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Implicitization and parametrization of quadratic and cubic surfaces by μ-bases

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Abstract

Parametric and implicit forms are two common representations of geometric objects. It is important to be able to pass back and forth between the two representations, two processes called parameterization and implicitization, respectively. In this paper, we study the parametrization and implicitization of quadrics (quadratic parametric surfaces with two base points) and cubic surfaces (cubic parametric surfaces with six base points) with the help of μ-bases – a newly developed tool which connects the parametric form and the implicit form of a surface. For both cases, we show that the minimal μ-bases are all linear in the parametric variables, and based on this observation, very efficient algorithms are devised to compute the minimal μ-bases either from the parametric equation or the implicit equation. The conversion between the parametric equation and the implicit equation can be easily accomplished from the minimal μ-bases.

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1. Introduction

A surface defined by an algebraic equation of degree two (resp. three) is called a quadric (resp. cubic surface). In geometric modeling systems, parametric representations of surfaces are preferred because of their simple analytical properties. On the other hand, these systems rely heavily on the ability to decide quickly whether a given point is inside or outside a given object. If that object is defined by a simple implicit surface, such a decision is quick and reliable [11]. Hence one possibly needs both the parametric representations and the implicit representations depending on different applications. The process of converting parametric forms into implicit forms is called *implicitization*, and the converse process is called *parameterization*.

In this paper, we study implicitization and parameterization of quadratic and cubic surfaces. A nonsingular quadric has a rational parameterization of degree two with two base points [17], and a nonsingular cubic surface has a parameterization of degree three with six base points [1]. A base point of a rational surface $\mathbf{P}(s,t) = (x(s,t), y(s,t), z(s,t), w(s,t))$ is a parameter value (s_0,t_0) such that $\mathbf{P}(s_0,t_0) = (0,0,0,0)$.

There are several methods to solve the implicitization and parameterization problems for a general surface. For implicitization, resultants and Gröbner bases are two mostly used methods [15], while for parameterization, the process is much harder [13]. For quadratic and cubic surfaces, special methods exist to solve the above problems. For example, Wang presented a method to parameterize a quadric by a stereographic projection [17]; while in [1], the implicitization and parametrization of a nonsingular cubic surface are unified with Hilbert-Burch matrices. The purpose of this paper is to deal with the implicitization and parametrization of quadratic and cubic surfaces by the μ-basis theory in a more unified and simple approach.

The μ-basis was first introduced in [9] to provide a compact representation for the implicit equation of a rational parametric curve. Then it was generalized by one of the present authors to general rational surfaces [2], [3], [5]. The μ-basis of a rational curve/surface can recover the parametric equation as well as derive the implicit equation of the curve/surface. Thus it serves as a connection between the implicit form and the parametric form of a curve/surface.

Unfortunately, except for rational ruled surfaces, little is known about the μ-bases of general parametric surfaces. For example, one does not have a degree bound for the μ-bases, and one does not know how to compute a μ-basis with minimal degree, though an algorithm to compute a non-minimal μ-basis of a rational surface was developed recently [10]. In this paper, we show that the minimal μ-bases of quadrics and cubic surfaces are linear in the parameter variables, and based on this fact, efficient algorithms are developed to compute the minimal μ-bases either from the parametric equation or the implicit equation. Furthermore, the implicit equation and the parametric equation can be derived straightforward from the minimal μ-bases.

The paper is organized as follows. In the next section, we present some preliminary knowledge about the μ-bases of rational surfaces. In Sects. 3 and 4, the minimal μ-bases of quadrics and cubic surfaces are derived, and conversion between the parametric form and the implicit form is accomplished by the minimal μ-bases. Section 5 concludes the paper with a summary and some further research problems.

2. Preliminaries

Let $\mathbb{R}[s,t]$ be the polynomial ring which consists of all the bivariate polynomials with real coefficients. A rational parametric surface in homogenous form is defined as

$$\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t)), \tag{2.1}$$

where $a, b, c, d \in \mathbb{R}[s,t]$ are polynomials with $\gcd(a, b, c, d) = 1$. The rational surface (2.1) is assumed to be nonsingular. In the later discussion, sometimes we need to homogenize the variables s, t to s, t, u. Hence $\mathbf{P}(s,t) = (a(s,t), b(s,t), c(s,t), d(s,t))$ is also written as $\mathbf{P}(s,t,u) = (a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u))$, where $a(s,t,u)$, $b(s,t,u)$, $c(s,t,u)$, and $d(s,t,u)$ are homogenous polynomials with the same total degree.

