

Computing self-intersection curves of rational ruled surfaces

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ABSTRACT

An algorithm is presented to compute the self-intersection curves of a rational ruled surface based on the theory of μ -bases. The algorithm starts by constructing the principal subresultants for a μ -basis of the rational ruled surface. The principal subresultant coefficients provide information about not only the parametric loci of the self-intersection curves, but also the orders of the self-intersection curves. Based on this observation, an efficient algorithm is provided to compute the parametric loci of the self-intersection curves as well as their corresponding orders. The isolated singular points of the rational ruled surface are also computed.

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1. Introduction

A *singular point* of a surface is a point on the surface where the tangent plane of the surface is not uniquely defined. The *singular locus* of a surface is the set of all the singular points on the surface. In general, the singular locus of a surface consists of a finite number of isolated points on the surface and some space curves, where the space curves are the *self-intersection curves* of the surface. Detecting the singularities helps to determine the shape and the topology of the surfaces, which have wide-ranging applications in Computer Aided Design and Geometric Modeling. In this paper, we are interested in computing the real singular loci of real rational ruled surfaces. Indeed, although the computations can be carried out in general over the field of complex numbers, we are interested in applications to Computer Graphics and Geometric Modeling where the emphasis is typically on real valued functions.

Ruled surfaces are one of the simplest types of surfaces that occur in many applications in the design and manufacturing industries (Aumann, 1991; Edge, 1931; Ravani and Chen, 1986), and many computational issues for ruled surfaces have previously been discussed (Chen et al., 2001; Chen and Wang, 2003b; Farin et al., 2002; Han et al., 2001; Elber and Fish, 1997; Andersson et al., 1998, 2006). The main purpose of this paper is to extend a recently developed method for computing the singular points of rational curves (Chen and Wang, 2003a; Chen et al., 2008) to rational ruled surfaces.

The basic idea of the method is as follows. First we calculate a μ -basis for the ruled surface and then we compute the principal subresultant coefficients for two elements of the μ -basis. The principal subresultant coefficients contain information about the singular locus of the ruled surface, as well as the orders of the singular loci. From these coefficients, we derive an efficient algorithm to compute the preimage of the self-intersection curves. Isolated singular points of the ruled surface can also be computed.

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The remainder of the paper is organized as follows. Section 2 recalls some preliminary results about the μ -basis of a rational ruled surface, and introduces the notion of the i th principal subresultant coefficient. Section 3 explores the principal subresultant coefficients derived from the μ -basis of a rational ruled surface and presents an algorithm to compute the self-intersection curves of a rational ruled surface. The isolated singular points of these surfaces are also computed. Section 4 provides some examples to demonstrate the algorithm for computing these self-intersection curves. Section 5 discusses the implicit equations of the self-intersection curves. We conclude the paper in Section 6 with some open problems for future research.

2. Preliminaries on μ -bases and principal subresultants

Let $\mathbb{R}[s]$ be the set of polynomials in s with real coefficients, and let $\mathbb{R}[s]^r$ be the set of r -dimensional row vectors with entries in $\mathbb{R}[s]$. For a given polynomial vector $\mathbf{p} \in \mathbb{R}[s]^r$, write

$$\mathbf{p} = \sum_{i=0}^k \mathbf{p}_i s^i,$$

where $\mathbf{0} \neq \mathbf{p}_k \in \mathbb{R}^r$ is called the *leading vector* of \mathbf{p} , denoted by $LV(\mathbf{p})$. The *degree* of \mathbf{p} is k , denoted by $\deg(\mathbf{p})$.

A rational ruled surface of degree n is defined in homogeneous form as

$$\mathbf{P}(s, t) := \mathbf{P}_0(s) + t\mathbf{P}_1(s) := (a(s, t), b(s, t), c(s, t), d(s, t)), \quad (1)$$

where

$$\mathbf{P}_i(s) := (a_i(s), b_i(s), c_i(s), d_i(s)) \in \mathbb{R}[s]^4, \quad i = 0, 1,$$

and the maximum degree of \mathbf{P}_0 and \mathbf{P}_1 is n . To avoid the degenerate case where $\mathbf{P}(s, t)$ parameterizes a curve, $\mathbf{P}_0(s)$ and $\mathbf{P}_1(s)$ are always assumed to be $\mathbb{R}[s]$ -linearly independent. Furthermore, we assume that the parametrization is generically one-to-one.

A *moving plane* $\mathbf{L}(s, t) := (A(s, t), B(s, t), C(s, t), D(s, t))$ is a family of planes $L(x, y, z, w; s, t) := A(s, t)x + B(s, t)y + C(s, t)z + D(s, t)w = 0$ with one plane corresponding to each parameter pair (s, t) (Chen et al., 2001; Chen and Wang, 2003b; Chen, 2005). A moving plane is said to *follow* the rational ruled surface $\mathbf{P}(s, t)$ if

$$\mathbf{L}(s, t) \cdot \mathbf{P}(s, t) = A(s, t)a(s, t) + B(s, t)b(s, t) + C(s, t)c(s, t) + D(s, t)d(s, t) \equiv 0. \quad (2)$$

Geometrically, a moving plane $\mathbf{L}(s, t)$ follows a surface $\mathbf{P}(s, t)$ means that for any parameter pair (s_0, t_0) , the point $\mathbf{P}(s_0, t_0)$ lies on the plane $\mathbf{L}(s_0, t_0)$. A moving plane $\mathbf{L}(s, t)$ is called an *axial* moving plane if $\mathbf{L}(s, t)$ passes through a fixed point \mathbf{Q}_0 or line l , i.e., $\mathbf{L} \cdot \mathbf{Q}_0 \equiv 0$ or $\mathbf{L} \cdot \mathbf{Q} \equiv 0$ for any point $\mathbf{Q} \in l$. The fixed point \mathbf{Q}_0 or line l is called the *axis* of the moving plane.

