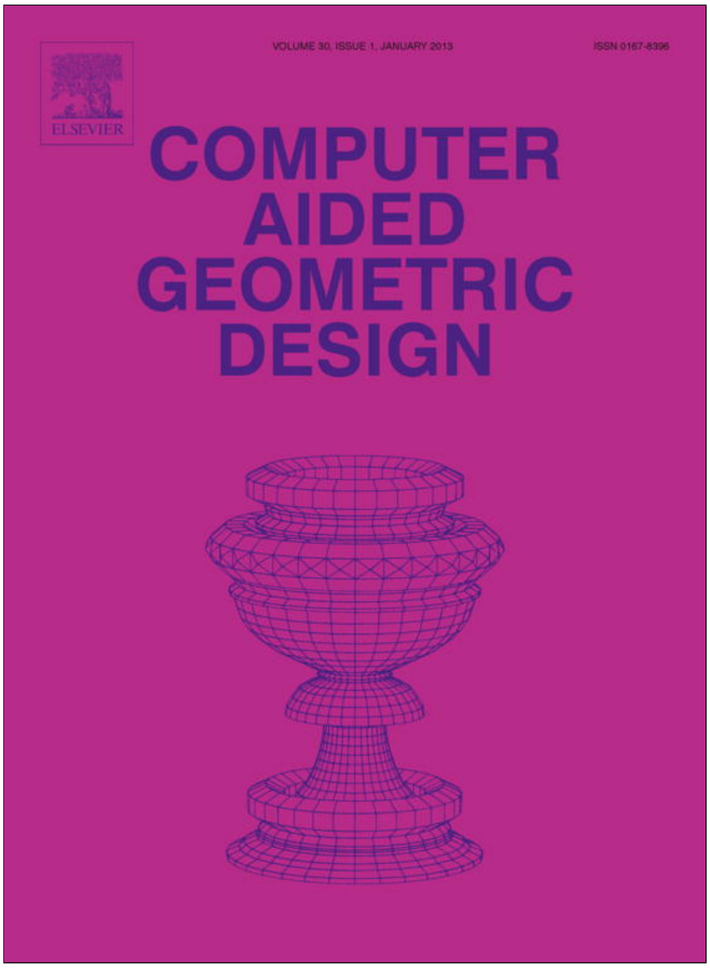
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Dimension of spline spaces with highest order smoothness over hierarchical T-meshes

Meng Wu, Jiansong Deng *, Falai Chen

School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui, 230026, PR China

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A B S T R A C T

This paper discusses the dimension of spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over a certain type of hierarchical T-mesh. The key step is to establish a bijection between the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ and a univariate spline space defined in terms of the l-edges of the extended T-mesh with respect to bi-degree (m, n). We decompose the univariate spline space into an isomorphic direct sum using the theory of short exact sequences from homological algebra. Using this decomposition, we can obtain a formula for the dimension of the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over the required type of hierarchical T-mesh. We also construct a set of basis functions for the spline space.

1. Introduction

Non-Uniform Rational B-Splines (NURBS) are a popular tool for representing surface models in Computer Aided Geometric Design (CAGD) and Computer Graphics. However, due to the tensor product structure of NURBS, local refinement of these surface models is impossible, and furthermore, NURBS models generally contain a large number of superfluous control points. To overcome the above drawbacks, Sederberg et al. introduced T-splines, whose control meshes allow T-junctions (Sederberg et al., 2003, 2004). T-splines provide a local refinement strategy and can reduce the large number of superfluous control points in NURBS models.

A few years ago, the authors of the current paper introduced the concept of a spline space over a T-mesh. In contrast to T-splines, a spline over a T-mesh is represented by a single polynomial within each cell of the T-mesh, and such a spline therefore achieves the specified smoothness across common edges. Splines over T-meshes can be applied in geometric modeling (Deng et al., 2008), and are suitable for applications in isogeometric analysis (Wang et al., 2011; Nguyen-Thanh et al., 2011a, 2011b).

One major focus in the theory of splines over T-meshes is the calculation of their dimension—a problem already studied by several researchers. In Deng et al. (2006), the dimension formula for the spline space $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ was obtained under the constraints $m \geqslant 2\alpha + 1$, $n \geqslant 2\beta + 1$. This result was further improved by Li et al. (2006). For spline spaces with higher order smoothness where $m \leqslant 2\alpha$ and $n \leqslant 2\beta$, we derived a dimension formula for C^1 biquadratic spline spaces (that is, $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$) over hierarchical T-meshes (Deng et al., 2008). Recently, B. Mourrain gave a general formula for the dimension of spline spaces $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ by applying homological techniques (Mourrain, 2010). Unfortunately, this formula includes a term that is very difficult to compute in practice.

In contrary with the above positive results, Li and Chen recently showed that the dimension of spline spaces over T-meshes with highest order smoothness may depend not only on their topology but also on their geometry (Li and Chen, 2011). This result suggests the futility of studying the dimension formula for splines over general T-meshes.

* Corresponding author.
E-mail address: dengjs@ustc.edu.cn (J. Deng).

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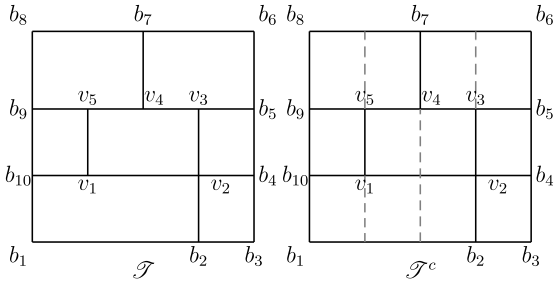


Fig. 1. A T-mesh $\mathscr{T}$ and its associated tensor product mesh $\mathscr{T}^c$.

In this paper, we will explore the dimension formula of spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over a certain type of hierarchical T-mesh.

Several methods exist for establishing the dimension of spline spaces, such as the B-net method (Deng et al., 2006), the smoothing cofactor method (Wang, 1979) and the homology method (Billera, 1988). In the smoothing cofactor method, the cofactor of a spline associated with the common edge of two adjacent cells is given by a univariate polynomial. This method recalls homology theory from the field of topology. Using the smoothing cofactor method and according to a result in Deng et al. (2008), we will construct an isomorphism between a spline space with highest order smoothness over a T-mesh and a univariate spline space satisfying some conditions. For a certain class of hierarchical T-mesh, by ordering the interior l-edges of the extended T-mesh, we can decompose the univariate spline space into a direct sum, yielding a dimension formula for the spline space over a hierarchial T-mesh. This l-edge based method is adopted to study the dimension. Moreover, this l-edge based method in this paper has varieties. In Wu et al. (2012), an edge-based method has been presented. Based on this variety, a set of hierarchical basis functions is constructed and used into Finite Element Method.

The rest of the paper is organized as follows. In Section 2, we review the definitions and some results regarding T-meshes and spline spaces over T-meshes. In Section 3, we establish an equivalence between the spline space over a T-mesh and a univariate spline space. In Section 4, we introduce a specific type of T-meshes and present the surjection lemma as a preparation for proving our dimension formula over the specific type of T-meshes. In Section 5, the proof of the dimension formula is presented, and a set of basis functions are constructed. In Section 6, we conclude the paper with some future research problems.

2. Spline spaces over T-meshes

In this section, we first review some concepts regarding T-meshes and spline spaces over T-meshes. For simplicity, we only consider regular T-meshes whose boundary grid lines form rectangles.

Definition 2.1. A *T-mesh* is a rectangular grid that allows T-junctions.

1. A grid point in a T-mesh is called a *vertex* of the T-mesh. Vertices of a T-mesh can be divided into different types. If a vertex is located on a boundary grid line, it is called a *boundary vertex*; otherwise, it is an *interior vertex*. For example, in Fig. 1, b_i is a boundary vertex and v_j is an interior vertex. A *crossing vertex* is the endpoint of four grid segments; a *T-vertex* is the endpoint of three grid segments. In Fig. 1, v_2 is a crossing vertex and v_3 is a T-vertex.
2. The line segment connecting two adjacent vertices on a grid line is called *an edge* of the T-mesh (e.g. v_4v_5, b_9b_{10}, v_2v_3 in Fig. 1). A *large edge* (l-edge for short) is the longest possible line segment consisting of several edges. The boundary of a regular T-mesh consists of four l-edges, which are called *boundary l-edges*. The other l-edges in a T-mesh are called *interior l-edges*. In Fig. 1, b_2v_3 is an interior l-edge.
3. Each rectangular grid element is called a *cell* of this T-mesh.

