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Dimensions of Spline Spaces over 3D Hierarchical T-Meshes ⋆

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Abstract

A 3D T-mesh is basically a partition of a cuboid such that every part is a smaller cuboid. In this paper we define the spline spaces over 3D T-meshes, which would play an important role in adaptive and dynamic implicit surface reconstruction from unorganized point clouds, and present a dimension formula about the spline space over a special kind of T-mesh, i.e., 3D hierarchical T-mesh. The formula holds when the smoothness is less than half of the degree of the spline functions, and it involves only the topological quantities of the T-meshes. The construction of basis functions of the spline space is briefly discussed.

Keywords: T-Spline; T-mesh; Bézier ordinate; Dimension; Hierarchical

1 Introduction

A 2D T-mesh is formed by a set of rectangles in such a way that there is no holes among these rectangles. 2D T-meshes are used extensively in geometric modeling. In [5] hierarchical B-splines were introduced, whose domain are T-meshes, and two concepts were defined: local refinement using an efficient representation and multi-resolution editing. In principle, hierarchical B-splines are the accumulation of tensor-product surfaces with different resolutions and domains. In [9] tensor-product splines with knot segments were discussed. In fact, they defined a spline space over a more general T-mesh, where crossing, T-junctional, and L-junctional vertices are allowed. But his research is just for the semi-regular mesh. In 2003, [7] invented T-spline, a spline defined over T-mesh. It is a point-based spline, i.e., for every vertex, a basis function of the spline space is defined. Each of the basis functions comes from some tensor-product spline spaces. This type of spline supports many valuable operations within a consistent framework, but some of them, say, local refinement, are not simple. In the T-spline theory, the local refinement is dependent on the structure of the mesh, and the complexity is uncertain ([6]). The reason leading to this problem

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is that the spline over every cell of the mesh is not a polynomial, but a piecewise polynomial. Hence, in order to make full use of adaptivity of T-meshes, two of the present authors defined spline spaces over T-meshes ([3]). In this type of the spline space, the function is a polynomial within each facet(cell) of T-meshes. A formula was proposed to calculate the dimension of the spline space in [3].

As a natural generalization of 2D T-meshes, in the paper, we will consider 3D T-meshes. A 3D T-mesh is basically a partition of a cuboid such that each part is a smaller cuboid and there are no holes among them. We will define the spline spaces over 3D T-meshes, where the spline function is a tensor-product polynomial within every cell, and achieves the specified smoothness across common edges of the cells. We expect this type of splines to play an important role in adaptive and dynamic implicit surface reconstruction from unorganized point clouds ([10]), where the target implicit surface will be the zero set of a spline over a 3D T-mesh. As one of the fundamental problems, in this paper we study the dimensions of spline function spaces over 3D T-meshes. A method based on Bézier nets (simply called **B-nets**) is proposed to solve the problem. If we consider with 3D hierarchical T-meshes, and when the smoothness is less than half of the degree of polynomials in the spline space, a dimension formula which involves only the topological quantities of the T-mesh is derived. The construction of basis functions of the spline space over the T-mesh is briefly discussed.

The organization of the rest parts of the paper is as follows. We give the definitions and notation of 3D T-meshes and the spline spaces over 3D T-meshes in Section 2. In Section 3, a method based on B-nets is proposed. When considered hierarchical mesh and the order of the smoothness is less than half of the degree of the polynomials in the spline space, a dimension formula is derived in Section 4. The construction of the basis functions is also briefly discussed. Finally, in Section 5, we conclude the paper with some future research problems.

2 Spline Spaces over 3D T-meshes

In this section, we first introduce 3D T-meshes and some concepts related with 3D T-meshes, and then present the spline spaces over T-meshes and 3D hierarchical T-meshes.

2.1 3D T-meshes

A 3D T-mesh is basically a partition of a cuboid such that every part is a smaller cuboid and there are no holes among them. See Fig. 1 for some examples.

Suppose Ω is a 3D T-mesh, which is the partition of a cuboid Ξ. Then the small cuboid is called the **cell** of the T-mesh. A vertex of a cell in Ω is also called a **vertex** of the T-mesh. If a vertex is on the boundary of Ξ, then it is called a **boundary vertex**. Otherwise, it is called an **interior vertex**. The line segment connecting two adjacent vertices on an edge of a cell is called an **edge** of the T-mesh. If an edge is on the boundary of Ξ, then it is called a **boundary edge**; otherwise it is called an **interior edge**. It is evident that the boundary edges can either lie in the faces or on the edges of Ξ. A face of any cell is called a **face** of Ω if it doesn't contain any faces of other cells. If a face is on the boundary of Ξ, then it is called a **boundary face**; otherwise it is called an **interior face**. As any edges and faces have three directions, so we might as well suppose that the horizontal direction is x, the vertical direction is y and the other perpendicular direction is z. We suppose the three directions of faces are xy, yz, zx which are parallel to the planes xy, yz, zx respectively. We call two edges (or two faces) are parallel when they have the same direction. Two parallel faces are called **adjacent** if they share a common rectangle. Two parallel edges are called **adjacent** if they share a common line segment. If a face of a cell is adjacent with a face of another cell, then the cells are called **adjacent**. Fig. 1 illustrates three examples of 3D T-meshes. In Fig. 1c, v_1, $i = 1, \cdots, 2$ are interior vertices and b_i, $i = 1, \cdots, 5$ are boundary vertices. b_1b_2 and b_2b_3 are boundary edges and b_3v_1 is an interior edge. $b_1b_2b_3b_4$ is a boundary face and $b_4b_3v_1b_5$ is an interior face. In Fig. 1a, faces $v_1v_4v_6v_1$ and $v_2v_7v_8v_3$ are adjacent, and edge v_4v_5, c-edge v_4v_6 are also adjacent.