A moving plane is a family of planes with parameter pair (s, t), defined by

$$L(s,t) := L(x,y,z;s,t) := A(s,t)x + B(s,t)y + C(s,t)z + D(s,t) = 0,$$

or, in vector form:

$$\mathbf{L}(s,t) := (A(s,t), B(s,t), C(s,t), D(s,t)) \in \mathbb{R}[s,t]^4.$$

A moving plane $\mathbf{L}(s,t)$ is said to *follow* the rational surface $\mathbf{P}(s,t)$ if

$$\mathbf{L}(s,t)\cdot\mathbf{P}(s,t) = a(s,t)A(s,t) + b(s,t)B(s,t) + c(s,t)C(s,t) + d(s,t)D(s,t) \equiv 0.$$

Let $\mathbf{L}_{s,t}$ be the set of the moving planes following the rational surface $\mathbf{P}(s,t)$. Then $\mathbf{L}_{s,t}$ is exactly the syzygy module $\mathrm{syz}(a,b,c,d)$ and is a free module of rank 3 [2].

Definition 1: *Let* $\mathbf{p}, \mathbf{q}, \mathbf{r} \in \mathbf{L}_{s,t}$ *be three moving planes of* $\mathbf{P}(s,t)$ *such that*

$$[\mathbf{p},\mathbf{q},\mathbf{r}] = \kappa\mathbf{P}(s,t) \tag{2.2}$$

for some nonzero constant κ*, where* $\mathbf{p} = (p_1, p_2, p_3, p_4)$*,* $\mathbf{q} = (q_1, q_2, q_3, q_4)$*, and* $\mathbf{r} = (r_1, r_2, r_3, r_4)$*. Then* $\{\mathbf{p},\mathbf{q},\mathbf{r}\}$ *are called to form a* **μ-basis** *of the rational surface* $\mathbf{P}(s,t)$*. Here* $[\mathbf{p},\mathbf{q},\mathbf{r}]$ *is the* **outer product** *of* $\mathbf{p}$*,* $\mathbf{q}$ *and* $\mathbf{r}$ *which is defined by*

$$[\mathbf{p},\mathbf{q},\mathbf{r}] = \left(\begin{vmatrix} p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{vmatrix}, -\begin{vmatrix} p_1 & q_1 & r_1 \\ p_3 & q_3 & r_3 \\ p_4 & q_4 & r_4 \end{vmatrix}, \begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_4 & q_4 & r_4 \end{vmatrix}, -\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} \right). \tag{2.3}$$

If, in addition, among all the triples of $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *satisfying* (2.2)*, the total degree* $\mathrm{tdeg}(\mathbf{p}) + \mathrm{tdeg}(\mathbf{q}) + \mathrm{tdeg}(\mathbf{r})$ *is smallest, then* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *are called to form a* **minimal μ-basis**.

The existence of μ-bases was proved in [2] and an algorithm was developed to compute a μ-basis in [10]. However, it is an unsolved problem to compute a minimal μ-basis for a general rational surface.

3. Quadrics

In this section, we first show that the minimal μ-bases of a quadric surface are linear in the variables s, t, u, and then we present a very simple algorithm to compute the minimal μ-bases. The conversion between the parametric form and the implicit form of a quadric is thus derived.

3.1. The form of the minimal μ-bases

For a quadratic parametric surface, its minimal μ-bases are linear in the parameter variables.