Let $\mathbf{L}[s]$ be the set of moving planes that involve only the parameter $s - \mathbf{L}(s) := (A(s), B(s), C(s), D(s)) \in \mathbb{R}[s]^4$, and for which $\mathbf{L}(s) \cdot \mathbf{P}(s, t) \equiv 0$. It is easy to check that $\mathbf{L}[s]$ is a module over $\mathbb{R}[s]$ (Chen et al., 2001).

Proposition 2.1. (See Chen et al. (2001), Chen and Wang (2003b).) Suppose m is the implicit degree of the rational ruled surface $\mathbf{P}(s, t)$. There exist two moving planes $\mathbf{p}(s) = (p_1(s), p_2(s), p_3(s), p_4(s)) \in \mathbf{L}[s]$ and $\mathbf{q}(s) = (q_1(s), q_2(s), q_3(s), q_4(s)) \in \mathbf{L}[s]$, of degree μ and $m - \mu$ ($\mu \leq \lfloor m/2 \rfloor$), such that $\mathbf{p}(s)$ and $\mathbf{q}(s)$ form a basis of $\mathbf{L}[s]$. Furthermore, there is a moving plane

$$\mathbf{r}(s, t) := \mathbf{u}(s) + t\mathbf{v}(s),$$

where

$$\mathbf{u}(s) = (u_1(s), u_2(s), u_3(s), u_4(s)) \in \mathbb{R}[s]^4,$$

$$\mathbf{v}(s) = (v_1(s), v_2(s), v_3(s), v_4(s)) \in \mathbb{R}[s]^4$$

such that

$$[\mathbf{p}, \mathbf{q}, \mathbf{r}] = \kappa \mathbf{P}(s, t) \quad (3)$$

for some nonzero constant κ . Here $[\mathbf{p}, \mathbf{q}, \mathbf{r}]$ is the outer product of \mathbf{p}, \mathbf{q} and \mathbf{r} .

Definition 2.1. The polynomial vectors $\mathbf{p}(s)$, $\mathbf{q}(s)$ and $\mathbf{r}(s, t)$ in Proposition 2.1 are called a μ -basis of the rational ruled surface $\mathbf{P}(s, t)$. Sometimes, we also call $p(x, y, z, w, s) = \mathbf{p}(s) \cdot \mathbf{X}$, $q(x, y, z, w, s) = \mathbf{q}(s) \cdot \mathbf{X}$ and $r(x, y, z, w, s, t) = \mathbf{r}(s, t) \cdot \mathbf{X}$ a μ -basis, where $\mathbf{X} = (x, y, z, w)$.

Geometrically, $\mathbf{p}(s)$ and $\mathbf{q}(s)$ are two moving planes, whose intersection gives the rulings of the ruled surface $\mathbf{P}(s, t)$. Furthermore, the moving plane $\mathbf{r}(s, t)$ intersects with the rulings to give points on $\mathbf{P}(s, t)$.

Chen and his collaborators provide efficient algorithms to compute a μ -basis for rational ruled surfaces in Chen et al. (2001), Chen and Wang (2003b), Chen (2003). They also prove the following results.

Proposition 2.2. (See Chen et al. (2001).) Let $\mathbf{p}(s)$ and $\mathbf{q}(s)$ be two elements of a μ -basis for the rational ruled surface $\mathbf{P}(s, t)$. Then for any parameter value s_0 , $\mathbf{p}(s_0)$ and $\mathbf{q}(s_0)$ are linearly independent. In particular, the leading vectors of $\mathbf{p}(s)$ and $\mathbf{q}(s)$ are linearly independent.

Definition 2.2. Let $P(s), Q(s) \in \mathbb{R}[s]$ be two polynomials with degree n and m respectively. Let

$$P(s) = p_0s^n + p_1s^{n-1} + \dots + p_n, \quad Q(s) = q_0s^m + q_1s^{m-1} + \dots + q_m.$$

The Sylvester matrix associated to $P(s)$ and $Q(s)$ is

$$\text{Syl}(P, Q) = \begin{pmatrix} \overbrace{p_0 \ p_1 \ \dots \ p_n}^{n+m} & & & \\ & \ddots & & \\ & & p_0 \ \dots \ p_n & \\ q_0 \ q_1 \ \dots \ q_m & & & \\ & \ddots & & \\ & & q_0 \ q_1 \ \dots \ q_m & \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} p_0 \\ \vdots \\ p_0 \end{matrix}} \right\} m \\ \left. \vphantom{\begin{matrix} q_0 \\ \vdots \\ q_0 \end{matrix}} \right\} n \end{matrix} \tag{4}$$

The resultant of the two polynomials $P(s)$ and $Q(s)$ is defined as $\text{Res}(P, Q) = \det(\text{Syl}(P, Q))$.

Proposition 2.3. (See Chen et al. (2001).) Let $p := p(x, y, z, w, s)$ and $q := q(x, y, z, w, s)$ be two elements of a μ -basis for the rational ruled surface $\mathbf{P}(s, t)$. Then the implicit equation (in homogeneous form) of $\mathbf{P}(s, t)$ is given by $\text{Res}(p, q)$ with respect to s .

Although all the discussions in this paper are valid under the assumption that the rational ruled surface is properly parametrized, Proposition 2.3 can also be extended to the non-proper case, which in fact gives a power of the irreducible implicit equation of the corresponding properly parametrized ruled surface (Chen et al., 2001).

A μ -basis not only gives the implicit equation, but also provides information about the singularities of a rational ruled surface. In order to find the key to connecting singularities and μ -bases of rational ruled surfaces, we need to explore the principal subresultant coefficients associated to two polynomials.

