Journal of Computational and Applied Mathematics 238 (2013) 68–94

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Dimensions of biquadratic spline spaces over T-meshes

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ARTICLE INFO

Article history:
Received 13 May 2011
Received in revised form 26 August 2012

MSC:
65D07

Keywords:
Spline spaces over T-meshes
Dimension
Extended T-meshes
Homogeneous boundary conditions

ABSTRACT

This paper discusses the dimensions of spline spaces over T-meshes of low degree. Two new concepts are proposed: an extension of T-meshes and spline spaces with homogeneous boundary conditions. In the dimensional analysis, the key strategy is linear space embedding with the operator of the mixed partial derivative. A lower bound on the dimension of the biquadratic spline spaces over general T-meshes is provided. Furthermore, by making full use of the level structure of hierarchical T-meshes, a dimension formula of biquadratic spline space over hierarchical T-meshes is proved. Additionally, a topological explanation of the dimension formula is provided.

1. Introduction

A T-mesh is a rectangular grid that allows T-junctions. T-splines, which represent a type of point-based spline defined over a T-mesh, were proposed in [1,2] and have become an important tool in geometric modeling, surface reconstruction, isogeometric analysis [3,4], and other applications. In the definition of T-splines, the T-mesh plays two roles: defining the parametric domain decomposition of a T-spline and representing the topological structure of the control net of a T-spline surface.

According to its definition, a T-spline is a piecewise polynomial rather than a single polynomial within a cell of a T-mesh. This definition is incompatible with the standard definition of classical splines, according to which, given a spline knot sequence, a univariate spline can be defined as a single polynomial between any two neighboring knots. Therefore, in [5], the authors introduced spline spaces over T-meshes, where every function in the space can be exactly represented by a polynomial within each cell of the T-mesh. We use $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ to denote a spline space over a T-mesh $\mathscr{T}$, where a function in the space is a polynomial of bi-degree (m, n) in each cell of $\mathscr{T}$ and continuous with order α along the horizontal direction and with order β along the vertical direction. A dimension formula can be proved with the B-net method [5] and the smoothing cofactor method [6] for the spline space $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ with $m \geqslant 2\alpha+1$ and $n \geqslant 2\beta+1$. In [7], we provided an approach to defining the basis functions of bicubic splines with first-order smoothness and examined applications of the basis functions in surface fitting.

According to the dimension formula provided in [5], specified dimension formulae can be obtained for certain spline spaces of low degree, such as $\mathbf{S}(1, 1, 0, 0, \mathscr{T})$, $\mathbf{S}(2, 2, 0, 0, \mathscr{T})$, $\mathbf{S}(3, 3, 0, 0, \mathscr{T})$, and $\mathbf{S}(3, 3, 1, 1, \mathscr{T})$. Furthermore, using approaches similar to those used in [7], the basis functions of spline spaces can be constructed with favorable properties, for example, compact support, nonnegativity, and the ability to form a partition of unity. To achieve higher-order smoothness with as low a degree as possible, we attempt to obtain the dimension formulae and basis functions construction for the

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doi:10.1016/j.cam.2012.08.020

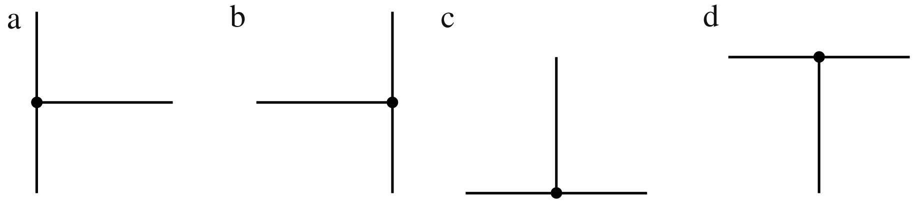


Fig. 1. Horizontal T-vertices ((a) and (b)) and vertical T-vertices ((c) and (d)).

spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$. The most interesting formulae are those for $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ and $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$. In this paper, we focus mainly on the dimension of the former space.

The paper begins by introducing two concepts: an extended T-mesh associated with a spline space and spline spaces with homogenous boundary conditions (HBC). Given a T-mesh $\mathscr{T}$, its extension $\mathscr{T}^{\varepsilon}$ associated with $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is a larger T-mesh properly constructed from $\mathscr{T}$. We use $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ to represent the spline space over $\mathscr{T}$ with HBC, which is a subset of $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$. A function $f \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ means that f smoothly vanishes along the boundary of $\mathscr{T}$. Suppose $\mathscr{T}^{\varepsilon}$ is an extended T-mesh of $\mathscr{T}$ associated with $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$. We will show that the dimension of $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ equals the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^{\varepsilon})$. Our practice shows that the dimensions of spline spaces with HBC can be analyzed with a simple and consistent approach. In this paper, we will use this transition to analyze the dimensions of biquadratic spline spaces over T-meshes.

As a foundation of the dimensional analysis, we discuss in detail the dimension formula and basis functions construction for the space $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. In [5], we showed that the dimension of a bilinear spline space is the sum of the numbers of crossing vertices and boundary vertices in the given T-mesh. Here, we propose a new proof that shares the same scheme as the analysis of a lower bound on the dimension of biquadratic spline spaces and avoids the problem of the recycling dependence of T-vertices, as stated in [8].

An important technique in dimensional analysis is linear space embedding by the operator of the mixed partial derivative. This operator embeds the space $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ into the space $\overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$. In this work, we only discuss cases of $m = n = 1$ or 2. Using this method, a lower bound on the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ is proved. Finally, by making use of the level structure of hierarchical T-meshes, a dimension formula is provided for biquadratic spline spaces over hierarchical T-meshes.

The paper is organized as follows. In Section 2, the spline spaces over T-meshes are reviewed, and two concepts are proposed: the extension of T-meshes and spline spaces with homogeneous boundary conditions. The dimension formula and basis function construction of bilinear spline spaces of zero-order smoothness over T-meshes are discussed in detail in Section 3. In Section 4, a lower bound on the dimension of biquadratic spline spaces over general T-meshes is provided. In Section 5, using the level structure of hierarchical T-meshes, a dimension formula of biquadratic spline space over hierarchical T-meshes is proved. Additionally, a topological explanation of the dimension formula is provided. Section 6 concludes the paper with some discussion.

2. T-meshes and spline spaces

A T-mesh is a rectangular grid that allows T-junctions.

Definition 2.1. Suppose $\mathscr{T}$ is a set of axis-aligned rectangles and the intersection of any two distinct rectangles in $\mathscr{T}$ either is empty or consists of points on the boundaries of the rectangles. Then, $\mathscr{T}$ is called a *T-mesh*. Furthermore, if the entire domain occupied by $\mathscr{T}$ is a rectangle, $\mathscr{T}$ is called a *regular T-mesh*. If some edges of $\mathscr{T}$ also form a T-mesh $\mathscr{T}'$, $\mathscr{T}'$ is called a *submesh* of $\mathscr{T}$.

In this paper, the T-meshes that we consider are regular, and we adopt the definitions of vertex, edge, and cell provided in [5].

T-vertices in a T-mesh can be classified into two types: *horizontal T-vertices* and *vertical T-vertices*. The T-vertices shown in Fig. 1(a) and (b) are horizontal, and those shown in Fig. 1(c) and (d) are vertical. In [5], edges and c-edges are defined. Here, we introduce a new type of edge, the l-edge ("long edge"). An *interior horizontal/vertical l-edge* is a continuous line segment that consists of interior horizontal/vertical edges and whose end points are boundary vertices or horizontal/vertical T-vertices. In other words, an l-edge is a longest possible straight line segment in the mesh. The four boundary straight line segments of a regular T-mesh are called *boundary l-edges*. For example, in Fig. 2, the given T-mesh $\mathscr{T}$ has three interior horizontal l-edges and three interior vertical l-edges.

Given two series of real numbers x_i, $i = 1, \ldots, m$ and y_j, $j = 1, \ldots, n$, where $x_i < x_{i+1}$ and $y_j < y_{j+1}$, a rectangular grid with vertices (x_i, y_j), $i = 1, \ldots, m$, and $j = 1, \ldots, n$ can be formed. This grid is called a *tensor-product mesh* and is denoted by $(x_1, \ldots, x_m) \times (y_1, \ldots, y_n)$, which is a special type of T-mesh. From a regular T-mesh $\mathscr{T}$, a tensor-product mesh $\mathscr{T}^c$ can be constructed by extending all the interior l-edges to the boundary. $\mathscr{T}^c$ is called the *associated tensor-product mesh* with $\mathscr{T}$. See Fig. 2 for an example.

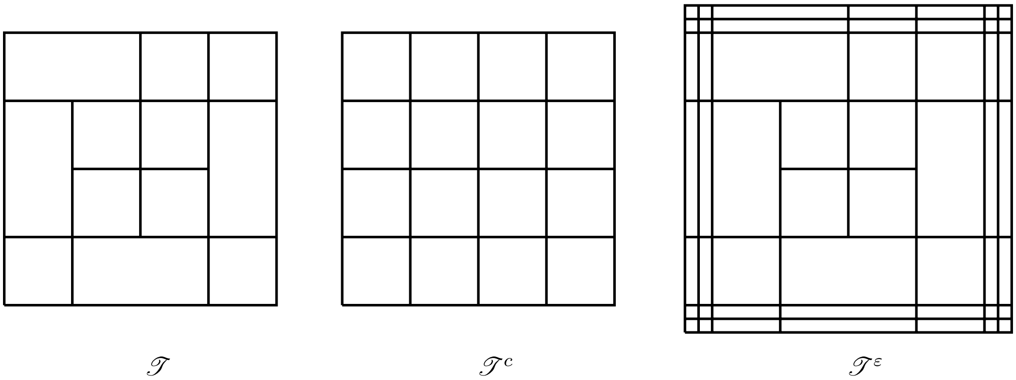


Fig. 2. A T-mesh $\mathscr{T}$ with its associated tensor-product mesh $\mathscr{T}^c$ and its extension $\mathscr{T}^\varepsilon$.

2.1. A topological equation of T-meshes

In the proof of the dimension formulae discussed above, we will use the following topological equation of T-meshes:

Lemma 2.2. *Given a regular T-mesh $\mathscr{T}$, suppose $\mathscr{T}$ has F cells, V^c crossing vertices, and E interior l-edges. Then,*

$$F = V^c + E + 1.$$

Proof. Suppose that in $\mathscr{T}$ there are V^T T-vertices and V^{bT} boundary vertices (excluding the four corner points). Because every cell has four vertices, running through all of the cells we will meet each crossing vertex four times, each interior T-vertex twice, each corner point once, and each other boundary vertex twice. It follows that

$$4F = 4V^c + 2V^T + 2V^{bT} + 4.$$

However, the end points of every interior l-edge are either interior T-vertices or boundary vertices that are not corner points. Therefore, we have $V^T + V^{bT} = 2E$. From these two equations, we have $F = V^c + E + 1$. □

2.2. Spline spaces over T-meshes

Given a T-mesh $\mathscr{T}$, we use $\mathscr{F}$ to denote all of the cells in $\mathscr{T}$ and Ω to denote the region occupied by the cells in $\mathscr{T}$. In [5], the following spline space definition is proposed:

$$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\Omega) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}\}, \tag{2.1}$$

where $\mathbb{P}_{mn}$ is the space of the polynomials with bi-degree (m, n) and $C^{\alpha,\beta}$ is the space consisting of all the bivariate functions continuous in Ω with order α along the x direction and with order β along the y direction. It is obvious that $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is a linear space.

Now, we introduce a new spline space over a T-mesh with the following definition:

$$\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\mathbb{R}^2) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}, \text{ and } f|_{\mathbb{R}^2\setminus\Omega} \equiv 0\}. \tag{2.2}$$

This new spline space is called a spline space over the given T-mesh $\mathscr{T}$ with *homogeneous boundary conditions (HBC)*.

$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ consists of certain functions defined over Ω and $\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{T})$ consists of some functions defined over $\mathbb{R}^2$. However, it is obvious that

$$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) \supset \overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{T})|_\Omega.$$

In this paper, we will analyze the dimension of the biquadratic spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ over a T-mesh $\mathscr{T}$ by investigating the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$, where $\mathscr{T}^\varepsilon$ is an extension of $\mathscr{T}$ associated with $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$. In the next subsection, we will define extensions of T-meshes.

2.3. Extensions of T-meshes

In univariate spline theory [9], to define basis functions for spline spaces over given knot sequences, we must form extended knot sequences by inserting more knots at the left and right ends of the given knot sequences.

In this subsection, this technique is adapted to T-meshes for dimensional analysis. To discuss the dimension of the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$, we also extend the given regular T-mesh $\mathscr{T}$ in the following fashion. A tensor-product mesh $\mathscr{M}$ with $2(m+1)$ vertical lines and $2(n+1)$ horizontal lines is produced such that the central rectangle of $\mathscr{M}$ is identical

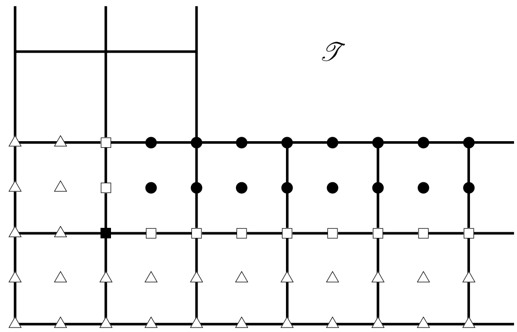


Fig. 3. Bézier ordinates in an extended T-mesh.

to Ω, which is the region occupied by $\mathscr{T}$. Next, the edges with an end point on the boundary of $\mathscr{T}$ are extended to reach the boundary of $\mathscr{M}$. The resulting mesh, which we denote as $\mathscr{T}^\varepsilon$, is an extension of the original T-mesh associated with the present spline space. $\mathscr{T}^\varepsilon$ is also called an extended T-mesh. Fig. 2 shows an example, in which the extended T-mesh is associated with $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$.

The extension described here is associated with a spline space. For different spline spaces, different extended T-meshes will be obtained. The following theorem shows that when an extension of the T-mesh is used, the dimensional analysis of $\bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$ is the same as the dimensional analysis of $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$.

Theorem 2.3. *Given a T-mesh $\mathscr{T}$, assume that $\mathscr{T}^\varepsilon$ is its extension associated with $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ and that Ω is the region occupied by the cells in $\mathscr{T}$. Then,*

$$\mathbf{S}(2, 2, 1, 1, \mathscr{T}) = \bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)|_\Omega, \tag{2.3}$$

$$\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) = \dim \bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon). \tag{2.4}$$

Proof. We first prove Eq. (2.3). Because $\mathscr{T}^\varepsilon$ is an extension of $\mathscr{T}$, it follows that

$$\bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)|_\Omega \subset \mathbf{S}(2, 2, 1, 1, \mathscr{T}).$$

In the following, we will prove that for any $f \in \mathbf{S}(2, 2, 1, 1, \mathscr{T})$, there exists $\bar{f} \in \bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$ such that $\bar{f}|_\Omega = f$.

Here, we define the function $\bar{f}$ over $\mathscr{T}^\varepsilon$ by assigning the function's Bézier ordinates in each cell of $\mathscr{T}^\varepsilon \setminus \mathscr{T}$. Because $\bar{f} \in C^{1,1}$ and there are no interior T-vertices in $\mathscr{T}^\varepsilon \setminus \mathscr{T}, \bar{f}$ shares the same ordinates along the common boundary between two neighboring cells. (These ordinates appear once in Fig. 3, which illustrates the distribution of the Bézier ordinates in the left-bottom section of the extended region $\mathscr{T}^\varepsilon \setminus \mathscr{T}$.) According to the specification of f over $\mathscr{T}$, it follows that the ordinates labeled "•" are determined. Because $\bar{f}$ meets the zero function along the boundary of the extended T-mesh $\mathscr{T}^\varepsilon$ with order one, the ordinates labeled "△" are also determined. The remaining ordinates, which are labeled "□", can be determined by the corresponding neighboring horizontal or vertical ordinates labeled "△" and "•". Here, the continuous condition is the collinear of the corresponding three points. The ordinates determined in this fashion automatically define a continuous spline.