Lemma 1: *Suppose* $\mathbf{P}(s, t, u)$ *is a rational parametric quadratic surface with two base points. Then it has a minimal* μ*-basis with the following form:*

$$\begin{aligned}\mathbf{p} &= (p_1, p_2, p_3, p_4) = \mathbf{p}_u u + \mathbf{p}_s s,\\ \mathbf{q} &= (q_1, q_2, q_3, q_4) = \mathbf{q}_u u + \mathbf{q}_t t,\\ \mathbf{r} &= (r_1, r_2, r_3, r_4) = \mathbf{r}_u u + \mathbf{r}_s s + \mathbf{r}_t t,\end{aligned} \tag{3.1}$$

where $\mathbf{p}_u$, $\mathbf{p}_s$, $\mathbf{q}_u$, $\mathbf{q}_t$, $\mathbf{r}_u$, $\mathbf{r}_s$, *and* $\mathbf{r}_t$ *are constant vectors in* $\mathbb{R}^4$, *and* $\mathbf{p}_s = \mathbf{q}_t$.

Proof: The details are very tedious and we just outline the proof. For the details, the reader is referred to [16]. By choosing a proper parameter transformation $(s, t, u) = (\tilde{s}, \tilde{t}, \tilde{u})\mathbf{M}$, where $\mathbf{M}$ is a three by three invertible matrix, the quadratic surface $\mathbf{P}(s, t, u)$ with two base points can be transformed into a ruled surface by mapping one of the base points to (0, 1, 0):

$$\tilde{\mathbf{P}}(\tilde{s}, \tilde{t}, \tilde{u}) = (p_1(\tilde{s}, \tilde{t}, \tilde{u}), p_2(\tilde{s}, \tilde{t}, \tilde{u}), p_3(\tilde{s}, \tilde{t}, \tilde{u}), p_4(\tilde{s}, \tilde{t}, \tilde{u})),$$

where $p_i(\tilde{s}, \tilde{t}, \tilde{u}) = p_{i0}\tilde{u}^2 + p_{i1}\tilde{s}\tilde{u} + p_{i2}\tilde{t}\tilde{u} + p_{i3}\tilde{s}\tilde{t} + p_{i4}\tilde{s}^2$. By the result in [3], the μ-bases of $\tilde{\mathbf{P}}(\tilde{s}, \tilde{t}, \tilde{u})$ have the form $\tilde{\mathbf{p}}(\tilde{s}, \tilde{u})$, $\tilde{\mathbf{q}}(\tilde{s}, \tilde{u})$, $\tilde{\mathbf{r}}(\tilde{t}, \tilde{u})$ with total degree one. Then it follows that

$$[\tilde{\mathbf{p}}, \tilde{\mathbf{q}}, \tilde{\mathbf{r}}] = k_1\tilde{u}\tilde{\mathbf{P}}(\tilde{s}, \tilde{t}, \tilde{u}), \quad \text{for some nonzero constant } k_1. \tag{3.2}$$

By applying the inverse transformation of $\mathbf{M}$, one does not usually obtain the μ-basis of $\mathbf{P}(s, t, u)$, since $\tilde{u}$ in Eq. (3.2) would be replaced by a linear combination of s, t, u. Hence some linear transformation should be taken on $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}$, $\tilde{\mathbf{r}}$ before applying the inverse transformation of $\mathbf{M}$. The details are omitted. □

3.2. Computing μ*-bases*

Based on the linear form of the minimal μ-basis of a quadric, we present efficient algorithms to compute the minimal μ-basis either from the parametric form or the implicit form of the quadric, respectively.

3.2.1. From parametric equations

The following lemma is essential to the μ-basis algorithm below.

Lemma 2: *Let* $\mathbf{P}(s, t)$ *be a quadratic parametric surface with two base points. Then there exist unique vectors (up to a scalar multiple)* $\mathbf{p}_u, \mathbf{p}_s, \mathbf{q}_u \in \mathbb{R}^4$ *such that*

$$(\mathbf{p}_u + \mathbf{p}_s s) \cdot \mathbf{P}(s, t) = (\mathbf{q}_u + \mathbf{p}_s t) \cdot \mathbf{P}(s, t) = 0.$$

Furthermore, there exist at least four linearly independent solutions (*with* $\mathbf{r}_u, \mathbf{r}_s, \mathbf{r}_t \in \mathbb{R}^4$ *being unknowns*) *for the following equation:*

$$(\mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t) \cdot \mathbf{P}(s, t) = 0.$$