Definition 2.2. Some special types of T-meshes:

1. *Sub-mesh*. If a T-mesh $\mathscr{T}_1$ is a subgraph of another T-mesh $\mathscr{T}_2$, then $\mathscr{T}_1$ is called a sub-mesh of $\mathscr{T}_2$.
2. *Associated tensor product meshes*. Each regular T-mesh has an associated tensor product mesh, which is the smallest tensor product mesh containing this T-mesh. In Fig. 1, $\mathscr{T}^c$ is the associated tensor product mesh of $\mathscr{T}$.
3. *Hierarchical T-meshes*. A *hierarchical T-mesh* is a special type of T-mesh that possesses a natural level structure (Deng et al., 2008). It is defined recursively. Initially, a tensor product mesh (level 0) is given. A cell at level k is then subdivided into four sub-cells at level $(k+1)$ by connecting the mid-points of opposite edges with two perpendicular segments. These edges are called $(k+1)$ edges at level $(k+1)$. Fig. 2 illustrates a sequence of hierarchical T-meshes. Based on the structure of hierarchical T-meshes, an l-edge consists only of edges from the same level; if an l-edge consists of edges from level k, then it is called an l-edge at level k. If a vertex is the intersection point of two l-edges at level l, then this vertex is called an (l, l)-*vertex*.

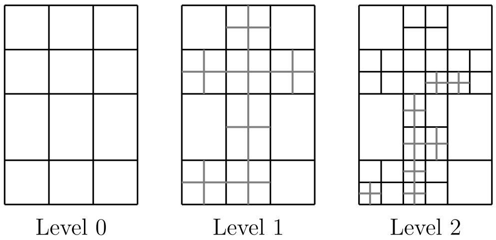


Fig. 2. Hierarchical T-meshes at level 0, level 1 and level 2.

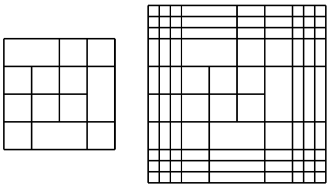


Fig. 3. A T-mesh $\mathscr{T}$ and its extended T-mesh $\mathscr{T}^{\varepsilon}$ with respect to bi-degree $(3,3)$.

4. *Extended T-meshes*. For a T-mesh $\mathscr{T}$, *the extended T-mesh* $\mathscr{T}^{\varepsilon}$ with respect to bi-degree (m,n) is an enlarged version of $\mathscr{T}$ formed by copying each horizontal boundary line of $\mathscr{T}$ m times, copying each vertical boundary line of $\mathscr{T}$ n times, and extending every line segment having an end-point on the boundary of $\mathscr{T}$ (Deng et al., 2008), where $m,n \in \mathbb{N}$ and they are important to spline spaces with bi-degree (m,n) over $\mathscr{T}$ (this point will be given in Theorem 2.1). This process can be seen in the following example. Fig. 3 illustrates a T-mesh (left) and its extended T-mesh (right) with respect to bi-degree $(3,3)$.

Definition 2.3. (See Deng et al., 2006.) Given a T-mesh $\mathscr{T}$, let $\mathscr{F}$ be the set of all the cells of $\mathscr{T}$ and Ω be the region occupied by the cells in $\mathscr{F}$. The *spline spaces over* $\mathscr{T}$ are defined as

$$\mathbf{S}(m,n,\alpha,\beta,\mathscr{T}) := \left\{ f(x,y) \in C^{\alpha,\beta}(\Omega):\ f(x,y)\big|_{\phi} \in \mathbb{P}_{m,n},\ \forall \phi \in \mathscr{F} \right\}, \tag{1}$$

where $\mathbb{P}_{m,n}$ is the space of all the polynomials with bi-degree (m,n), and $C^{\alpha,\beta}$ is the space consisting of bivariate functions that are continuous in Ω with order α in the x direction and order β in the y direction.

In this paper, we are interested in the spline space $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$.

Definition 2.4. (See Deng et al., 2008.) Let $\mathscr{T}$ be a T-mesh. A spline space over $\mathscr{T}$ with homogeneous boundary conditions is defined as

$$\bar{\mathbf{S}}(m,n,\alpha,\beta,\mathscr{T}) := \left\{ f(x,y) \in C^{\alpha,\beta}\left(\mathbb{R}^2\right):\ f(x,y)\big|_{\phi} \in \mathbb{P}_{m,n},\ \forall \phi \in \mathscr{F} \text{ and } f\big|_{\mathbb{R}^2\setminus\Omega} \equiv 0 \right\},$$

where $\mathbb{P}_{m,n}$, $\mathscr{F}$, $C^{\alpha,\beta}(\mathbb{R}^2)$ are defined as before.

One important observation made in Deng et al. (2008) is that the two spline spaces $\mathbf{S}(m,n,m-1,n-1,\mathscr{T})$ and $\bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon})$ are closely related.

Theorem 2.1. *(See Deng et al., 2008.) Given a T-mesh $\mathscr{T}$, let $\mathscr{T}^{\varepsilon}$ be its extended T-mesh with respect to bi-degree (m,n). Then,*

$$\mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \bar{\mathbf{S}}\left(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}\right)\big|_{\mathscr{T}}, \tag{2}$$

$$\dim \mathbf{S}(m,n,m-1,n-1,\mathscr{T}) = \dim \bar{\mathbf{S}}\left(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}\right). \tag{3}$$

By the above theorem, we need only consider spline spaces over T-meshes with homogeneous boundary conditions.

3. Equivalent spline spaces

In this section, we give an equivalent description of spline spaces $\bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon})$ by using the smoothing cofactor method.

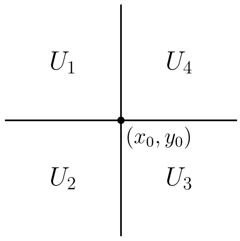


Fig. 4. Smoothing conditions in adjacent cells.

3.1. Smoothing cofactor method

As before, let $\mathscr{T}$ and $\mathscr{T}^{\varepsilon}$ be a T-mesh and its extension respectively. Let $\mathscr{F} = \{C_1, C_2, \ldots, C_j\}$ be the set of cells in $\mathscr{T}^{\varepsilon}$ and Ω_i be the region occupied by $C_i \in \mathscr{F}$. Set $\Omega_{j+1} = \mathbb{R}^2 \setminus \bigcup_{i=1}^{j} \Omega_i$ and $\mathscr{F}^{\varepsilon} = \{\Omega_1, \Omega_2, \ldots, \Omega_{j+1}\}$.

Let $U_1, U_2, U_3, U_4 \in \mathscr{F}^{\varepsilon}$ be four regions in adjacent positions, as shown in Fig. 4. For a spline function $f(x, y) \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon})$, let $f_i(x, y)$ be a bivariate polynomial that coincides with $f(x, y)$ on U_i, $i = 1, 2, 3, 4$. Note that if $f_i(x, y) = f_j(x, y)$ for two adjacent cells U_i and U_j, then U_i and U_j can be merged into a single region, making (x_0, y_0) into a T-junction.

The smoothing cofactor method yields the following relationship between the $f_i(x, y)$:

Lemma 3.1. *(See Li et al., 2006; Li and Chen, 2011.) Let $f_i(x, y)$, $i = 1, 2, 3, 4$ be defined as in the preceding paragraph. Then there exist a constant $k \in \mathbb{R}$ and polynomials $a(y) \in \mathbb{P}_n[y]$, $b(x) \in \mathbb{P}_m[x]$ such that*

$$\begin{aligned}
f_1(x, y) &= f_2(x, y) + b(x)(y - y_0)^n, \\
f_3(x, y) &= f_2(x, y) + a(y)(x - x_0)^m, \\
f_4(x, y) &= f_2(x, y) + a(y)(x - x_0)^m + b(x)(y - y_0)^n + k(x - x_0)^m (y - y_0)^n,
\end{aligned}$$

where $v(x_0, y_0)$ is the vertex of $\mathscr{T}^{\varepsilon}$ in Fig. 4. Furthermore, $a(y)$, $b(x)$ and k are uniquely determined by $f_i(x, y)$ for $i = 1, 2, 3, 4$. Specifically,

$$a(y) = \frac{1}{m!}\left(\frac{\partial^m f_3}{\partial x^m}(x_0, y) - \frac{\partial^m f_2}{\partial x^m}(x_0, y)\right), \tag{4}$$

$$b(x) = \frac{1}{n!}\left(\frac{\partial^n f_1}{\partial y^n}(x, y_0) - \frac{\partial^n f_2}{\partial y^n}(x, y_0)\right), \tag{5}$$

$$k = \frac{1}{m!}\frac{1}{n!}\frac{\partial^{m+n}(f_2(x, y) + f_4(x, y) - f_1(x, y) - f_3(x, y))}{\partial x^m \partial y^n}. \tag{6}$$

Here, $a(y)$ and $b(x)$ are smoothing cofactors associated to edges between adjacent cells. The constant k is called the conformality factor associated with the common vertex $v(x_0, y_0)$.

By the above lemma, we may convert the problem of calculating the dimension of a spline space into the study of a univariate spline space. We first introduce the following definition.