After determining the ordinates described above, the ordinate labeled "■" can be determined as follows. It follows that the eight ordinates neighboring the ordinate "■" define a bilinear function. When the ordinate corresponding to "■" also lies on the bilinear function, the continuation of order one in two directions can be guaranteed. In fact, if we meet cases of recycling determinations when addressing the determination of all of the ordinates in the extension region around $\mathscr{T}$, we can address the ordinates using this method of bilinear functions construction.

Therefore, we obtain the specification of the function $\bar{f}$ in $\mathbf{S}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$, which satisfies $\bar{f}|_\Omega = f$. According to the former analysis, it follows that

$$\mathbf{S}(2, 2, 1, 1, \mathscr{T}) = \bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)|_\Omega.$$

To prove Eq. (2.4), we will prove that for any nonzero function $\bar{f} \in \bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)$, it follows that $\bar{f}|_\Omega \not\equiv 0$. Suppose $\bar{f}|_\Omega \equiv 0$, i.e., there exists $p = (x_0, y_0) \in \mathscr{T}^\varepsilon \setminus \mathscr{T}$ such that $\bar{f}(x_0, y_0) \neq 0$. In addition, assume $\Omega = (x_l, x_r) \times (y_b, y_t)$. If $x_l \leqslant x_0 \leqslant x_r, y_0 < y_b$, then $\bar{f}(x_0, y), y \leqslant y_b$ is a quadratic spline whose support has only three breakpoints. This result contradicts the minimal support of quadratic splines (the support of a nonzero quadratic spline should have at least four breakpoints). Similarly, we can show that p is outside the regions $x_l \leqslant x_0 \leqslant x_r, y_0 > y_t$; $y_b \leqslant y_0 \leqslant y_t, x_0 < x_l$; and $y_b \leqslant y_0 \leqslant y_t, x_0 > x_r$. Therefore, without loss of generality, we assume $x_0 < x_l, y_0 < y_b$. However, this assumption also results in a nonzero quadratic spline with three breakpoints in its support. Therefore, we obtain $\bar{f}|_\Omega \not\equiv 0$ and have proven that

$$\dim \mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) = \dim \bar{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{T}^\varepsilon),$$

which completes the proof of the theorem. □

According to Theorem 2.3, we will consider only the biquadratic spline spaces over T-meshes with HBC.

Remark 2.4. A similar analysis can show that for any m and n,

$$\mathbf{S}(m, n, m-1, n-1, \mathscr{T}) = \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})|_{\Omega}, \tag{2.5}$$

$$\dim \mathbf{S}(m, n, m-1, n-1, \mathscr{T}) = \dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}), \tag{2.6}$$

where $\mathscr{T}^{\varepsilon}$ is an extension of $\mathscr{T}$ associated with $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ and Ω is the region occupied by the cells in $\mathscr{T}$. Therefore, we can replace the dimensional discussion for $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ with that for $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$.

2.4. Space embedding with the operator of the mixed partial derivative

In this subsection, we will introduce the linear space embedding with the operator of the mixed partial derivative. Before proceeding, we propose the following definition.

Definition 2.5. Suppose we are given a set S of equations $\{f_i(x_1, \ldots, x_m),\ i = 1, \ldots, n\}$, where for any i, f_i is a linear combination of variables $x_1, \ldots, x_m$. If the number of a maximal linearly independent subset in S is r, then r is called the *rank* of S, and $n - r$ is called the *defective rank* of S.

According to linear algebra, the rank of a given equation set does not depend on the choice of the set's maximal linearly independent subset.

The operator of the mixed partial derivative is defined as follows:

$$\mathcal{D} := \frac{\partial^2}{\partial x \partial y} : \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) \to \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T}), \tag{2.7}$$

where $m, n \geqslant 1$. $\mathcal{D}$ is linear, i.e.,

$$\mathcal{D}(\alpha f + \beta g) = \alpha \mathcal{D}(f) + \beta \mathcal{D}(g),$$

where $f, g \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ and α, β are constants. However, because applied functions satisfy the HBC, $\mathcal{D}$ is injective, i.e., for any $f_1, f_2 \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, if $f_1 \neq f_2$, then $\mathcal{D}(f_1) \neq \mathcal{D}(f_2)$. However, $\mathcal{D}$ is not an onto mapping. For example, for any nonzero $g \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, it follows that $\mathcal{D}(g)$ must attain positive values in certain parts and negative values in others. Therefore, a nonnegative function in $\overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$ has no pre-image under the operator $\mathcal{D}$.

Define

$$\mathcal{I}(g)(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} g(s, t) ds dt. \tag{2.8}$$

$\mathcal{I}$ is also linear and injective. For any $f \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, $\mathcal{I} \circ \mathcal{D}(f) = f$. Therefore, $\mathcal{I}$ can be considered a left inverse operator of $\mathcal{D}$.

It follows that for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, $\mathcal{I}(g)$ is a piecewise polynomial of degree (m, n) with smoothness $C^{m-1,n-1}$. With respect to functions in $\overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$, $\partial^m \mathcal{I}(g)/\partial x^m$ or $\partial^n \mathcal{I}(g)/\partial y^n$ may be discontinuous inside certain cells of $\mathscr{T}$. However, this discontinuity must occur on the extension of certain l-edges of $\mathscr{T}$. Let

$$d_{m,n} = \dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}).$$

If we know $d_{m-1,n-1}$, i.e., the dimension of $\overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, and the number $r_{m-1,n-1}$ of linear-independent constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, it follows that

$$d_{m,n} = d_{m-1,n-1} - r_{m-1,n-1}. \tag{2.9}$$

Here, the constraints are equations of linear combinations of certain variables. However, if there are $r'_{m-1,n-1}$ constraints (these constraints may be linearly dependent) proposed for ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(m-1, n-1, m-2, n-2, \mathscr{T})$, it follows that

$$d_{m,n} \geqslant d_{m-1,n-1} - r'_{m-1,n-1}. \tag{2.10}$$

Suppose the rank of these $r'_{m-1,n-1}$ constraints is $\alpha_{m-1,n-1}$. Then, we have

$$d_{m,n} = d_{m-1,n-1} - \alpha_{m-1,n-1}. \tag{2.11}$$

In the remainder of the paper, Eqs. (2.9)–(2.11) are used to prove the dimension formulae of bilinear and biquadratic spline spaces over T-meshes.

3. Dimensions and basis functions of $\bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$

In [5], we proved the following dimension formula of the spline space $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ over a T-mesh:

$$\dim \mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) = F(m+1)(n+1) - E_h(m+1)(\beta+1) - E_v(n+1)(\alpha+1) + V(\alpha+1)(\beta+1), \tag{3.1}$$

where $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$, F is the number of cells in $\mathscr{T}$; E_h and E_v are the numbers of horizontal and vertical interior edges, respectively; and V is the number of interior vertices (including crossing and T-vertices). Specifically, when $m = n = 1$ and $\alpha = \beta = 0$, it follows that

$$\dim \mathbf{S}(1, 1, 0, 0, \mathscr{T}) = V^c + V^b, \tag{3.2}$$

where V^c is the number of crossing vertices, and V^b the number of boundary vertices. Using an extension of the T-meshes associated with $\mathbf{S}(1, 1, 0, 0, \mathscr{T})$, it follows that

$$\dim \bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}^\varepsilon) = \dim \mathbf{S}(1, 1, 0, 0, \mathscr{T}) = V^c_\varepsilon, \tag{3.3}$$

where V^c_ε is the number of crossing vertices in the extended T-mesh $\mathscr{T}^\varepsilon$, which equals the total number of crossing vertices and boundary vertices in $\mathscr{T}$.

Next, we will prove that the former formula holds for a general regular T-mesh and an extended T-mesh. The method applied in the proof is similar to the method proposed in the next section for proving a lower bound on the dimension of $\bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$. This method solves the problem of the recycling dependence of T-vertices, which occurs in the proof proposed in [5].

3.1. Notation

In the given T-mesh $\mathscr{T}$, let E denote the number of interior l-edges and V^c and V^b denote the number of crossing vertices and boundary vertices, respectively. Define $\bar{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$ as the space consisting of functions that are constant within each cell of the T-mesh $\mathscr{T}$ and have no smoothness requirement between neighboring cells. It is obvious that the space's dimension is the number of cells in the T-mesh.

In the following analysis, we use the standard notation for one-sided limits from calculus. For a function $f(x)$ over some interval I, we write $f(x_0^-)$ for the limit as x increases in value approaching x_0 and $f(x_0^+)$ for the limit as x decreases in value approaching x_0.

3.2. Some lemmas

When $m = n = 1$, the following lemma proposes the constraints ensuring $\mathcal{I}(g) \in \bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ for any $g \in \bar{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$.

Lemma 3.1. *Given a regular T-mesh $\mathscr{T}$, let $g \in \bar{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$. Then,*

$$\mathcal{I}(g)(x, y) \in \bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$$

if and only if the following two sets of conditions are satisfied simultaneously:

1. *For any horizontal l-edge l^h,*

$$\int_{x_0}^{x_1} g(s, y_0^-)ds = \int_{x_0}^{x_1} g(s, y_0^+)ds, \tag{3.4}$$

where the two end points of l^h have the coordinates (x_0, y_0) and (x_1, y_0);

2. *For any vertical l-edge l^v,*

$$\int_{y_0}^{y_1} g(x_0^-, t)dt = \int_{y_0}^{y_1} g(x_0^+, t)dt, \tag{3.5}$$

where the two end points of l^v have the coordinates (x_0, y_0) and (x_0, y_1).

Proof. We first prove the necessity of the lemma. For any horizontal l-edge l^h, its right end point (x_1, y_0) is either a T-vertex or a boundary vertex. Suppose the edge through the vertex (x_1, y_0) and perpendicular with l^h is e^v, as shown in Fig. 4. If $\mathcal{I}(g)(x, y) \in \bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, then $\mathcal{I}(g)|_{e^v}$ is a linear polynomial in a neighborhood of (x_1, y_0). Therefore, it follows that

$$\frac{\partial}{\partial y}\mathcal{I}(g)(x_1, y_0^-) = \frac{\partial}{\partial y}\mathcal{I}(g)(x_1, y_0^+). \tag{3.6}$$

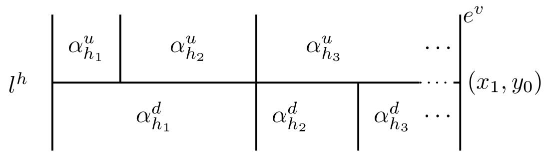


Fig. 4. A horizontal l-edge (1).

According to the definition of $\mathcal{I}$ in Eq. (2.8), Eq. (3.6) can be rewritten as

$$\int_{-\infty}^{x_1} g(s, y_0^-)ds = \int_{-\infty}^{x_1} g(s, y_0^+)ds. \tag{3.7}$$

Extend l^h to reach the left boundary of the T-mesh. If there are no other horizontal l-edges on the extension edge, then in every cell that intersects the extension edge g is a pure constant. Then, $g(s, y_0^-) = g(s, y_0^+)$ when $s \leqslant x_0$, and

$$\int_{-\infty}^{x_0} g(s, y_0^-)ds = \int_{-\infty}^{x_0} g(s, y_0^+)ds. \tag{3.8}$$

According to Eqs. (3.7) and (3.8), it follows that Eq. (3.4) in the first condition holds for any horizontal l-edge.

If additional horizontal l-edges on the extension edge exist, we proceed through these horizontal l-edges from left to right. For the first such l-edge, we can use the former approach to prove that Eq. (3.4) holds. Then, we consider the other l-edges one by one and prove that the corresponding equation (3.8) holds. Then, Eq. (3.4) holds. Therefore, we finish by proving that Eq. (3.4) holds for every horizontal l-edge.

For any vertical l-edge, we can prove that the corresponding equation (3.5) holds in a similar way. Therefore, the necessity of the lemma is proved.

We will now prove the sufficiency of the lemma. Suppose $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$ satisfies the two sets of conditions in the lemma, however, $\mathcal{I}(g) \notin \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, i.e., there exists a cell for which $\partial\mathcal{I}(g)/\partial x$ or $\partial\mathcal{I}(g)/\partial y$ has at least one point of discontinuity. Define two sets consisting of the cells of $\mathscr{T}$ as follows. Let $\mathscr{B}_x$ denote all of the cells for which $\partial\mathcal{I}(g)/\partial x$ has at least one point of discontinuity, and let $\mathscr{B}_y$ denote all of the cells for which $\partial\mathcal{I}(g)/\partial y$ has at least one point of discontinuity. Then, according to the assumption, $\mathscr{B}_x \cup \mathscr{B}_y$ is not empty. Without loss of generality, we assume that $\mathscr{B}_y$ is not empty. Consider a cell in $\mathscr{B}_y$ whose left-bottom corner has the minimal y coordinate in $\mathscr{B}_y$. If more than one such cell exists, select the cell whose left-bottom corner has the minimal x coordinate. We have now selected a unique cell c. In c, $\partial\mathcal{I}(g)/\partial y$ has at least one point of discontinuity $(\hat{x}, \hat{y})$. Therefore, it follows that

$$\int_{-\infty}^{\hat{x}} g(s, \hat{y}^-)ds \neq \int_{-\infty}^{\hat{x}} g(s, \hat{y}^+)ds. \tag{3.9}$$

Consider the horizontal straight line through $(\hat{x}, \hat{y})$. If there is no l-edge of $\mathscr{T}$ on the left side of the straight line through $(\hat{x}, \hat{y})$, then Eq. (3.9) fails to hold. If there is at least one l-edge on the left side of $(\hat{x}, \hat{y})$, then, according to the first condition, a contradiction also arises. Therefore, $\mathscr{B}_y$ is empty. In the same fashion, it follows that $\mathscr{B}_x$ is also empty. This result contradicts the assumption that $\mathscr{B}_x \cup \mathscr{B}_y$ is not empty. Therefore, we finish the proof of the sufficiency. □

There are $E + 4$ conditions in Lemma 3.1, where $E + 4$ is the total number of the interior l-edges and the four boundary l-edges. The following lemma is a fundamental result that shows that these $E+4$ conditions are equivalent to $E+2$ conditions in another form.

Lemma 3.2. *Given a regular T-mesh $\mathscr{T}$, the rectangle occupied by $\mathscr{T}$ is $(x_l, x_r) \times (y_b, y_t)$, where the different y coordinates of the rectangle's horizontal l-edges are $y_0 < y_1 < \cdots < y_n$. Suppose $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$. Then,*

$$\int_{x_l}^{x_r} g(s, y_i^-)ds = \int_{x_l}^{x_r} g(s, y_i^+)ds, \quad i = 0, 1, \ldots, n \tag{3.10}$$

is equivalent to

$$\int_{x_l}^{x_r} g(s, y)ds = 0, \quad y \in (y_i, y_{i+1}), \quad i = 0, \ldots, n-1. \tag{3.11}$$

Because g is a piecewise constant, $\int_{x_l}^{x_r} g(s, y)ds$, $y \in (y_i, y_{i+1})$ is a constant independent of y.

