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# Optimal Degree Reduction of Interval Polynomials and Interval Bézier Curves under $\mathbf{L}_1$ Norm

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## Abstract

In this paper, we develop an analytic solution for the optimal degree reduction of interval polynomials and interval Bézier curves under $\mathbf{L}_1$ norm. The key ingredient is a characterization for the nonnegative least deviation polynomials from zero based on orthogonal polynomials. The direct application of the characterization leads to the optimal degree reduction algorithm. Analytic results and examples show our algorithm is much better than the previous method.

**Keywords**: Degree reduction, Interval polynomials, Interval Bézier curves, orthogonal polynomials

## 1 Introduction

Interval Bézier curves which were firstly introduced by Sederberg etal. [14] are new representation forms for parametric curves. Such representation form can embody a complete description about coefficient errors along with the curve, and it is convenient for tolerance analyse in geometric modeling.

Inspired by Sederberg etal.'s work, Hu etal. ([7, 8, 9, 10, 15]) recently appeal to interval form of geometric objects and rounded interval arithmetic to deal with the boolean operations in solid modeling. Their works indicate that using interval representations of geometric objects will substantially increase the numerical stability in geometric computations and thus enhance the robustness of current CAD/CAM systems.

In this paper, we will discuss the problem of bounding interval polynomials/Bézier curves with lower degree interval polynomials/Bézier curves. The degree reduction of Bézier curves has been a hot spot in CAGD in the past decade, and there are a lot of research literatures focusing on this problem([1, 2, 4, 5, 11, 16]). However, these works mainly concern how well the approximation is, and none has ever dealt with the approximation errors, or in other words, none has ever considered to approximate an interval Bézier curve with a lower degree interval Bézier curve. As the authors are aware, the only work which relates with degree reduction of interval polynomials is J. Rokne's paper [13], in which the author used (the graph of) an interval polynomial to bound (the graph of) a higher degree interval polynomial. In this paper, we will present a totally different approach to solve this problem. This new algorithm is optimal in the sense that the area of the degree reduction interval polynomial is the smallest in all the interval polynomials which bound a given interval polynomial. and the solution is analytic.

The organization of this paper is as follows. In Section 2, we give a characterization for the nonnegative least deviation polynomials from zero under $\mathbf{L}_1$ norm. Then we apply the result to the degree reduction of interval polynomials/Bézier curves in the following section. Finally in section 4, we provide some examples to demonstrate our algorithm. Theoretical results and examples show our algorithm is much better than the previous method(Rokne's method).

## 2 Nonnegative Least Deviation Polynomials

Let $\mathbb{R}[x]$ be the set of all polynomials with real coefficients, and

$$\Pi_n = \{p(x) \in \mathbb{R}[x] \,|\, \deg(p) \leqslant n\}.$$
$$\mathbf{P}_n = \{p(x) \in \mathbb{R}[x] \,|\, \deg(p) = n\}.$$
$$\mathbf{G}_n = \{p(x) \in \mathbf{P}_n \,|\, \mathrm{LC}(p) = 1\}.$$
$$\mathbf{H}_n = \{p(x) \in \mathbf{P}_n \,|\, \mathrm{LC}(p) = -1\}$$

where $\deg(p)$ and $\mathrm{LC}(p)$ denote the degree of $p$ and the leading coefficient of $p$ respectively.

For any polynomial $f(x) \in \mathbb{R}[x]$, we define the

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... norm of $f$ by

$$||f||_p = \left(\int_0^1 |f(x)|^p \, dx\right)^{\frac{1}{p}} \tag{2.1}$$

The *inner product* of $f, g \in \mathbb{R}[x]$ with respect to weight function $w(x)$ is defined by

$$\langle f, g\rangle = \int_0^1 w(x)f(x)g(x)dx. \tag{2.2}$$

Two polynomials $f$ and $g$ are said *orthogonal* to each other if

$$\langle f, g\rangle = 0 \tag{2.3}$$

Let S be a base of polynomial space $\mathbb{R}[x]$, if any two elements of S are orthogonal to each other, then we call S an *orthogonal polynomial system*, and the elements of S are called *orthogonal polynomials*. There are many special types of orthogonal polynomials such as Chebyshev polynomials(of second kind whose weight function $w(x) = 1/\sqrt{x(1-x)}$)