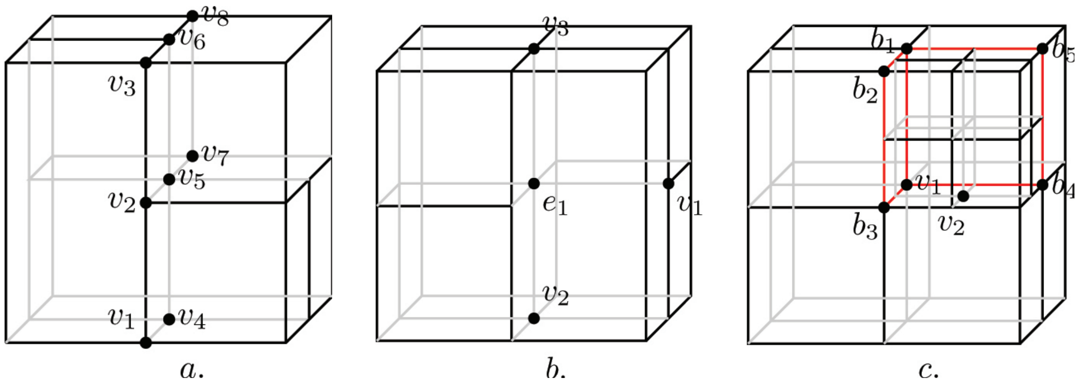


Fig. 1: Three examples of 3D T-meshes.

Now we will define **composite face**(shortly, **c-face**) and **composite edge**(shortly, **c-edge**) for a T-mesh Ω, which are the keys to the proof of the dimension formula in Section 4. The same as the edge and face, the c-face or c-edge are also have three directions. A **composite face** of some direction is a set of faces of this direction which has the following property: if a face of some cell is adjacent to the c-face, then it belongs to the c-face, but this is not the case for any subsets of the c-face. A **composite edge** of some direction is a set of edges of this direction who holds the following property: if an edge of some cell is adjacent to the c-edge, then it belongs to the c-edge, but any subsets of the c-edge don't hold it. It is easy to see that all the boundary faces are c-faces. For example, in Fig. 1c, $b_1b_2b_3v_1$ is a c-face of yz and $b_1v_1b_4b_5$ is a c-face of xy. b_1v_1 and b_3v_1 are both c-edges. But b_1v_1 is the direction of y and b_3v_1 is the direction of z.

There are some differences among c-edges. A c-edge may belong to two c-faces (Fig. 1b, e_1v_1), it also may belong to four c-faces(Fig. 1b, v_2v_3). If a c-edge belongs to two c-faces, it must lie in a face of some cells. If a c-edge belongs to four c-faces, then it is only on the edges of cells in Ω, which is called a **plus edge** (shortly, **p-edge**). A vertex of T-mesh Ω can be a vertex of some cells, it also can lie in the face of some cells. If a vertex doesn't lie on any faces of all the cells, then we call it a **plus vertex**(shortly, **p-vertex**). It is evident that the vertex of Ξ and the vertex on the boundary edge of Ξ are all p-vertices. In Fig. 1c, b_1v_1 and b_3v_1 are p-edges and v_1, b_3, b_4 are all p-vertices. In the end, we will introduce a special kind of vertex which is called **e-vertex**. It lies on a p-edge, which is the vertex of four cells and the end vertex of six edges. In Fig. 1b, the vertex e_1 is an e-vertex.

2.2 Definition of spline spaces

A 3D T-mesh is a natural generalization of 2D T-mesh which is described in [3, 7] in detail. We assume a 3D **T-mesh** is basically a partition of a solid Ω that each cell is a cuboid and there exist no holes among them. Given a 3D T-mesh $\mathcal{T}$, we denote by $\mathcal{C}$ all the cells and by Ξ the region occupied by all the cells in $\mathcal{T}$. Let

$$\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T}) := \Big\{ s(x,y,z) \in C^{\alpha,\beta,\gamma}(\Xi) \Big| s(x,y,z)|_\phi \in \mathbb{P}_{mnk}, \forall \phi \in \mathcal{C} \Big\} \tag{1}$$

where $\mathbb{P}_{mnk}$ is the space of all polynomials with tri-degree (m,n,k), and $C^{\alpha,\beta,\gamma}(\Xi)$ is the space consisting of all tri-variate functions which are continuous in Ξ with order α, β, γ along x, y, z directions. It is obvious that $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ is a linear space. We call it **the spline space over the T-mesh** $\mathcal{T}$.

As the complexity of the topological quantities of the 3D T-meshes (See Fig. 1a for an example), so we define a special type of 3D T-mesh, which is named as 3D Hierarchical T-mesh in the next subsection. We assume the T-spline over this kind of T-mesh is almost enough for the applications.

2.3 3D hierarchical T-meshes

Suppose we are given a tri-variate tensor-product B-spline on a cube with three knot sequences $r_0, \cdots, r_0 < r_1, \cdots, r_1 < \cdots < r_l, \cdots, r_l$, $s_0, \cdots, s_0 < s_1, \cdots, s_1 < \cdots < s_m, \cdots, s_n$, and $t_0, \cdots, t_0 < t_1, \cdots, t_1 < \cdots < t_n, \cdots, t_n$ along three coordinate axes x, y, and z. Then the plane families $x = r_i$, $i = 0, \cdots, l$, $y = s_j$, $j = 0, \cdots, m$, $z = t_k$, $k = 0, \cdots n$ will form a 3D mesh. We call it a standard tensor-product(TP for short) mesh.

Given a 3D TP mesh Ω_1, we can construct a series of special 3D T-meshes Ω_k, $k = 2, 3, ...$, in the following iterative fashion. For any $k \geq 2$, select some cells of level k from Ω_k and divide each of them equally into eight parts with the three planes which are respectively parallel to xy, yz and zx. Then we can get the mesh Ω_{k+1} and all the new cells are labeled with level $(k+1)$. Then for any $k \geq 1$, the mesh Ω_k is called a **3D hierarchical T-mesh** of level k. In Fig. 1, the mesh c is a 3D hierarchical T-mesh of level 2, but the meshes a and b are not hierarchical T-meshes.

In the following section, we will propose a method to calculate the dimension of the spline space over T-mesh when $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$ and $k \geqslant 2\gamma + 1$. If we consider with 3D hierarchical T-meshes, the dimension formula which involves only the topological quantities of the T-mesh is proved in Section 4.

3 B-net Method

In the theory of multivariate splines, in order to calculate the dimension of some specified spline space, we first need to transfer the smoothness conditions into algebraic forms. There are many approaches to address this problem. Among them, B-net method ([4]) is a dominant one. We will apply this method to calculate the dimensions of spline spaces over T-meshes.