Definition 3.1. Let E^h be a horizontal l-edge in $\mathscr{T}^{\varepsilon}$, and $\{v_1, v_2, \ldots, v_r\}$ be vertices on E^h. Assume without loss of generality that the x-coordinates of the vertices satisfy $x_1 < x_2 < \cdots < x_r$. The univariate spline space with homogeneous boundary conditions associated with E^h is defined as

$$\bar{\mathbf{S}}\big(m, m-1, E^h\big) := \big\{p(x) \in C^{m-1}(\mathbb{R}):\ p(x)\big|_{[x_i, x_{i+1}]} = p_i(x) \in \mathbb{P}_m[x],\ i = 1, 2, \ldots, r-1, \text{ and } p(x)\big|_{\mathbb{R} \setminus [x_1, x_r]} \equiv 0\big\}. \tag{7}$$

Similarly, given a vertical l-edge E^v, we can define a univariate spline space $\bar{\mathbf{S}}(n, n-1, E^v)$ associated with E^v.

By the homogeneous boundary conditions and smoothness requirement in Definition 3.1, $p(x) \in \bar{\mathbf{S}}(m, m-1, E^h)$ can be expressed explicitly as $\sum_{j=1}^{i} k_j (x - x_j)^m$ on $[x_i, x_{i+1}]$; if $x > x_r$, then $\sum_{j=1}^{r} k_j (x - x_j)^m \equiv 0$, where $k_j \in \mathbb{R}$, $j = 1, 2, \ldots, r$ are constants depending on $p(x)$, i.e., $p(x) \in \bar{\mathbf{S}}(m, m-1, E^h)$ if and only if there exist constants $k_1, \ldots, k_r$ such that

$$\sum_{i=1}^{r} k_i (x - x_i)^m \equiv 0. \tag{8}$$

Eq. (8) is equivalent to a linear system associated with E^h:

$$\begin{cases} \sum_{i=1}^{r} k_i = 0, \\ \sum_{i=1}^{r} k_i x_i = 0, \\ \cdots, \\ \sum_{i=1}^{r} k_i x_i^m = 0. \end{cases} \tag{9}$$

For a vertical l-edge E^v, there is a similar equation

$$\sum_{i=1}^{s} k_i (y - y_i)^n \equiv 0, \tag{10}$$

which is also equivalent to a linear system associated with E^v:

$$\begin{cases} \sum_{i=1}^{s} k_i = 0, \\ \sum_{i=1}^{s} k_i y_i = 0, \\ \cdots, \\ \sum_{i=1}^{s} k_i y_i^n = 0. \end{cases} \tag{11}$$

By (9) and (11), we immediately have the following lemma.

Lemma 3.2. *(See Li et al., 2006.)*

$$\dim \bar{\mathbf{S}}\big(m, m-1, E^h\big) = (r - m - 1)_+, \qquad \dim \bar{\mathbf{S}}\big(n, n-1, E^v\big) = (s - n - 1)_+.$$

In the above equations, we have $u_+ = \max(0, u)$.

3.2. Conformality vector spaces

To calculate the dimension of the spline space $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$, we need to consider the conformality conditions for all the (horizontal and vertical) l-edges. Thus, we introduce the following definition.

Definition 3.2. Let E_i^h $(i = 1, 2, \ldots, p)$ form all the horizontal l-edges of $\mathscr{T}^\varepsilon$ and E_j^v $(j = 1, 2, \ldots, q)$ form all the vertical l-edges of $\mathscr{T}^\varepsilon$. Define a linear space $W[\mathscr{T}^\varepsilon]$ by

$$W\big[\mathscr{T}^\varepsilon\big] := \big\{\mathbf{k} = (k_1, k_2, \ldots, k_v)^T \colon L_i^h = 0,\ L_j^v = 0,\ i = 1, \ldots, p,\ j = 1, \ldots, q\big\},$$

where v is the number of vertices of $\mathscr{T}^\varepsilon$, k_i is the conformality factor corresponding to the i-th vertex of $\mathscr{T}^\varepsilon$, and $L_i^h = 0$, $L_j^v = 0$ are linear systems associated with l-edges E_i^h and E_j^v, respectively. $W[\mathscr{T}^\varepsilon]$ is called the conformality vector space of $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$. Similarly, one can define the conformality vector space $W_{\mathscr{T}^\varepsilon}[E^h]$ (or $W_{\mathscr{T}^\varepsilon}[E^v]$) of $\bar{\mathbf{S}}(m, m-1, E^h)$ (or $\bar{\mathbf{S}}(n, n-1, E^v)$) associated with the vertices on the l-edge E^h (E^v) of $\mathscr{T}^\varepsilon$, where $W_{\mathscr{T}^\varepsilon}[E^h]$ $(W_{\mathscr{T}^\varepsilon}[E^v])$ is determined by the vertices on E^h (E^v) of $\mathscr{T}^\varepsilon$.

The following facts should be noted regarding $W[\mathscr{T}^\varepsilon]$.

1. $W[\mathscr{T}]$ can also be defined over a general T-mesh $\mathscr{T}$ (besides an extended T-mesh). Every $f(x, y) \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T})$ corresponds to a unique vector $\mathbf{k} \in W[\mathscr{T}]$ called *conformality vector*.
2. k_i (the i-th component of $\mathbf{k}$) is the conformality factor corresponding to the i-th vertex in $\mathscr{T}^\varepsilon$, which is the intersection point of two l-edges. Thus, k_i must satisfy Eqs. (9) and (11) for the two l-edges. Once $\mathbf{k}$ has been determined, the smoothing cofactors $a(y)$ and $b(x)$ can be constructed accordingly: $a(y)$ is determined by the conformality factors whose corresponding vertices lie beneath the edge associated with $a(y)$; $b(x)$ is determined by the conformality factors

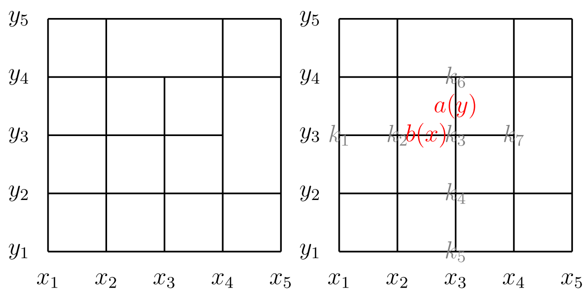


Fig. 5. Reconstruction of smoothing cofactors from a conformality vector.

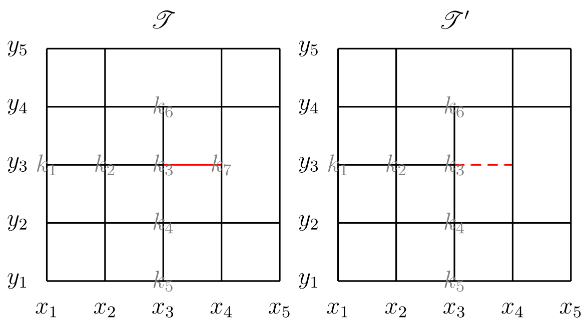


Fig. 6. The naught value of a conformality factor.

whose corresponding vertices lie to the left of the edge corresponding to $b(x)$. Fig. 5 illustrates an example, where $a(y) = k_5(y - y_1)^n + k_4(y - y_2)^n + k_3(y - y_3)^n$, $b(x) = k_1(x - x_1)^m + k_2(x - x_2)^m$.

3. A zero value of k_i, which is the conformality factor corresponding to a T-vertex, results in the vanishing of the corresponding vertex in $\mathscr{T}$. An example is illustrated in Fig. 6, where $k_7 = 0$; in this case, $\mathscr{T}$ degenerates to its sub-mesh, denoted as $\mathscr{T}'$ or $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}) = \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}')$.

As an example, we discuss the conformality vectors of B-spline functions.

Example 3.1. Let Δ: $-\infty < x_1 < x_2 < \cdots < x_r < \infty$ $(r > m+1)$ be a partition of $\mathbb{R}$, and $\bar{\mathbf{S}}(m, m-1, \Delta)$ be the B-spline function space defined over Δ; that is, $\bar{\mathbf{S}}(m, m-1, \Delta)$ consists of piecewise polynomials of degree m with C^{m-1} continuity over Δ. The B-spline basis function $N[x_i, x_{i+1}, \ldots, x_{i+m+1}](x)$ is an element of $\bar{\mathbf{S}}(m, m-1, E_i)$, where E_i is the interval $[x_i, x_{i+m+1}]$ having partition $x_i < x_{i+1} < \cdots < x_{i+m+1}$. Here, i satisfies the inequalities $1 \leqslant i \leqslant r - m - 1$. The conformality vector $\mathbf{k} = (k_i, k_{i+1}, \ldots, k_{i+m+1})$ of $N[x_i, x_{i+1}, \ldots, x_{i+m+1}](x)$ can be obtained by solving the associated linear system (9). Note that $k_j k_{j+1} < 0$, $j = i, i+1, \ldots, i+m$ and $k_i > 0$.

Example 3.2. Let $\bar{\mathbf{S}}(m, m-1, \Delta_x)$ be the B-spline function space defined over Δ_x: $-\infty < x_1 < x_2 < \cdots < x_r < \infty$ $(r > m+1)$, and let $\bar{\mathbf{S}}(n, n-1, \Delta_y)$ be the B-spline function space defined over Δ_y: $-\infty < y_1 < y_2 < \cdots < y_s < \infty$ $(s > n+1)$. Let $\mathscr{T}_\otimes$ denote a tensor product mesh $\Delta_x \times \Delta_y$. It is easy to see that if $f(x) \in \bar{\mathbf{S}}(m, m-1, \Delta_x)$ and $g(y) \in \bar{\mathbf{S}}(n, n-1, \Delta_y)$, then $f(x)g(y) \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_\otimes)$.