$$U_n(x) = \frac{\sin[(n+1)\arccos\,(2x-1)]}{\sin[\arccos\,(2x-1)]}. \tag{2.4}$$

and Legendre polynomials(with weight function $w(x) \equiv 1$)

$$L_n(x) = \frac{1}{2^{2n}n!}\frac{d^n}{dy^n}\{((2x-1)^2-1)^n\} \tag{2.5}$$

These orthogonal polynomials are special cases of the Jacobi polynomials

$$J_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n}\binom{n+a}{i}\binom{n+b}{n-i}(x-1)^{n-i}x^i \tag{2.6}$$

which are orthogonal with respect to weight function $w(x) = (1-x)^\alpha x^\beta$. Jacobi polynomials can be represented in Bézier form as

$$\begin{aligned} J_n^{(\alpha,\beta)}(x) &= \sum_{i=0}^{n}(-1)^{(n+i)}\frac{\binom{n+a}{i}\binom{n+b}{n-i}}{\binom{n}{i}}B_i^n(x) \\ &:= \sum_{i=0}^{n} c_{n,i}B_i^n(t) \end{aligned} \tag{2.7}$$

where $B_i^n(x) = \binom{n}{i}x^i(1-x)^{n-i}$ are Bernstein bases.

Orthogonal polynomials are a very important class of polynomials in approximation theory, and they have many nice properties, one of which is the following

**Lemma 1** *Let $p_n(x) \in \mathbf{G}_n$, $n = 0, 1, \ldots,$ be an orthogonal system with respect to some weight function $w(x)$. If $p_*(x) \in \mathbf{G}_n$ be the polynomial such that*

$$\int_0^1 w(x)p_*^2(x)\,dx = \min_{p\in\mathbf{G}_n}\int_0^1 w(x)p^2(x)\,dx \tag{2.8}$$

*then* $p_*(x) = p_n(x)$.

Now we are ready to characterize the nonnegative least deviation polynomials from zero.

**Theorem 1** *Let*

$$\mathbf{G}_n^+ = \{p(x) \in \mathbf{G}_n | p(x) \geqslant 0 \text{ over } [0,1]\}, \tag{2.9}$$

*and $p_n^* \in \mathbf{G}_n^+$ be the polynomial which minimizes integral*

$$\min_{p(x)\in\mathbf{G}_n^+}\int_0^1 p(x)\,dx \tag{2.10}$$

*Then*

$$p_n^*(x) = \begin{cases} \tilde{L}_k^2(x), & \text{if } n = 2k, \\ x\tilde{J}_k^2(x), & \text{if } n = 2k+1, \end{cases} \tag{2.11}$$

*where $\tilde{L}_k(x) = 2^n/\binom{2n}{n}L_k(x) \in \mathbf{G}_k$ is Legendre polynomial(multiplied by a constant), and $\tilde{J}_k(x) \in \mathbf{G}_k$ is the Jacobi polynomial $J^{(0,1)}(x)$ (multiplied by a constant). We call $p_n^*$ the nonnegative least deviation polynomial from zero(with leading coefficient 1).*

**Proof:** We will prove the theorem in following four steps when $n$ is even. The proof is similar for odd $n$.

**(1)** Polynomial $p_{2k}^*(x)$ has only real roots.

Suppose $p_{2k}^*$ has complex roots, so $p_{2k}^*$ has the following form:

$$p_{2k}^*(x) = (x^2 + ax + b)h(x),$$

where $a, b \in \mathbb{R}$, $a^2 < 4b$, and $h(x) \in \mathbf{G}_{2k-2}^+$. Thus

$$\int_0^1 p_{2k}^*(x)\,dx > \int_0^1 \left(x^2 + ax + \frac{a^2}{4}\right)h(x)\,dx$$

But $p_{2k}^*(x)$ minimizes integral (2.10), an impossibility.

**(2)** All the roots of polynomial $p_{2k}^*$ are in $[0,1]$.

Assume there exists a root $r$ of $p_{2k}^*$ is outside of $[0,1]$. Without loss of generality, we assume that $r < 0$. Then we can write

$$p_{2k}^* = (x - r)h(x),$$

where $h(x) \in \mathbf{G}_{2k-1}^+$. So

$$\int_0^1 p_{2k}^*(x)\,dx > \int_0^1 xh(x)\,dx.$$

which contradicts (2.10).