Definition 2.3. (See Abdeljaoued et al. (2004).) Let $P(x), Q(x) \in \mathbb{R}[x]$, with $n = \deg(P) \geq m = \deg(Q)$:

$$P(x) = p_0x^n + p_1x^{n-1} + \dots + p_n, \quad Q(x) = q_0x^m + q_1x^{m-1} + \dots + q_m.$$

The i th principal subresultant coefficient associated with $P(x)$ and $Q(x)$, denoted by $\mathbf{psc}_i(P, Q)$, is the determinant of the square matrix:

$$\begin{pmatrix} \overbrace{p_0 \ \dots \ p_{n-2i+m-1}}^{n+m-2i} & & & \\ & \ddots & & \\ & & p_0 \ \dots \ p_{n-i} & \\ q_0 \ \dots \ q_{m-2i+n-1} & & & \\ & \ddots & & \\ & & q_0 \ \dots \ q_{m-i} & \end{pmatrix} \begin{matrix} \left. \vphantom{\begin{matrix} p_0 \\ \vdots \\ p_0 \end{matrix}} \right\} m-i \\ \left. \vphantom{\begin{matrix} q_0 \\ \vdots \\ q_0 \end{matrix}} \right\} n-i \end{matrix}$$

where $i \in \{0, \dots, m\}$, and $p_i = q_j = 0$ for $i > n, j > m$.

Proposition 2.4. (See Abdeljaoued et al. (2004).) Let $P(x), Q(x) \in \mathbb{R}[x]$, with $n = \deg(P) \geq m = \deg(Q)$:

$$P(x) = p_0x^n + p_1x^{n-1} + \dots + p_n, \quad Q(x) = q_0x^m + q_1x^{m-1} + \dots + q_m.$$

Let $\mathbf{psc}_i(P, Q)$ be the i th principal subresultant coefficient defined above. Then

$$\deg(\text{gcd}(P, Q)) = i \iff \begin{cases} \mathbf{psc}_0(P, Q) = \dots = \mathbf{psc}_{i-1}(P, Q) = 0 \\ \mathbf{psc}_i(P, Q) \neq 0. \end{cases} \tag{5}$$

3. Self-intersection curves of rational ruled surfaces

To compute the self-intersection curves of a rational ruled surface, we first consider the singular points on the surface, since all the points on the self-intersection curves are necessarily singular points.

Definition 3.1. Let $F(x, y, z, w) = 0$ be the implicit equation of the rational ruled surface $\mathbf{P}(s, t)$ in homogeneous form. A point $\mathbf{Q}_0 = (x_0, y_0, z_0, w_0)$ is an order r singular point on the surface if and only if

$$\frac{\partial^{r-1} F}{\partial x^i \partial y^j \partial z^k \partial w^l} (x_0, y_0, z_0, w_0) = 0, \quad i + j + k + l = r - 1, \tag{6}$$

and at least one r th partial derivative of F at \mathbf{Q}_0 does not vanish.

To allow for $s = \infty$ and/or $t = \infty$, we shall homogenize the parameters s and t to $s : u$ and $t : v$. Thus the ruled surface $\mathbf{P}(s, t)$ becomes

$$\mathbf{P}(s, u; t, v) = \mathbf{P}_0(s, u)v + \mathbf{P}_1(s, u)t := (a(s, u; t, v), b(s, u; t, v), c(s, u; t, v), d(s, u; t, v)). \tag{7}$$

Similarly, the μ -basis $\mathbf{p}(s), \mathbf{q}(s)$ and $\mathbf{r}(s, t)$ is homogenized to $\mathbf{p}(s, u), \mathbf{q}(s, u)$ and $\mathbf{r}(s, u; t, v)$.

Since even a real singular point may correspond to a pair of complex conjugate parameter values, we need to perform our analysis using complex parameters. Therefore, throughout this section, we shall assume that the parametric domain is $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$, where $\mathbb{P}^1(\mathbb{C})$ is one dimensional projective space over the field of complex numbers. Nevertheless, even though our parameter values may be complex, we shall be concerned only with points with real coordinates.

3.1. Singular points on rational ruled surfaces

Let $\mathbf{p}(s, u) = (p_1, p_2, p_3, p_4)$ and $\mathbf{q}(s, u) = (q_1, q_2, q_3, q_4)$ be two elements of a μ -basis for the rational ruled surface $\mathbf{P}(s, u; t, v)$ whose degrees are μ and $m - \mu$ respectively ($\mu \leq m - \mu$), where m is the implicit degree of $\mathbf{P}(s, u; t, v)$ and p_i, q_i are homogeneous polynomials in $s, u, i = 1, 2, 3, 4$. Let $p(s, u) = \mathbf{p}(s, u) \cdot \mathbf{X}$ and $q(s, u) = \mathbf{q}(s, u) \cdot \mathbf{X}$, where $\mathbf{X} = (x, y, z, w)$. Furthermore, let $\mathbf{Q}_0 = (x_0, y_0, z_0, w_0)$ be a singular point of $\mathbf{P}(s, u; t, v)$. The following theorem provides a relationship between the order of the singular point \mathbf{Q}_0 and the degree of the greatest common divisor (GCD) of $\mathbf{p} \cdot \mathbf{Q}_0$ and $\mathbf{q} \cdot \mathbf{Q}_0$, which is the key to connecting singularities and μ -bases for rational ruled surfaces.

Theorem 3.1. Let $\mathbf{p}(s, u), \mathbf{q}(s, u)$ be two elements of a μ -basis for the rational ruled surface $\mathbf{P}(s, u; t, v)$. Then \mathbf{Q}_0 is an order r singular point of $\mathbf{P}(s, u; t, v)$ if and only if

$$\deg(\gcd(\mathbf{p}(s, u) \cdot \mathbf{Q}_0, \mathbf{q}(s, u) \cdot \mathbf{Q}_0)) = r, \tag{8}$$

provided that the point \mathbf{Q}_0 is not a common axis point of $\mathbf{p}(s, u)$ and $\mathbf{q}(s, u)$.