Proof. Because g is a piecewise constant in $\mathscr{T}$ and vanishes out of $\mathscr{T}$, it follows that

$$\int_{x_l}^{x_r} g(s, y_0^-)ds = \int_{x_l}^{x_r} g(s, y)ds = 0, \quad y < y_0,$$

$$\int_{x_l}^{x_r} g(s, y_0^+)ds = \int_{x_l}^{x_r} g(s, y)ds, \quad y_0 < y < y_1.$$

Therefore, according to Eq. (3.10), when $i = 0$, we obtain

$$\int_{x_l}^{x_r} g(s, y)ds = 0, \quad y \in (y_0, y_1).$$

This result means that Eq. (3.11) holds when $i = 0$. Similar discussions guarantee that Eq. (3.11) holds when $i = 1, 2, \dots, n-1$.

However, for any $i = 0, 1, 2, \dots, n-1$, if all the equations in (3.11) hold, it is straightforward to prove that all the equations in (3.10) hold. □

Remark 3.3. The statement in Lemma 3.2 applies to horizontal l-edges. A similar statement can be made for vertical l-edges.

As mentioned, for a given T-mesh $\mathscr{T}$, an associated tensor-product mesh $\mathscr{T}^c$ can be obtained by extending all the interior l-edges to the boundary of $\mathscr{T}$. Let $\mathscr{T}^c = (x_0, x_1, \dots, x_m) \times (y_0, y_1, \dots, y_n)$, where $x_0 = x_l, x_m = x_r$, and $y_0 = y_b, y_n = y_t$. Suppose there are E' interior l-edges in $\mathscr{T}^c$ and E interior l-edges in $\mathscr{T}$, where $E' \leqslant E$ (because it is possible that more than one l-edge in $\mathscr{T}$ is on the same l-edge in $\mathscr{T}^c$). Select any horizontal l-edge ℓ from $\mathscr{T}^c$, and suppose the horizontal l-edges $\ell_1, \dots, \ell_k$ in $\mathscr{T}$ lie on ℓ. It follows that the constraints

$$\int_{\ell_i} g(s, \hat{y}^-)ds = \int_{\ell_i} g(s, \hat{y}^+)ds, \quad i = 1, \dots, k$$

are equivalent to the constraints

$$\int_{\ell_i} g(s, \hat{y}^-)ds = \int_{\ell_i} g(s, \hat{y}^+)ds, \quad i = 1, \dots, k-1, \qquad \int_{\ell} g(s, \hat{y}^-)ds = \int_{\ell} g(s, \hat{y}^+)ds, \tag{3.12}$$

where $\hat{y}$ is the vertical coordinate of ℓ. When we run through all of the horizontal l-edges in $\mathscr{T}^c$, the last constraint in Eq. (3.12) will form a subset of constraints with $n+1$ elements as follows:

$$\left\{ \int_{x_0}^{x_m} g(s, y_j^-)ds = \int_{x_0}^{x_m} g(s, y_j^+)ds,\ j = 0, \dots, n \right\}.$$

According to Lemma 3.2, these $n+1$ constraints are equivalent to the following n constraints:

$$\left\{ \int_{x_0}^{x_m} g(s, y)ds = 0,\ y \in (y_j, y_{j+1}),\ j = 0, \dots, n-1 \right\}.$$

A similar equivalence can be produced for the vertical l-edges. Here, we should note that for any boundary l-edge, there is only one constraint stated in Lemma 3.1. Therefore, the necessary and sufficient conditions that $\mathcal{I}(g) \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ for $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$ are $E+2$ constraints in this fashion. However, because g is a piecewise constant function, it follows that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} g(s, t)dsdt = \sum_{i=0}^{m-1} (x_{i+1} - x_i)C_i = \sum_{j=0}^{n-1} (y_{j+1} - y_j)D_j, \tag{3.13}$$

where

$$C_i = \int_{y_0}^{y_n} g(x, y)dy, \quad x \in (x_i, x_{i+1}),$$

$$D_j = \int_{x_0}^{x_m} g(x, y)dx, \quad y \in (y_j, y_{j+1}).$$

By Eq. (3.13), the m constraints $\{C_i = 0, i = 0, \dots, m-1\}$ and the n constraints $\{D_j = 0, j = 0, \dots, n-1\}$ are linearly dependent. These $m+n$ constraints are in the $E+2$ constraints, which ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ for $g \in \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$. Therefore, these $E+2$ constraints have a defective rank of at least one. The latter dimension Theorem 3.4 will show that the defective rank is exactly one.

3.3. Dimension theorem

Theorem 3.4. *Given a regular T-mesh $\mathscr{T}$ with V^c crossing vertices, it follows that*

$$\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) = V^c.$$

Proof. First, we prove that $\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) \leqslant V^c$. For this purpose, suppose a function $f \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ that reaches zero at all of the crossing vertices. We intend to prove that $f \equiv 0$.

Suppose $f \not\equiv 0$. Without loss of generality, we assume f is greater than zero in certain regions in $\mathscr{T}$. There then exists a point p in $\mathscr{T}$ such that $f(p) = \delta = \max f > 0$. Let p be in the cell c. Only the following two cases will occur:

1. there exists an edge e of c such that f is a constant δ along e;
2. for any edge e of c, f is not a constant along e.

Consider Case 1. Let l_0 denote the l-edge on which e lies, and define a set L that consists of the l-edges on which f are constants δ. P consists of the end points of the l-edges in L. Because $l_0 \in L$, both L and P are non-empty. Now, we categorize the vertices in P into two types. If a vertex in P is also an interior vertex on another l-edge in L, then the vertex is called a flat vertex. Otherwise, it is called a non-flat vertex. In the following, we will prove that there must exist at least one non-flat vertex in P. If at least one non-flat vertex exists in P, select one non-flat vertex q. Then, q must lie on an l-edge l_1 that is not in L. It follows that q is an interior point on l_1. Because $f(q) = \delta$ and $f|_{l_1}$ is not a constant, there exists a point r on l_1 such that $f(r) > \delta$, which contradicts that δ is the maximum of f over $\mathscr{T}$.

In fact, the non-flat vertex q can be selected as the vertex in P with the minimal y coordinate. If there exists more than one vertex with the minimal y coordinate, the vertex with the minimal x coordinate is selected. Such a selection ensures that q is a non-flat vertex. If q is not a non-flat vertex, q lies on two l-edges, l_2 and l_3, in L, where l_2 is horizontal and l_3 is vertical. If q is a horizontal T-vertex, the bottom end point of l_3 has a y coordinate that is smaller than q. If q is a vertical T-vertex, the left end point of l_2 has the same y coordinate as q but a smaller x coordinate than q. Both cases contradict the selection of q. Therefore, q is a non-flat vertex, and we have proved $f \equiv 0$ for Case 1.

We now consider Case 2. Because $f|_c$ is a bilinear function and f is not a constant along any edges of c, f reaches its maximum only at one of the corners of c. Therefore, p is a corner of c and a T-vertex, for example a horizontal T-vertex. The vertical l-edge through p is assumed to be l_4, which takes p as its interior point. Because $f(p) = \delta$ and $f|_{l_4}$ is not a constant, there exists a point s on l_4 such that $f(s) > \delta$, which contradicts that δ is the maximum of f over $\mathscr{T}$.

If the considerations for both cases are summarized, it follows that $f \equiv 0$. Then, all the crossing vertices in $\mathscr{T}$ form a determining set of the spline space $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. According to the theory of the determining sets in spline functions [10,11], we obtain

$$\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) \leqslant V^c. \tag{3.14}$$

However, in the following, we will prove that $\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) \geqslant V^c$. For this purpose, we consider the operator of the mixed partial derivative defined in Eq. (2.7) as $m = n = 1$:

$$\mathscr{D} : \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) \to \overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T}).$$

The spline space $\overline{\mathbf{S}}(0, 0, -1, -1, \mathscr{T})$ has dimension F, which represents the number of cells in $\mathscr{T}$. According to the analysis presented in the preceding section, to ensure that $\mathscr{I}(g) \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, we must satisfy $E + 2$ constraints, which have a defective rank of at least one. Then, according to Lemma 2.2,

$$\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) \geqslant F - (E + 2) + 1 = V^c. \tag{3.15}$$

If Eqs. (3.14) and (3.15) are combined, it follows that the dimension theorem is proved, i.e.,

$$\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) = V^c. \quad \square$$

3.4. Basis functions

We can construct a set of basis functions $\{b_i(x, y)\}_{i=1}^{V^c}$ for the spline space $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. Similar to standard B-splines, the basis functions should have the following properties:

1. *Compact support*: For any i, $b_i(x, y)$ has as small a support as possible;
2. *Nonnegativity*: For any i, $b_i(x, y) \geqslant 0$;
3. *Partition of unity*: If $\mathscr{T}$ is an extension of some T-mesh $\mathscr{T}_0$ associated with the spline space $\mathbf{S}(1, 1, 0, 0, \mathscr{T}_0)$ and the region occupied by $\mathscr{T}_0$ is Ω, then

$$\sum_{i=1}^{V^c} b_i(x, y) = 1, \quad (x, y) \in \Omega.$$

These properties will facilitate their applications in geometric modeling. In fact, when representing a geometric shape with a linear combination of basis functions satisfying these properties, we can implement local shape control, affine invariance, and variational diminishing. See [12] for details.

Basis functions with these properties can be constructed as follows. Suppose the crossing vertices in $\mathscr{T}$ are v_i with coordinate (x_i, y_i), $i = 1, \ldots, V^c$. Then, we require that the function $b_i(x, y)$ satisfy $b_i(x_j, y_j) = \delta_{ij}$. According to the dimension Theorem 3.4 and the first part of its proof, $b_i(x, y)$ is uniquely determined. All of the functions $b_i(x, y)$ form a set of basis functions of the spline space and have the former three properties. It is straightforward to show that Properties 1 and 2 are satisfied. Now, we prove Property 3.

Theorem 3.5. *Suppose $\mathscr{T}$ is an extension of some T-mesh $\mathscr{T}_0$ associated with the spline space $\mathbf{S}(1, 1, 0, 0, \mathscr{T}_0)$, and the region occupied by $\mathscr{T}_0$ is Ω. The crossing vertices in $\mathscr{T}$ are v_i with the coordinate (x_i, y_i), $i = 1, \ldots, V^c$. The function set $\{b_i(x, y)\}_{i=1}^{V^c} \subset \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ satisfies $b_i(x_j, y_j) = \delta_{ij}$. Then,*

$$\sum_{i=1}^{V^c} b_i(x, y) = 1, \quad (x, y) \in \Omega. \tag{3.16}$$

Proof. Let

$$f(x, y) = \sum_{i=1}^{V^c} b_i(x, y)$$

and ℓ denote the boundary of $\mathscr{T}_0$. To show that Eq. (3.16) holds, we first prove that $f|_\ell \equiv 1$. Because the vertices on ℓ are crossing vertices in $\mathscr{T}$, it follows that f reaches 1 on these vertices. Because $f|_\ell$ is a piecewise linear function with knots being these vertices, it follows that $f|_\ell \equiv 1$. Then, in the following, we prove that for any $(x, y) \in \Omega, f(x, y) = 1$. Consider the function

$$g(x, y) = \begin{cases} f(x, y) - 1 & (x, y) \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

Because $f|_\ell \equiv 1$, we have $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. However, g is zero at the crossing vertices of $\mathscr{T}$. According to the proof of Theorem 3.4, it follows that $g \equiv 0$, i.e., $f(x, y) = 1, (x, y) \in \Omega$. □

In this section, certain interesting questions remain.

1. How can we directly specify the former basis functions in every cell of a general T-mesh?
2. How can we evaluate the function or the surface represented in the linear combination of the former basis functions?
3. What is the "knot" insertion algorithm in this spline space?

These questions also apply to higher degree spline spaces over T-meshes. The answers to these questions are required if we want to effectively represent, evaluate, and modify spline functions over T-meshes in geometric modeling and other areas. Because of their simplicity, bilinear spline spaces over T-meshes should be among our first choices to answer these questions. We hope the solutions for bilinear cases may help us to find answers regarding higher degree spline spaces over T-meshes.

4. A lower bound on the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$

We can apply a method similar to that proposed in the preceding section to the dimensional analysis of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ and obtain a lower bound on the dimension, i.e.,

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \geqslant V^c - E + 1. \tag{4.1}$$

4.1. Some lemmas

We consider the operator of the mixed partial derivative as follows:

$$\mathscr{D} := \frac{\partial^2}{\partial x \partial y} : \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \to \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}).$$

The operator $\mathscr{I}(g)$ is defined in the same manner as Eq. (2.8). The following lemmas discuss constraints that ensure $\mathscr{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$.

Lemma 4.1. *Given a regular T-mesh $\mathscr{T}$, let the coordinates of the end points of any horizontal l-edges l_i^h be (x_{i1}^h, y_i^h) and (x_{i2}^h, y_i^h), $i = 0, 1, \ldots, m$, and the coordinates of the end points of any vertical l-edge l_j^v be (x_j^v, y_{j1}^v) and (x_j^v, y_{j2}^v), $j = 0, 1, \ldots, n$. For any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, it follows that*

$$\mathscr{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \Leftrightarrow \begin{cases} \displaystyle\int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^{h-}) ds = \int_{x_{i1}^h}^{x_{i2}^h} \frac{\partial}{\partial y} g(s, y_i^{h+}) ds, \quad i = 0, 1, \ldots, m, \quad \text{and} \\ \displaystyle\int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_j^{v-}, t) dt = \int_{y_{j1}^v}^{y_{j2}^v} \frac{\partial}{\partial x} g(x_i^{v+}, t) dt, \quad j = 0, 1, \ldots, n. \end{cases}$$

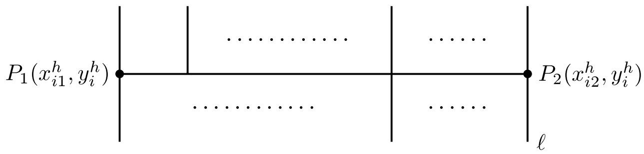


Fig. 5. A horizontal l-edge (2).

Proof. We first prove the necessity $\Longrightarrow$. Let $f(x, y) = \mathcal{I}(g)(x, y)$. Without loss of generality, we only prove that when $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ the constraints corresponding to horizontal l-edges are satisfied. As shown in Fig. 5, the two end points of the horizontal l-edge are $P_1(x^h_{i1}, y^h_i)$ and $P_2(x^h_{i2}, y^h_i)$. The vertical edge on which P_2 lies is ℓ. Because $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, it follows that $f(x^h_{i2}, y)$ is a quadratic polynomial with respect to the variable y in a neighborhood of P_2 along ℓ. Therefore,

$$f(x^h_{i2}, y^{h-}_i) = f(x^h_{i2}, y^{h+}_i),$$
$$\frac{\partial}{\partial y} f(x^h_{i2}, y^{h-}_i) = \frac{\partial}{\partial y} f(x^h_{i2}, y^{h+}_i),$$
$$\frac{\partial^2}{\partial y^2} f(x^h_{i2}, y^{h-}_i) = \frac{\partial^2}{\partial y^2} f(x^h_{i2}, y^{h+}_i).$$

According to the definition of f and the continuity of g, the first two equations hold trivially. Substituting the definition of f into the last equation, we have

$$\int_{-\infty}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h-}_i) ds = \int_{-\infty}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h+}_i) ds. \tag{4.2}$$

Extend the current l-edge to the left boundary of the T-mesh. If no other l-edges exist on the extension, then in every cell that intersects the extension on the left side of (x^h_{i1}, y^h_i), g is a single bilinear function. It follows that

$$\int_{-\infty}^{x^h_{i1}} \frac{\partial}{\partial y} g(s, y^{h-}_i) ds = \int_{-\infty}^{x^h_{i1}} \frac{\partial}{\partial y} g(s, y^{h+}_i) ds. \tag{4.3}$$

Subtracting Eq. (4.3) from Eq. (4.2), we have

$$\int_{x^h_{i1}}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h-}_i) ds = \int_{x^h_{i1}}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h+}_i) ds.$$

This result proves that the equation in the lemma corresponding to the current l-edge holds. If other l-edges exist on the extension, we consider these l-edges one by one from left to right. We can prove that the former equations hold for these l-edges. Therefore, for the current l-edge, according to Eq. (4.2), the same equation also holds.