**(3)** There exists polynomial $h(x) \in \mathbf{G}_k$, such that $p_{2k}^*(x) = h^2(x)$.

Since $p_{2k}^*(x) \geqslant 0$ over interval $[0,1]$ and the leading coefficient of $p_{2k}^*(x)$ is positive , any root of $p_{2k}^*(x)$ has even multiplicity. Thus there exists $h(x) \in \mathbf{G}_k$ such that $p_{2k}^*(x) = h^2(x)$.

**(4)** $p_{2k}^*(x)$ has required form (2.11).

Let $p_{2k}^*(x) = h^2(x)$, where $h(x) \in \mathbf{G}_k$. By the definition of $p_{2k}^*(x)$, $h(x)$ is the polynomial which minimize integral

$$\min_{g(x)\in\mathbf{G}_k} \int_0^1 g^2(x)\, dx$$

From Lemma 1, we know $h(x) = \tilde{L}_k(x)$ is Legendre polynomial multiplied by a constant. This completes the proof ■

**Theorem 2** *Let*

$$\mathbf{H}_n^+ = \{q(x) \in \mathbf{H}_n | q(x) \geqslant 0 \ over\ [0,1]\}, \tag{2.12}$$

*and $q_n^* \in \mathbf{H}_n^+$ be the polynomial which minimizes integral*

$$\min_{q(x)\in\mathbf{H}_n^+} \int_0^1 q(x)\, dx \tag{2.13}$$

*then*

$$q_n^*(x) = \begin{cases} (1-x)\tilde{J}_{k,1}^2(x), & n = 2k+1, \\ x(1-x)\tilde{J}_{k,2}^2(x), & n = 2k+2, \end{cases} \tag{2.14}$$

*where $\tilde{J}_{k,1}(x) \in \mathbf{G}_k$ is the Jacobi polynomial $J_k^{(1,0)}$(multiplied by a constant), and $\tilde{J}_{k,2}(x) \in \mathbf{G}_k$ is the Jacobi polynomial $J_k^{(1,1)}$(multiplied by a constant). $q_n^*(x)$ is called the nonnegative least deviation polynomial from zero(with leading coefficient $-1$).*

**Proof:** Similar to the proof of theorem 1. ■

Next we consider the nonnegative least deviation polynomial from zero with constraints.

**Theorem 3** *Let*

$$\mathbf{G}_{n,m}^+ = \left\{ p(x) \in \mathbf{G}_n \middle| p^{(j)}(0) = 0, p^{(j)}(1) = 0, \right.$$
$$j = 0,1,\ldots,m-1, 2m < n$$
$$\left. p(x) \geqslant 0 \ over\ [0,1] \right\}, \tag{2.15}$$

*If $p_{n,m}^* \in \mathbf{G}_{n,m}^+$ satisfies*

$$||p_{n,m}^*||_1 = \min_{p\in\mathbf{G}_{n,m}^+} ||p||_1 \tag{2.16}$$

*then $p_{n,m}^*(x) = W_{n,m}(x) J_{n,m}^2(x)$, where*

$$W_{n,m}(x) = (1-x)^\lambda x^\mu, \tag{2.17}$$

$$(\lambda,\mu) = \begin{cases} (m, m+1), & n \text{ odd, } m \text{ even} \\ (m,m) & n \text{ even, } m \text{ even} \\ (m+1, m), & n \text{ odd, } m \text{ odd} \\ (m+1, m+1), & n \text{ even, } m \text{ odd} \end{cases}, \tag{2.18}$$

*and $J_{n,m}(x) \in \mathbf{G}_k(k = (n-\lambda-\mu)/2)$ is the Jacobi polynomial $J_k^{(\lambda,\mu)}$(multiplied by a constant).*

**Theorem 4** *Let Let*

$$\mathbf{H}_{n,m}^+ = \left\{ p(x) \in \mathbf{H}_n \middle| p^{(j)}(0) = 0, p^{(j)}(1) = 0, \right.$$
$$j = 0,1,\ldots,m-1, 2m < n$$
$$\left. p(x) \geqslant 0 \ over\ [0,1] \right\}, \tag{2.19}$$