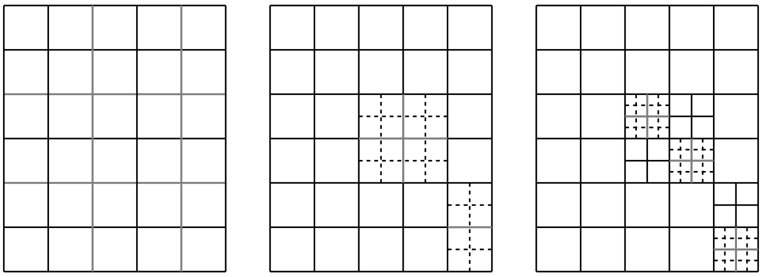
Now assume that the B-spline basis functions $N[x_i, x_{i+1}, \ldots, x_{i+m+1}](x) \in \bar{\mathbf{S}}(m, m-1, \Delta_x)$ and $N[y_j, y_{j+1}, \ldots, y_{j+n+1}](y) \in \bar{\mathbf{S}}(n, n-1, \Delta_y)$ have conformality vectors given by $\mathbf{k}^1 := (k_i^1, k_{i+1}^1, \ldots, k_{i+m+1}^1)$ and $\mathbf{k}^2 := (k_j^2, k_{j+1}^2, \ldots, k_{j+n+1}^2)$, respectively. Then, by Eq. (6), the B-spline basis function $N[x_i, x_{i+1}, \ldots, x_{i+m+1}](x)N[y_j, y_{j+1}, \ldots, y_{j+n+1}](y) \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}_\otimes)$ has conformality vector $\mathbf{k}^1 \otimes \mathbf{k}^2$, which is a vector of dimension $(m+2)(n+2)$ with elements $k_p^1 k_q^2$, $p = i, i+1, \ldots, i+m+1$, $q = j, j+1, \ldots, j+n+1$.

3.3. Equivalence of spline spaces

Following the above preparations, we obtain a mapping $\mathscr{K}$ between the spline space $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$ and the conformality vector space $W[\mathscr{T}^\varepsilon]$:

$$\mathscr{K} : \bar{\mathbf{S}}\left(m, n, m-1, n-1, \mathscr{T}^\varepsilon\right) \to W\left[\mathscr{T}^\varepsilon\right]. \tag{12}$$

By Eq. (6), $\mathscr{K}$ is linear due to the linear property of the operator $\partial^{m+n}/\partial x^m \partial y^n$. In fact, $\mathscr{K}$ is an isomorphism.



(a) level 0 (b) level 1 (c) level 2

Fig. 7. $\mathscr{T}_{3,3}$ at level 0, 1, 2 in the case of $p = 5$, $q = 6$.

Theorem 3.3. *The mapping* $\mathscr{K} : \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon) \to W[\mathscr{T}^\varepsilon]$ *is bijective.*

Proof. We first prove that $\mathscr{K}$ is injective. It suffices to show that $\mathscr{K} f = \mathbf{0}$ implies $f \equiv 0$ for $f \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$. If the conformality vector of $f(x, y)$ is a zero vector, first consider a T-vertex. By Remark 3 following Definition 3.2, $\mathscr{T}^\varepsilon$ must degenerate to a sub-mesh by removing the edges with this T-vertex as an endpoint, with $f(x, y)$ defined over this sub-mesh. Repeating this process, and considering vertices located on the corner of $\mathscr{T}^\varepsilon$ as T-vertices (analogous to the method to do with v in Lemma 3.1), eventually no more edge of $\mathscr{T}^\varepsilon$ will remain. By the homogeneous boundary conditions, $f(x, y) \equiv 0$. Thus, $\mathscr{K}$ is injective.

Next, we show that $\mathscr{K}$ is surjective. By Remark 2 following Definition 3.2, for a given conformality vector $\mathbf{k} \in W[\mathscr{T}^\varepsilon]$, we can construct a smoothing cofactor for each edge of $\mathscr{T}^\varepsilon$ and then construct the appropriate polynomial for each cell based on Lemma 3.1 and the homogeneous boundary conditions. Thus, we obtain a spline function $f(x, y) \in \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$ corresponding to $\mathbf{k}$; that is, $\mathscr{K}$ is surjective, and so the mapping $\mathscr{K}$ is bijective. □

By the above theorem, the spline space $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$ is isomorphic to the conformality vector space $W[\mathscr{T}^\varepsilon]$, and the dimension of $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)$ satisfies the following

$$\dim \bar{\mathbf{S}}\bigl(m, n, m-1, n-1, \mathscr{T}^\varepsilon\bigr) = \dim W\bigl[\mathscr{T}^\varepsilon\bigr]. \tag{13}$$

Similarly, one can show that for a horizontal l-edge E^h and a vertical l-edge E^v, the following relationships must hold

$$\bar{\mathbf{S}}\bigl(m, m-1, E^h\bigr) \cong W_{\mathscr{T}^\varepsilon}\bigl[E^h\bigr], \qquad \bar{\mathbf{S}}\bigl(n, n-1, E^v\bigr) \cong W_{\mathscr{T}^\varepsilon}\bigl[E^v\bigr]. \tag{14}$$

We should make sure to define trivial l-edges.

Definition 3.3. E is a *trivial l-edge* of $\mathscr{T}^\varepsilon$ if $W_{\mathscr{T}^\varepsilon}[E] = \mathbf{0}$.

In what follows, we only need to analyze the structure of $W[\mathscr{T}^\varepsilon]$.

4. Definition of (m, n)-subdivided T-meshes

To study the applications of spline spaces with highest order smoothness over T-meshes, we will need to define a hierarchical T-mesh with bi-degree (m, n), where $m \geqslant 2$, $n \geqslant 2$.

Definition 4.1. Let $\mathscr{T}_\otimes = [x_0, x_1, \ldots, x_p] \times [y_0, y_1, \ldots, y_q]$ be a tensor product mesh. Note that there exist unique integers s and t such that $0 < p - s(m-1) \leqslant m-1$, $0 < q - t(n-1) \leqslant n-1$. Denote $X = \{x_0, x_{m-1}, \ldots, x_{s(m-1)}, x_p\}$ and $Y = \{y_0, y_{n-1}, \ldots, y_{t(n-1)}, y_q\}$. The domain $\Omega = [x_0, x_p] \times [y_0, y_q]$ is subdivided by the lines $\{x = x_i,\ y = y_j\colon\ x_i \in X,\ \ y_j \in Y\}$ into sub-domains, each of which is occupied by a local tensor product mesh of size at most $(m-1) \times (n-1)$ (may be smaller near the right or upper boundary lines). Each such subdomain is called an (m, n)-subdomain of $\mathscr{T}_\otimes$. An (m, n)-subdomain is a boundary subdomain if the intersection of its boundary lines and the boundary lines of the T-mesh is nonempty. A subdomain is subdivided if each cell in the subdomain is subdivided (a cell is subdivided by connecting the mid points of opposite sides of the cell with two perpendicular line segments). Two subdomains are adjacent if they share a common boundary line segment. A subdomain that is not a boundary subdomain is called isolated if it is subdivided and its adjacent subdomains are not subdivided.

Fig. 7(a) illustrates an example in which a tensor product mesh has been subdivided by gray line into $(3, 3)$-subdomains. In Fig. 7(b), two subdomains are further subdivided by lines of dashes, and one subdomain is isolated.

Definition 4.2. An (m,n)-subdivided T-mesh, denoted as $\mathscr{T}_{m,n}$, is defined level by level. At level 0, $\mathscr{T}_{m,n}$ is a tensor product mesh denoted by $\mathscr{T}_{\otimes}$. To construct level 1, from level 0, subdivide $\mathscr{T}_{\otimes}$ into (m,n)-subdomains, some of which are further subdivided. In general, to construct a level $(k+1)$ mesh, subdivide some local tensor product meshes at level k which are obtained by subdividing subdomains at level $(k-1)$ into (m,n)-subdomains at level k.

An (m,n)-subdivided T-mesh is a special type of hierarchical T-mesh (Deng et al., 2008). In particular, it is a general hierarchical T-mesh when $m=n=2$. Fig. 7 illustrates the process of generating $\mathscr{T}_{3,3}$. Fig. 7(a) illustrates a tensor product mesh which is subdivided into subdomains at level $k=0$. Fig. 7(b) shows the construction of a level 1 mesh by subdividing the mesh at 0. Fig. 7(c) demonstrates the subdivision of two local tensor product meshes (corresponding to the two subdivided subdomains in Fig. 7(b)) into (3, 3)-subdomains at level $k=1$, which are further subdivided to get a mesh at level $k=2$.