Before proving this theorem, we need a lemma.

Lemma 3.2. Given polynomial vectors $\mathbf{p} = (p_1, p_2, p_3, p_4) \in \mathbb{R}[s]^4$ and $\mathbf{q} = (q_1, q_2, q_3, q_4) \in \mathbb{R}[s]^4$, let $p = \mathbf{p} \cdot \mathbf{X}$ and $q = \mathbf{q} \cdot \mathbf{X}$, where $\mathbf{X} = (x, y, z, w)$. Suppose that the leading vectors of \mathbf{p} and \mathbf{q} are linearly independent and let

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix},$$

where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a unimodular matrix of polynomials in $\mathbb{R}[s]$. Then

$$\text{Res}(\hat{p}, \hat{q}) = \lambda \cdot l^{\deg(\hat{p}) + \deg(\hat{q}) - \deg(p) - \deg(q)} \cdot \text{Res}(p, q). \tag{9}$$

Here $\lambda \neq 0$ is a constant and l is the leading coefficient of \hat{p} (or \hat{q}) with respect to s (denoted by $lc(\hat{p})$ or $lc(\hat{q})$).

Proof. Since a unimodular matrix can be factored into the product of elementary matrices of the following types:

$$\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

where α, β are nonzero constants and $\gamma, \delta \in \mathbb{R}[s]$, we only need to prove the statement for the above four types of matrices. In the following, we only prove the case:

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p + \gamma q \\ q \end{pmatrix},$$

where $\gamma \in \mathbb{R}[s]$; the other cases can be treated similarly. There are two situations for the first case:

1. $\deg(\gamma q) \leq \deg(p)$. Since $LV(\mathbf{p})$ and $LV(\mathbf{q})$ are linearly independent, the leading terms of p and γq cannot cancel each other, hence $\deg(\hat{p}) = \deg(p)$. By properties of resultants, $\text{Res}(\hat{p}, \hat{q}) = \text{Res}(p, q)$. Thus (9) holds.
2. $\deg(\gamma q) > \deg(p)$. Without loss of generality, assume that γ is a monic polynomial. Let $\deg(p) = m$, $\deg(q) = n$ and $\deg(\gamma) = k$ ($k + n > m$) temporarily. Then $\deg(\hat{p}) = n + k$, and $lc(\hat{p}) = lc(\hat{q}) = lc(q)$. Furthermore, let $p = \sum_{i=0}^m p_i s^i$ and $\hat{q} = q = \sum_{i=0}^n q_i s^i$, where p_i and q_i are linear in x, y, z, w . Regard p as a polynomial of degree $n+k$: $\tilde{p} := p = \sum_{i=0}^{n+k} p_i s^i$, where $p_{m+1} = \dots = p_{n+k} = 0$. Then $\text{Res}(\hat{p}, \hat{q}) = \text{Res}(\tilde{p}, q)$. The Sylvester matrix of \tilde{p} and q is

$$\text{Syl}(\tilde{p}, q) = \begin{pmatrix} p_0 & \dots & p_m & \dots & p_{n+k} & & & & \\ & \ddots & & \ddots & & \ddots & & & \\ & & p_0 & \dots & p_m & \dots & p_{n+k} & & \\ q_0 & \dots & q_n & & & & & & \\ & \ddots & & \ddots & & & & & \\ & & q_0 & \dots & q_n & & & & \\ & & & \ddots & & \ddots & & & \\ & & & & q_0 & \dots & q_n & & \end{pmatrix} = \begin{pmatrix} \text{Syl}(p, q) & 0 \\ * & Q \end{pmatrix},$$

where $Q \in \mathbb{R}[x, y, z, w]^{(n+k-m) \times (n+k-m)}$ is a lower triangular matrix

$$Q = \begin{pmatrix} q_n & & & \\ q_{n-1} & q_n & & \\ \vdots & & \ddots & \\ \dots & \dots & \dots & q_n \end{pmatrix}.$$

Thus

$$\text{Res}(\hat{p}, \hat{q}) = \text{Res}(\tilde{p}, q) = \det(\text{Syl}(p, q)) \cdot q_n^{n+k-m} = l^{\deg(\hat{p}) - \deg(p)} \cdot \text{Res}(p, q).$$

This completes the proof. □

Now we are ready to prove Theorem 3.1.

Without loss of generality, we assume $\mathbf{Q}_0 = (0, 0, 0, 1)$. Then $h := \gcd(\mathbf{p}(s, u) \cdot \mathbf{Q}_0, \mathbf{q}(s, u) \cdot \mathbf{Q}_0) = \gcd(p_4, q_4)$, and \mathbf{Q}_0 is a singular point of order r if and only if the implicit equation $F(x, y, z, w)$ of $\mathbf{P}(s, u; t, v)$ takes the form:

$$F(x, y, z, w) = f_r(x, y, z)w^{m-r} + f_{r+1}(x, y, z)w^{m-r-1} + \dots + f_m(x, y, z),$$

where $f_i(x, y, z)$ is a homogeneous polynomial of degree i in x, y, z and $f_r(x, y, z) \neq 0$, i.e., the lowest degree term of $F(x, y, z, 1)$ has degree r .

Let $p(s, u) = \mathbf{p}(s, u) \cdot \mathbf{X}$ and $q(s, u) = \mathbf{q}(s, u) \cdot \mathbf{X}$, where $\mathbf{X} = (x, y, z, 1)$. Let $S := S(x, y, z)$ be the Sylvester matrix of p and q with respect to s, u . Since $F(x, y, z, 1) = \text{Res}(p, q) = \det(S)$ and $h = \gcd(p_4, q_4)$, we only have to prove that the lowest degree term of $\text{Res}(p, q) = \det(S)$ in x, y, z has degree r if and only if $\deg(h) = r$.

