using (38)–(41) recursively we get

$$\max_{i,j} \|\Delta^k_{i,j,1}\| \leqslant \left(\sum_{r=0}^{m} |a_r| \sum_{s=0}^{m} |c_s| \right)^k \max_{i,j} \|\Delta^0_{i,j,1}\|,$$

$$\max_{i,j} \|\Delta^k_{i,j,1}\| \leqslant \left(\sum_{r=0}^{m} |a_r| \sum_{s=0}^{m} |d_s| \right)^k \max_{i,j} \|\Delta^0_{i,j,1}\|,$$

$$\max_{i,j} \|\Delta^k_{i,j,1}\| \leqslant \left(\sum_{r=0}^{m} |b_r| \sum_{s=0}^{m} |c_s| \right)^k \max_{i,j} \|\Delta^0_{i,j,1}\|,$$

$$\max_{i,j} \|\Delta^k_{i,j,1}\| \leqslant \left(\sum_{r=0}^{m} |b_r| \sum_{s=0}^{m} |d_s| \right)^k \max_{i,j} \|\Delta^0_{i,j,1}\|.$$

From above inequalities and (20) we get

$$\max_{i,j} \|\Delta^k_{i,j,1}\| \leqslant (\delta)^k \max_{i,j} \|\Delta^0_{i,j,1}\|. \tag{48}$$

Using (42) and (43) recursively we get

$$\max_{i,j} \|\Delta^k_{i,j,3}\| \leqslant (\delta)^k \max_{i,j} \|\Delta^0_{i,j,3}\|. \tag{49}$$

Similarly from (44)–(47) we get

$$\max_{i,j} \|\Delta^k_{i,j,2}\| \leqslant (\delta)^k \max_{i,j} \|\Delta^0_{i,j,2}\|. \tag{50}$$

From (37) and (48)–(50) we get

$$\|P^{k+1} - P^k\|_\infty \leqslant (\eta\beta_1 + \tau\beta_2 + \xi\beta_3)\delta^k,$$

where

$$\beta_t = \max_{i,j} \|\Delta^0_{i,j,t}\|, \quad t = 1, 2, 3.$$

Using triangle inequality we get

$$\|P^k - P^\infty\|_\infty \leqslant (\eta\beta_1 + \tau\beta_2 + \xi\beta_3) \left(\frac{\delta^k}{1-\delta} \right).$$

This completes the proof. □

Corollary 8. *Given initial control polygon* $p^0_{i,j} = p_{i,j},\ i, j \in \mathbb{Z}$, *let the values* $p^k_{i,j},\ k \geqslant 0$ *be defined recursively by tensor product form of the Chaikin's binary subdivision process. Suppose* P^k *be the piecewise linear interpolation to the values* $p^k_{i,j}$ *and* P^∞ *be the limit surface of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \{\tfrac{17}{16}\beta_1 + \tfrac{5}{4}\beta_2 + \tfrac{13}{16}\beta_3\}(\tfrac{1}{2})^k,$$

where

$$\beta_t = \max_{i,j} \|\Delta^0_{i,j,t}\|, \quad t = 1, 2, 3.$$

Corollary 9. *Given initial control polygon* $p^0_{i,j} = p_{i,j},\ i, j \in \mathbb{Z}$, *let the values* $p^k_{i,j},\ k \geqslant 0$ *be defined recursively by tensor product form of the cubic B-spline subdivision process. Suppose* P^k *be the piecewise linear interpolation to the values* $p^k_{i,j}$ *and* P^∞ *be the limit surface of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \{\tfrac{11}{16}\beta_1 + 2\beta_2 + \tfrac{7}{4}\beta_3\}(\tfrac{1}{2})^k,$$