3.1 Review of B-net method

Let $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ be two polynomials with tri-degree (m, n, k), defined over two adjacent domains $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$ and $[x_1, x_2] \times [y_0, y_1] \times [z_0, z_1]$, respectively. They can be expressed in **Bernstein-Bézier** forms

$$\begin{aligned}\pi_1(x,y,z) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{l=0}^{k} b^1_{i,j,l} B^m_i\left(\frac{x-x_0}{x_1-x_0}\right) B^n_j\left(\frac{y-y_0}{y_1-y_0}\right) B^k_l\left(\frac{z-z_0}{z_1-z_0}\right),\\ \pi_2(x,y,z) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\sum_{l=0}^{k} b^2_{i,j,l} B^m_i\left(\frac{x-x_1}{x_2-x_1}\right) B^n_j\left(\frac{y-y_0}{y_1-y_0}\right) B^k_l\left(\frac{z-z_0}{z_1-z_0}\right),\end{aligned} \tag{2}$$

where $B^m_i(t)$, $B^n_j(t)$ and $B^k_l(t)$ are **Bernstein polynomials**. $\{b^1_{i,j,l}\}$ and $\{b^2_{i,j,l}\}$ are called the **Bézier ordinates** of $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ respectively. It is well known that $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ are r times differentiable across their common boundary if and only if ([4])

$$\frac{1}{(x_1-x_0)^i}\Delta^{i,0,0} b^1_{m-i,j,l} = \frac{1}{(x_2-x_1)^i}\Delta^{i,0,0} b^2_{0,j,l}, \quad j=0,\cdots,n,\ i=0,\cdots,r,\ l=0,\cdots,k. \tag{3}$$

Here the difference operators are defined by

$$\Delta^{p,0,0} b_{i,j,l} = \Delta^{p-1,0} b_{i+1,j,l} - \Delta^{p-1,0,0} b_{i,j,l} \tag{4}$$

with $\Delta^{0,0,0} b_{i,j,l} = b_{i,j,l}$.

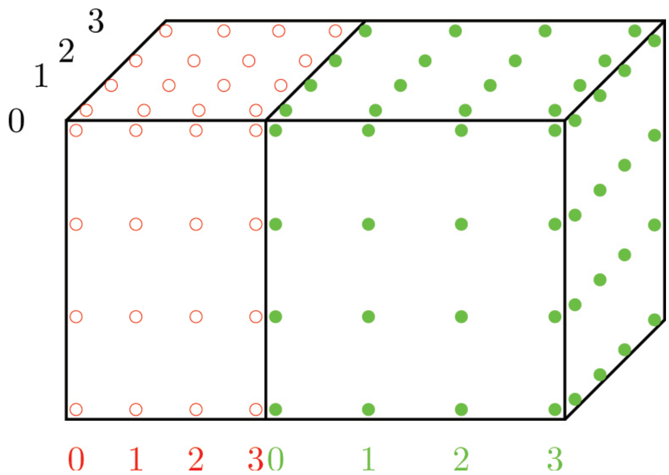


Fig. 2: Bézier ordinates of two tri-cubic polynomials.

The geometric meaning of the above conditions are illustrated in Fig. 2. The Bézier ordinates of two tri-cubic Bézier functions are shown in circle and solid, respectively. Suppose $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ are C^0 continuous along their common boundary, then the 3rd column of the circle ordinates should be coincide with the 0-th column of the solid ordinates. Hence the whole function defined over $[x_0, x_2] \times [y_0, y_1] \times [z_0, z_1]$ has $(3 + 1)^3 + 3 \times (3 + 1)^2 = 112$ free coefficients.

Now we assume the two polynomials $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ are C^1 continuous across their common boundary, then the 1-st column of solid ordinates are determined by the 2-nd and 3-rd columns of the circle ordinates. In this case, the whole function has 96 free coefficients. If we define a polynomial of tri-degree $(1, 3, 3)$ with the 2-nd and 3-rd columns of the circle ordinates as its Bézier ordinates, and similarly define a polynomial of tri-degree $(1, 3, 3)$ with the 0-th and 1-st columns of the solid ordinates as its Bézier ordinates, then these two polynomials are the same.

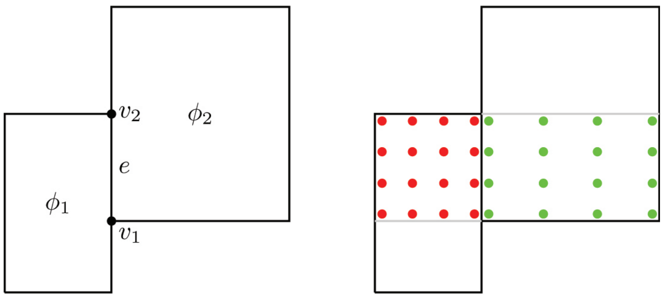
Similarly, for second order continuity conditions, the 2-nd column of the solid ordinates are determined by the 1-st, 2-nd and 3-rd columns of the circle ordinates, and the whole function has 80 free coefficients. For third order continuity conditions, the two polynomials $\pi_1(x, y, z)$ and $\pi_2(x, y, z)$ are identical.

3.2 Applications in T-meshes

Given a T-mesh $\mathcal{T}$, let $\bar{\mathcal{S}}(m, n, k, \mathcal{T}) = \mathcal{S}(m, n, k, -1, -1, -1, \mathcal{T})$ be the piecewise polynomial function space without continuity constraints between adjacent cells. Then it is easy to verify that

$$\dim \bar{\mathcal{S}}(m, n, k, \mathcal{T}) = C(m + 1)(n + 1)(k + 1),$$

where C is the number of the cells in $\mathcal{T}$. Any function in $\mathcal{S}(m, n, k, \alpha, \beta, \gamma, \mathcal{T})$ can be thought of as a function in $\bar{\mathcal{S}}(m, n, k, \mathcal{T})$ that satisfies additional smoothness conditions.



a. b.

Fig. 3: Two horizontal adjacent cells.