First, assume $\deg(h) = r$. Then $p_4 = h\hat{p}_4$, $q_4 = h\hat{q}_4$, where \hat{p}_4 and \hat{q}_4 are homogeneous polynomials of degree $\mu - r$ and $m - \mu - r$ in s, u , and $\gcd(\hat{p}_4, \hat{q}_4) = 1$. Now we dehomogenize the variables s, u by setting $u = 1$ in each polynomial, and for brevity we use the same symbol to denote the polynomials after dehomogenization. For example, $p = p(s) = p(s, 1)$. Note that after dehomogenization, the degree of each polynomial (in s) may decrease.

Since $\gcd(\hat{p}_4, \hat{q}_4) = 1$, then $\deg(\hat{p}_4) = \mu - r$ or $\deg(\hat{q}_4) = m - \mu - r$, and there exist polynomials $\alpha, \beta \in \mathbb{R}[s]$ such that $\alpha\hat{p}_4 + \beta\hat{q}_4 = 1$ with $\deg(\alpha) \leq m - \mu - r - 1$ and $\deg(\beta) \leq \mu - r - 1$. Let

$$\hat{p} = \alpha p + \beta q, \quad \hat{q} = -\hat{q}_4 p + \hat{p}_4 q.$$

By the properties of μ -basis, $\deg(p) = \mu$ and $\deg(q) = m - \mu$, and the leading terms of $\hat{q}_4 p$ and $\hat{p}_4 q$ do not cancel each other. Hence $\deg(\hat{q}) = \max(\deg(\hat{q}_4 p), \deg(\hat{p}_4 q)) = m - r$. Denote $\deg(\hat{p}) = k$.

Since $\begin{pmatrix} \alpha & \beta \\ -\hat{q}_4 & \hat{p}_4 \end{pmatrix}$ is a unimodular matrix, by Lemma 3.2,

$$\text{Res}(\hat{p}, \hat{q}) = l^{k-r} \cdot \text{Res}(p, q), \tag{*}$$

where l is the leading coefficient of \hat{p} (and \hat{q}), which is a degree one polynomial in x, y, z .

On the other hand,

$$\hat{p} = (\alpha p_1 + \beta q_1)x + (\alpha p_2 + \beta q_2)y + (\alpha p_3 + \beta q_3)z + h,$$

$$\hat{q} = (\hat{p}_4 q_1 - \hat{q}_4 p_1)x + (\hat{p}_4 q_2 - \hat{q}_4 p_2)y + (\hat{p}_4 q_3 - \hat{q}_4 p_3)z.$$

Hence $l = lc(\hat{p}) = lc(\hat{q}) = l_1 x + l_2 y + l_3 z$, where $l_i, i = 1, 2, 3$ are constants. Denote

$$\hat{\mathbf{p}} = (\alpha p_1 + \beta q_1, \alpha p_2 + \beta q_2, \alpha p_3 + \beta q_3, h) := (\hat{p}_1, \hat{p}_2, \hat{p}_3, h),$$

$$\hat{\mathbf{q}} = (\hat{p}_4 q_1 - \hat{q}_4 p_1, \hat{p}_4 q_2 - \hat{q}_4 p_2, \hat{p}_4 q_3 - \hat{q}_4 p_3, 0) := (\hat{q}_1, \hat{q}_2, \hat{q}_3, 0).$$

By properties of μ -basis, $\mathbf{p}(s_0)$ and $\mathbf{q}(s_0)$ are linearly independent for any parameter value s_0 ; therefore $\hat{\mathbf{p}}(s_0)$ and $\hat{\mathbf{q}}(s_0)$ are also linearly independent. Hence $\gcd(\hat{q}_1, \hat{q}_2, \hat{q}_3, h) = 1$, and consequently $\text{Res}(\hat{q}, h) \neq 0$. Thus the lowest degree term of $\text{Res}(\hat{p}, \hat{q})$ in x, y, z is $\text{Res}(\hat{q}, \hat{h})$, where \hat{h} is the polynomial h regarded as a polynomial of degree k . Obviously, $\text{Res}(\hat{q}, \hat{h}) = \det(\text{Syl}(\hat{q}, \hat{h}))$ is a homogeneous polynomial in x, y, z of degree k . From Eq. (*), the lowest degree term of $\text{Res}(p, q)$ in x, y, z has degree $k - (k - r) = r$.

Conversely, suppose that the lowest degree term of $\text{Res}(p, q)$ is x, y, z is r . Let $\deg(h) = r'$. By the argument of the first part, $r' = r$. The theorem is proved. \square

Theorem 3.1 is valid under the assumption that $r \leq \mu$. Now we consider the cases: $\mu < r \leq m - \mu$ and $m - \mu < r \leq m$. We still assume $\mathbf{Q}_0 = (0, 0, 0, 1)$.

First, we assume that $\mu < r = \deg(h) \leq m - \mu$. Then $\mathbf{p} \cdot \mathbf{Q}_0 \equiv 0$ and $h = \mathbf{q} \cdot \mathbf{Q}_0$, so $r = m - \mu$, and the μ -basis takes the form

$$p(s, u) = p_1x + p_2y + p_3z, \quad q(s, u) = q_1x + q_2y + q_3z + h,$$

where $p_i, q_i, i = 1, 2, 3$ and h are homogeneous polynomials in s, u . Now it is easy to show that the lowest degree term of the implicit equation $F(x, y, z, 1) = \text{Res}(p, q)$ is $\text{Res}(p, h)$, which is a homogeneous polynomial of degree $r = m - \mu$ in x, y, z . Hence \mathbf{Q}_0 is a singular point of order $m - \mu$. Thus Theorem 3.1 is still valid for $\mu < r \leq m - \mu$.