4.1. The order of interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$

In this subsection, the order of interior l-edges with different levels is given, and then the order inside the set of interior l-edges with the same level is defined. We denote the set of interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$ by $\mathbf{E}$.

4.1.1. The order between l-edges with different levels

The set of interior l-edges $\mathbf{E}$ of $\mathscr{T}^{\varepsilon}_{m,n}$ can be divided into disjoint sets $\mathbf{E}^i$ for $i=0,1,\ldots,l$, i.e., $\mathbf{E}=\bigcup_{i=0}^{l}\mathbf{E}^i$ with $\mathbf{E}^i\cap\mathbf{E}^j=\emptyset$, $i\neq j$. $\mathbf{E}^i$ is the set of interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$ defined at level i for $i=0,1,\ldots,l$, where l is the highest level of $\mathscr{T}_{m,n}$. The order "$\prec$" relating $\mathbf{E}^p$ and $\mathbf{E}^q$ is then defined as follows: $\forall e^p\in\mathbf{E}^p$, $\forall e^q\in\mathbf{E}^q$, if $q<p$, then $e^q\prec e^p$.

4.1.2. The order inside the set of interior l-edges with the same level

We now extend the order "$\prec$" on pairs of elements within $\mathbf{E}^p$. Let $\widetilde{\mathscr{T}}^p$ be obtained by deleting the l-edges $\bigcup_{i=p+1}^{l}\mathbf{E}^i$ from $\mathscr{T}^{\varepsilon}_{m,n}$. Without loss of generality, it suffices to consider the case $p=l$.

As a preparation, we will define a deleting operator on a T-mesh $\mathscr{T}$. Given an l-edge E_i, the deleting operator $\mathbf{D}$ on $\mathscr{T}$ produces $\mathbf{D}_{E_i}(\mathscr{T})=\mathscr{T}_1$, where $\mathscr{T}_1$ is the T-mesh obtained by deleting E_i from $\mathscr{T}$.

Now we are going to define the order inside $\mathbf{E}^l$. Algorithms for generating an l-edge sequence are presented. Based on this l-edge sequence, we can define the order inside $\mathbf{E}^l$ naturally. In the following, the algorithms in the cases $l=0$ and $l>0$ are given respectively.

1. $l=0$.

In this case, $\mathscr{T}^{\varepsilon}_{m,n}$ is a tensor product mesh denoted by $\mathscr{T}_{\otimes}$. We now present Algorithm 1 for choosing a sequence of interior l-edges.

Algorithm 1.

```
𝒯 ← 𝒯⊗
S_E ← the set of interior l-edges of 𝒯
i ← 1
while S_E ≠ ∅ do
  if no trivial interior l-edge exists then
    choose any interior l-edge E_i^0
  else
    choose a trivial interior l-edge E_i^0 (i.e., W_𝒯[E_i^0] = 0)
  end if
  𝒯 ← D_{E_i^0}(𝒯)
  S_E ← the set of interior l-edges of 𝒯
  i ← i + 1
end while
```

We make some remarks about Algorithm 1.

- According to the structure of a tensor product mesh, when an interior l-edge is removed from this tensor product mesh, other l-edges are still interior l-edges of the current T-mesh which is still a tensor product mesh.
- The number of l-edges in S_E decreases by 1 at each step of the while loop. Thus, this algorithm will terminate after a finite number of steps.
- With the help of this algorithm, we put in the set of interior l-edges $\mathbf{E}^0$ of $\mathscr{T}_{\otimes}$ and obtain a sequence of $\mathbf{E}^0$ denoted by $E_1^0, E_2^0, \ldots$, where E_i^0 is chosen earlier than E_{i+1}^0.
- The desired order inside $\mathbf{E}^0$ is $E_{i+1}^0 \prec E_i^0$.

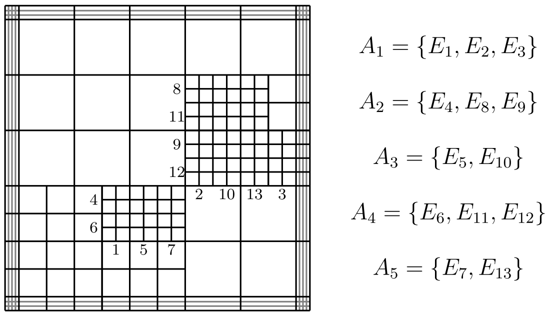


Fig. 8. An example of $\mathbf{E}^2$, where $m = 4$, $n = 3$, $p = 6$, $q = 6$.

2. $l > 0$.

In this part, a position index for each interior l-edge is defined firstly and the interior l-edges in $\mathbf{E}^l$ $(l > 0)$ are classified based on their position index. Then Algorithm 2 is presented.

Definition 4.3. Given an interior l-edge $E \in \mathbf{E}^l$ $(l > 0)$, then E must intersect with an (m, n)-subdomain Σ at level $(l - 1)$. If E is vertical, then we arrange from left to right all the vertical l-edges in $\mathbf{E}^l$ that intersect with Σ. As a result of this arrangement, E is the i-th l-edge. By the structure of $\mathscr{T}^{\varepsilon}_{m,n}$, E is also the i-th l-edge in another (m, n)-subdomain at level $(l - 1)$ that intersects with E. The position index $L(E)$ of E is defined as i.

If E is horizontal, then $L(E)$ can be defined in a similar way by arranging the horizontal l-edges in $\mathbf{E}^l$ from top to bottom.

By the definition of $L(E)$ and the structure of $\mathscr{T}^{\varepsilon}_{m,n}$, if E is vertical (or horizontal), $L(E) < m$ (or n).

According to the position index of l-edge in $\mathbf{E}^l$, $\mathbf{E}^l$ is divided into the set of l-edges A_1, A_2, A_3, A_4, A_5:

$$A_1 = \{E \in \mathbf{E}^l\text{: if } E \text{ is vertical and } L(E) < m - 2\text{; if } E \text{ is horizontal and } L(E) < n - 2\},$$
$$A_2 = \{E \in \mathbf{E}^l\text{: } E \text{ is horizontal and } L(E) = n - 2\},$$
$$A_3 = \{E \in \mathbf{E}^l\text{: } E \text{ is vertical and } L(E) = m - 2\},$$
$$A_4 = \{E \in \mathbf{E}^l\text{: } E \text{ is horizontal and } L(E) = n - 1\},$$
$$A_5 = \{E \in \mathbf{E}^l\text{: } E \text{ is vertical and } L(E) = m - 1\}.$$

As an example, in Fig. 8 we consider the case $l = 2$. Here we have the parameters $m = 4$, $n = 3$, $p = 6$, $q = 6$. We label these interior l-edges in $\mathbf{E}^2$ in Fig. 8 and present the members $A_1, A_2, \ldots, A_5$ on the right side of this figure, where as on the left side, we label E_i with i.

We now give details of Algorithm 2 for choosing an interior l-edge sequence.

Algorithm 2.

$\mathscr{T}^0 \leftarrow \mathscr{T}^{\varepsilon}_{m,n}$
for $i = 1$ to 5 **do**
  $\widetilde{\mathscr{T}} \leftarrow \mathscr{T}^{i-1}$
  $S_E^{A_i} \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$ in A_i
  $S_E \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$
  $j \leftarrow 1$
  while $S_E \neq \emptyset$ and $S_E^{A_i} \neq \emptyset$ **do**
    if no trivial l-edge in S_E exists **then**
      choose any interior l-edge $E^l_{i,j}$ from $S_E^{A_i}$
    else
      choose any trivial l-edge $E^l_{i,j}$ from S_E
    end if
    $\widetilde{\mathscr{T}} \leftarrow \mathbf{D}_{E^l_{i,j}}(\widetilde{\mathscr{T}})$
    $S_E^{A_i} \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$ in A_i
    $S_E \leftarrow$ the set of interior l-edges of $\widetilde{\mathscr{T}}$
    $j \leftarrow j + 1$
  end while
  $\mathscr{T}^i \leftarrow \widetilde{\mathscr{T}}$
end for

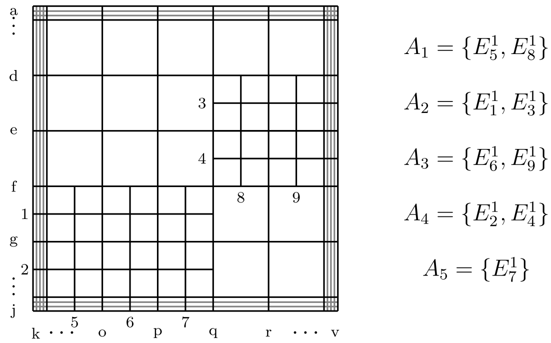


Fig. 9. An example of $\mathbf{E}^1$ and $\mathbf{E}^0$, where $m = 4$, $n = 3$, $p = 6$, $q = 6$.

We make some remarks for Algorithm 2.