Now we consider smoothness conditions. Select an arbitrary function $s \in \bar{\mathcal{S}}(m, n, k, \mathcal{T})$. Let ϕ_1 and ϕ_2 be two horizontal adjacent cells in the T-mesh $\mathcal{T}$ as shown in Fig. 3a. Assume ϕ_1 and ϕ_2 are defined over $[x_0, x_1] \times [y_0, y_1] \times [z_0, z_1]$ and $[x_1, x_2] \times [y_2, y_3] \times [z_2, z_3]$ respectively. Suppose $s|_{\phi_r} = s_r$, and the Bézier ordinates of s_r over ϕ_r are b^r_{ijl}, $i = 0, \cdots, m$, $j = 0, \cdots, n$, $l = 0, \cdots, k$, $r = 1, 2$. Then, according to the subdivision algorithm for tensor-product polynomials ([4]), we compute the Bézier ordinates for polynomial s_r restricted over $[x_r, x_{r+1}] \times [\underline{y}, \bar{y}] \times [\underline{z}, \bar{z}]$, $r = 1, 2$, where $\bar{y} = \max(y_1, y_3)$, $\underline{y} = \min(y_0, y_2)$ and $\bar{z} = \max(z_1, z_3)$, $\underline{z} = \min(z_0, z_2)$ (see Fig. 3b). Assume the new Bézier ordinates are $\bar{b}^r_{ijl}$, $i = 0, \cdots, m$, $j = 0, \cdots, n$, $l = 0, \cdots, k$, $r = 1, 2$. According to Eq. (3), we obtain the smoothness conditions satisfied by these two sets of the Bézier ordinates. These conditions lead to the smoothness conditions on the original two sets of Bézier ordinates b^r_{ijl}, $i = 0, \cdots, m$, $j = 0, \cdots, n$, $l = 0, \cdots, k$, $r = 1, 2$.

The similar smoothness conditions can be derived for any two vertical adjacent cells in $\mathcal{T}$. All the smooth conditions can be collected into a homogeneous linear system

$$Ac = 0,$$

where A is a matrix and c is the vector with all the Bézier ordinates collected in some order. Since the number of elements in c is $F(m+1)(n+1)(k+1)$, it follows that

$$\dim \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T}) = C(m+1)(n+1)(k+1) - \operatorname{rank} A. \tag{5}$$

To facilitate the computation of $\operatorname{rank} A$, we apply the concept of determining sets introduced by Alfeld and Schumaker ([2]) in the following subsection.

3.3 Determining sets

Suppose that $\mathcal{T}$ is an arbitrary T-mesh, and the cells of $\mathcal{T}$ are denoted by $\phi_1, \cdots, \phi_c$, where $\phi_r = [x_0^r, x_1^r] \times [y_0^r, y_1^r] \times [z_0^r, z_1^r]$. For any $s \in \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$, let s_r denote the polynomial $s|_{\phi_r}$. Assume the Bézier ordinates of s_r over ϕ_r are b_{ijl}^r, $i = 0, \cdots, m$, $j = 0, \cdots, n$, $l = 0, \cdots, k$. Each of the Bézier ordinate b_{ijl}^r is associated with a **domain point**

$$P_{ijl}^r = \Big(\frac{(m-i)x_0^r + ix_1^r}{m}, \frac{(n-j)y_0^r + jy_1^r}{n}, \frac{(k-l)y_0^r + ly_1^r}{k}\Big).$$

Let $\mathcal{B}(m,n,k,\mathcal{T})$ denote the set of domain points. For $t = P_{ijl}^r \in \mathcal{B}(m,n,k,\mathcal{T})$, let λ_t denote the linear functional on $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ defined by $\lambda_t s = b_{ijl}^r$. A set of domain points $\mathcal{P} \subset \mathcal{B}(m,n,k,\mathcal{T})$ is called a **determining set** ([2]) for the spline space $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ if for $s \in \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$,

$$\lambda_t s = 0, \forall t \in \mathcal{P} \Longrightarrow s = 0.$$

If $\mathcal{P}$ is a determining set for $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$, then it follows that $\dim \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T}) \leqslant \#\mathcal{P}$, where $\#\mathcal{P}$ denotes the cardinality of the set $\mathcal{P}$. If $\mathcal{Q}$ is a determining set, and each of its nontrivial subsets is not a determining set, then it is called a **minimal determining set**. It follows from basic linear algebra that the number of points in any minimal determining set equals the dimension of the spline space. The Bézier ordinates corresponding to the points in a minimal determining set can be chosen arbitrarily and they uniquely determine a spline s in $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$. For the details about determining sets, the reader is referred to ([2]). Though the results in [2] were obtained for spline spaces over triangular meshes, similar conclusions hold for the spline spaces over T-meshes.

The following lemma states a key fact which will be used in the proof of the dimension formula in Section 4.

Lemma 1 *Given a 3D hierarchical T-mesh $\mathcal{T}$ and a spline space $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ defined over $\mathcal{T}$, consider a interior c-face which is parallel to xy plane. Then the $\gamma+1$ rows of the Bézier ordinates near the c-face in each of these cells will define an identical polynomial of tri-degree (m,n,γ) (see Fig. 4). The similar result can be considered the other c-faces.*

Proof Suppose ϕ_1 and ϕ_2 are the two leftmost and nearest adjacent cells near the c-face with ϕ_2 beneath ϕ_1. With the similar analysis in Subsection 3.1, $\gamma+1$ rows of the Bézier ordinates near the c-face in ϕ_1 and ϕ_2 will define the same polynomial with tri-degree (m,n,γ). If the number of the cell near the c-face is more than 1, then we select the leftmost and nearest cell in the rest of adjacent cells of the c-edge. The new cell will be adjacent vertically to ϕ_1 or ϕ_2. Therefore $\gamma+1$ rows of the Bézier ordinates near the c-edge in the new cell will define the same polynomial. By this fashion, we can run through all the cells, and thus all the $\gamma+1$ rows of the Bézier ordinates near the c-edge define an identical polynomial.

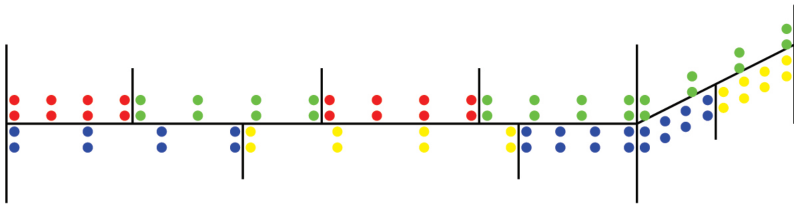


Fig. 4: Bézier ordinates in every cell near a c-face ($m = n = k = 3$, $\alpha = \beta = \gamma = 1$).