*If $q_{n,m}^* \in \mathbf{H}_{n,m}^+$ satisfies*

$$||q_{n,m}^*||_1 = \min_{q\in\mathbf{H}_{n,m}^+} ||q||_1 \tag{2.20}$$

*then $q_{n,m}^*(x) = \tilde{W}_{n,m}(x) \tilde{J}_{n,m}^2(x)$, where*

$$\tilde{W}(x) = (1-x)^\lambda x^\mu, \tag{2.21}$$

$$(\lambda,\mu) = \begin{cases} (m+1, m), & n \text{ odd, } m \text{ even} \\ (m+1, m+1), & n \text{ even, } m \text{ even} \\ (m, m+1), & n \text{ odd, } m \text{ odd} \\ (m,m), & n \text{ even, } m \text{ odd} \end{cases} \tag{2.22}$$

*and $\tilde{J}_{n,m}(x) \in \mathbf{G}_k((k = (n-\lambda-\mu)/2)$ is the Jacobi polynomial $J_k^{(\lambda,\mu)}$(multiplied by a constant).*

# 3 Degree Reduction of Interval Polynomials and Interval Bézier Curves

The direct application of the results presented in the last section yields an algorithm for the optimal degree reduction of interval polynomials and interval Bézier curves.

## 3.1 Interval polynomials and interval Bézier curves

An *interval polynomial*([14]) is a polynomial whose coefficients are intervals:

$$[p](t) = \sum_{k=0}^{n} [a_k, b_k] B_k^n(t), \qquad 0 \leqslant t \leqslant 1 \tag{3.1}$$

$$p_{\min}(t) = \sum_{k=0}^{n} a_k B_k^n(t) \text{ and } p_{\max}(t) = \sum_{k=0}^{n} b_k B_k^n(t) \tag{3.2}$$

are called *lower bound*(denoted by $\mathrm{lb}([p](t))$) and *upper bound*(denoted by $\mathrm{ub}([p](t))$) of $[p](t)$ respectively. The *width* of an interval polynomial is defined

$$\begin{aligned} w([p](t)) &= \int_0^1 (p_{\max}(t) - p_{\min}(t))\, dx \\ &= \frac{1}{n+1}\sum_{k=0}^{n}(b_k - a_k) \end{aligned} \tag{3.3}$$

Using interval arithmetic([12]), we can carry the usual rules of polynomial arithmetic to interval polynomials.

An *interval Bézier curve*[14] is a curve whose control points are vector-valued intervals:

$$[\mathbf{P}](t) = \sum_{k=0}^{n}[\mathbf{P}_k]B_k^n(t) \tag{3.4}$$

where

$$[\mathbf{P}_k] = [a_k, b_k] \times [c_k, d_k] = ([a_k, b_k], [c_k, d_k]) \tag{3.5}$$

which describes a rectangular region in the plane.

Like ordinary Bézier curves, interval Bézier curves can also be degree elevated, subdivided, etc.

## 3.2 Degree reduction of interval polynomial

The problem of degree reduction of interval polynomials can be put in the following way:

**Problem 1** *Given an interval polynomial $[p](t)$ of degree $n$ as defined in (3.1), find an interval polynomial $[q](t)$ of degree $m = n-1$:*

$$[q](t) = \sum_{k=0}^{m}[\bar{a}_k, \bar{b}_k]B_k^n(t) \tag{3.6}$$

*such that*

$$[p](t) \subset [q](t), \qquad t \in [0,1] \tag{3.7}$$

*and the width of $[q](t)$ is the smallest in all the interval polynomials of degree $m$ which bound $[p](t)$.*

**Theorem 5** *Given a polynomial of degree $n$*

$$p(x) = \sum_{i=0}^{n} a_i x^i, a_i \in \mathbb{R}, \tag{3.8}$$

*Let*

$$\mathrm{Up}(p)(x) = \begin{cases} p(x) + a_n q_n^*(x), & a_n > 0, \\ p(x) - a_n p_n^*(x), & a_n < 0, \end{cases} \tag{3.9}$$

$$\mathrm{Lo}(p)(x) = \begin{cases} p(x) - a_n p_n^*(x), & a_n > 0, \\ p(x) + a_n q_n^*(x), & a_n < 0, \end{cases} \tag{3.10}$$