Proof: In the proof, all the vectors are in column form. Assume $\mathbf{P}(s,t) = \mathbf{P}_1 + \mathbf{P}_s s + \mathbf{P}_t t + \mathbf{P}_{s^2} s^2 + \mathbf{P}_{st} st + \mathbf{P}_{t^2} t^2$. It follows that $(\mathbf{p}_u + \mathbf{p}_s s)\cdot\mathbf{P}(s,t) = (\mathbf{q}_u + \mathbf{p}_s t)\cdot\mathbf{P}(s,t) \equiv 0$ if and only if

$$(\mathbf{p}_u^T, \mathbf{p}_s^T, \mathbf{q}_u^T)\mathbf{M} = \mathbf{0},$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{P}_s & \mathbf{P}_t & \mathbf{P}_{s^2} & \mathbf{P}_{st} & \mathbf{P}_{t^2} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_1 & \mathbf{0} & \mathbf{P}_s & \mathbf{P}_t & \mathbf{0} & \mathbf{P}_{s^2} & \mathbf{P}_{st} & \mathbf{P}_{t^2} & \mathbf{0} & \mathbf{0} & \mathbf{P}_1 & \mathbf{0} & \mathbf{P}_s & \mathbf{P}_t \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{P}_1 & \mathbf{P}_s & \mathbf{P}_t & \mathbf{P}_{s^2} & \mathbf{P}_{st} & \mathbf{P}_{t^2} \end{pmatrix}$$

is a matrix with size 12×15. We know $\mathbf{P}(s,t)$ has two base points. If these two base points are different, denote them as (s_i, t_i, u_i), $i = 1, 2$. Then it follows that $u_i^2\mathbf{P}_1 + u_i s_i \mathbf{P}_s + u_i t_i \mathbf{P}_t + s_i^2 \mathbf{P}_{s^2} + s_i t_i \mathbf{P}_{st} + t_i^2 \mathbf{P}_{t^2} = \mathbf{0}$, $i = 1, 2$. From these relationships, we can show that rank($\mathbf{M}$) is exactly 11 by applying column transformations on $\mathbf{M}$. If the two base points are a double base point with multiplicity two, the same result holds. This proves the first part of the lemma. The second part can be proved by a similar technique. □

Now the algorithm to compute minimal μ-bases is described as follows.

Algorithm (PAR-MU-BASIS)

Input: A quadratic parametric surface $\mathbf{P}(s,t)$ with two base points.
Output: A minimal μ-basis of the quadratic surface.
Steps:

1. Solve the linear system of equations:

$$(\mathbf{p}_u + \mathbf{p}_s s)\cdot\mathbf{P}(s,t) = (\mathbf{q}_u + \mathbf{p}_s t)\cdot\mathbf{P}(s,t) = 0$$

   with $\mathbf{p}_u, \mathbf{q}_u, \mathbf{p}_s$ being unknown vectors. By Lemma 2, there exists one solution.
2. Solve the linear system

$$(\mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t)\cdot\mathbf{P}(s,t) = 0$$

   with $\mathbf{r}_u, \mathbf{r}_s, \mathbf{r}_t$ being unknown vectors. By Lemma 2, we can select $\mathbf{r}_u, \mathbf{r}_s, \mathbf{r}_t$ such that $\mathbf{p}_u + \mathbf{p}_s s$, $\mathbf{q}_u + \mathbf{p}_s t$ and $\mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t$ are linearly independent.
3. Output $\mathbf{p}_u + \mathbf{p}_s s$, $\mathbf{q}_u + \mathbf{p}_s t$ and $\mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t$. By Lemma 1 and 2, they are a minimal μ-basis of the quadratic surface $\mathbf{P}(s,t)$.

We provide some examples to illustrate the above algorithm.