4 The Dimension Formula

In this section, we will derive a dimension formula for the spline space $\mathcal{S}(m, n, k, \alpha, \beta, \gamma, \mathcal{T})$ over a given hierarchical T-mesh $\mathcal{T}$ when $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$ and $k \geqslant 2\gamma + 1$. The formula, in the form of weighted Euler formula, depends on only the topological quantities of $\mathcal{T}$.

4.1 Notations for T-meshes

Before deriving the dimension formula, we introduce some notations for a T-mesh as shown in Table 1.

Table 1: Notations for a 3D hierarchical T-mesh.

C	number of cells of the mesh
F^i_{str}	number of interior faces which parallel the plane str, str=xy, yz, zx
F^{ic}_{str}	number of interior c-faces which parallel the plane str, str=xy, yz, zx
F^f_{str}	number of boundary faces(c-faces) which parallel the plane str, str=xy, yz, zx
F^c_{str}	number of c-faces which parallel the plane str, str=xy, yz, zx, $(= F^f_{str} + F^{ic}_{str})$
E^i_{dir}	number of interior edges along the direction of dir, dir=x, y, z
E^b_{dir}	number of edges on the edges of Ξ along the direction of dir, dir=x, y, z
E^f_{dir}	number of edges in the faces of Ξ along the direction of dir, dir=x, y, z
E^c_{dir}	number of c-edges in the faces of Ξ along the direction of dir, dir=x, y, z
E^+_{dir}	number of p-edges along the direction of dir, dir=x, y, z
V_{dir}	number of e-vertices on the p-edges along the direction of dir, dir=x, y, z
V^i	number of interior vertices
V^+	number of p-vertices

Lemma 2 *Given a hierarchical T-mesh with the notations in Table* 1*, then*

1)

$$2C = F^i_{xy} + F^{ic}_{xy} + F^f_{xy} = F^i_{yz} + F^{ic}_{yz} + F^f_{yz} = F^i_{zx} + F^{ic}_{zx} + F^f_{zx}. \tag{6}$$

2)

$$\begin{aligned} E^b_x + E^c_x - F^f_{xy} - F^f_{zx} &= -1/2(E^f_x - E^c_x), \\ E^b_y + E^c_y - F^f_{yz} - F^f_{xy} &= -1/2(E^f_y - E^c_y), \\ E^b_z + E^c_z - F^f_{yz} - F^f_{zx} &= -1/2(E^f_z - E^c_z). \end{aligned} \tag{7}$$

3)

$$\begin{aligned} F^i_{xy} + F^i_{zx} - F^{ic}_{xy} - F^{ic}_{zx} &= E^i_x + V_x - (E^+_x - E^b_x - E^c_x) - 1/2(E^c_x - E^f_x), \\ F^i_{xy} + F^i_{yz} - F^{ic}_{xy} - F^{ic}_{yz} &= E^i_y + V_y - (E^+_y - E^b_y - E^c_y) - 1/2(E^c_y - E^f_y), \\ F^i_{yz} + F^i_{zx} - F^{ic}_{yz} - F^{ic}_{zx} &= E^i_z + V_z - (E^+_z - E^b_z - E^c_z) - 1/2(E^c_z - E^f_z). \end{aligned} \tag{8}$$

4)

$$\begin{aligned} 4C + E^+_x - 2F^c_{xy} - 2F^c_{zx} &= E^i_x + V_x, \\ 4C + E^+_y - 2F^c_{xy} - 2F^c_{yz} &= E^i_y + V_y, \\ 4C + E^+_z - 2F^c_{yz} - 2F^c_{zx} &= E^i_z + V_z. \end{aligned} \tag{9}$$

5)

$$8C + 2E^+_x + 2E^+_y + 2E^+_z - 4F^c_{xy} - 4F^c_{yz} - 4F^c_{zx} - V^+ = V_i + V_x + V_y + V_z. \tag{10}$$

Proof

1) Since every cell has two faces that are parallel to xy plane, each of which is a part of some c-face that is parallel to xy plane. Let λ_i denote the number of faces of the ith c-face. Then the c-face has $\lambda_i + 1$ adjacent cells, by the contribution of boundary edges, it follows that:

$$2C = F^f_{xy} + \sum_{i=1}^{F^{ic}_{xy}} (\lambda_i + 1) = F^i_{xy} + F^{ic}_{xy} + F^f_{xy}.$$

Similarly, we can get the other two equations.

2) Observe that each face of Ω is a regular T-mesh in R^2. Use the lemma in [3] on the four faces which is parallel to xy or zx plane and plus all the equations, then the edges on the edge of Ξ will be computed twice, so we can get:

$$2F^f_{zx} + 2F^f_{xy} = 2E^b_x + E^f_x - E^c_x$$

After simplifying follows Eq. (7).

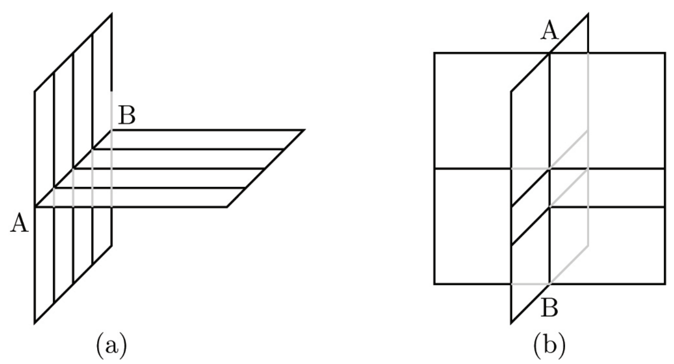


Fig. 5: two kinds of c-edges.