*then* $\mathrm{Up}(p) \in \Pi_{n-1}$, $\mathrm{Lo}(p) \in \Pi_{n-1}$,

$$\mathrm{Lo}(p)(x) \leqslant p(x) \leqslant \mathrm{Up}(p)(x), \quad \forall x \in [0,1] \tag{3.11}$$

*and*

$$||\mathrm{Up}(p) - p||_1 = \min_{\substack{q \in \Pi_{n-1}, \\ q \geqslant p}} ||q - p||_1, \tag{3.12}$$

$$||p - \mathrm{Lo}(p)||_1 = \min_{\substack{q \in \Pi_{n-1}, \\ q \leqslant p}} ||q - p||_1, \tag{3.13}$$

*where $p_n^*(x)$ and $q_n^*(x)$ are defined as (2.11),(2.14).*

**Proof:** (3.11) can be obtained by the definitons of $p_n^*(x)$ and $q_n^*(x)$ Now we prove (3.12) for the case $a_n > 0$. The other cases are similar.

For any polynomial $q(x) \in \Pi_{n-1}$, $q(x) \geqslant p(x)$ for $x \in [0,1]$.

$$\begin{aligned} ||q(x) - p(x)||_1 &= a_n \left\| -x^n + \left(\frac{1}{a_n}q(x) - \sum_{i=0}^{n-1}\frac{a_i}{a_n}x^i\right)\right\|_1 \\ &\geqslant a_n||q_n^*(x)||_1 = ||\mathrm{Up}(p) - p(x)||_1 \end{aligned}$$

The theorem is thus proved. ■

Using relationship (2.7), It is easy to convert the above results to Bernstein form. The details are omitted.

Based on the above theorem, we can device an algorithm to solve **Problem 1** as follows.

Given an interval polynomial $[p](t)$ of degree $n$, we construct an interval polynomial $[q](t)$ of degree $n-1$ whose lower bound is $\mathrm{Lo}(\mathrm{lb}([p](t)))$ and upper bound is $\mathrm{Up}(\mathrm{ub}([p](t)))$. Obviously, $[q](t)$ bounds $[p](t)$ and $[q](t)$ is optimal in the sense that the upper bound and the lower bound are optimal respectively.

## 3.3 Degree reduction of interval Bézier curves

To solve the problem of the degree reduction of interval Bézier curves is to solve:

**Problem 2** *Given an interval Bézier curve $[\mathbf{P}](t)$ of degree $n$, find an interval Bézier curve $[\mathbf{Q}](t)$ of degree $m = n-1$, such that*

$$[\mathbf{P}](t) \subset [\mathbf{Q}](t), \qquad t \in [0,1] \tag{3.14}$$

*and the region occupied by $[\mathbf{Q}](t)$ is as small as possible.*

The solution to the degree reduction of interval polynomials directly yields the algorithm to solve **Problem 2**.

**Theorem 6** *Given an interval Bézier curve of degree n:*

$$[\mathbf{P}](t) = ([x](t), [y](t)) = \sum_{i=0}^{n} ([a_i, b_i], [c_i, d_i]) B_i^n(t) \tag{3.15}$$

*Let*

$$[\bar{x}](t) = \sum_{i=0}^{m} [\bar{a}_i, \bar{b}_i] B_i^m(t), \tag{3.16}$$

$$[\bar{y}](t) = \sum_{i=0}^{m} [\bar{c}_i, \bar{d}_i] B_i^m(t) \tag{3.17}$$

*are the interval polynomial bounds of* $[x](t)$ *and* $[y](t)$ *of degree* $m < n$ *respectively, then interval Bézier curve*

$$[\mathbf{Q}](t) = ([\bar{x}](t), [\bar{y}](t)) = \sum_{i=0}^{m} ([\bar{a}_i, \bar{b}_i], [\bar{c}_i, \bar{d}_i]) B_i^m(t) \tag{3.18}$$

*bounds interval Bézier curve* $[\mathbf{P}](t)$, *i.e.,* $[\mathbf{P}](t) \subset [\mathbf{Q}](t)$.