where

$$\beta_t = \max_{i,j} \|\Delta^0_{i,j,t}\|, \quad t = 1, 2, 3.$$

Corollary 10. *Given initial control polygon* $p^0_{i,j} = p_{i,j},\ i, j \in \mathbb{Z}$, *let the values* $p^k_{i,j},\ k \geqslant 0$ *be defined recursively by tensor product form of the interpolatory* 4*-point binary subdivision process. Suppose* P^k *be the piecewise linear interpolation to the values* $p^k_{i,j}$ *and* P^∞ *be the limit surface of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \{\eta\beta_1 + \tau\beta_2 + \xi\beta_3\}\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\delta = \max\left\{2|w| + \tfrac{1}{2}, (4|w|+1)(|w| + |\tfrac{1}{2} + w|)\right\},$$

$$\eta = \max\left\{\tfrac{1}{2}, |w|, |w|(2\zeta + \tfrac{1}{2}) + \tfrac{1}{4}\right\},$$

$$\tau = \max\left\{1, 2\zeta + 1, \zeta + |\tfrac{1}{2} + w| + 2\zeta(\tfrac{1}{2} + |w|) + \tfrac{1}{2}\right\},$$

$$\xi = \max\left\{1, 2\zeta + \tfrac{1}{2}, \left(\zeta + \left|\tfrac{1}{2} + w\right|\right)\left(2\zeta + \tfrac{1}{2}\right) + \tfrac{1}{4}\right\},$$

$$\zeta = |w| + \left|\frac{1}{2} + w\right|, \qquad \beta_t = \max_{i,j} \|\Delta^0_{i,j,t}\|, \quad t = 1, 2, 3.$$

Corollary 11. *Given initial control polygon* $p^0_{i,j} = p_{i,j},\ i, j \in \mathbb{Z}$, *let the values* $p^k_{i,j},\ k \geqslant 0$ *be defined recursively by tensor product form of* 6*-point interpolatory binary subdivision process. Suppose* P^k *be the* *piecewise linear interpolation to the values* $p^k_{i,j}$ *and* P^∞ *be the limit surface of the subdivision process. Then*

$$\|P^k - P^\infty\|_\infty \leqslant \{\eta\beta_1 + \tau\beta_2 + \xi\beta_3\}\left(\frac{\delta^k}{1-\delta}\right),$$

where

$$\delta = \max\{\eta_3(\eta_1 + |\theta|), \eta_3\},$$

$$\eta = \max\left\{|\theta|(\eta_1+\eta_2) + \tfrac{1}{4}, 2|\theta|, \tfrac{1}{2}\right\},$$

$$\tau = \max\left\{2, \eta_1+\eta_2+\tfrac{1}{2}, 1+\eta_1+|\theta|, \eta_1 + (\eta_1+|\theta|)\eta_2 + \tfrac{1}{2}\right\},$$

$$\xi = \max\left\{2, \eta_1+\eta_2, 2\eta_1, \eta_1(\eta_1+\eta_2)+\tfrac{1}{4}\right\},$$

$$\eta_1 = 2\left|\tfrac{1}{16}+3\theta\right| + 2\left|\tfrac{9}{16}+2\theta\right| + |\theta|,$$

$$\eta_2 = |\theta| + |\tfrac{17}{16}+2\theta| + \tfrac{1}{2} + |\tfrac{1}{16}+2\theta|,$$

$$\eta_3 = 2|\theta| + 2\left|\tfrac{1}{16}+2\theta\right| + \tfrac{1}{2},$$

$$\beta_t = \max_{i,j}\|\Delta^0_{i,j,t}\|, \quad t = 1,2,3.$$

Remark 12. For Corollary 10 the range of w for which condition (20) satisfied is approximately $-0.2499 \leqslant w \leqslant 0.1035$ while for Corollary 11 the range of θ is approximately $-0.0683 \leqslant \theta \leqslant 0.0136$.

4. Conclusions and further work

We have estimated error bounds for binary subdivision curves/surfaces in terms of the maximal differences of the initial control point sequences and constants that depend on the subdivision mask. The bound is independent of the process of subdivision and can be evaluated without recursive subdivision. Our technique is independent of parameterizations therefore it can be easily and efficiently implemented. Estimation of error bounds for ternary subdivision curves/surfaces is our forthcoming work. It is yet to be investigated whether we can use above technique for estimating error bounds for subdivision surfaces on arbitrary topological meshes.

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