3) There are two kinds of c-edges. One kind is p-edges(Fig. 5b), the other kind is **t-edge**(Fig. 5a) which belongs to two c-faces. We only prove the first equation. Given the ith c-edge of x direction, if it is a p-edge, denote by λ_i^+ the number of edges on the p-edge, μ_i the number of e-vertex on the p-edge and ω_i^+ the number of c-faces adjacent to the p-edge. Observe that each interior c-face is a regular T-mesh in R^2 and each x c-edge must lie on some c-faces of xy and zx. Using the lemma in [3] on the c-faces, we can see that the number of the interior faces adjacent to the p-edge is $2\lambda_i^+ + 2\mu_i - 2 + \omega_i^+$. If the c-edge is a t-edge, suppose $\lambda_i^\perp$ is the number of edges on the t-edge and $\omega_i^\perp$ is the number of c-faces adjacent to the t-edge. Then the number of the interior faces which are adjacent to the t-edge is $2\lambda_i^\perp + \omega_i^\perp$. Thus:

$$\begin{aligned} 2F_{xy}^i + 2F_{zx}^i &= \sum(2\lambda_i^+ + 2\mu_i - 2 + \omega_i^+) + \sum(2\lambda_i^\perp + \omega_i^\perp) + E_x^f, \\ 2F_{xy}^{ic} + 2F_{zx}^{ic} &= \sum(\omega_i^+) + \sum(\omega_i^\perp) + E_x^c. \end{aligned} \tag{11}$$

In the Eq. (11), the first sum is for all the p-edges of x direction and the second is for all the t-edges of this direction. Subtracting the first equation from the second, we can get:

$$2F_{xy}^i + 2F_{zx}^i - 2F_{xy}^{ic} - 2F_{zx}^{ic} = \sum(2\lambda_i^+ + 2\mu_i - 2) + \sum(2\lambda_i^\perp) + E_x^f - E_x^c$$

On the right of the top equation, the interior c-edges and edges are computed twice. So the right of the equation equals to:

$$2E_x^i + 2V_x - 2(E_x^+ - E_x^b - E_x^c) - (E_x^c - E_x^f).$$

Divided by 2, we can get the Eq. (8).

4) From Eq. (6), (7) and Eq. (8), we can easily get Eq. (9).

5) As we have mentioned, each vertex must be a vertex of some cell. So it can be parted according to the number of cells which contain the vertex. In a 3D hierarchical T-mesh, we denote by P_i the set of the vertices which belongs to i cells and p_i the number of the set. Then, the vertices of P_1 must be the vertices of Ξ and the vertices of P_6 and P_8 must be the interior vertices of the T-mesh. The vertices of P_8 must be p-vertices. For $j = 2, 4$, we denote by p_j^b the number of vertices in P_j which are on the edges of Ξ and p_j^i the number of others which are the interior vertices. The others are denoted as p_j^f which are on the faces of Ξ. It is easy to see that p_4^b is 0. As an example in Fig. 1b, we can see that $p_2^b = 15$, $p_2^f = 6$, $p_2^i = 3$, $p_4^f = 9$, $p_4^i = 3$, $p_6 = 0$ and $p_8 = 2$. For an arbitrary hierarchical T-mesh, we can get the following equations.

$$\begin{aligned} 8C &= 2p_2 + 4p_4 + 6p_6 + 8p_8 + 8, \\ 2E_x^+ + 2E_y^+ + 2E_z^+ &= 4p_2^b + p_2^f + 5p_4^f + p_4^i - V_x - V_y - V_z + 2p_6 + 6p_8 + 24, \\ 4F_{xy}^c + 4F_{yz}^c + 4F_{zx}^c &= 5p_2^b + p_2^i + 3p_2^f + 8p_4^f + 4p_4^i - 2V_x - 2V_y - 2V_z + 7p_6 + 12p_8 + 24, \\ V^+ &= p_2^b + p_4^f + p_8 + 8. \end{aligned} \tag{12}$$

Summing all the equations in Eq. (12), we can see that the left of the Eq. (10) equals to:

$$p_2^i + p_4^i + p_6 + p_8 + V_x + V_y + V_z = V^i + V_x + V_y + V_z$$

So the Eq. (10) holds.

4.2 Dimension formula

Now we are ready to prove the dimension formula for the spline space $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ when $m \geqslant 2\alpha+1$, $n \geqslant 2\beta+1$ and $k \geqslant 2\gamma+1$ and $\mathcal{T}$ is a hierarchical T-mesh. It involves only the topological quantities of the T-mesh.

Theorem 1 *Given a regular 3D hierarchical T-mesh, suppose $m \geqslant 2\alpha+1$, $n \geqslant 2\beta+1$ and $k \geqslant 2\gamma+1$. Then the dimension of the corresponding spline space $\mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T})$ is*

$$\begin{aligned}\dim \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T}) = {} & C(m+1)(n+1)(k+1) - F_{xy}^i(m+1)(n+1)(\alpha+1) - F_{yz}^i(m+1) \\ & (\beta+1)(k+1) - F_{zx}^i(\alpha+1)(n+1)(k+1) + (E_x^i+V_x)(m+1) \\ & (\beta+1)(\gamma+1) + (E_y^i+V_y)(\alpha+1)(n+1)(\gamma+1) + (E_z^i+V_z)(\alpha+1) \\ & (\beta+1)(k+1) - (V^i+V_x+V_y+V_z)(\alpha+1)(\beta+1)(\gamma+1).\end{aligned} \quad (13)$$

Proof For any cell in the given T-mesh, since $m \geqslant 2\alpha+1$, $n \geqslant 2\beta+1$ and $k \geqslant 2\gamma+1$, we can divide the Bézier ordinates in the cell into twenty-seven parts as shown in Fig. 6.