1. According to the structure of $\mathscr{T}^{\varepsilon}_{m,n}$, the endpoints of each interior l-edge must lie on l-edges at lower levels. Thus, an l-edge of $\mathscr{T}^{\varepsilon}_{m,n}$ is still an l-edge of the current T-mesh, which we generate by removing l-edges with larger orders.
2. The number of l-edges in S_E decreases by 1 at each step of the while loop. Thus, this algorithm will terminate after a finite number of steps.
3. Algorithm 2 generates an interior l-edge sequence of $\mathbf{E}^l$ which is $E^l_{1,1}, E^l_{1,2}, \ldots, E^l_{1,j_1}, E^l_{2,1}, E^l_{2,2}, \ldots, E^l_{2,j_2}, \ldots, E^l_{5,j_5}$.
4. If one l-edge e_1 is chosen earlier than another l-edge e_2 in the interior l-edges sequence, the order is defined as $e_2 \prec e_1$.

Now, we can use Algorithms 1 and 2 to give an order of $\mathbf{E}$. For the example in Fig. 8, we remove all interior l-edges in $\mathbf{E}^2$. In Fig. 9, we label the interior l-edges in $\mathbf{E}^1$ of the T-mesh from Fig. 8 and $A_1, A_2, \ldots, A_5$ have been presented in the right side. $E^1_7 \prec E^1_2 \prec E^1_6 \prec E^1_1 \prec E^1_9 \prec E^1_4 \prec E^1_3 \prec E^1_8 \prec E^1_5$ is an interior l-edge sequence of $\mathbf{E}^1$ by Algorithm 2.

Now remove $\mathbf{E}^1$ from the current T-mesh, then there is a tensor product mesh with the interior l-edge set $\mathbf{E}^0$. By Algorithm 1, an interior l-edges sequence of $\mathbf{E}^0$ with the order is presented as follows

$$E^0_i \prec E^0_j \prec E^0_v \prec E^0_u \prec E^0_t \cdots \prec E^0_k \prec E^0_h \prec E^0_g \cdots \prec E^0_a,$$

where $a, b, \ldots, u, v$ are the index of interior l-edges in $\mathbf{E}^0$ labeled in Fig. 9.

Thus, the order of all the interior l-edges of the T-mesh in Fig. 9 is

$$E^0_i \prec E^0_j \prec E^0_v \prec E^0_u \prec E^0_t \cdots \prec E^0_k \prec E^0_h \prec E^0_g \cdots \prec E^0_a \prec E^1_7 \prec E^1_2 \prec E^1_6 \prec E^1_1 \prec E^1_9 \prec E^1_4 \prec E^1_3 \prec E^1_8 \prec E^1_5.$$

4.2. The surjection lemma and splines associated with interior l-edges

We now present a key lemma called the surjection lemma for proving our main theorem. This lemma will show that there exists a set of spline functions over $\mathscr{T}^{\varepsilon}_{m,n}$ associated with the interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$, in other words, we can construct spline functions for each interior l-edge. Considering Section 3.3, the surjection lemma is shown in "conformality factor" form.

Lemma 4.1. *Let $\mathscr{T}_{m,n}$ be an (m,n)-subdivided T-mesh and $\mathscr{T}^{\varepsilon}_{m,n}$ be its extended T-mesh with respect to bi-degree (m,n). $\mathbf{E}$ is the set of all the interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$. Based on the order on $\mathbf{E}$ defined in Section 4.1, the projection mapping*

$$\pi_{E_i} : W[\mathscr{T}_i] \to W_{\mathscr{T}_i}[E_i]$$

is surjective for every $i = 1, 2, \ldots, t$, where $W_{\mathscr{T}_i}[E_i]$ is the conformality vector space defined by the vertices of E_i on $\mathscr{T}_i$ and π_{E_i} is defined by keeping the cofactors associated with the E_i's vertices of a vector in $W[\mathscr{T}_i]$.

Proof. Before going to the details, we give an outline of the proof.

First, we choose a set of bases $\{\mathbf{k}^i_j\}$ for $W_{\mathscr{T}_i}[E_i]$. Then for each basis, we construct a spline function $B^i_j(x, y)$ such that $\pi_{E_i} \circ \mathscr{K}(B^i_j(x, y)) = \mathbf{k}^i_j$. According to Examples 3.1 and 3.2, we try to construct a B-spline function for $\mathbf{k}^i_j$ based on the local tensor product structure of $\mathscr{T}^{\varepsilon}_{m,n}$. However, this construction does not always work. In this case, we are able to construct $B^i_j(x, y)$ as the linear combination of two B-spline functions.

Choose a set of bases for $W_{\mathscr{T}_i}[E_i]$

If E_i is horizontal, then let Δ_x: $-\infty < x_1 < x_2 < \cdots < x_{r_i} < \infty$ be the x-coordinates of the vertices of E_i in $\mathscr{T}_i$. The B-spline functions $\{N[x_j, x_{j+1}, \ldots, x_{j+m+1}](x)\}_{j=1}^{r_i-m-1}$ is a set of bases of $\bar{\mathbf{S}}(m, m-1, \Delta_x)$ and $\{\mathbf{k}^i_j\}_{j=1}^{r_i-m-1}$ is a set of bases of $W_{\mathscr{T}_i}[E_i]$, where $\mathbf{k}^i_j$ is the conformality vector of $N[x_j, x_{j+1}, \ldots, x_{j+m+1}](x)$. For a given interior vertical l-edge, we can select a set of bases of $W_{\mathscr{T}_i}[E_i]$ similarly.

Construct a function $B^i_j(x, y)$ for $\mathbf{k}^i_j$

Let l be the highest level of $\mathscr{T}_{m,n}$ and e be a nontrivial interior l-edge in A_1 (or A_2, A_3).

When $l > 0$, the B-spline function $N^i_j(x, y)$ can be constructed by the tensor product mesh over the (m, n)-subdomains containing e at level $(l-1)$. Here, $N^i_j(x, y)$ is related to the knots of e, as it defines a tensor product B-spline function, which equals $B^i_j(x, y)$ up to a nonzero constant. For trivial interior l-edges, no function is introduced. Suppose that we have already defined the functions associated with all the l-edges whose orders are larger than the order of $E \in A_4$. We now construct the functions associated with E, where E is a nontrivial horizontal l-edge, and we denote the x-coordinates of E as $\{x_1, x_2, \dots, x_d\}$. Let the number of (l, l)-vertices be given by α. The i-th knot's sequence x-coordinates from E is $\{x_i, \dots, x_{i+(m+1)}\}$. According to the structure of the current mesh, we must have that $\alpha \in \{0, 1, 2\}$.

1. $\alpha = 0$. If $l \geqslant 2$, then to construct a B-spline, we use a tensor product mesh over all the (m, n)-subdomains containing E's i-th knot sequence and found at level $(l-2)$. If $l = 1$, then we can construct a B-spline according to the tensor product mesh from level 0. The required function $B^i_j(x, y)$ is identical to the B-spline function up to a nonzero constant.

2. $\alpha = 1$. If $l \geqslant 2$, consider the unique (l, l)-vertex, denoted by Q. Assume that Σ is the (m, n)-subdomain containing Q at level $(l-2)$. If the interior vertical l-edge containing Q passes through Σ, then we can construct a B-spline using the same method applied to the case $\alpha = 0$. If this interior vertical l-edge does not pass through Σ, then one of its endpoints is contained within Σ, and the other lies outside of Σ according to the structure of the special hierarchical T-mesh. Let R be the intersection point of the vertical l-edge and Σ. Then, we extend the l-edge from the endpoint within Σ to intersect the boundary of Σ at P. Then, we can use the $n + 2$ vertices to construct a B-spline $N_1(x, y)$ by using a tensor product mesh over all (m, n)-subdomains containing the given $(m + 2)$ vertices. These $n + 2$ vertices consist of P, R, and n vertices from the original vertical l-edge within Σ. Therefore, the conformality vector of $N_1(x, y)$ is nontrivial for these $(m + 2)$ vertices and trivial at the new vertices created by extension of the l-edge (except for P). Say that k_1 is the conformality factor at P. If another function $g^i_j(x, y)$ is constructed whose conformality factor at P is $-k_1$ and whose conformality factors at both the newly created vertices (excluding P) and the given $(m + 2)$ vertices are zero, then, by the linear property of $\mathscr{K}$ and Example 3.2, there is $\lambda(\neq 0)$ such that

$$\lambda B^i_j(x, y) = N_1(x, y) + g^i_j(x, y).$$

Denote $f^i_j(x, y) = N_1(x, y) + g^i_j(x, y)$. We now discuss the construction of $g^i_j(x, y)$. If we remove Q from the set consisting of P and all vertices in the original vertical l-edge, then the number of remaining vertices is not less than $(n + 2)$. Thus we choose $(n + 2)$-vertices from these remaining vertices, then P must be one of them. As in the case that $\alpha = 0$, no (l, l)-vertex appears in these $(n + 2)$-vertices. Thus, we have constructed a B-spline $N_2(x, y)$ and we denote by k_2 which is the conformality factor of $N_2(x, y)$ at P. By Example 3.2, we have $k_2 \neq 0$, and the conformality vector of $N_2(x, y)$ is trivial at the new vertices except for P, as well as the $(m + 2)$ vertices remaining after the removal of Q. We then define $g^i_j(x, y) = -k_1/k_2 N_2(x, y)$.