Next, we assume that $m - \mu < r = \deg(h) \leq m$. Then $\mathbf{p} \cdot \mathbf{Q}_0 \equiv 0$ and $\mathbf{q} \cdot \mathbf{Q}_0 \equiv 0$, and the μ -basis takes the form

$$p(s, u) = p_1x + p_2y + p_3z, \quad q(s, u) = q_1x + q_2y + q_3z.$$

Therefore the implicit equation $\text{Res}(p, q)$ is a homogeneous polynomial of degree m in x, y, z . In this case, the ruled surface $\mathbf{P}(s, u; t, v)$ is a cone with vertex \mathbf{Q}_0 , and \mathbf{Q}_0 is a singular point of order m . Summarizing the above analysis, we have:

Theorem 3.3. Let $\mathbf{Q}_0 = (x_0, y_0, z_0, w_0)$ is a singular point of order $r > \mu$ ($\mu < m - \mu$) of the rational ruled surface $\mathbf{P}(s, u; t, v)$. Then $r = m - \mu$ or $r = m$. Moreover,

1. \mathbf{Q}_0 is a singular point of order $m - \mu$ if and only if $\mathbf{p} \cdot \mathbf{Q}_0 \equiv 0$ and $\mathbf{q} \cdot \mathbf{Q}_0 \neq 0$, i.e., $p(s, u)$ is an axial moving plane with axis \mathbf{Q}_0 , but \mathbf{Q}_0 is not an axis of $q(s, u)$. In addition, a line l is a self-intersection line of order $m - \mu$ if and only if l is the axis of $p(s, u)$.
2. \mathbf{Q}_0 is a singular point of order m if and only if $\mathbf{p} \cdot \mathbf{Q}_0 \equiv 0$ and $\mathbf{q} \cdot \mathbf{Q}_0 \equiv 0$, i.e., $p(s, u)$ and $q(s, u)$ are axial moving planes, both with axis \mathbf{Q}_0 . Furthermore, the ruled surface $\mathbf{P}(s, u; t, v)$ has at most one singular point of order m .

Corollary 3.1. $\mu = 1$ if and only if the rational ruled surface $\mathbf{P}(s, u; t, v)$ has a self-intersection line of order $m - 1$.

Based on Theorem 3.3, it is easy to find the high order singularities of a rational ruled surface. One just computes the axes of the μ -basis elements $p(s, u)$ and $q(s, u)$, which tell everything about the singularities of order $m - \mu$ and m .

Example 3.1. Consider a rational ruled surface:

$$\mathbf{P}(s, t) = (s^2 + t, s^3 + 2 + t, s^2 - s + t, 1 + t).$$

The first two elements of a μ -basis of $\mathbf{P}(s, t)$ are

$$\mathbf{p}(s) = (-2s, 1, 1 + s, s - 2),$$

$$\mathbf{q}(s) = (2s + 2, -s, s^2 - s - 2, -s^2),$$

$\mathbf{p}(s)$ has an axis line l :

$$\begin{cases} 2x - z - w = 0, \\ 2x + y - 3w = 0. \end{cases}$$

$\mathbf{q}(s)$ has an axis point $\mathbf{Q} = (1, 1, 1, 1)$. We can check that the point \mathbf{Q} also lies on the axis line of $\mathbf{p}(s)$. According to Theorem 3.3, the line l is a self-intersection curve of order $m - \mu = 2$, while the point \mathbf{Q} is a singular point of order $m = 3$.

To find singular points of order $r \leq \mu$, we must apply Theorem 3.1. In the next subsection, we will explore a method to compute the singularities of a rational ruled surface based on the principal subresultant coefficients of the μ -basis.

3.2. Computing the parametric locus of self-intersection curves

For brevity of description, we dehomogenize the parameters $(s, u; t, v)$ hereafter, that is, we set $u = v = 1$. For example, $\mathbf{P}(s, t)$ stands for $\mathbf{P}(s, 1; t, 1)$. The case $u = 0$ and/or $v = 0$ can be treated without any significant changes.

To compute the self-intersection curves of a rational ruled surface $\mathbf{P}(s, t)$, we shall first need to examine the principal subresultant coefficients \mathbf{psc}_i of $\mathbf{p}(\bar{s}) \cdot \mathbf{X}$ and $\mathbf{q}(\bar{s}) \cdot \mathbf{X}$, where $\mathbf{X} = (x, y, z, w)$ is a point on the surface. Define

$$F_i(\mathbf{X}) := \mathbf{psc}_i(\mathbf{p}(\bar{s}) \cdot \mathbf{X}, \mathbf{q}(\bar{s}) \cdot \mathbf{X}; \bar{s}),$$

$$f_i(s, t) := F_i(\mathbf{P}(s, t)), \tag{10}$$

for $i \in \{0, \dots, \mu\}$. It is easy to see that $f_0 \equiv 0$, since $\mathbf{psc}_0(\mathbf{p}(\bar{s}) \cdot \mathbf{X}, \mathbf{q}(\bar{s}) \cdot \mathbf{X}) = \text{Res}(\mathbf{p}(\bar{s}) \cdot \mathbf{X}, \mathbf{q}(\bar{s}) \cdot \mathbf{X})$, which is the implicit equation of $\mathbf{P}(s, t)$.

Combining Proposition 2.4 and Theorem 3.1 leads to

Theorem 3.4. A point $\mathbf{Q}_0 = \mathbf{P}(s_0, t_0)$ on the ruled surface $\mathbf{P}(s, t)$ is of order r if and only if $f_1(s_0, t_0) = \dots = f_{r-1}(s_0, t_0) = 0$, and $f_r(s_0, t_0) \neq 0$.