Example 1: *Suppose the quadratic surface is defined by*

$$\mathbf{P}(s,t) = (-3 - 2t + 3s - st, 1 + 2t - s, 3 - 3t + 2s + 2st, -3 - t + 3s - 3st)$$

which has two base points $(1, 0, 0)$*,* $(0, 1, 0)$*. A minimal* μ*-basis is computed by our algorithm as*

$$\begin{bmatrix} 17+13s & 15+13t & 5-3s \\ 27-2s & 30-2t & 3-6s \\ 7-7s & -7t & 0 \\ -1-9s & -5-9t & -4+s \end{bmatrix},$$

where the three columns of the matrix are the three elements of the μ*-basis.*

Example 2: *Let the quadratic surface be*

$$\mathbf{P}(s,t) = (1+4t+3s^2, -4+3t-s, -4-4t-5s-4s^2, -2-5t+5s-5s^2)$$

which has a double base point $(0, 1, 0)$*. A minimal* μ*-basis is computed as follows:*

$$\begin{bmatrix} 36+135s & -168+135t & -858+13842s \\ -12-45s & 45-45t & 99-4614s \\ 17+50s & -61+50t & 5189s \\ 8+41s & -52+41t & -627+4154s \end{bmatrix}.$$

Example 3: *Consider a quadratic surface defined by*

$$\mathbf{P}(s,t) = (1-s^2-t^2, 2t, 2s, s^2+t^2+1).$$

It has a pair of conjugate complex base points $(\pm i, 1, 0)$*. A minimal* μ*-basis by our algorithm is:*

$$\begin{bmatrix} s & t & 1 \\ 0 & -1 & t \\ -1 & 0 & s \\ s & t & -1 \end{bmatrix}.$$

3.2.2. From implicit equations

Although the μ-bases are defined for parametric equations, we can also design μ-bases for implicit equations.

Lemma 3: *Given a quadratic implicit equation* $f(x, y, z) = 0$*, let* $Q_0 = (x_0, y_0, z_0)$ *be a point on the quadric surface, and* p_u*,* q_u *and* $p_s = q_t$ *be three linearly independent planes passing through* Q_0*. Then there exist three linear functions (in* x, y, z*)* r_u*,* r_s *and* r_t *such that*

$$p_u r_s + q_u r_t - p_s r_u = f(x, y, z). \tag{3.3}$$

Proof: Consider the ideal $I = \langle p_u, q_u, p_s \rangle$. Since p_u, q_u and $p_s = q_t$ are independent planes passing through Q_0, $V(I) = Q_0$. Since I is a maximal ideal and $f(Q_0) = 0$, by Hilbert's Nullstallensatz [8], $f \in I$. That is, there exist polynomials (in x, y, z) r_u, r_s, r_t such that Eq. (3.3) holds. Furthermore, it can be shown that these polynomials can be linear in x, y, z. □

Theorem 4: *Let $p_u, p_s, q_u, q_t, r_u, r_s, r_t$ be defined as in Lemma 3. Define*

$$p = p_u + p_s s, \quad q = q_u + q_t t, \quad r = r_u + r_s s + r_t t,$$

and let **p**, **q**, **r** *be the vector forms of p, q, r. Then*

$$\mathbf{P}(s,t) := [\mathbf{p}, \mathbf{q}, \mathbf{r}]$$

is a parametric representation of $f(x, y, z) = 0$.

Proof: For any parameter values (s_0, t_0), let $\mathbf{X} = (x_0, y_0, z_0, 1) = \mathbf{P}(s_0, t_0)$. From $\mathbf{P}(s,t)\cdot\mathbf{p} \equiv 0$, $\mathbf{P}(s,t)\cdot\mathbf{q} \equiv 0$ and $\mathbf{P}(s,t)\cdot\mathbf{r} \equiv 0$, one has

$$p(x_0, y_0, z_0; s_0, t_0) = 0, \quad q(x_0, y_0, z_0; s_0, t_0) = 0, \quad r(x_0, y_0, z_0; s_0, t_0) = 0.$$

Thus

$$\begin{pmatrix} p_u(x_0, y_0, z_0) & p_s(x_0, y_0, z_0) & 0 \\ q_u(x_0, y_0, z_0) & 0 & q_t(x_0, y_0, z_0) \\ r_u(x_0, y_0, z_0) & r_s(x_0, y_0, z_0) & r_t(x_0, y_0, z_0) \end{pmatrix} \begin{pmatrix} 1 \\ s_0 \\ t_0 \end{pmatrix} = \mathbf{0}.$$