When $l = 1$, we construct a B-spline associated with the given $m + 2$ vertices from the tensor product mesh of level 0.

3. $\alpha = 2$. If $l \geqslant 2$, then there are two (l, l)-vertices appearing among the given $(m + 2)$ vertices, which we denote by P_1 and P_2. One of the two vertical l-edges at P_1, P_2 must pass through Σ' which is occupied by the (m, n)-subdomains containing P_1 and P_2 at level $(l-2)$. Therefore, we can use this method to construct the function in the case that $\alpha = 1$. When $l = 1$, we can construct the desired function using the tensor product mesh at level 0 and the case $l \geqslant 2$.

After deleting all the l-edges in the sequence associated with A_4, no (l, l)-vertices belong to the l-edges in A_5. Accordingly, we can construct the desired function by following the method presented above for the case of $\alpha = 0$.

When $l = 0$, using the order defined on $\mathbf{E}^0$ and the tensor product mesh at level 0, we will always be able to construct a B-spline function. Thus, as stated previously, π_{E_i} is surjective. □

We will give an example for constructing $B^i_j(x, y)$. According to the constructing process, $B^i_j(x, y)$ can be chosen as a B-spline function when $E_i \in A_1 \cup A_2 \cup A_3 \cup A_5$.

When E_i belongs to A_4, it is possible that $B^i_j(x, y)$ is not a B-spline (see Fig. 10, where $m = 3$, $n = 3$). To construct a local tensor product mesh, we extend an l-edge chosen during the construction algorithm to a point P that is not a vertex of the original T-mesh, which is label with "□" in the figure. We form $N_1(x, y)$ over the tensor product mesh found in the gray domain with the triangles as its knots found in the upper right part of this figure. Similarly, we construct $N_2(x, y)$ over the tensor product mesh in the gray domain with the triangles as its knots in the lower right part of this figure. We can compute k_1 from $N_1(x, y)$ and k_2 from $N_2(x, y)$ at P, as we have already demonstrated in Examples 3.1 and 3.2. As a result, $f^i_j(x, y)$ is given by the following formula:

$$f^i_j(x, y) = N_1(x, y) - \frac{k_1}{k_2} N_2(x, y). \tag{15}$$

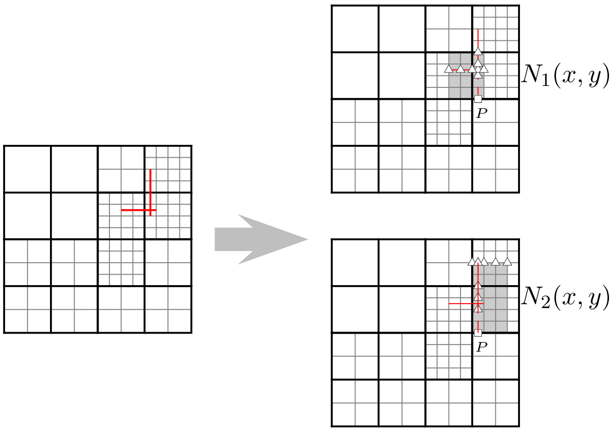


Fig. 10. Construction of a linear combination of B-splines for the case that $m = n = 3$.

5. The dimension formula

Let $\mathscr{T}_{m,n}$ be an (m,n)-subdivided T-mesh. We are now able to state our main result, which gives the dimension formula for the spline space $\bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}_{m,n})$, where $\mathscr{T}^{\varepsilon}_{m,n}$ is the extended T-mesh $\mathscr{T}_{m,n}$ with respect to bi-degree (m,n).

Theorem 5.1. *Let V^+ be the number of crossing vertices of $\mathscr{T}^{\varepsilon}_{m,n}$, and assume that $\mathscr{T}^{\varepsilon}_{m,n}$ has E_H interior horizontal l-edges and E_V interior vertical l-edges. Then we have the following formula*

$$\dim \bar{\mathbf{S}}\big(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}_{m,n}\big) = (m-1)(n-1) + V^+ - (m-1)E_H - (n-1)E_V + \delta, \tag{16}$$

where δ is the number of isolated subdomains of $\mathscr{T}^{\varepsilon}_{m,n}$ at all levels.

Before proving the theorem, we need to provide some additional facts. By Lemma 4.1, we can decompose the conformality vector space $W[\mathscr{T}^{\varepsilon}_{m,n}]$ into a direct sum of $W_{\mathscr{T}_i}[E_i]$.

Theorem 5.2. *Let the notations be the same as in Lemma 4.1. Then,*

$$W[\mathscr{T}_i] \cong W[\mathscr{T}_{i+1}] \oplus W_{\mathscr{T}_i}[E_i], \quad i = 1,2,\ldots,t.$$

Proof. Because $\mathscr{T}_i = \mathscr{T}_{i+1} \cup E_i$, it follows that $\bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}_{i+1}) \subset \bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}_i)$. Similarly, $W[\mathscr{T}_{i+1}]$ can be regarded as a subspace of $W[\mathscr{T}_i]$ by considering every vector $\mathbf{k} \in W[\mathscr{T}_{i+1}]$ to be a vector of $W[\mathscr{T}_i]$ whose components corresponding to the vertices of E_i are all zero, and whose remaining components agree with those of $\mathbf{k}$.

Now, consider the sequence

$$0 \longrightarrow W[\mathscr{T}_{i+1}] \xrightarrow{\ \mathcal{I}\ } W[\mathscr{T}_i] \xrightarrow{\ \pi_{E_i}\ } W_{\mathscr{T}_i}[E_i] \longrightarrow 0, \tag{17}$$

where $\mathcal{I}$ is the natural inclusion mapping, and π_{E_i} is the projection mapping defined in Lemma 4.1. We will show that the sequence is exact. Because $\mathcal{I}$ is injective and π_{E_i} is surjective by Lemma 4.1, it suffices to show that $\mathrm{Im}(\mathcal{I}) = \mathrm{Ker}(\pi_{E_i})$.

On the one hand, for any $\mathbf{k} \in W[\mathscr{T}_{i+1}]$, $\mathcal{I}(\mathbf{k}) = \mathbf{k}'$ is a vector in $W[\mathscr{T}_i]$ whose components (conformality factors) corresponding to the vertices of E_i are all zero. Thus, $\pi_{E_i}(\mathcal{I}(\mathbf{k})) = \mathbf{0} \in W[E_i]$, that is, $\mathrm{Im}(\mathcal{I}) \subset \mathrm{Ker}(\pi_{E_i})$.

On the other hand, for any $\mathbf{k} \in \mathrm{Ker}(\pi_{E_i})$, we have $\pi_{E_i}(\mathbf{k}) = \mathbf{0}$, that is, the components of $\mathbf{k}$ corresponding to the vertices of E_i are all zero. Therefore, $\mathbf{k} \in \mathrm{Im}(\mathcal{I})$, i.e., $\mathrm{Ker}(\pi_{E_i}) \subset \mathrm{Im}(\mathcal{I})$.

We can conclude that sequence (17) is exact. Because $W[\mathscr{T}_{i+1}]$, $W[\mathscr{T}_i]$ and $W_{\mathscr{T}_i}[E_i]$ are linear spaces, sequence (17) forms a split short exact sequence, and therefore $W[\mathscr{T}_i] \cong W[\mathscr{T}_{i+1}] \oplus W_{\mathscr{T}_i}[E_i]$, as required. □

The above theorem immediately implies a corollary.

Corollary 5.3. *The set of functions $\bigcup_{i=1}^{t}\{B^i_j(x,y)\}_{j=1}^{j=r_i-m-1}$ defined in the proof of Lemma 4.1 form a basis for the spline space $\bar{\mathbf{S}}(m,n,m-1,n-1,\mathscr{T}^{\varepsilon}_{m,n})$.*

Now we are ready to prove dimension formula (16).

Proof of Theorem 5.1. By Theorem 5.2, we can decompose $W[\mathscr{T}^{\varepsilon}_{m,n}]$ into a direct sum of $W_{\mathscr{T}_i}[E_i]$:

$$W\big[\mathscr{T}^{\varepsilon}_{m,n}\big] \cong \bigoplus_{i=1}^{t} W_{\mathscr{T}_i}[E_i].$$

We can then partition the interior l-edges $\mathbf{E} = \{E_1, E_2, \ldots, E_t\}$ of $\mathscr{T}^{\varepsilon}_{m,n}$ into disjoint sets $\mathbf{E}^i$, $i = 0, 1, \ldots, l$, i.e., $\mathbf{E} = \bigcup_{i=0}^{l} \mathbf{E}^i$ and $\mathbf{E}^i \cap \mathbf{E}^j = \emptyset$, $i \neq j$. $\mathbf{E}^i$ forms the set of interior l-edges of $\mathscr{T}^{\varepsilon}_{m,n}$ defined at level i, for $i = 0, 1, \ldots, l$. In particular, $\mathbf{E}^0$ is the set of interior l-edges of the initial extended tensor product mesh $\mathscr{T}^{\varepsilon}_{\otimes}$. By the order defined in Lemma 4.1, any element in $\mathbf{E}^i$ precedes any element in $\mathbf{E}^j$ for $i > j$.