Next, we prove the sufficiency $\Longleftarrow$. We will show that for any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, if the following equations hold

$$\int_{x^h_{i1}}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h-}_i) ds = \int_{x^h_{i1}}^{x^h_{i2}} \frac{\partial}{\partial y} g(s, y^{h+}_i) ds, \quad i = 0, 1, \ldots, m, \tag{4.4}$$

$$\int_{y^v_{j1}}^{y^v_{j2}} \frac{\partial}{\partial x} g(x^{v-}_j, t) dt = \int_{y^v_{j1}}^{y^v_{j2}} \frac{\partial}{\partial x} g(x^{v+}_i, t) dt, \quad j = 0, 1, \ldots, n, \tag{4.5}$$

then f is a single biquadratic polynomial in each cell of $\mathscr{T}$.

In contrast, suppose that in some cell c_i, f is piecewise with the horizontal lines of discontinuity $y = y_{i_1}, \ldots, y = y_{i_{n_i}}$ and the vertical lines of discontinuity $x = x_{i_1}, \ldots, x = x_{i_{m_i}}$ of the partial second-order derivatives. Let $\mathscr{C}_y = \bigcup_i \{y_{i_1}, \ldots, y_{i_{n_i}}\}$, $\mathscr{C}_x = \bigcup_i \{x_{i_1}, \ldots, x_{i_{m_i}}\}$. Then, according to the assumption, $\mathscr{C}_x \bigcup \mathscr{C}_y$ is non-empty. Without loss of generality, we assume $\mathscr{C}_y$ is non-empty. Let $\hat{y} = \min_{y \in \mathscr{C}_y} y$. Suppose c_i is the leftmost cell that takes $y = \hat{y}$ as its inner line of discontinuity of the partial second-order derivatives with respect to x. Denote the x coordinate of the left boundary edge of c_i as x_0, and let $(x_1, \hat{y})$ represent a point on the line of discontinuity, where $x_1 > x_0$. It follows that

$$\int_{-\infty}^{x_0} \frac{\partial}{\partial y} g(s, \hat{y}^-) ds = \int_{-\infty}^{x_0} \frac{\partial}{\partial y} g(s, \hat{y}^+) ds.$$

Because $(x_1, \hat{y})$ is one point of discontinuity, it follows that

$$\int_{-\infty}^{x_1} \frac{\partial}{\partial y} g(s, \hat{y}^-) ds \neq \int_{-\infty}^{x_1} \frac{\partial}{\partial y} g(s, \hat{y}^+) ds.$$

Therefore,

$$\int_{x_0}^{x_1} \frac{\partial}{\partial y} g(s, \hat{y}^-)ds \neq \int_{x_0}^{x_1} \frac{\partial}{\partial y} g(s, \hat{y}^+)ds.$$

This result contradicts that $\frac{\partial}{\partial y} g(x, y)$ is continuous inside the cell c_i. Therefore, we have proved that f is a single biquadratic polynomial in each cell of $\mathscr{T}$.

To prove $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, we still must verify that f satisfies the HBC. In fact, using the same approach, we can prove that f is a single biquadratic polynomial outside $\mathscr{T}$. However, $f(x, y)$ is zero when $x \leqslant x_0, y \leqslant y_0$, where (x_0, y_0) is the coordinate of the left-bottom corner of the T-mesh $\mathscr{T}$. Therefore, f is zero everywhere outside $\mathscr{T}$, which ensures that f satisfies the HBC. Therefore, $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$. □

According to Lemma 4.1 and because $\dim \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T}) = V^c$, to ensure $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, there are $E + 4$ constraints with V^c under-determining coefficients, where E is the number of interior l-edges. However, these constraints are not linearly independent, as stated in the following lemma.

Lemma 4.2. *Given a regular T-mesh $\mathscr{T}$ whose occupied rectangle is $(x_l, x_r) \times (y_b, y_t)$, assume that the different y coordinates from all the horizontal l-edges are $y_0 < y_1 < \cdots < y_n$. For $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, it follows that*

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^-)ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^+)ds, \quad i = 0, \ldots, n \tag{4.6}$$

is equivalent to

$$\int_{x_l}^{x_r} g(s, y_i)ds = 0, \quad i = 1, \ldots, n-1. \tag{4.7}$$

Proof. First, we prove the necessity. Beginning from the bottom boundary l-edge l_0, we have

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0^-)ds = 0, \qquad \int_{x_l}^{x_r} g(s, y_0)ds = 0.$$

Then, according to Eq. (4.6), when $i = 0$, we have

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0^+)ds = 0.$$

However, according to the piecewise bilinear definition of g, we have

$$\begin{aligned}\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_0+)ds &= \int_{x_l}^{x_r} \frac{\partial}{\partial y}\left(\frac{y_1 - y}{y_1 - y_0} g(s, y_0) + \frac{y - y_0}{y_1 - y_0} g(s, y_1)\right)ds \\ &= \frac{1}{y_1 - y_0}\left(\int_{x_l}^{x_r} g(s, y_1)ds - \int_{x_l}^{x_r} g(s, y_0)ds\right) \\ &= \frac{1}{y_1 - y_0}\int_{x_l}^{x_r} g(s, y_1)ds. \end{aligned} \tag{4.8}$$

Therefore,

$$\int_{x_l}^{x_r} g(s, y_1)ds = 0.$$

Recursively, we can prove that Eq. (4.7) holds for every horizontal l-edge.

To prove the sufficiency, we follow a deductive process similar to the one described in Eq. (4.8). We have

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^-)ds = \frac{1}{y_i - y_{i-1}}\left(\int_{x_l}^{x_r} g(s, y_i)ds - \int_{x_l}^{x_r} g(s, y_{i-1})ds\right),$$

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^+)ds = \frac{1}{y_{i+1} - y_i}\left(\int_{x_l}^{x_r} g(s, y_{i+1})ds - \int_{x_l}^{x_r} g(s, y_i)ds\right).$$

Therefore, Eq. (4.7) ensures that Eq. (4.6) holds. □

Remark 4.3. From the proof of Lemma 4.2, we can also conclude that for horizontal l-edges,

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^-) ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^+) ds, \quad i = 0, \ldots, n$$

is equivalent to

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^-) ds = 0, \quad i = 1, \ldots, n-1.$$

Lemma 4.2 states that there are at least two respective redundant constraints among those corresponding to horizontal l-edges and vertical l-edges. The following lemma indicates that we can consider the constraints along the four boundary l-edges redundant.

Lemma 4.4. *Given a regular T-mesh $\mathscr{T}$, the rectangle that it occupies is $(x_l, x_r) \times (y_b, y_t)$, where the different y coordinates from the horizontal l-edges are $y_0 < y_1 < \cdots < y_n$. Let $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. Then, Eq. (4.6) is equivalent to*

$$\int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^-) ds = \int_{x_l}^{x_r} \frac{\partial}{\partial y} g(s, y_i^+) ds, \quad i = 1, \ldots, n-1. \tag{4.9}$$

Proof. The necessity is obvious. To prove the sufficiency, we show that Eq. (4.9) implies that Eq. (4.7) holds. In fact, suppose

$$I_i = \int_{x_l}^{x_r} g(s, y_i) ds, \quad i = 0, 1, \ldots, n.$$

Then, $I_0 = I_n = 0$. Our object is to prove that for any $i = 1, 2, \ldots, n-1$, $I_i = 0$. By a similar deduction as that used to obtain Eq. (4.8), we have

$$\int_{x_l}^{x_r} g(s, y_i) ds = \frac{y_{i+1} - y_i}{y_{i+1} - y_{i-1}} \int_{x_l}^{x_r} g(s, y_{i-1}) + \frac{y_i - y_{i-1}}{y_{i+1} - y_{i-1}} \int_{x_l}^{x_r} g(s, y_{i+1}) ds,$$

i.e.,

$$I_i = \frac{y_{i+1} - y_i}{y_{i+1} - y_{i-1}} I_{i-1} + \frac{y_i - y_{i-1}}{y_{i+1} - y_{i-1}} I_{i+1}$$

holds for any $i = 1, 2, \ldots, n-1$. Therefore, the point set $\{(y_i, I_i)\}_{i=0}^n$ is collinear, which indicates that a linear function $f(y)$ exists such that $f(y_i) = I_i$, $i = 0, \ldots, n$. Because $y_0 = y_n = 0$, it follows that $f(y) \equiv 0$, which indicates that $I_i = 0$, $i = 1, 2, \ldots, n$. Thus, the sufficiency is proved. □

Remark 4.5. The statements in Lemmas 4.2 and 4.4 apply to horizontal l-edges. Similar statements can be made for vertical l-edges.

4.2. A lower bound on dimensions

For a given regular T-mesh $\mathscr{T}$ that has E interior l-edges, we assume that its associated tensor-product mesh $\mathscr{T}^c$ has E' interior l-edges, where $E' \leqslant E$ and $\mathscr{T}^c = (x_0, x_1, \ldots, x_m) \times (y_0, y_1, \ldots, y_n)$. Suppose the horizontal l-edges $\ell_1, \ldots, \ell_k$ in $\mathscr{T}$ lie on the horizontal l-edge ℓ in $\mathscr{T}^c$, where the vertical coordinate of ℓ is $\hat{y}$. Then, we have that

$$\int_{\ell_i} \frac{\partial}{\partial y} g(s, \hat{y}^-) ds = \int_{\ell_i} \frac{\partial}{\partial y} g(s, \hat{y}^+) ds, \quad i = 1, \ldots, k$$

are equivalent to

$$\int_{\ell_i} \frac{\partial}{\partial y} g(s, \hat{y}^-) ds = \int_{\ell_i} \frac{\partial}{\partial y} g(s, \hat{y}^+) ds, \quad i = 1, \ldots, k-1, \qquad \int_{\ell} \frac{\partial}{\partial y} g(s, \hat{y}^-) ds = \int_{\ell} \frac{\partial}{\partial y} g(s, \hat{y}^+) ds.$$

When we consider all of the horizontal l-edges in $\mathscr{T}^c$, the last constraint in the former equation will form a subset of constraints with $n+1$ elements:

$$\left\{ \int_{x_0}^{x_m} \frac{\partial}{\partial y} g(s, y_i^-) ds = \int_{x_0}^{x_m} \frac{\partial}{\partial y} g(s, y_i^+) ds, \ i = 0, \ldots, n \right\}.$$

According to Lemma 4.2, these constraints are equivalent to the following $n-1$ constraints:

$$\left\{ \int_{x_0}^{x_m} g(s, y_i) ds = 0, \ i = 1, \ldots, n-1 \right\}.$$

A similar conclusion can be made for the vertical l-edges. Therefore, the number of sufficient and necessary constraints that ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ for $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is simply E.

Moreover, in the tensor-product mesh $\mathscr{T}^c$, because g is a piecewise linear function, it follows that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(s,t)dsdt = \sum_{i=1}^{m-1}(x_{i+1}-x_{i-1})E_i = \sum_{j=1}^{n-1}(y_{j+1}-y_{j-1})F_j, \tag{4.10}$$

where

$$E_i = \int_{y_0}^{y_n} g(x_i, y)dy, \qquad F_j = \int_{x_0}^{x_m} g(x, y_j)dx.$$

Therefore, among the former E constraints, the element number in any maximally linearly independent subset is at most $E - 1$ when these constraints are not empty (i.e., $V^c > 0$).

Based on the foregoing analysis, a lower bound on the dimensions of the spline space $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ can be obtained as follows:

Theorem 4.6. *Given a regular T-mesh $\mathscr{T}$ with $V^c > 0$ crossing vertices and E interior l-edges, it follows that*

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \geqslant V^c - E + 1. \tag{4.11}$$

Proof. Because $V^c > 0$, the constraints are not empty, and $E > 1$. According to the former analysis and Theorem 3.4, the dimension of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is V^c. For any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, the constraints ensuring $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ have maximally linearly independent subsets with element numbers of $E - 1$ at most. Here, both $\mathcal{D}$ and $\mathcal{I}$ are linear and injective (see Section 2.4. Therefore, $\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \geqslant V^c - E + 1$. □

The lower bound in the theorem is sharp for spline spaces over certain T-meshes. For example, consider the T-mesh $\mathscr{T}$ shown in Fig. 2. Use of the B-net method in [5] can show that $\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = 0$. However, in $\mathscr{T}$, $V^c = 5$, $E = 6$. Then, $V^c - E + 1 = 0$, which indicates that the lower bound is reached in this T-mesh.

It is straightforward to verify that the former lower bound is exactly the dimension of the biquadratic spline space over a tensor-product mesh if $V^c - E + 1 \geqslant 0$. In the next section, we will prove that for certain special hierarchical T-meshes, the lower bound is also sharp.

To conclude this section, we use Theorems 2.3 and 4.6 to provide a lower bound on the dimensions of spline spaces over T-meshes.

Corollary 4.7. *Given a regular T-mesh $\mathscr{T}$ with V^c crossing vertices, E interior l-edges, and V^b boundary vertices, it follows that*

$$\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) \geqslant 2V^b + V^c - E + 1. \tag{4.12}$$

Proof. Let $\mathscr{T}^\varepsilon$ be an extension of $\mathscr{T}$ associated with the spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$. Then, according to the construction rule of an extended T-mesh, a crossing vertex of $\mathscr{T}$ is also a crossing vertex of $\mathscr{T}^\varepsilon$; a boundary vertex (but not a corner vertex) of $\mathscr{T}$ results in two crossing vertices of $\mathscr{T}^\varepsilon$; and a corner vertex of $\mathscr{T}$ results in four crossing vertices of $\mathscr{T}^\varepsilon$. Therefore, there are $V^c + 2(V^b - 4) + 16 = 2V^b + V^c + 8$ crossing vertices in $\mathscr{T}^\varepsilon$. However, it is obvious that there are $(E + 8)$ l-edges in $\mathscr{T}^\varepsilon$.

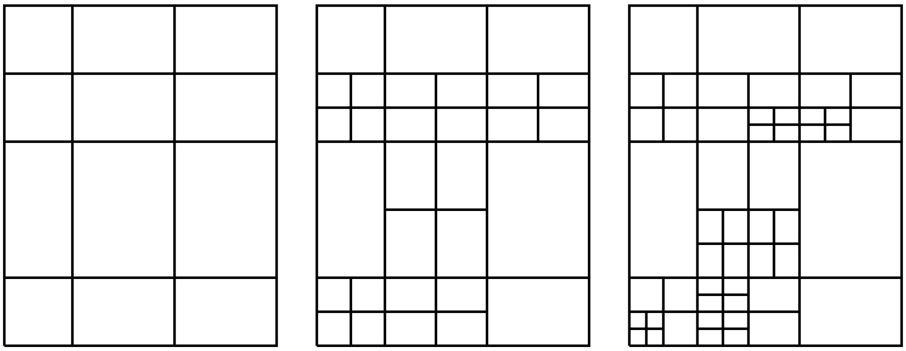
According to Theorems 2.3 and 4.6, we have

$$\begin{aligned}\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) &= \dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)\\ &\geqslant (2V^b + V^c + 8) - (E + 8) + 1\\ &= 2V^b + V^c - E + 1. \quad \square\end{aligned}$$

5. The dimensions of spline spaces $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ over hierarchical T-meshes

In this section, a careful analysis of the constraints in Section 4 will help us construct a dimension formula of the spline space $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ over a hierarchical T-mesh. Here, the key procedure consists of the following components:

1. A general hierarchical T-mesh is divided into crossing-vertex-connected branches (Definition 5.3). Then, the spline space over the hierarchical T-mesh is divided into the direct sum of some subspaces, each of which is defined over a crossing-vertex-connected hierarchical branch, which is also a T-mesh (see Section 5.6).
2. In a crossing-vertex-connected hierarchical T-mesh, the constraint set is proved to have a defective rank (see Definition 2.5) of exactly one by the following process:
   (a) The constraints are converted into a new form to reflect the level structure of the hierarchical T-mesh (see Section 5.2).
   (b) A new set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ is defined according to the structure of the T-mesh such that the occurrence of the basis function coefficients in the constraints is regularized (see Section 5.3 and Proposition 5.12).