Therefore

$$\begin{aligned} f(x_0, y_0, z_0) &= (p_u r_s + q_u r_t - p_s r_u)|_{(x,y,z)=(x_0,y_0,z_0)} \\ &= \begin{vmatrix} p_u(x_0, y_0, z_0) & p_s(x_0, y_0, z_0) & 0 \\ q_u(x_0, y_0, z_0) & 0 & q_t(x_0, y_0, z_0) \\ r_u(x_0, y_0, z_0) & r_s(x_0, y_0, z_0) & r_t(x_0, y_0, z_0) \end{vmatrix} = 0. \end{aligned}$$

On the other hand, one can show that

$$r_s p_u + r_t q_u - p_s r_u \equiv 0 \Leftrightarrow [\mathbf{p}, \mathbf{q}, \mathbf{r}] \equiv \mathbf{0}.$$

Thus $\mathbf{P}(s,t) := [\mathbf{p}, \mathbf{q}, \mathbf{r}]$ gives a parameterization of $f(x, y, z) = 0$. □

Based on the above theorem, an algorithm for computing a μ-basis of a quadratic implicit surface is outlined as follows.

Algorithm (IMP-MU-BASIS)

Input: A quadratic implicit equation $f(x, y, z) = 0$.
Output: A μ-basis of the quadric surface.

Step:

1. Choose a point $Q_0 = (x_0, y_0, z_0)$ on the surface $f(x, y, z) = 0$, and three independent planes p_u, q_u and $p_s = q_t$ passing through the point Q_0.
2. Use polynomial division algorithm to divide $f(x, y, z)$ by p_u, q_u and p_s. Let the quotients be r_s, r_t and $-r_u$, i.e.,

$$f(x, y, z) = p_u r_1 + q_u r_2 - p_s r_u.$$

3. Let $p = p_u + p_s s, \quad q = q_u + q_t t, \quad r = r_u + r_s s + r_t t$. Output p, q, r.

Example 4: *Suppose the implicit equation of a quadric surface is* $f(x, y, z) = -116x^2 - 212xy + 117x - 156y^2 + 52y + 14xz - 21yz - 21z - 37 = 0$. *Choose a point* $\mathbf{Q}_0 = (1/3, 0, -2/3)$ *on the quadric, and three planes passing through* $\mathbf{Q}_0$ *are* $p_u = 17x + 27y + 7z - 1$, $q_u = 15x + 30y - 5$ *and* $p_s = 13x - 2y - 7z - 9$. *Using the polynomial division algorithm to find the quotients* r_u, r_s *and* r_t *of* f *divided by* p_u, q_u *and* p_s*:*

$$r_u = 5x + 3y - 4, \quad r_s = -3x - 6y + 1, \quad r_t = 0.$$

Then

$$\begin{aligned} \mathbf{p} &= (17 + 13s, 27 - 2s, 7 - 7s, -1 - 9s), \\ \mathbf{q} &= (15 + 13t, 30 - 2t, -7t, -5 - 9t), \\ \mathbf{r} &= (5 - 3s, 3 - 6s, 0, -4 + s). \end{aligned} \tag{3.4}$$

3.3. Applications of μ-bases

As we have observed in the above section, μ-bases serve as a connection between the parametric form and the implicit form of a quadric surface. Thus conversion between the parametric form and the implicit surface can be easily achieved. Furthermore, the inversion formula (given a point on a parametric surface, find corresponding parameter value) is directly obtained from the μ-bases.

3.3.1. Implicitization

Given the parametric equation $\mathbf{P}(s, t)$ of a quadric surface, we can compute a minimal μ-basis $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ of the surface by the algorithm PAR-MU-BASIS. Then the implicit equation of the surface is given by the following theorem.

Theorem 5: *Let*

$$p = p_u + p_s s, \quad q = q_u + q_t t, \quad r = r_u + r_s s + r_t t$$

be a minimal μ-basis of a quadratic surface with two base points. Then the implicit equation of the quadric surface is given by

$$f(x, y, z) := p_u r_s + q_u r_t - p_s r_u = 0.$$

Proof: Similar to the proof of Theorem 4. □

We illustrate an example.