Theorem 3.5. Let $f_i(s, t)$ be the polynomial defined in (10), $i = 1, 2, \dots, \mu$. Let

$$\bar{f}_1(s, t) = f_1(s, t), \quad \bar{f}_i(s, t) := \text{gcd}(\bar{f}_{i-1}(s, t), f_i(s, t)),$$

$i = 2, \dots, \mu$, and let $D_i(s, t)$ be the polynomial obtained by eliminating all the factors of $\bar{f}_{i+1}(s, t)$ from $\bar{f}_i(s, t)$, $i = 1, 2, \dots, \mu - 1$. Then $D_i(s, t) = 0$ corresponds to the singular curves on $\mathbf{P}(s, t)$ with order $i + 1$, that is, except for a finite number of parameter pairs (s, t) , for each (s_0, t_0) satisfying $D_i(s_0, t_0) = 0$, $\mathbf{P}(s_0, t_0)$ is a singular point of order $i + 1$, $i = 1, 2, \dots, \mu - 1$.

Proof. Notice that $\bar{f}_i(s, t) = \text{gcd}(f_1(s, t), f_2(s, t), \dots, f_i(s, t))$, so except for a finite number of parameter pairs (s, t) , $D_i(s, t) = 0$ if and only if $\bar{f}_i(s, t) = 0$ and $\bar{f}_{i+1}(s, t) \neq 0$, or equivalently if and only if $f_1(s, t) = \dots = f_i(s, t) = 0$ and $f_{i+1}(s, t) \neq 0$. Thus the assertion follows directly from Theorem 3.4. \square

Definition 3.2. $D_i(s, t)$ defined in Theorem 3.5 is called the parametric locus (or preimage) of the singular curves of $\mathbf{P}(s, t)$ with order $i + 1$, $i = 1, 2, \dots, \mu - 1$.

Remark 3.1. To eliminate all the factors of a polynomial $f(s, t)$ from another polynomial $g(s, t)$, the following computation can be performed:

```
e := gcd(f, g)
while e <> 1 do
  while e|g do
    g := g/e
  end while
  e := gcd(e, g)
end while
output: g.
```

For example, let $f(s, t) = (s + t - 1)^3(s - 2t - 2)$ and $g(s, t) = (s + t - 1)^2(s - 2t - 2)^3(s - 3)$. By eliminating all the factors of $f(s, t)$ from $g(s, t)$, we get $\hat{g}(s, t) = s - 3$.

3.3. Computing isolated singular points

So far we have computed the self-intersection curves of $\mathbf{P}(s, t)$. Note that on each self-intersection curve there still might be a finite number of higher order singularities, which are usually formed by the intersections of self-intersection curves. Next we provide a way to find such higher order singular points.

Theorem 3.6. Let $f_i(s, t)$ and $\bar{f}_i(s, t)$ be defined as in Theorem 3.5 and let

$$U_i = \{(s, t) \mid f_1(s, t) = f_2(s, t) = \dots = f_{i-1}(s, t) = 0, f_i(s, t) \neq 0\}, \tag{11}$$

$$V_i = \{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, f_i(s, t) \neq 0\}, \tag{12}$$

$$S_i = \left\{ (s, t) \mid \frac{f_1(s, t)}{\bar{f}_{i-1}(s, t)} = \frac{f_2(s, t)}{\bar{f}_{i-1}(s, t)} = \dots = \frac{f_{i-1}(s, t)}{\bar{f}_{i-1}(s, t)} = 0, f_i(s, t) \neq 0, \bar{f}_{i-1}(s, t) \neq 0 \right\}. \tag{13}$$

Then $U_i = V_i \cup S_i, i = 2, 3, \dots, \mu$.

Proof. It is easy to see that $V_i \subset U_i$ and $S_i \subset U_i$. Thus $V_i \cup S_i \subset U_i$. Conversely, suppose $(s_0, t_0) \in U_i$ for some i . Then we must have $\bar{f}_{i-1}(s_0, t_0) = 0$ or $\frac{f_1}{f_{i-1}}(s_0, t_0) = \frac{f_2}{f_{i-1}}(s_0, t_0) = \dots = \frac{f_{i-1}}{f_{i-1}}(s_0, t_0) = 0$, so $U_i \subset V_i \cup S_i$. Hence $U_i = V_i \cup S_i, i = 2, 3, \dots, \mu$. \square

Remark 3.2. In the above theorem, U_i is the set of the parameter pairs that give all the singular points of order i , V_i corresponds to self-intersection curves of order i , and S_i corresponds to the isolated singular points of order i .

Theorem 3.7. Let V_i be defined in Eq. (12), and let $D_i(s, t)$ be as defined in Theorem 3.5. Then

$$\{(s, t) \mid D_{i-1}(s, t) = 0\} = \mathbb{Z}(V_i), \tag{14}$$

where $\mathbb{Z}(V_i)$ is the Zariski closure of $V_i, i = 2, 3, \dots, \mu$.

Proof. Recall that $D_{i-1}(s, t)$ is obtained by eliminating all the factors of $\bar{f}_i(s, t)$ from $\bar{f}_{i-1}(s, t), i = 2, \dots, \mu$, so $\{(s, t) \mid D_{i-1}(s, t) = 0\} = \{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0\}$. It is easy to verify that

$$\{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0\} = \{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, f_i(s, t) \neq 0\} \cup \{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0, f_i(s, t) = 0\}.$$

Since the last set in the above equation is finite, the assertion follows immediately. \square

According to the proof of Theorem 3.7, there could be a finite number of singular points of order higher than i on the self-intersection curves of order $i, i = 2, \dots, \mu$. In fact, we have

Theorem 3.8. For each $2 \leq i \leq \mu$,

$$\{(s, t) \mid D_{i-1}(s, t) = 0\} \setminus V_i = \{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0, f_i(s, t) = 0\}$$

gives all the parameter pairs that correspond to the singular points of order higher than i on the self-intersection curves of order i . More specifically,

$$\{(s, t) \mid \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0, f_i(s, t) = \dots = f_{i+j-1}(s, t) = 0, f_{i+j}(s, t) \neq 0\} \tag{15}$$

gives the parameter pairs corresponding to the singular points of order $i + j$ that lie on the order i self-intersection curves, $1 \leq j \leq \mu - i$.