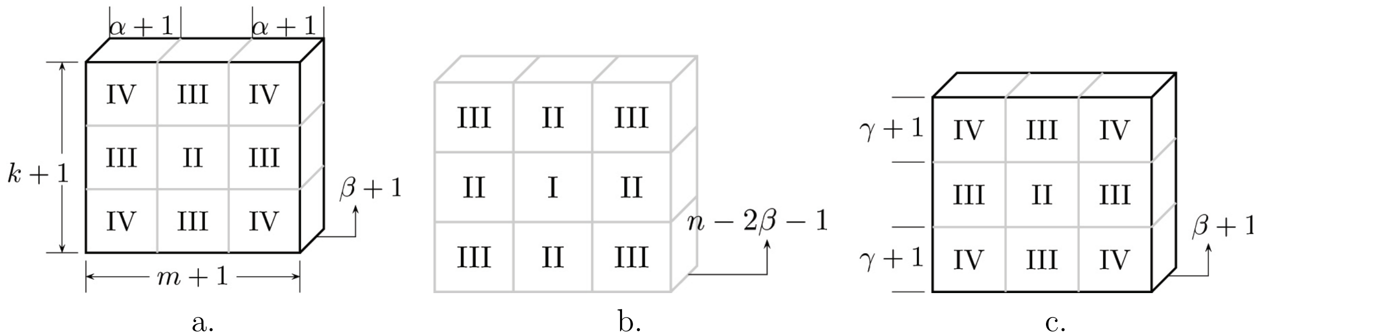


Fig. 6: A cell in T-mesh.

The Bézier ordinates in part I are free since no constraints are imposed on them. Hence we have obtained the first component in the dimension formula

$$\begin{aligned} d_1 &= C(m-2\alpha-1)(n-2\beta-1)(k-2\gamma-1) \\ &= C(m+1)(n+1)(k+1) - 2C(m+1)(n+1)(\alpha+1) - 2C(m+1)(\beta+1)(k+1) - \\ &\quad 2C(\alpha+1)(n+1)(k+1) + 4C(m+1)(\beta+1)(\gamma+1) + 4C(\alpha+1)(n+1)(\gamma+1) + \\ &\quad 4C(\alpha+1)(\beta+1)(k+1) - 8C(\alpha+1)(\beta+1)(\gamma+1). \end{aligned} \quad (14)$$

Now we will consider how many free Bézier ordinates of the parts labeled with II, III and IV according to the Lemma 1.

Firstly, we consider the free ordinates in parts labeled with II. Notice that for any c-face, there is only one part free and the part will not lie in the other c-faces (the parts with diagonal lines in Fig. 7). In every part, there are $(m-2\alpha-1)(n-2\beta-1)(\gamma+1)$ free ordinates. As there are F_{xy}^c c-faces who are parallel to xy-plane, so the free ordinates of this direction are

$$F_{xy}^c(m-2\alpha-1)(n-2\beta-1)(\gamma+1).$$

The same result can be got of the other two directions. So it is easy to show that the free Bézier ordinates of part II are

$$d_2 = F^c_{xy}(m-2\alpha-1)(n-2\beta-1)(\gamma+1) + F^c_{yz}(\alpha+1)(n-2\beta-1)(k-2\gamma-1) + F^c_{zx}(m-2\alpha-1)(\beta+1)(k-2\gamma-1). \quad (15)$$

Next, we will come to the part III. Notice that each part III is around an edge of a cell. The part III corresponds an edge one to one. For any c-edge of Ω, by the subdivision algorithm of tensor product surfaces ([4]), we can prove that the determination of the Bézier ordinates of each edge of the c-edge is equivalent. So the number of the free parts of part III at most is the number of the c-edges. In the following, we will prove that the number of the free parts is the number of the p-edges.

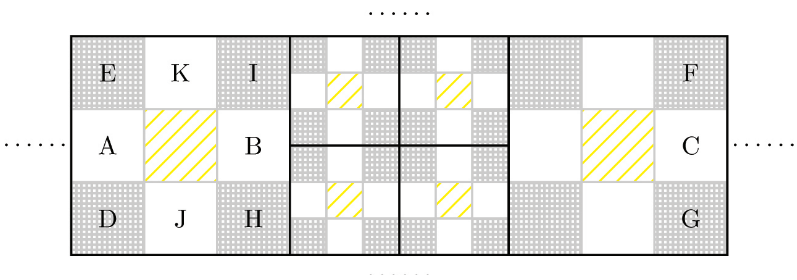


Fig. 7: An interior c-face of xy.

As every edge has three directions, we assume that the free parts also have three directions. In any face of a cell, there are four parts III, two are in one direction and the other two are in another direction. In a c-face, there are two free parts labeled with III along some direction (For example the direction of x). Referring to Fig. 7, according to the analysis for calculating d_2, if the Bézier ordinates which is labeled with A , B, J and K are fixed, then all the Bézier ordinates of the white in Fig. 7 will be fixed as well. By the subdivision algorithm of tensor product surfaces ([4]), we can prove that the determination of the Bézier ordinates of A and B is equivalent to determining A and C. So the frames A and C (the direction of x) in the part need to be determined. Therefore we have proved that if the Bézier ordinates in part III around the two end c-edges of a c-face are fixed, then the Bézier ordinates in part III around every interior c-edge on the c-face are determined correspondingly.

But all these parts of the two end c-edge of a c-face may belong to another c-face. For example, a c-edge of direction x may belong to a c-face of xy and a c-face of zx. If an end c-edge of a c-face is not a p-edge, it must lie in a c-face of another direction. So it can be determined by the two end c-edges of the c-face. Keeping on this process, we can see that if the Bézier ordinates around all the p-edges are filled, then the Bézier ordinates around all the edges will be determined correspondingly. Furthermore, the Bézier ordinates around the p-edges are independent. Thus, the free ordinates of this direction are

$$E^+_x(m-2\alpha-1)(\beta+1)(\gamma+1).$$

The same result will be got of the other two directions. So the free ordinates of part III are

$$d_3 = E^+_x(m-2\alpha-1)(\beta+1)(\gamma+1) + E^+_y(\alpha+1)(n-2\beta-1)(\gamma+1) + E^+_z(\alpha+1)(\beta+1)(k-2\gamma-1).$$

This part will also be counted into the dimension formula.