By the direct sum decomposition of $W[\mathscr{T}^{\varepsilon}_{m,n}]$, its dimension can be represented as follows

$$\dim W\big[\mathscr{T}^{\varepsilon}_{m,n}\big] = \sum_{i=0}^{l} \sum_{E_j \in \mathbf{E}^i} \dim W_{\mathscr{T}_j}[E_j]. \tag{18}$$

We now calculate $\dim W_{\mathscr{T}_j}[E_j]$ for each level j from the highest level l to the lowest level 0.

By Lemma 3.2 and Eq. (14), if E_j is a horizontal l-edge, then we have

$$\dim W_{\mathscr{T}_j}[E_j] = \dim \bar{\mathbf{S}}(m, m-1, E_j) = \big(v_j^+ - m + 1\big)_+,$$

and if E_j is a vertical l-edge, then

$$\dim W_{\mathscr{T}_j}[E_j] = \dim \bar{\mathbf{S}}(n, n-1, E_j) = \big(v_j^+ - n + 1\big)_+.$$

Here, v_j^+ is the number of crossing vertices of E_j contained in the mesh $\mathscr{T}_j$.

Consider all the interior l-edges $\mathbf{E}^i$ at a fixed level i ($i = l, l-1, \ldots, 0$). $\mathbf{E}^i$ is obtained by subdividing some subdomains at level $i-1$ (except when $i = 0$, in which case $\mathbf{E}^0$ is the initial extended tensor product mesh). As shown in the proof of Lemma 4.1, if no isolated subdomain at level $i-1$ exists, then we can order the l-edges in $\mathbf{E}^i$ such that

$$v_j^+ \geqslant \begin{cases} m-1, & \text{if } E_j \text{ is a horizontal l-edge,} \\ n-1, & \text{if } E_j \text{ is a vertical l-edge.} \end{cases}$$

If a subdomain $\mathscr{S}$ is isolated, then one can order the l-edges in the subdomain $\mathscr{S}$ such that the above equality holds for all but one vertical l-edge in $\mathscr{S}$. For the exceptional l-edge,

$$v_j^+ = n - 2.$$

From the above facts, we have

$$\sum_{E_j \in \mathbf{E}^i} \dim W_{\mathscr{T}_j}[E_j] = \sum_{E_j \in \mathbf{E}^i_h} \big(v_j^+ - m + 1\big) + \sum_{E_j \in \mathbf{E}^i_v} \big(v_j^+ - n + 1\big) + \delta_i, \quad i \geqslant 1, \tag{19}$$

where $\mathbf{E}^i_h$ is the set of horizontal l-edges in $\mathbf{E}^i$, $\mathbf{E}^i_v$ is the set of vertical l-edges in $\mathbf{E}^i$, and δ_i is the number of isolated subdomains at level $i-1$. For the case that $i = 0$, because $\mathbf{E}^0$ is the set of l-edges of a tensor product mesh $\mathscr{T}^{\varepsilon}_{\otimes}$, one can easily check that the following equation must hold

$$\sum_{E_j \in \mathbf{E}^0} \dim W_{\mathscr{T}_j}[E_j] = \sum_{E_j \in \mathbf{E}^0_h} \big(v_j^+ - m + 1\big) + \sum_{E_j \in \mathbf{E}^0_v} \big(v_j^+ - n + 1\big) + (m-1)(n-1). \tag{20}$$

Observe that

$$\sum_{i=0}^{l} \sum_{E_j \in \mathbf{E}^i} v_j^+ = V^+.$$

As a result, the dimension formula given in (16) follows from (13), (18)–(20). $\square$

5.1. Examples

In this subsection, we give two examples in which we count the dimensions of spline spaces $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^{\varepsilon}_{m,n})$.

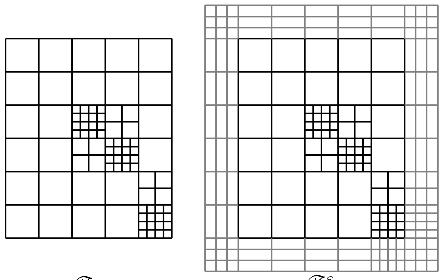


Fig. 11. A (3, 3)-subdivided T-mesh $\mathscr{T}_{3,3}$ and its extension.

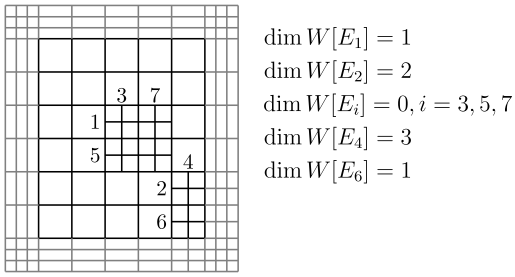


Fig. 12. $\mathscr{T}_{3,3}$ at level 1, where $W[E_i]$ is determined by the vertices on E_i of the current T-mesh by following the order $E_7 \prec E_6 \prec E_5 \prec E_4 \prec E_3 \prec E_2 \prec E_1$.

Example 5.1. In the case $m = 2$, $n = 2$, $\mathscr{T}_{m,n}$ is a general hierarchical T-mesh, which we denote as $\mathscr{T}$. By the dimension formula in (16),

$$\begin{aligned}\dim \mathbf{S}(2, 2, 1, 1, \mathscr{T}) &= \dim \bar{\mathbf{S}}\big(2, 2, 1, 1, \mathscr{T}^{\varepsilon}\big)\\ &= V^{+} - E + \delta + 1\end{aligned}$$

where V^{+} is the number of crossing vertices, E is the number of interior l-edges, and δ is the number of isolated cells of $\mathscr{T}^{\varepsilon}$. This formula has been already derived in Deng et al. (2008).

Next, we give a concrete example for the case $m = 3$, $n = 3$.

Example 5.2. In this example, Fig. 11 illustrates the hierarchical T-mesh $\mathscr{T}_{3,3}$ (which is the same as the T-mesh in Fig. 7(c)) and its extension $\mathscr{T}^{\varepsilon}_{3,3}$. Three isolated subdomains (one at level 0 and two at level 1) appear in the middle of $\mathscr{T}^{\varepsilon}_{3,3}$. Thus, we have $\delta = 3$. One can verify that $V^{+} = 166$, $E_H = 21$ and $E_V = 19$. Hence,

$$\begin{aligned}\dim \mathbf{S}(3, 3, 2, 2, \mathscr{T}_{3,3}) &= \dim \bar{\mathbf{S}}\big(3, 3, 2, 2, \mathscr{T}^{\varepsilon}_{3,3}\big)\\ &= V^{+} - (3-1)E_H - (3-1)E_V + \delta + (3-1)(3-1)\\ &= 93.\end{aligned}$$

We can also obtain the dimension of the space by counting the number of basis functions.

At level 0, we have $(12 - 4)(13 - 4) = 72$ basis functions over the 11×12 tensor product mesh. For level $k > 0$, we can count the number of basis functions by applying Eq. (19). For example, at level 1, in Fig. 12, the order defined in Section 4.1 implies that $E_7 \prec E_6 \prec E_5 \prec E_4 \prec E_3 \prec E_2 \prec E_1$, where E_i is labeled with i in Fig. 12. Thus, we add 7 basis functions to our set by subdividing two subdomains at level 0.

Similarly, at level 2, we add 14 basis functions to the basis functions set by subdividing three subdomains. In total, we have $72 + 7 + 14 = 93$ basis functions, and so we conclude that the dimension of $\mathbf{S}(3, 3, 2, 2, \mathscr{T}_{3,3})$ is 93. In this example, the dimension of the space achieves the upper bound of the dimension formula given in Mourrain (2010). Nevertheless, under this order of these maximal interior segments (or interior l-edges in our paper), $\mathscr{T}_{3,3}$ is not the T-mesh as required in the dimension formula in Mourrain (2010). In other words, $\mathscr{T}_{3,3}$ does not satisfy the conditions in Mourrain (2010) such that the inequality meets its upper bound.

6. Conclusions and future work

This paper has presented a dimension formula for a spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$ over a certain type of hierarchical T-mesh. Using the smoothing cofactor method, we established a bijection between the spline space $\mathbf{S}(m, n, m-1,$

$n-1, \mathscr{T})$ and a conformality vector space $W[\mathscr{T}]$. Then, we applied a homological technique to decompose $W[\mathscr{T}]$ into a direct sum of simple linear spaces, thus obtaining a dimension formula. As a by-product, we also obtain a set of basis functions for the spline space $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$.

In applications, it is important to construct a set of basis functions having certain desirable properties such as nonnegativity, partition of unity and local support. Generally, the basis functions constructed in Section 4.2 just have local support and do not satisfy two other properties. Thus, in the future, we intend to study this topic by modifying the set of basis functions constructed in this paper.

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