(a) Level 0. (b) Level 1. (c) Level 2.

Fig. 6. A hierarchical T-mesh.

(c) The l-edges and the corresponding constraints are ordered according to the structure of the T-mesh. For each l-edge and constraint, a characteristic vertex is introduced. Write all of the constraints in vector form $(C_0, C_1, \ldots, C_T)^T$ in increasing order. The constraint C_0 can be removed because we know that all of the constraints have a defective rank of at least one. Therefore, we must show that $(C_1, \ldots, C_T)^T$ has full rank (see Section 5.1.2).

(d) Assume the characteristic vertex of C_i is V_i, $i = 1, \ldots, T$. Arrange the coefficients into a vector $(\beta_1, \ldots, \beta_T, \beta_{T+1}, \ldots, \beta_M)^T$ such that β_i is the coefficient of the basis function corresponding to V_i, $i = 1, 2, \ldots, T$, in $\bar{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. There then exists a non-singular upper-triangular matrix $\mathbf{M} = (m_{ij})_{T\times M}$ such that $(C_1, \ldots, C_T)^T = \mathbf{M}(\beta_1, \ldots, \beta_M)^T$.

Therefore, the complete set of constraints has a defective rank of one.

5.1. Hierarchical T-meshes

A hierarchical T-mesh [7] is a special type of T-mesh that has a natural level structure. It is defined recursively. Generally, we start from a tensor-product mesh (level 0). Then, the individual cells at level k are each divided into four subcells, which are cells at level $k+1$. For simplicity, we subdivide each cell by connecting the middle points of the opposite edges with two straight lines. Fig. 6 illustrates the process of generating a hierarchical T-mesh. To emphasize the level structure of a hierarchical T-mesh $\mathscr{T}$, in certain cases, we denote the T-mesh of level k as $\mathscr{T}^k$. The maximal level number that appears is called the *level* of the hierarchical T-mesh.

Definition 5.1. In a regular T-mesh, a *cross-cut l-edge* is an l-edge with two end points on the boundary of the T-mesh. The four boundary l-edges are cross-cut l-edges. The horizontal and vertical cross-cut l-edges form a tensor-product mesh, which is called the *maximal tensor-product submesh* of the given T-mesh.

Suppose a hierarchical T-mesh $\mathscr{T}$ is formed from a tensor-product mesh $\mathscr{T}^0$. It is obvious that $\mathscr{T}^0$ is a submesh of the maximal tensor-product submesh of $\mathscr{T}$.

A given hierarchical T-mesh $\mathscr{T}$ can be extended to obtain an extended T-mesh $\mathscr{T}^\varepsilon$ associated with the spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$. In the following, *the term "hierarchical T-meshes" refers to both classical hierarchical T-meshes and their extensions*.

Over a hierarchical T-mesh $\mathscr{T}$, the dimension of $\bar{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ may be greater than the lower bound provided in Theorem 4.6. For example, consider the hierarchical T-mesh shown in Fig. 7 in which the mesh of level 0 is a tensor-product mesh with size 3×3, and in the level 1 mesh, only one cell is subdivided. In this mesh, $V^c = 5$, $E = 6$. The lower bound is $V^c - E + 1 = 0$. However, the dimension of the biquadratic spline space over the mesh is obviously at least one.

5.1.1. Level numbers of edges, l-edges, and crossing-vertices

We assign each edge or l-edge in a hierarchical T-mesh to the level number of the T-mesh on which the edge or l-edge appears. An l-edge of level k consists of edges on level k. The extension of an edge in the extended T-mesh is assigned to the same level as its source. The newly added l-edges in the extended T-mesh are assigned to level 0.

A crossing vertex is assigned two level numbers denoted (k_h, k_v). These numbers correspond to the level number k_h of the horizontal l-edge and the level number k_v of the vertical l-edge where the vertex lies, respectively, and k_h and k_v are called the horizontal level and the vertical level, respectively, of the crossing vertex.

For example, in the hierarchical T-mesh $\mathscr{T}$ shown in Fig. 8, suppose the middle horizontal l-edge is of level p_0 and that the upper and lower neighboring l-edges are of levels p_1 and p_2, respectively. Here, $p_0 > p_1$, $p_0 > p_2$. Because v_1 is the intersection of two l-edges with level p_0, its level is (p_0, p_0). Similarly, because v_2 is the intersection of a horizontal l-edge of level p_0 and a vertical l-edge of level p_1, its level is (p_0, p_1).

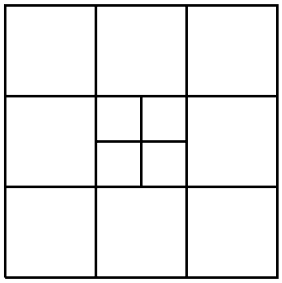


Fig. 7. A hierarchical T-mesh in which the dimension is greater than the lower bound.

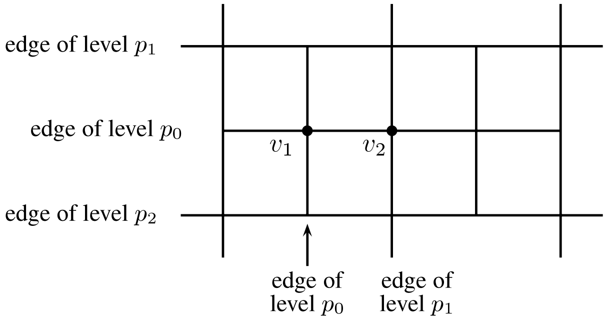


Fig. 8. Level numbers of edges, l-edges, and crossing vertices.

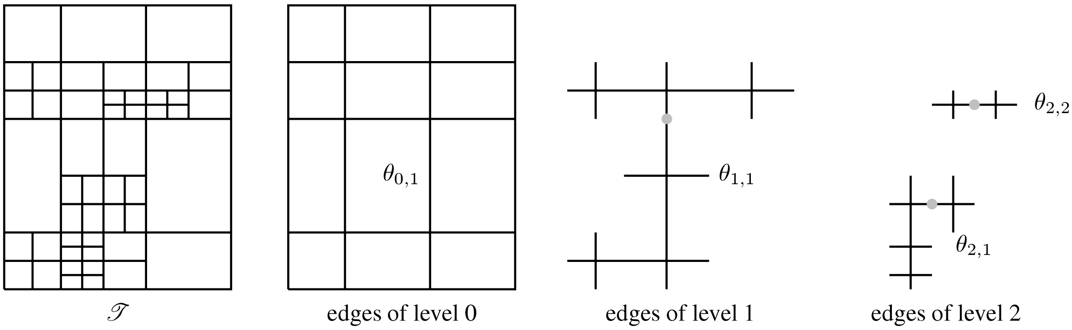


Fig. 9. A hierarchical T-mesh showing its decomposition according to the edge levels.

5.1.2. Ordering on interior l-edges

To reasonably sort the constraints, we introduce a partial ordering on interior l-edges in a hierarchical T-mesh. This order will be used in the proof of Theorem 5.13 to facilitate the rank analysis of the constraints. First, however, we propose the following definition:

Definition 5.2. In a hierarchical T-mesh $\mathscr{T}$, two interior l-edges are *continuous* if they intersect in a crossing vertex of $\mathscr{T}$. An l-edge set S is *connected* if for any two l-edges ℓ_0 and ℓ_1 in S there exists a *continuous series of l-edges* $l_0, l_1, \ldots, l_k$ in S between ℓ_0 and ℓ_1, i.e., $l_0 = \ell_0, l_k = \ell_1$, and l_i and l_{i+1} are continuous for any $i = 0, 1, \ldots, k-1$.

Consider a crossing-vertex-connected hierarchical T-mesh $\mathscr{T}$. Fix a level number $k \geqslant 0$. Then, all l-edges of level k are possibly not connected (see Fig. 9 for an example in which all of the edges of level 2 are not connected). We assume that the l-edges of level k form some maximal connected branches. Denote these branches as $\theta_{k,i}, i = 1, \ldots, T_k$. Therefore, in the example of Fig. 9, we have $T_0 = 1, T_1 = 1$, and $T_2 = 2$. Because $\mathscr{T}$ is crossing-vertex-connected, when $k > 1$ there exists at least one crossing vertex of level number (k, j) or (j, k) on some l-edge in the branch $\theta_{k,i}$, where $j < k$. (For example, see Fig. 9, in which the specified crossing vertex is shown in light gray.) This crossing vertex is the intersection between two l-edges of levels k and j. The l-edge of level j is called an *entering l-edge* of the branch that is connected to each l-edge in $\theta_{k,i}$. The "entering l-edge" of all the l-edges of level zero is defined as any l-edge of level zero.

Fix an entering l-edge ℓ_0 of $\theta_{k,i}$. In $\theta_{k,i}$, there are many continuous series of l-edges connecting $\ell \in \theta_{k,i}$ and ℓ_0. The number of l-edges in a series is called the length of the series, and the minimal length of the series connecting ℓ and ℓ_0, which is called the *distance between ℓ and ℓ_0*, is denoted $\text{dist}(\ell, \ell_0)$. Suppose $e_0 = \ell_0, e_1, \ldots, e_s = \ell$ is an l-edge series of minimal

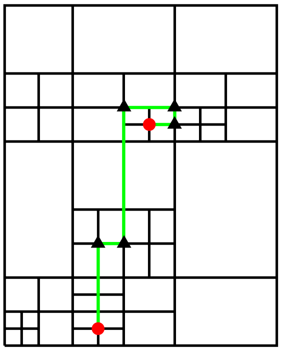


Fig. 10. A hierarchical T-mesh with a path between two connected crossing vertices.

length within the continuous series connecting ℓ and ℓ_0. Then, the intersection between $e_s = \ell$ and e_{s-1} is a crossing vertex on ℓ, which is called a *characteristic vertex of the l-edge* ℓ, and the vertex's level is (k, j) or (j, k), where $j \leqslant k$.

After selecting the entering l-edges for all the connected branches $\theta_{k,i}$, we introduce a partial ordering $<_1$ of the interior l-edges in $\mathscr{T}$. For any two interior l-edges ℓ_1 and ℓ_2 of levels k_1 and k_2, respectively, where ℓ_j is in the branch $\theta_{k_j,i_j}, j = 1, 2$, we define $\ell_1 <_1 \ell_2$ if

1. $k_1 < k_2$, or
2. $k_1 = k_2$ and $i_1 < i_2$, or
3. $k_1 = k_2$ and $i_1 = i_2$, $\mathrm{dist}(\ell_1, \ell_0) < \mathrm{dist}(\ell_2, \ell_0)$, where ℓ_0 is the entering l-edge of the connected branch in which both ℓ_1 and ℓ_2 lie.

This order is incomplete, because, in Case 3, it is possible for two different interior l-edges that $\mathrm{dist}(\ell_1, \ell_0) = \mathrm{dist}(\ell_2, \ell_0)$.

5.1.3. Crossing-vertex-connected hierarchical T-meshes

To construct examples of spline spaces over hierarchical T-meshes where the lower bound in Eq. (4.11) is sharp, we focus on a special type of hierarchical T-meshes.

Definition 5.3. For a regular T-mesh, if between any two different crossing vertices a continuous poly line consisting of edges in the mesh exists such that every joint between two neighboring horizontal and vertical edges on the poly line is a crossing vertex in the mesh, then the T-mesh is called *crossing-vertex-connected*. The poly lines in such a mesh are called *paths* between the crossing vertices.

For example, two crossing vertices labeled with red dots are selected in the T-mesh shown in Fig. 10. A path between these two vertices is illustrated in green, and the joints are shown with black triangles.

Definition 5.4. In forming a hierarchical T-mesh, from level k to level $k + 1$, if there exists a cell of level k to be subdivided for which all horizontal and vertical neighboring cells of level k remain unchanged, the cell is called an *isolated subdivided cell*. Here, we require that at least one neighboring cells exists. Furthermore, if an isolated subdivided cell is not a boundary cell (whose four boundary edges are interior edges of the mesh), the cell is called a *non-boundary isolated subdivided cell*.

The following proposition states the relationship between crossing-vertex-connected hierarchical T-meshes and isolated subdivided cells.

Proposition 5.5. *Suppose $\mathscr{T}$ is a hierarchical T-mesh whose maximal tensor-product submesh has at least one crossing vertex. Then, $\mathscr{T}$ is crossing-vertex-connected if and only if in any level of forming the hierarchical T-mesh, no isolated subdivided cells exist.*

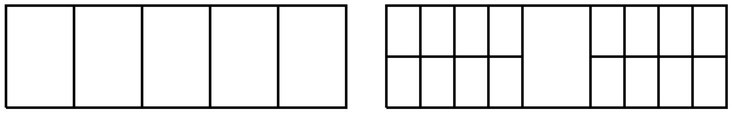
Proof. Assume the maximal tensor-product submesh of $\mathscr{T}$ is $\mathscr{M}$ and the level of $\mathscr{T}$ is k_0. Define

$$\mathscr{M}^k = \mathscr{T}^k \cup \mathscr{M},$$

where $A \cup B$ represents a T-mesh with edges in either A or B. It follows that $\mathscr{M}^{k_0} = \mathscr{T}^{k_0} = \mathscr{T}$.

To finish the proof, we will use an important characteristic of a hierarchical T-mesh whose maximal tensor-product submesh has at least one crossing vertex: For any k $(0 \leqslant k \leqslant k_0)$, any interior l-edge ℓ in $\mathscr{M}^k$ has at least one crossing vertex in $\mathscr{M}^k$. In fact, according to the status of ℓ, we have the following two possibilities:

- ℓ is in $\mathscr{M}$: Because $\mathscr{M}$ has at least one crossing vertex, $\mathscr{M}$ has at least one interior horizontal cross-cut l-edge and at least one interior vertical cross-cut l-edge. Therefore, ℓ has at least one crossing vertex in $\mathscr{M}$;



(a) A tensor-product mesh. (b) Four cells are subdivided.

Fig. 11. Forming a hierarchical T-mesh from a tensor-product mesh without an interior crossing vertex.

- ℓ is in $\mathscr{T}^k$ with $k > 0$: ℓ consists of some edges with level k. As previously mentioned, the edges of level k appear when we subdivide certain cells of level $k-1$ by connecting the middle points of the opposite edges of each cell with straight lines. Therefore, ℓ has at least one crossing vertex of level (k, k).

Here, we do not need to consider the case that ℓ is in $\mathscr{T}^0$ because $\mathscr{T}^0$ is a submesh of $\mathscr{M}$.