Example 5: *Consider the quadratic surface in Example 3. A μ-basis is obtained as*

$$p = z - (x+1)s,$$
$$q = y - (x+1)t,$$
$$r = 1 - x - zs - yt.$$

So the implicit equation is $f(x, y, z, w) := p_u r_s + q_u r_t - r_u p_s = 1 - x^2 - y^2 - z^2 = 0.$

3.3.2. Parameterization

Given the implicit equation $f(x, y, z) = 0$ of a quadric surface, we can compute a μ-basis **p**, **q**, **r** of the surface by the algorithm IMP-MU-BASIS. Then a parametrization of the surface is given by $\mathbf{P}(s, t) = [\mathbf{p}, \mathbf{q}, \mathbf{r}]$.

The following example illustrates the parametrization procedure from μ-bases.

Example 6: *Considering the surface in Example 4, we have construct a minimal μ-basis as shown in Eq.* (3.4)*. Then the parametric equation of the surface is*

$$\mathbf{P}(s, t) = [\mathbf{p}, \mathbf{q}, \mathbf{r}] = (-3 - 2t + 3s - st, 1 + 2t - s, 3 - 3t + 2s + 2st, -3 - t + 3s - 3st).$$

3.3.3. Inversion formula

Given a point (x_0, y_0, z_0) on the quadratic parametric surface $\mathbf{P}(s, t)$, we compute a μ-basis:

$$p = p_y + p_s s, \quad q = q_u + q_t t, \quad r = r_u + r_s s + r_t t.$$

From $p(x_0, y_0, z_0; s, t) = 0$, $p(x_0, y_0, z_0; s, t) = 0$ and $p(x_0, y_0, z_0; s, t) = 0$, we obtain

$$s = -\frac{p_u(x_0, y_0, z_0)}{p_s(x_0, y_0, z_0)}, \quad t = -\frac{q_u(x_0, y_0, z_0)}{q_t(x_0, y_0, z_0)},$$

which is the inversion formula for the parametric equation $\mathbf{P}(s, t)$.

For the parametric surface in Example 3, the inversion formula is

$$s = \frac{z}{x+1}, \quad t = \frac{y}{x+1}.$$

4. Cubic surfaces

The results in the above section can be similarly generalized to cubic surfaces which can be parameterized by cubic rational parametric surfaces with six base points. We just summarize the results in the following theorems. We refer the reader to [16] for details.

Theorem 6: *The minimal μ-bases of a cubic rational parametric surface $\mathbf{P}(s,t)$ with six base points take the form*

$$\mathbf{p} = \mathbf{p}_u + \mathbf{p}_s s, \quad \mathbf{q} = \mathbf{q}_u + \mathbf{q}_t t, \quad \mathbf{r} = \mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t.$$

Based on the above theorem, the minimal μ-bases can be found using undetermined coefficients method from $\mathbf{p}\cdot\mathbf{P} = \mathbf{q}\cdot\mathbf{P} = \mathbf{r}\cdot\mathbf{P} = 0$.

Theorem 7: *Let*

$$\mathbf{p} = \mathbf{p}_u + \mathbf{p}_s s, \quad \mathbf{q} = \mathbf{q}_u + \mathbf{q}_t t, \quad \mathbf{r} = \mathbf{r}_u + \mathbf{r}_s s + \mathbf{r}_t t.$$

be a minimal μ-bases of $\mathbf{P}(s,t)$. Then the implicit equation of $\mathbf{P}(s,t)$ is given by

$$f(x,y,z) = \begin{vmatrix} p_s & 0 & p_u \\ 0 & q_t & q_u \\ r_s & r_t & r_u \end{vmatrix} = 0,$$

where $p_u = \mathbf{p}_u\cdot\mathbf{X}$, $p_s = \mathbf{p}_s\cdot\mathbf{X}$, $q_u = \mathbf{q}_u\cdot\mathbf{X}$, $q_t = \mathbf{q}_t\cdot\mathbf{X}$, $r_u = \mathbf{r}_u\cdot\mathbf{X}$, $r_s = \mathbf{r}_s\cdot\mathbf{X}$, $r_t = \mathbf{r}_t\cdot\mathbf{X}$, and $\mathbf{X} = (x,y,z,1)$.