Till now, only the Bézier ordinates around vertices in parts labeled with IV have to be determined. Similarly, in a c-face, there are four free parts labeled with IV. Referring to Fig. 7, according to the analysis for calculating d_2 and d_3, if the Bézier ordinates which is labeled with D, E, H and I (the number of the Bézier ordinates in every frame is $(\alpha+1)(\beta+1)(\gamma+1)$) are fixed, then all the Bézier ordinates of the grey in Fig. 7 will be fixed as well. By the subdivision algorithm of tensor product surfaces ([4]), we can prove that the determination of the Bézier ordinates in the frames D, E, H and I is equivalent to the determination of the Bézier ordinates in the frames of D, E, F and G. Given a vertex, if it is not a p-vertex, then it must lie in the interior of a c-face, so it can be determined by the Bézier ordinates around a vertex of another c-face. Keeping on the process, we can see that if the Bézier ordinates around all the p-vertices are filled, then the Bézier ordinates around all the vertices will be determined correspondingly. Furthermore, the Bézier ordinates around the p-vertices are independent. As there are V^+ p-vertices, so the free ordinates for this part are

$$d_4 = V^+(\alpha+1)(\beta+1)(\gamma+1).$$

This part will be also counted into the dimension formula.

Now, all the free parts have been confirmed. So the dimension is the sum of $d_i, i = 1, 2, 3, 4$

$$\begin{aligned}
& \dim \mathcal{S}(m,n,k,\alpha,\beta,\gamma,\mathcal{T}) = d_1 + d_2 + d_3 + d_4 \\
= \; & C(m+1)(n+1)(k+1) - (2C - F^c_{xy})(m+1)(n+1)(\alpha+1) - (2C - F^c_{yz})(m+1) \\
& (\beta+1)(k+1) - (2C - F^c_{zx})(\alpha+1)(n+1)(k+1) + (4C + E^+_x - 2F^c_{xy} - 2F^c_{zx}) \\
& (m+1)(\beta+1)(\gamma+1) + (4C + E^+_y - 2F^c_{xy} - 2F^c_{yz})(\alpha+1)(n+1)(\gamma+1) \\
& +(4C + E^+_z - 2F^c_{zx} - 2F^c_{yz})(\alpha+1)(\beta+1)(k+1) \\
& -(8C + 2E^+_x + 2E^+_y + 2E^+_z - 4F^c_{xy} - 4F^c_{yz} - 4F^c_{zx} - V^+)(\alpha+1)(\beta+1)(\gamma+1) \\
= \; & C(m+1)(n+1)(k+1) - F^i_{xy}(m+1)(n+1)(\alpha+1) - F^i_{yz}(\beta+1)(k+1) \\
& -F^i_{zx}(\alpha+1)(n+1)(k+1) + (E^i_x + V_x)(m+1)(\beta+1)(\gamma+1) \\
& +(E^i_y + V_y)(\alpha+1)(n+1)(\gamma+1) + (E^i_z + V_z)(\alpha+1)(\beta+1)(k+1) \\
& -(V^i + V_x + V_y + V_z)(\alpha+1)(\beta+1)(\gamma+1).
\end{aligned}$$

The second equal mark is because of the Lemma 2. So this completes the proof of the theorem.

Denote F, E, V as the number of all the interior faces, interior edges and interior vertices of a hierarchical T-mesh respectively. For the special case where $m = n = k$ and $\alpha = \beta = \gamma$, we have the following corollary.

Corollary 1 *Suppose $m \geqslant 2\alpha + 1$. Then*

$$\begin{aligned}
\dim \mathcal{S}(m,m,m,\alpha,\alpha,\alpha,\mathcal{T}) = & (m+1)^3 + (C-1)(m+1)(m-\alpha)(m+\alpha+2) - E(m+1) \\
& (\alpha+1)(m-\alpha) + (V - V_x - V_y - V_z)(\alpha+1)^2(m-\alpha). \qquad (16)
\end{aligned}$$

Proof The difference between Eqs. (13) and (16) is

$$(m+1)(\alpha+1)^2(C - F + E - V - 1).$$

By Euler's formula, it follows that $C - F + E - V - 1 = 0$. Hence the Eq. (16) holds.

If $m = 2\alpha + 1$, $n = 2\beta + 1$ and $k = 2\gamma + 1$, we can give the dimension formulae of the spline space over arbitrary T-mesh. It is only associated with the number of p-vertex and m, n, k.

Corollary 2 *If $m = 2\alpha + 1$, $n = 2\beta + 1$ and $k = 2\gamma + 1$, and for arbitrary T-mesh $\mathcal{T}$, we have*

$$\dim \mathcal{S}(2\alpha + 1, 2\beta + 1, 2\gamma + 1, \alpha, \beta, \gamma, \mathcal{T}) = (\alpha + 1)(\beta + 1)(\gamma + 1)V^+. \quad (17)$$

Proof Notice that the proof in the Section 4.2 is not especial for hierarchical T-mesh. So the dimension of the spline space over arbitrary T-mesh is also the sum of $d_i, i = 1, 2, 3, 4$. When $m = 2\alpha+1$, $n = 2\beta+1$ and $k = 2\gamma+1$, all the d_i are zero except d_4. That is to say the dimension is $d_4 = (\alpha + 1)(\beta + 1)(\gamma + 1)V^+$.

4.3 Remarks

According to the proof of Theorem 1, we have an approach to define the basis functions for the spline space over a T-mesh. We have totally $\sum_{i=1}^{4} d_i$ free Bézier ordinates and for each of them, we let its value be one and all the other free ordinates be zero. The rest of Bézier ordinates can be determined by continuity conditions. In this way, we can define a spline function for each free ordinate, which can serve as the basis function of the spline space. Generally, most of the basis functions have compact supports. But this type of basis functions may not be evenly distributed. In the future we will work on how to construct more evenly distributed basis functions.

As we have provided a representation of implicit surface. We expect this type of splines to play an import role in the following fields, such as implicit surface reconstruction, blending, spatial movement or deformation of a surface and so on, which will be discussed in our future work.

5 Conclusions and Future Work

In this paper, we introduced the spline spaces over 3D T-meshes, which are a natural generalization of the work in [3]. Based on the B-net method, an algorithm to compute the dimension of a spline space over a T-mesh is presented, and in the case that $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$ and $k \geqslant 2\gamma + 1$, a dimension formula of the spline over hierarchical T-mesh is derived. The expectation to introduce the new splines is that they can not only inherit the attractive properties of T-splines but also provide more simplicity and flexibility in adaptive and dynamic implicit surface reconstruction from unorganized point clouds.

The paper leaves several open problems for further research. For example, currently the dimension formula is obtained under the constraints that $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$, $k \geqslant 2\gamma + 1$ and hierarchical T-meshes. A general dimension formula without these constraints is still unavailable.

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