Next, we prove that if in any level of forming $\mathscr{T}$, no isolated subdivided cells exist, then $\mathscr{T}$ is crossing-vertex-connected. It is obvious that $\mathscr{M}^0 = \mathscr{M}$ is crossing-vertex-connected. Suppose v_0 is a crossing vertex of $\mathscr{M}$. According to the definition of crossing-vertex-connection, we must only prove that for any $k > 0$, if v is a crossing vertex in $\mathscr{M}^k$, a path exists between v and v_0. If this supposition is true for k, we will prove that it is also true for $k+1$. From $\mathscr{M}^k$ to $\mathscr{M}^{k+1}$, let v be a new crossing vertex, which is the intersection of two l-edges $\ell_1, \ell_2 \in \mathscr{M}^{k+1}$ with levels k_1 and k_2, respectively. Without loss of generality, we assume $k_1 \leqslant k_2$. It follows that $k_2 = k+1$. We have the following two cases:

1. $k_1 < k_2 = k+1$:
   In this case, ℓ_1 is in $\mathscr{M}^k$ and has a crossing vertex v' in $\mathscr{M}^k$. Therefore, a path exists between v and v'. Consequently, a path exists between v and v_0.
2. $k_1 = k_2 = k+1$:
   This case indicates that a cell c of $\mathscr{M}^k$ is subdivided by connecting the middle points of its opposite edges with two straight line segments. The two straight line segments are on ℓ_1 and ℓ_2, respectively, and their intersection is v. However, c is not an isolated subdivided cell. Therefore, either ℓ_1 or ℓ_2 must intersect an l-edge of a level less than $k+1$. Suppose the intersection is v'. According to Case 1, a path exists between v' and v_0. Therefore, a path exists between v and v_0.

This result indicates that for any $k > 0$, if v is a crossing vertex in $\mathscr{M}^k$, a path exists between v and v_0. Therefore, $\mathscr{M}^k$ is crossing-vertex-connected, which completes the proof of the sufficiency of the lemma.

Finally, we prove the necessity of the lemma, i.e., if $\mathscr{T}$ is crossing-vertex-connected, then in any level of forming $\mathscr{T}$, no isolated subdivided cells exist. We assume that c is an isolated subdivided cell of level k. Suppose v is the new crossing vertex of level $(k+1, k+1)$ inside c. Then, v is not in $\mathscr{M}$, and we can find another crossing vertex v^1 in $\mathscr{M}$. Therefore a path does not exist between v and v^1, which contradicts that $\mathscr{T}$ is crossing-vertex-connected. □

In Proposition 5.5, we only consider a hierarchical T-mesh whose maximal tensor-product submesh has at least one crossing vertex. If we form a hierarchical T-mesh from a tensor-product mesh without a crossing vertex, we might obtain a mesh that is not crossing-vertex-connected, even if in any level of forming the mesh, no isolated subdivided cells exist. See Fig. 11, where the tensor-product mesh in Fig. 11(a) has no crossing vertex and the mesh in Fig. 11(b) is not crossing-vertex-connected, although no isolated subdivided cells exist.

In practice, we rarely begin from a tensor-product mesh without a crossing vertex because we usually use a hierarchical T-mesh that is an extension of some hierarchical T-mesh associated with a biquadratic spline space. However, if we must calculate the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, where the maximal tensor-product submesh of $\mathscr{T}$ has no crossing vertex, then a set of hierarchical T-meshes $\{\mathscr{T}_i\}$ exists where $\mathscr{T}_i$ is a submesh of $\mathscr{T}$ and its maximal tensor-product submesh has at least one crossing vertex such that

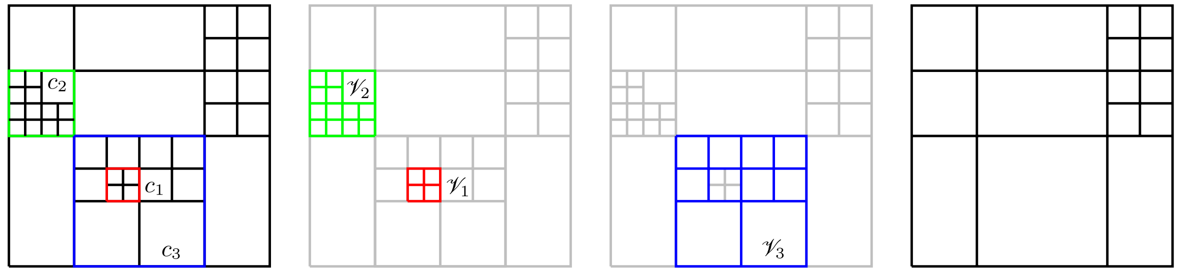
$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = \sum_i \dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}_i). \tag{5.1}$$

Because the maximal tensor-product submesh of $\mathscr{T}$ has no crossing vertex, $\mathscr{T}$ has some cells of level 0 that remain unchanged in the latter subdivision when forming $\mathscr{T}$. Each of those cells has a pair of opposite edges, which are on the boundary of $\mathscr{T}$. Therefore, according to the HBC of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, for any $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, $f \equiv 0$ over these cells. After deleting these cells, we obtain hierarchical T-meshes $\{\mathscr{T}_i\}$, where $\mathscr{T}_i$ is a submesh of $\mathscr{T}$ and its maximal tensor-product submesh has at least one crossing vertex. Here, $\mathscr{T}_i \cap \mathscr{T}_j = \emptyset$ for any $i \neq j$. Therefore, Eq. (5.1) holds.

Remark 5.6. In the remainder of the paper, we only consider a hierarchical T-mesh whose maximal tensor-product submesh has at least one crossing vertex.

5.1.4. Division of a general hierarchical T-mesh

In this subsection, we discuss how to divide a general hierarchical T-mesh into the union of some crossing-vertex-connected hierarchical T-meshes. Suppose the given hierarchical T-mesh is $\mathscr{T}$, and let c_i, $i = 1, \ldots, C$ be all the isolated subdivided cells of level $k > 0$ in forming $\mathscr{T}$. Then, the subdivision occurring in c_i to form $\mathscr{T}$ will also form a hierarchical



(a) $\mathscr{T}$. (b) $\mathscr{V}_i = \mathscr{U}_i, i = 1, 2$. (c) $\mathscr{V}_3 = \mathscr{U}_3 \setminus \{\mathscr{U}_1\}$. (d) $\mathscr{V}_0 = \mathscr{T} \setminus \{\mathscr{U}_1, \mathscr{U}_2, \mathscr{U}_3\}$.

Fig. 12. Dividing a general hierarchical T-mesh into the union of some crossing-vertex-connected hierarchical T-meshes.

T-mesh, denoted $\mathscr{U}_i$. Here, $\mathscr{U}_i$ occupies the same region as the cell c_i. In the following, we will apply a T-mesh operation "\". Suppose $\mathscr{T}_i, i = 0, 1, \ldots, k$ are hierarchical T-meshes. Then, $\mathscr{T}_0 \setminus \{\mathscr{T}_1, \ldots, \mathscr{T}_k\}$ is a new T-mesh that consists of edges and vertices in $\mathscr{T}_0$ but not in the interior of $\mathscr{T}_i, i = 1, \ldots, k$. Let $\mathscr{U}_0 = \mathscr{T}$, and

$$\mathscr{V}_i = \mathscr{U}_i \setminus \{\mathscr{U}_j : \mathscr{U}_j \text{ is a submesh of } \mathscr{U}_i, j = 1, \ldots, C, j \neq i\}, \quad i = 0, 1, \ldots, C.$$

Then, it is straightforward to verify that $\mathscr{V}_i$ is a crossing-vertex-connected hierarchical T-mesh. Here, $\mathscr{V}_0$ derives from $\mathscr{T}$. The other $\mathscr{V}_i$'s derive from the isolated subdivided cells of level $k > 0$. It follows that $\mathscr{T}$ can be viewed as the disjoint union of $\mathscr{V}_i$, $i = 0, 1, \ldots, C$.

See Fig. 12 for an example. Fig. 12(a) is a hierarchical T-mesh $\mathscr{T}$ with three isolated subdivided cells c_1, c_2 and c_3, whose boundaries are in red, green, and blue, respectively. Then, for $i = 1, 2, 3$, $\mathscr{U}_i$ is a hierarchical T-mesh formed by all the edges of $\mathscr{T}$ within c_i. $\{\mathscr{V}_i\}_{i=0}^4$ are given as in Fig. 12(b)–(d).

In Sections 5.2–5.5, we prove that the lower bound on the dimension proposed in Theorem 4.6 is exactly the dimension of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ over a crossing-vertex-connected hierarchical T-mesh. Then, in Section 5.6, a dimension theorem can be proposed to calculate the dimension of the spline space over a general hierarchical T-mesh by dividing the hierarchical T-mesh into the union of some crossing-vertex-connected hierarchical branches.

5.2. Conversion of constraints

In Section 4, we proposed different versions of the necessary and sufficient conditions that ensure $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ for any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. To facilitate the latter analysis, we use the following notation and constraints.

For any regular T-mesh $\mathscr{T}$, denote its occupied rectangle as $(x_l, x_r) \times (y_b, y_t)$. $\mathscr{T}^c$ is an associated tensor-product mesh with $\mathscr{T}$. Assume the y coordinates of the horizontal l-edges in $\mathscr{T}^c$ are $y_b = y_0 < y_1 < \cdots < y_m < y_{m+1} = y_t$ and the x coordinates of the vertical l-edges in $\mathscr{T}^c$ are $x_l = x_0 < x_1 < \cdots < x_n < x_{n+1} = x_r$. For $i = 0, 1, \ldots, m + 1$, assume the horizontal l-edges $l_{i1}^h, \ldots, l_{i\alpha_i}^h$ have the y coordinate $y = y_i$. Here, $l_{i1}^h, \ldots, l_{i\alpha_i}^h$ are sorted from left to right. However, for $j = 0, 1, \ldots, n + 1$, assume the vertical l-edges $l_{j1}^v, \ldots, l_{j\beta_j}^v$ have the x coordinate $x = x_j$. Here, $l_{j1}^v, \ldots, l_{j\beta_j}^v$ are sorted from bottom to top. It follows that $\alpha_0 = \alpha_{m+1} = \beta_0 = \beta_{n+1} = 1, \alpha_1 + \cdots + \alpha_m + \beta_1 + \cdots + \beta_n = E$.

With this notation, we can allocate the E constraints on the interior l-edges in $\mathscr{T}^c$ according to Lemma 4.4 in the following manner. For any interior horizontal l-edge $y = y_i$, the corresponding constraints are

$$\int_{l_{ik}^h} \frac{\partial}{\partial y} g(s, y_i^-) ds = \int_{l_{ik}^h} \frac{\partial}{\partial y} g(s, y_i^+) ds, \quad k = 1, \ldots, \alpha_i. \tag{5.2}$$

For any interior vertical l-edge $x = x_j$, the corresponding constraints are

$$\int_{l_{jk}^v} \frac{\partial}{\partial x} g(x_j^-, t) dt = \int_{l_{jk}^v} \frac{\partial}{\partial x} g(x_j^+, t) dt, \quad k = 1, \ldots, \beta_j. \tag{5.3}$$

Furthermore, by applying a deduction similar to that used in Eq. (4.8), Eqs. (5.2) and (5.3) are equivalent to

$$(y_{i+1} - y_{i-1}) \int_{x_{ik0}^h}^{x_{ik1}^h} g(s, y_i) ds = (y_{i+1} - y_i) \int_{x_{ik0}^h}^{x_{ik1}^h} g(s, y_{i-1}) ds + (y_i - y_{i-1}) \int_{x_{ik0}^h}^{x_{ik1}^h} g(s, y_{i+1}) ds, \quad k = 1, \ldots, \alpha_i, \tag{5.4}$$

$$(x_{j+1} - x_{j-1}) \int_{y_{jk0}^v}^{y_{jk1}^v} g(x_j, t) dt = (x_{j+1} - x_j) \int_{y_{jk0}^v}^{y_{jk1}^v} g(x_{j-1}, t) dt + (x_j - x_{j-1}) \int_{y_{jk0}^v}^{y_{jk1}^v} g(x_{j+1}, t) dt, \quad k = 1, \ldots, \beta_j, \tag{5.5}$$

respectively, where x_{ik0}^h and x_{ik1}^h are the x coordinates of the two end points of l_{ik}^h, and y_{jk0}^v and y_{jk1}^v are the y coordinates of the two end points of l_{jk}^v.

We now focus on hierarchical T-meshes, for which the corresponding constraints can be converted into a form that reflects the level structure of the T-mesh. First, we introduce two definitions.

In a hierarchical T-mesh $\mathscr{T}$, select any horizontal l-edge ℓ of level $k > 0$. Therefore, ℓ appears in $\mathscr{T}$ since $\mathscr{T}^k$. On *l*, one or more crossing vertices exist with vertical level *k*. These crossing vertices are the center of some inserted crossing from $\mathscr{T}^{k-1}$ to $\mathscr{T}^k$. The l-edge ℓ consists of the horizontal edges of this inserted crossing. It follows that the vertical edges of the inserted crossing intersect the two l-edges ℓ^t and ℓ^b in $\mathscr{T}^{k-1}$. Here, we assume that ℓ^b of level k^b lies under ℓ and that ℓ^t of level k^t lies above ℓ.

Definition 5.7. For a horizontal l-edge *l* of level $k > 0$, the l-edges ℓ^b and ℓ^t are defined as described above. Then, ℓ^b and ℓ^t are called the *support l-edges* of ℓ. The support l-edges of a horizontal l-edge ℓ of level 0 are defined as the two nearest horizontal l-edges of level 0 that lie above and below ℓ, respectively. For vertical l-edges, the support l-edges are defined similarly.

It is obvious that for any horizontal/vertical l-edge ℓ, two vertical/horizontal l-edges that pass through the two end points of ℓ intersect the two support l-edges of ℓ in cases where there is no other crossing vertex between the intersection points and the end points along the two vertical/horizontal l-edges.

In Eqs. (5.4) and (5.5), a constraint along an l-edge is represented by the linear combination of three integrations. These integrations are conducted along the current l-edge and its two neighboring horizontal/vertical lines, respectively, with the same integration limits. Here, the horizontal/vertical lines share the same *y*/*x* coordinates as the two-sided nearest horizontal/vertical l-edges to the current l-edge. In the following lemma, we will convert these constraints into a new form such that every constraint along an l-edge is a linear combination of three integrations along the current l-edge and its support l-edges. With this form, every constraint will involve undetermined coefficients in a manner that simplifies the determination of rank.