**Proof:** Straight forward. ■

The above theorem states, the degree reduction of interval Bézier curve can be obtained by seperately finding degree reduction interval polynomial for each component.

# 4 Examples

In this section we will provide some examples to illustrate the approximation results of the algorithm in this paper, and compare it with Rokne's algorithm.

**Example 1** In this example, we consider a degree seven polynomial whose magnitude drastically changes on $[0, 1]$

$$[p](t) = 43\,t - 368\,t^2 + 537\,t^3 + 2168\,t^4 - 7514\,t^5 + 8046\,t^6 - 2193\,t^7$$

We use an interval polynomial of degree six to bound $[p](t)$. Fig 1 and Fig 2 illustrate the approximation results by our algorithm and Rokne's method respectively. The areas of the bounding interval polynomials by these two methods are 0.59, 165.03 respectively.

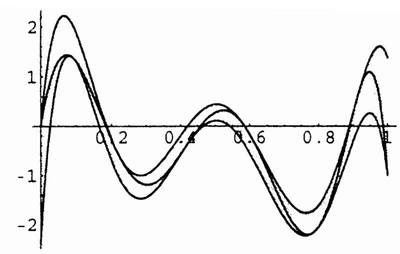


Figure 1: Using our algorithm

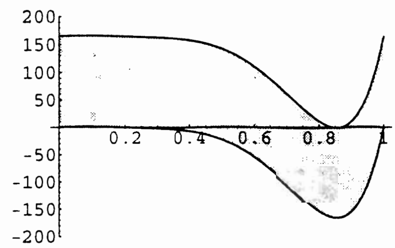


Figure 2: Using Rokne's algorithm

**Example 2** Let $[\mathbf{P}](t)$ be an interval Bézier curve of degree six with control points as follows:

$$\begin{aligned}
[\mathbf{P}_0] &= ([50, 70], [140, 155]), \\
[\mathbf{P}_1] &= ([130, 150], [350, 370]), \\
[\mathbf{P}_2] &= ([190, 220], [110, 120]), \\
[\mathbf{P}_3] &= ([250, 260], [290, 300]), \\
[\mathbf{P}_4] &= ([340, 360], [100, 120]), \\
[\mathbf{P}_5] &= ([390, 400], [150, 165]), \\
[\mathbf{P}_6] &= ([430, 455], [360, 375]).
\end{aligned}$$

We bound $[\mathbf{P}](t)$ with a degree five interval Bézier curve using our method and Rokne's method. The results are shown in Fig. 3(Rokne's method), Fig. 4(our method without constraints) and Fig. 5(our method with constraints).

From the above examples, we can see our method produces much better approximation results than Rokne's method.

# 5 Conclusions

In this paper, we present an analytic solution to the problem of how to bound an interval polynomial/Bézier curve with a lower degree interval polynomial/Bézier curve. The solution is optimal in the sense that the area between the upper bounds and the area between the lower bounds of the two interval polynomials/curves attain minimum values seperately. Theoretical results and examples show that our algorithm provides much tighter bounds than Rokne's algorithm.

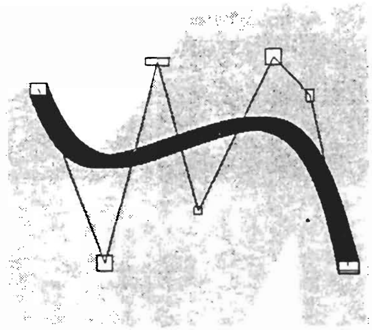

Figure 3:

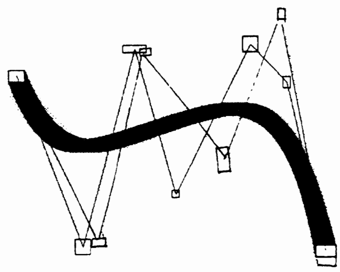

Figure 4:

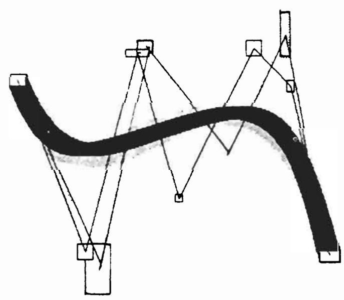

Figure 5:

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