Remark 1: *For a cubic surface, the implicit equation is exactly the resultant of the minimal μ-basis. The reason is that a μ-basis gives a homogeneous μ-basis in this case. On the other hand, for a quadratic surface, the resultant of the minimal μ-basis is a multiple of the implicit equation (with the multiple being $p_s = q_t$) since a μ-basis fails to give a homogeneous μ-basis.*

We illustrate an example.

Example 7: $\mathbf{P}(s,t) = (3s^2+s+s^2t-5st^2, 2s^2+st-3st^2, (-5s+2t+3s^2)/2, (s^2-3s+2t^2)/2)$ *is a cubic rational parametric surface with six base points* $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,1)$, $(1,2,3)$, *and* $(2,-1,1)$. *One of its minimal μ-bases is* $\mathbf{p} = (0,-1,s,-3s)$, $\mathbf{q} = (3,-5,-2t,2t)$, $\mathbf{r} = (9t, 3s-15t-5, -4s, 0)$. *Hence the implicit equation of $\mathbf{P}(s,t)$ is*

$$\begin{aligned} f(x,y,z) &= \begin{vmatrix} z-3 & 0 & -y \\ 0 & -2z & 3x-5y+2 \\ 3y-4z & 9x-15y & -5y \end{vmatrix} \\ &= 2yz^2 - 3x^2z + 10xyz - 9y^2z - 2xz + 9x^2 - 30xy + 25y^2 + 6x - 10y \\ &= 0. \end{aligned}$$

Conversely, if we are given the implicit equation of a cubic surface, we can also devise a minimal μ-basis and thus find a parameterization of the cubic surface.

Theorem 8: *Let $L_1 = p_s \cap q_t$, $L_2 = p_s \cap p_u$, $L_3 = q_t \cap q_u$, where p_s, p_u, q_t, q_u are defined in Theorem 7. Then L_1, L_2 and L_3 lie on the cubic surface $\mathbf{P}(s,t)$. Conversely, suppose that $f(x,y,z) = 0$ is a cubic surface, and $L_1 = p_s \cap q_t$, $L_2 = p_s \cap p_u$, $L_3 = q_t \cap q_u$ are three lines on the cubic surface. Then there are linear functions r_u, r_s and r_t such that*

$$f = p_s q_t r_u - p_u q_t r_s - p_s q_u r_t.$$

Based on the above theorem, the parametrization algorithm of a nonsingular cubic surface is stated as follows.

Input: A cubic implicit equation $f(x,y,z) = 0$.
Output: A parametric representation of the cubic surface.
Steps:

1. Find a line L_1 on the cubic surface.
2. Find planes p_s and q_t passing through L_1 such that they intersect the cubic surface into three lines, respectively. Let $L_2 = p_s \cap p_u$ and $L_3 = q_t \cap q_u$ be one of the three lines, respectively.
3. Find r_u, r_s and r_t using polynomial division algorithm such that
$$f = p_s q_t r_u - p_u q_t r_s - p_s q_u r_t.$$
4. Let $p = p_u + p_s s$, $q = q_u + q_t t$, $r = r_u + r_s s + r_t t$, and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be the vector forms of p, q, r. Then $\mathbf{P}(s,t) := [\mathbf{p}, \mathbf{q}, \mathbf{r}]$ is a parameterization of the cubic surface.

Remark 2: *The details in Step 1 and Step 2 can be found in [14].*

Remark 3: *The three rows of the Hilbert-Burch matrix in [1] is in fact the minimal μ-basis of a cubic surface.*

5. Conclusion

In this paper, the parametrization and implicitization of a quadric surface and a cubic surface are studied with the the help of μ-bases theory. In both cases, we proved that the minimal μ-bases of quadratic and cubic surfaces are all linear in the parametric variables. Based on this result, fast algorithms are proposed to compute the minimal μ-bases from either parametric representations or implicit representations, and the following conversion diagram between parametric forms and implicit forms is achieved:

A quadratic/cubic parametric surface	$\Longleftrightarrow$	μ-basis: three families of planes	$\Longleftrightarrow$	An implicit quadric/cubic surface

In the future, we will try to generalize the above diagram to higher degree curves and surfaces. We will also study the problem of finding "good" parametrization for a given implicit equation. Applications of μ-bases in other problems such as singular locus computation are also worthy of further exploration.

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