Lemma 5.8. *Given a hierarchical T-mesh $\mathscr{T}$, select an interior horizontal l-edge ℓ_i^h. Suppose the y coordinate of ℓ_i^h is y_i^h, the x coordinates of its two end points are x_{i1}^h and x_{i2}^h, and the y coordinates of its support l-edges are y_i^{hb} and y_i^{ht}. Then, Eq. (5.4) holds for the interior horizontal l-edges if and only if for each interior horizontal l-edge the following equation holds:*

$$(y_i^{ht} - y_i^{hb}) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^h) ds = (y_i^{ht} - y_i^h) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^{hb}) ds + (y_i^t - y_i^{hb}) \int_{x_{i1}^h}^{x_{i2}^h} g(s, y_i^{ht}) ds. \tag{5.6}$$

Proof. Necessity. Suppose the interior horizontal l-edge ℓ^h is of level *k*, and its two support l-edges are ℓ^b and ℓ^t. Between ℓ^h and ℓ^b, other l-edges exist. Suppose the different *y* coordinates of these l-edges are $\bar{y}_0 < \bar{y}_1 < \cdots < \bar{y}_{\bar{n}}$, where $\bar{y}_0$ and $\bar{y}_{\bar{n}}$ correspond to ℓ^b and ℓ^h, respectively. Suppose again that the different *y* coordinates of the l-edges between ℓ^h and ℓ^t are $\bar{y}_{\bar{n}} < \bar{y}_{\bar{n}+1} < \cdots < \bar{y}_{\bar{n}+\bar{m}}$, where $\bar{y}_{\bar{n}+\bar{m}}$ corresponds to ℓ^t. According to the definition of the support l-edges, the horizontal l-edges, excluding ℓ^h, ℓ^b and ℓ^t, are of a level greater than *k*. Therefore, no l-edge exists whose vertical projection onto ℓ^h takes one of the end points of ℓ^h as its interior point. Therefore, according to the constraints set forth in Eq. (5.4) that correspond to these l-edges, we can conclude that $(\bar{y}_{i-1}, I_{i-1})$, $(\bar{y}_i, I_i)$, and $(\bar{y}_{i+1}, I_{i+1})$ are collinear, where $i = 1, \ldots, \bar{m} + \bar{n} - 1$, and

$$I_i = \int_{x_{i1}^h}^{x_{i2}^h} g(s, \bar{y}_i) ds.$$

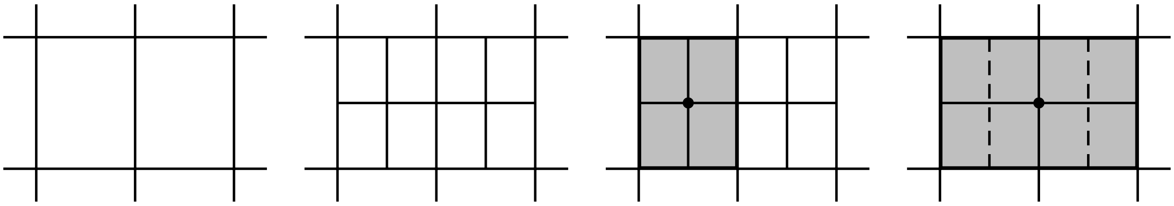
Then, $(\bar{y}_0, I_0)$, $(\bar{y}_{\bar{n}}, I_{\bar{n}})$, and $(\bar{y}_{\bar{n}+\bar{m}}, I_{\bar{n}+\bar{m}})$ are collinear, which indicates that the corresponding constraint in Eq. (5.6) holds.

Sufficiency. We prove the sufficiency inductively from the highest level to the lowest level. We will show that for any given level k_0, if the constraints defined in Eq. (5.6) that correspond to the l-edges of level $k \geqslant k_0$ hold, then the constraints in Eq. (5.4) that correspond to all the l-edges of level k_0 also hold.

Suppose the maximal level number in the hierarchical T-mesh $\mathscr{T}$ is *M*. For an interior horizontal l-edge of level *M*, it follows from the fact that other l-edges between the current l-edge and its support l-edges do not exist that the corresponding constraint defined in Eq. (5.4) is the same as the constraint defined in Eq. (5.6).

We now assume that for the interior horizontal l-edges of levels greater than *k*, if the corresponding constraints in Eq. (5.6) hold, then the corresponding constraints in Eq. (5.4) also hold. Select an arbitrary interior horizontal l-edge ℓ^h of level *k* whose support l-edges are ℓ^b and ℓ^t. The two-sided nearest l-edges of ℓ^h have the *y* coordinates h_0 and h_1. Because the level numbers of the other horizontal l-edges between ℓ^h and ℓ^b are greater than *k*, according to the inductive assumption, the integration values along these horizontal lines with the integration interval $[x_{i1}^h, x_{i2}^h]$ are collinear with respect to their *y* coordinates. In particular, the corresponding integration values along ℓ^h, ℓ^b and $y = h_0$ are collinear with respect to their *y* coordinates. Similarly, the integration values along ℓ^h, ℓ^t and $y = h_1$ are collinear with respect to their *y* coordinates. Therefore, the integration values along ℓ^h, $y = h_0$ and $y = h_1$ are collinear with respect to their *y* coordinates, which completes the proof of the sufficiency. □

Remark 5.9. The statement in Lemma 5.8 applies to horizontal l-edges. A similar statement can be made for vertical l-edges.



(a) A part in $\mathscr{T}^{k-1}$. (b) After inserting crosses. (c) Support example 1. (d) Support example 2.

Fig. 13. The support of the hierarchical basis functions.

In the next subsection, we will specify a proper set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ such that the coefficients of g appear in these constraints in a regular form under these basis functions.

5.3. Hierarchical basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$

In Section 3.4, an approach was proposed for specifying a set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ with many favorable properties. Now, we specify a new set of basis functions for $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ that do not possess the partition of unity. However, under this set, the constraints in Lemma 5.8 appear in a regular form, which facilitates the rank determination of the constraints.

The new set of basis functions is defined level by level when forming the hierarchical T-mesh. Every basis function is associated with a crossing vertex. At level 0, we consider the level 0 T-mesh $\mathscr{T}^0$ and introduce the functions as standard bilinear tensor-product B-splines. Therefore, every function can be associated with a crossing vertex in $\mathscr{T}^0$ in such a way that the function reaches one at the crossing vertex and zero at the other crossing vertices of $\mathscr{T}^0$. Use the set $\mathscr{B}^0$ to denote all these functions. Suppose the current level number is $k \geqslant 1$. Consider a newly appearing crossing vertex in this level.

1. If the level of the crossing vertex is (k, k), then the crossing vertex must be the center of an inserted cross in a cell c of $\mathscr{T}^{k-1}$. Therefore, a function associated with the crossing vertex can be defined such that the function reaches one at the vertex and the support of the function is c (see Fig. 13(c); in the figure, the light-gray region is the support of the specified function associated with the new crossing vertex labeled "•"). Here, the discontinuity of the derivatives $\partial/\partial x$ and $\partial/\partial y$ appears on the edges of the inserted cross.
2. Otherwise, the crossing vertex is the middle point of an edge e in $\mathscr{T}^{k-1}$, where e is the common edge of two neighboring cells c_1 and c_2, each of which is subdivided by inserting a cross from level $k-1$ to k. In this case, a function can be defined such that the function reaches one at the current crossing vertex, its support is exactly $c_1 \cup c_2$, and its derivative discontinuity in the support lies only on the lines through the current crossing vertex (see Fig. 13(d) in which the derivatives $\partial/\partial x$ and $\partial/\partial y$ are continuous along the dashed edges).

All of the functions introduced at level k are denoted as $\mathscr{B}^k$.

Lemma 5.10. *Suppose the maximal level number of the hierarchical T-mesh $\mathscr{T}$ is M. Define $\mathscr{B} = \bigcup_{k=0}^{M} \mathscr{B}^k$. Then, $\mathscr{B}$ is a set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$.*

Proof. By the definition of the functions in $\mathscr{B}^k$, it follows that the number of functions in $\mathscr{B}$ is exactly the number of crossing vertices in $\mathscr{T}$. Moreover, these functions are in the spline space $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. Therefore, to prove that $\mathscr{B}$ forms a set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, we only must show that the functions in $\mathscr{B}$ are linearly independent.

Assume the region occupied by $\mathscr{T}$ is Ω. Let $\mathscr{B}^k = \{b_1^k(x, y), \ldots, b_{n_k}^k(x, y)\}$. Suppose that a set of coefficients of α_i^k ensures that

$$f(x, y) := \sum_{k=0}^{M} \sum_{i=0}^{n_k} \alpha_i^k b_i^k(x, y) = 0, \quad (x, y) \in \Omega.$$

Now, we will prove that $\alpha_i^k = 0$ for any i and k.

Consider the values of f at every crossing vertex v_i^0 of level (0, 0). Because $f \equiv 0$, it follows that $f(v_i^0) = 0$. However, the function $b_i^0(x, y)$ in $\mathscr{B}^0$ associated with v_i^0 is one at v_i^0, and the other functions vanish at v_i^0. Therefore, $f(v_0^i) = \alpha_i^0 b_i^0(v_0^i) = \alpha_i^0$, i.e., $\alpha_i^0 = 0$, and we obtain

$$f(x, y) = \sum_{k=1}^{M} \sum_{i=0}^{n_k} \alpha_i^k b_i^k(x, y).$$

Suppose we have proved that $\alpha_i^j = 0, i = 0, \ldots, n_j, j = 0, 1, \ldots, k-1$. Now, we consider the functions and their coefficients introduced at level k. Each such function is associated with a crossing vertex. First, we consider the functions associated with crossing vertices whose vertical and horizontal level numbers differ. On such a crossing vertex, all the functions are zero except the function associated with the current crossing vertex. Therefore, the corresponding coefficient is zero. Next, we consider the other functions that are associated with crossing vertices of the same vertical and horizontal level numbers. In a similar manner, we can conclude that their coefficients are also zero. Therefore, all of the coefficients are zero, and the functions in $\mathscr{B}$ are linearly independent. Thus they form a set of basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. □

We will call this set of basis functions *the hierarchical basis functions* of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$.

5.4. Concretion of the constraints

For a given hierarchical T-mesh $\mathscr{T}$, suppose the hierarchical basis functions of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$ are

$$\{b_i^k(x, y)\} = \{c_j(x, y)\}_{j=1}^{V^c},$$

where $b_i^k(x, y)$ is associated with a crossing vertex appearing in $\mathscr{T}$ since level k.

Represent g as

$$g(x, y) = \sum_{j=1}^{V^c} \alpha_j c_j(x, y). \tag{5.7}$$

Because there is a one-to-one mapping between the basis functions and the crossing vertices, the coefficient of a basis function is also called the *coefficient of the corresponding crossing vertex*.

We now state the characteristics of the constraints after substituting the representation of g defined in Eq. (5.7) into Eq. (5.6), which ensures $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$. First, we consider the constraints corresponding to the interior l-edges of level 0.

Proposition 5.11. *After substituting Eq. (5.7) into Eq. (5.6) and performing a proper transformation, the constraints corresponding to the interior horizontal l-edges of level 0 are in such a form that the nonzero terms in a constraint consist of those associated with the crossing vertices on the same l-edge.*

Proof. Each interior horizontal l-edge of level 0 must traverse the entire mesh. The support l-edges of a horizontal l-edge of level 0 are also of level 0. Over the T-mesh $\mathscr{T}^0$, we apply Lemma 4.2 and find that all the constraints corresponding to the interior horizontal l-edges of level 0 are equivalent to the constraints that the integration of g along each interior horizontal l-edge is zero. For every l-edge of level 0, only the coefficients of the crossing vertices on the current l-edge are nonzero. Therefore, the integration can be represented by a linear combination of the coefficients of the crossing vertices on the current l-edge. □

Next, we consider the constraints corresponding to the interior l-edges of a level greater than 0.

Proposition 5.12. *Suppose the current interior horizontal l-edge is of level $k > 0$. After substituting Eq. (5.7) into Eq. (5.6), the nonzero coefficients of the basis functions in the corresponding constraint consist of two parts, as follows:*

A. *the coefficients of the crossing vertices on the current l-edge;*
B. *the possible coefficients of the crossing vertices of a horizontal level less than k and a vertical level greater than k.*

Proof. Select an interior horizontal l-edge ℓ^h of level $k > 0$. Suppose its support l-edges are ℓ^b and ℓ^t, which are of levels k_1 and k_2, respectively. Assume that the x coordinates of the two end points of ℓ^h are x_0^h and x_1^h and that the vertical l-edges through the two end points are ℓ_l^v and ℓ_r^v. Then, the two vertical l-edges ℓ_l^v and ℓ_r^v intersect ℓ_b and ℓ_t. Consider the crossing vertices whose associating basis functions are nonzero on ℓ^h, $\ell^b|_{[x_0^h, x_1^h]}$ or $\ell^t|_{[x_0^h, x_1^h]}$. These crossing vertices can be classified into the following cases with respect to their levels (k_0^h, k_1^h):

1. $k_0^h < k$:
   (a) $k_1^h \leqslant k$. Because for any $x \in [x_0^h, x_1^h]$ the corresponding three points on ℓ^h, ℓ^t, and ℓ^b with the horizontal coordinates x appear simultaneously in a cell of $\mathscr{T}^{k-1}$ (including its boundary), it follows that for the current basis function $b(x, y)$, Eq. (5.6) holds as $g = b(x, y)$. This result indicates that the coefficient of $b(x, y)$ does not appear in the constraint after simplification.
   (b) $k_1^h > k$. This type of coefficient can appear in the constraint.
2. $k_0^h = k$. This type of crossing vertex does not contribute to the constraint unless the crossing vertex lies on l^h.
3. $k_0^h > k$. The basis function associated with this type of crossing vertex vanishes on ℓ^h, ℓ^b and ℓ^t between x_0^h and x_1^h. Therefore, the corresponding coefficient does not appear in the constraint.

Therefore, nonzero coefficients occur in Cases 1(b) and 2, which correspond to Cases B and A, respectively, in the proposition description. □

We can similarly classify the coefficients' appearance in the constraints corresponding to interior vertical l-edges.

Following this classification, we know that the coefficient of a crossing vertex of level (k_1, k_2) can appear in any of the following three locations:

1. The constraints corresponding to the horizontal l-edge through the vertex;
2. The constraints corresponding to the vertical l-edge through the vertex;
3. If $k_1 < k_2 - 1$ (or $k_1 - 1 > k_2$), the coefficient may appear in the constraints corresponding to the horizontal (or vertical) l-edges of the levels less than k_2 (or k_1). If $k_1 = k_2$ or $k_2 \pm 1$, this situation does not occur.

In the first two cases, the involved l-edges are called *the naturally appearing l-edges of the coefficient*. In the third case, the involved l-edges are called *the unnaturally appearing l-edges of the coefficient*. For simplicity, a constraint corresponding to a horizontal/vertical l-edge of level k is simply called the *horizontal/vertical constraint of level k*. For two given interior l-edges ℓ_1 and ℓ_2, the corresponding constraints are c_1 and c_2, respectively. If $\ell_1 <_1 \ell_2$, then we also define $c_1 <_1 c_2$.

5.5. A dimension theorem for crossing-vertex-connected hierarchical T-meshes

With this preparation, we now state and prove the dimension theorem of biquadratic spline spaces over crossing-vertex-connected hierarchical T-meshes.

Theorem 5.13. *Over a crossing-vertex-connected hierarchical T-mesh $\mathscr{T}$, it follows that*

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = V^c - E + 1.$$

Proof. First, certain facts must be restated. For any $g \in \overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$, to ensure that $\mathcal{I}(g) \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, the constraints corresponding to every interior horizontal l-edge (defined in Eq. (5.6)) and vertical l-edge should be satisfied. Here, we apply the hierarchical basis functions $\mathscr{B}$ of $\overline{\mathbf{S}}(1, 1, 0, 0, \mathscr{T})$. These constraints can be represented by linear combinations of the coefficients of g under the basis functions $\mathscr{B}$. The coefficients appear in these constraints as stated in Propositions 5.11 and 5.12.

Suppose that the l-edges of level k in $\mathscr{T}$ form some connected branch $\theta_{k,i}, i = 1, 2, \ldots, T_k$ and that the entering l-edges for all of the branches $\theta_{k,i}$ have been selected and denoted as $\ell_{k,i}$. Specifically, the interior l-edges of level zero form a connected branch. The entering l-edge of this branch is selected to any one interior l-edge of level zero. Then, we can introduce a partial ordering $<_1$ on the interior l-edges and the corresponding constraints, as stated in Section 5.1.2.

At the beginning, all of the constraints are linearly dependent because Eq. (4.10) shows a linear combination of these constraints with a result of zero. In this linear combination, each coefficient of level zero is nonzero. Therefore, the rank of the constraints remains unchanged after deleting any level zero constraint. Without loss of generality, we assume that the deleted constraint corresponds to the entering l-edge $\ell_{0,1}$ of $\theta_{0,1}$. Then, we focus on the remaining constraints and show that they are linearly independent.

Sort the remaining constraints into a non-decreasing series $C_1, C_2, \ldots, C_T$ according to the ordering $<_1$, where $T = E-1$, and sort the coefficients $\{\alpha_i\}$ in Eq. (5.7) into a series $\beta_1, \beta_2, \ldots, \beta_{V^c}$ such that β_i is a characteristic vertex of the l-edge whose corresponding constraint is C_i, $i = 1, \ldots, T$. The remaining variables $\beta_{T+1}, \ldots, \beta_{V^c}$ are arranged randomly. Then, we can write these constraints into the following matrix form:

$$(C_1, C_2, \ldots, C_T)^T = \mathbf{M}\,(\beta_1, \beta_2, \ldots, \beta_{V^c})^T,$$

where $\mathbf{M} = (m_{ij})$. Because the characteristic vertex of an l-edge ℓ of level k is of level (k, j) or (j, k), where $j \leqslant k$, it follows according to Propositions 5.11 and 5.12 that the coefficient β_i does not appear in the constraints $C_{i+1}, \ldots, C_T$, where $i = 1, \ldots, T-1$. Therefore, $m_{ij} = 0$, $i > j$. However, the matrix $\mathbf{M}$ shows $m_{ii} \neq 0$, $i = 1, \ldots, T$. This result indicates that matrix $\mathbf{M}$ is of full row rank, i.e., rank $\mathbf{M} = T$. Therefore, the dimension of the spline space $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$ is $V^c - T = V^c - E + 1$, which completes the proof of the theorem. □

5.6. A general dimension theorem

In this subsection, we consider the dimension formula of $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, where $\mathscr{T}$ is a general hierarchical T-mesh.

Lemma 5.14. *Suppose the hierarchical T-mesh $\mathscr{T}$ and its disjoint union of $\mathscr{V}_i$, $i = 0, 1, \ldots, C$ are defined as stated in Section 5.1.4. It follows that*

$$\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i). \tag{5.8}$$

Proof. First, we prove that the intersection of any two different subspaces among $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i)$, $i = 0, 1, \ldots, C$ is $\{0\}$. Therefore, we can define the direct sum of these subspaces. Suppose we select the submeshes $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$, where $i_0 \neq i_1$. The submeshes $\mathscr{U}_{i_0}$ and $\mathscr{U}_{i_1}$ are defined as before. If any submesh of $\mathscr{U}_{i_0}$ and $\mathscr{U}_{i_1}$ is not a submesh of the other, then the regions occupied by the two are disjoint, and it follows that the intersection of the subspaces over $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$ is simply $\{0\}$. Otherwise, we assume without loss of generality that $\mathscr{U}_{i_0}$ is a submesh of $\mathscr{U}_{i_1}$. In this case, the region occupied by $\mathscr{U}_{i_0}$ is inside a cell of $\mathscr{U}_{i_1}$. Denote the cell as c. In $\mathscr{V}_{i_1}$, the zero function is a unique function whose function values and two first-order partial derivatives are zero along the boundary of c. It follows that the intersection of the subspaces over $\mathscr{V}_{i_0}$ and $\mathscr{V}_{i_1}$ is also $\{0\}$.

It is obvious that

$$\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \supset \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i). \tag{5.9}$$

However, for any $f \in \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T})$, we can construct its component in each subspace as follows. For any $j = 0, 1, \ldots, C$, we can arrange all the meshes $\{\mathscr{U}_i\}_{i=1}^{C}$, each of which takes $\mathscr{U}_j$ as a submesh in an ascending chain as follows:

$$\mathscr{U}_j = \mathscr{U}_{i_j} \subset \mathscr{U}_{i_{j-1}} \subset \cdots \subset \mathscr{U}_{i_0} = \mathscr{U}_0.$$

Because $\mathscr{V}_0$ is obtained from $\mathscr{U}_0 = \mathscr{T}$ by deleting the subdivisions in certain isolated subdivided cells, we can define a new function $\mathbb{P}_0 f$ that meets f with order one along all the edges in $\mathscr{V}_0$. Then, $f - \mathbb{P}_0 f$ vanishes out of those isolated subdivided cells of $\mathscr{U}_0$. We define recursively that $\mathbb{P}_j f$ is a function in $\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_j)$ that meets the function $f - \sum_{k=i_0}^{i_j-1} \mathbb{P}_k f$ with order one along all the edges in $\mathscr{V}_j$. Then, $f - \sum_{k=i_0}^{i_j} \mathbb{P}_k f$ vanishes in $\mathscr{V}_j$ except in the isolated subdivided cells of $\mathscr{U}_j$.

According to the definition of $\mathbb{P}_k$, it follows that $f - \sum \mathbb{P}_k f$ vanishes everywhere in $\mathscr{T}$. Therefore, $f \in \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i)$, which indicates that

$$\overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) \subset \bigoplus_{i=0}^{C} \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i). \tag{5.10}$$

Combining Eqs. (5.9) and (5.10), we have Eq. (5.8). □

With this lemma, we have the following theorem concerning the dimension formula of spline spaces over general hierarchical T-meshes.

Theorem 5.15. *Suppose $\mathscr{T}$ is a hierarchical T-mesh with $\delta - 1$ isolated subdivided cells. Then,*

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = V^c - E + \delta.$$

Proof. Suppose $\mathscr{U}_j$ and $\mathscr{V}_j$, $j = 0, 1, \ldots, C$ are defined as in the proof of Lemma 5.14. It follows that $C = \delta - 1$ and, according to Lemma 5.14,

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = \sum_{i=0}^{C} \dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{V}_i).$$

Assume that in $\mathscr{V}_i$ there are V_i^+ crossing vertices and E_i interior l-edges. Then,

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = \sum_{i=0}^{C} (V_i^+ - E_i + 1).$$

Because any two different meshes among V_i do not share any common crossing vertices or interior l-edges, it follows that

$$\sum_{i=0}^{C} (V_i^+ - E_i + 1) = V^c - E + C + 1 = V^c - E + \delta.$$

Therefore, we have

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = V^c - E + \delta. \quad \square$$

Corollary 5.16. *Suppose $\mathscr{T}$ is a hierarchical T-mesh with $\delta' - 1$ non-boundary isolated subdivided cells. Then,*

$$\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) = 2V^b + V^c - E + \delta'.$$

Proof. Let $\mathscr{T}^\varepsilon$ be an extension of $\mathscr{T}$ associated with the spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$. Then, according to the construction rule of an extended T-mesh, a crossing vertex of $\mathscr{T}$ is a crossing vertex of $\mathscr{T}^\varepsilon$; a boundary vertex (but not a corner vertex) of $\mathscr{T}$ results in two crossing vertices of $\mathscr{T}^\varepsilon$; and a corner vertex of $\mathscr{T}$ results in four crossing vertices of $\mathscr{T}^\varepsilon$. Therefore, there are $V^c + 2(V^b - 4) + 16 = 2V^b + V^c + 8$ crossing vertices in $\mathscr{T}^\varepsilon$. However, it is obvious that there are $(E + 8)$ l-edges in $\mathscr{T}^\varepsilon$.

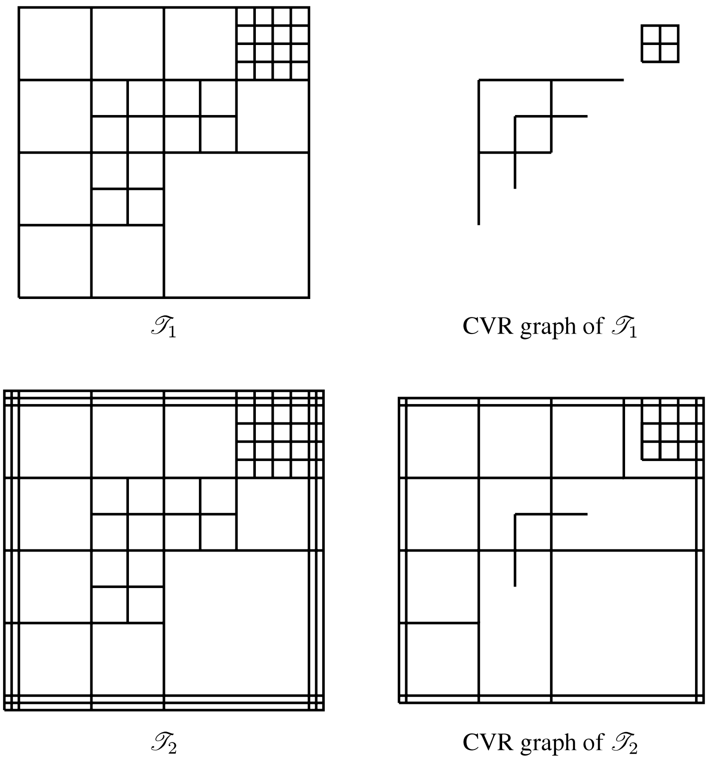


Fig. 14. Two examples of CVR graphs.

A non-boundary isolated subdivided cell of $\mathscr{T}$ is an isolated subdivided cell of $\mathscr{T}^\varepsilon$, and in $\mathscr{T}^\varepsilon$, we cannot find other isolated subdivided cells.

According to Theorems 2.3 and 5.15, we have

$$\begin{aligned}\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) &= \dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}^\varepsilon)\\ &= (2V^b + V^c + 8) - (E + 8) + \delta'\\ &= 2V^b + V^c - E + \delta'. \quad \square\end{aligned}$$

For example, we consider the hierarchical T-mesh in Fig. 6(c), which has 22 boundary vertices, 28 crossing vertices, and 22 interior l-edges. $\mathscr{T}$ has only one isolated subdivided cell, which appears in the left-lower corner cell of level 1. Therefore, this cell is not a non-boundary isolated subdivided cell, and $\delta = 2$, $\delta' = 1$. Then we have

$$\dim \overline{\mathbf{S}}(2, 2, 1, 1, \mathscr{T}) = 28 - 22 + 2 = 8,$$
$$\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) = 2 \times 22 + 28 - 22 + 1 = 51.$$

5.7. *Notes on construction of basis functions*

After obtaining the dimension formulae of biquadratic spline spaces over hierarchical T-meshes, we naturally consider how to construct their basis functions with desirable properties, as stated for the bilinear basis functions presented in Section 3.4.

For this purpose, we first must clarify the topological meaning of $V^c - E + \delta$.

Definition 5.17. Given a hierarchical T-mesh $\mathscr{T}$, we can construct a graph $\mathscr{G}$ by retaining the crossing vertices and the line segments with two end points that are crossing vertices and removing the other vertices and the edges in $\mathscr{T}$. $\mathscr{G}$ is called the *crossing-vertex-relationship graph* (CVR graph for short) of $\mathscr{T}$.

See Fig. 14 for two examples of hierarchical T-meshes and their corresponding CVR graphs.

According to graph theory [13], a CVR graph is a plane graph that partitions the plane into some bounded regions and an unbounded region. Here, we call a bounded region a cell of the CVR graph. The unbounded region is not included within the cells of the CVR graph.

Given a plane graph $\mathscr{G}$, we consider one of its connected components (a connected component is a maximal connected subgraph). The Euler formula states that $F_C - E_C + V_C = 1$, where F_C, E_C, and V_C are the numbers of cells, edges, and vertices in the current connected part, respectively. After considering all of the connected components in $\mathscr{G}$, we then have $F_{\mathscr{G}} - E_{\mathscr{G}} + V_{\mathscr{G}} = \delta$, where $F_{\mathscr{G}}$, $E_{\mathscr{G}}$, and $V_{\mathscr{G}}$ are the numbers of cells, edges, and vertices in $\mathscr{G}$, and δ is the number of connected parts in $\mathscr{G}$. The Euler formula holds for a CVR graph.

According to the definition of a CVR graph, if a hierarchical T-mesh $\mathscr{T}$ has $\delta - 1$ isolated subdivided cells, then its CVR graph $\mathscr{G}$ has δ connected components (see the examples in Fig. 14). $\mathscr{T}_1$ has an isolated subdivided cell at its top-right corner. Therefore, $\delta = 2$. The CVR graph of $\mathscr{T}_1$ has two disconnected components. $\mathscr{T}_2$ is an extended T-mesh of $\mathscr{T}_1$ associated with the spline space $\mathbf{S}(2, 2, 1, 1, \mathscr{T}_1)$. $\mathscr{T}_2$ does not have any isolated subdivided cells. The corresponding CVR graph is connected.

For a hierarchical T-mesh $\mathscr{T}$ and its CVR graph $\mathscr{G}$, the following theorem states that $V^c - E + \delta$ in $\mathscr{T}$ is exactly the cell number $F_{\mathscr{G}}$ in $\mathscr{G}$.

Theorem 5.18. *Given a hierarchical T-mesh $\mathscr{T}$ with V^c crossing vertices, E interior l-edges and $\delta - 1$ isolated subdivided cells, suppose there are $F_{\mathscr{G}}$ cells in its CVR graph $\mathscr{G}$. It follows that*

$$V^c - E + \delta = F_{\mathscr{G}}. \tag{5.11}$$

Proof. Suppose $\mathscr{G}$ has $E_{\mathscr{G}}$ edges and $F_{\mathscr{G}}$ cells. According to the definition of a CVR graph, $\mathscr{G}$ has V^c vertices and δ connected components. Therefore, with the Euler formula, we have

$$F_{\mathscr{G}} - E_{\mathscr{G}} + V^c = \delta.$$

To prove Eq. (5.11), we merely must show that $2V^c = E + E_{\mathscr{G}}$. Consider any l-edge ℓ_i with V_i^c crossing vertices. Then, ℓ_i generates $V_i^c - 1$ edges in $\mathscr{G}$. On checking all the l-edges in $\mathscr{T}$, we observe that each of the crossing vertices is met twice. It follows that $2V^c = E + E_{\mathscr{G}}$, which completes the proof of the theorem. □

Theorem 5.18 indicates the possibility that the basis functions of the biquadratic spline space over a hierarchical T-mesh could be constructed around the cells in its CVR graph. Experiments based on this idea have been conducted, and the idea will be more fully examined in the future.

We expect CVR graphs to play an important role in the dimensional analysis and basis function construction of higher degree spline spaces over hierarchical T-meshes. This expectation includes the following conjecture:

Conjecture 1. *Suppose $\mathscr{T}$ is a hierarchical T-mesh whose CVR graph is $\mathscr{G}$. When $m, n \geqslant 2$, it follows that*

$$\dim \overline{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) = \dim \overline{\mathbf{S}}(m-2, n-2, m-3, n-3, \mathscr{G}),$$

where the spline space $\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{G})$ is defined in a similar way as the spline space over a T-mesh.

Theorem 5.18 states that the conjecture holds for $m = n = 2$. For the case in which $m = n = 3$, we have tested many examples, which also support this conjecture.

6. Conclusions

In this paper, the dimensions of bilinear and biquadratic spline spaces over T-meshes are discussed. The strategy used is based on linear space embedding with an operator of the mixed partial derivative. We obtained the dimension formula of bilinear spline spaces over general T-meshes and that of biquadratic spline spaces over hierarchical T-meshes. Only a lower bound on the dimension was constructed for biquadratic spline spaces over general T-meshes.

In the future, the basis function construction of biquadratic spline spaces and the proposed conjecture will be examined.

Acknowledgments

The authors are supported by 973 Program 2011CB302400, the NSF of China (No. 60873109, 11031007 and 61073108), the Program for New Century Excellent Talents in Universities (No. NCET-08-0514), and the 111 Project (No. b07033). Moreover, we are grateful for the helpful suggestions from our anonymous reviewers.

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