Theorem 3.9. Let (s_0, t_0) be a parameter pair corresponding to a singular point of order $2 \leq i \leq \mu$. Then we must have $D_{j-1}(s_0, t_0) = 0$ for some $2 \leq j \leq \mu$.

Proof. First, (s_0, t_0) must satisfy $f_1(s_0, t_0) = 0$. By the definition of $D_i(s, t), D_{i-1}(s, t) \mid f_1(s, t)$ for all $2 \leq i \leq \mu$, and $\gcd(D_i, D_j) = 1$ for $i \neq j$. Therefore, we have $D_1 \dots D_{\mu-1} \mid f_1$. Moreover, there exist irreducible polynomials $h_1, \dots, h_k \in \mathbb{R}[s, t]$ such that $f_1 = D_1 h_1^{l_1} \dots h_k^{l_k}$ for positive integers $l_i, i = 1, \dots, k$, and each h_i must be a factor of some $D_{j-1} (2 \leq j \leq \mu)$. Therefore, we must have $D_{j-1}(s_0, t_0) = 0$ for some $2 \leq j \leq \mu$. \square

Remark 3.3. In the complex space \mathbb{C}^3 , Theorem 3.9 implies that there does not exist any isolated singularity which does not lie on any self-intersection curve. Thus any isolated singular point in S_i must be a higher order (order i) singularity on some lower order (order j) self-intersection curve. Of course, this is not true in the real space \mathbb{R}^3 .

3.4. Algorithms

We shall now outline an algorithm to compute the self-intersection curves and the singular points of higher order on the self-intersection curves of a rational ruled surface based on the results in previous sections.

Algorithm 1 (SELF-INTERSECTION-CURVES).

Input: Rational ruled surface $\mathbf{P}(s, t) = (a(s, t), b(s, t), c(s, t), d(s, t))$.

Output: Parametric locus of the self-intersection curves of $\mathbf{P}(s, t)$.

- Steps:**
1. Compute the μ -basis elements $\mathbf{p}(s), \mathbf{q}(s)$ of $\mathbf{P}(s, t)$.
 2. Compute the principal subresultant coefficients $\mathbf{psc}_i(\mathbf{p}(\bar{s}) \cdot \mathbf{X}, \mathbf{q}(\bar{s}) \cdot \mathbf{X}), i = 1, 2, \dots, \mu$.
 3. Substitute $\mathbf{P}(s, t)$ for \mathbf{X} in \mathbf{psc}_i to get $\{f_i(s, t)\}_{i=1}^\mu$.
 4. Compute $\{\bar{f}_i(s, t)\}_{i=1}^\mu$.
 5. Eliminate the factors of $\bar{f}_{i+1}(s, t)$ from $\bar{f}_i(s, t)$ to get $\{D_i(s, t)\}_{i=1}^{\mu-1}$.
 6. Output $\{D_i(s, t) = 0\}_{i=1}^{\mu-1}$, which are the parametric loci of the order $i + 1$ self-intersection curves of $\mathbf{P}(s, t)$.

Algorithm 2 (HIGHER-ORDER-SINGULARITIES-ON-SELF-INTERSECTION-CURVES).

Input: $\{\bar{f}_i(s, t) = 0\}_{i=1}^{\mu-1}, \{f_i(s, t)\}_{i=2}^\mu$ computed by Algorithm 1.

Output: Parametric pairs corresponding to singular points of order higher than i on the self-intersection curves of order i .

Steps: For each $2 \leq i \leq \mu - 1$ and $1 \leq j \leq \mu - i$, compute the set

$$\{(s, t) | \bar{f}_{i-1}(s, t) = 0, \bar{f}_i(s, t) \neq 0, f_i(s, t) = \dots = f_{i+j-1}(s, t) = 0, f_{i+j}(s, t) \neq 0\}$$

whose elements correspond to the singular points of order $i + j$ on order i self-intersection curves. To do so, first compute the Gröbner basis g_1, g_2, \dots, g_k of the elimination ideal

$$J = \langle \bar{f}_{i-1}, f_i, \dots, f_{i+j-1}, 1 - \bar{f}_i f_{i+j} \rangle \cap \mathbb{R}[s, t].$$

Then solve the system of equations

$$g_1(s, t) = g_2(s, t) = \dots = g_k(s, t)$$

to find the singularities.

4. Examples

In this section, we provide some examples to illustrate the algorithms presented in the previous section for computing the singular locus of a rational ruled surface.

Example 4.1. Consider the rational ruled surface given by

$$\mathbf{P}(s, t) = \mathbf{P}_0(s) + t\mathbf{P}_1(s),$$

where

$$\mathbf{P}_0(s) = (s + 3, 1, s^2 - 3s + 1, s), \quad \mathbf{P}_1(s) = (1, s^2 + 1, 2s, s + 3).$$

The first two elements of a μ -basis for $\mathbf{P}(s, t)$ are

$$\mathbf{p}(s) = (81s - 59, 25s + 104, 25s + 73, -25s^2 - 79s - 15),$$

$$\mathbf{q}(s) = (-17s^2 + 14s + 11, -28s - 41, -11s + 8, 28s^2 - 4s + 10).$$

Computing the principal subresultant coefficients of the μ -basis yields

$$f_1(s, t) = 1887s^2t^2 + 2363s^2t + 1003s^2 + 8211st^2 - 6001st - 7021s + 10727t^2 - 26537t + 14943,$$

$$D_1(s, t) = \bar{f}_1(s, t) = f_1(s, t).$$

Therefore $f_1(s, t) = 0$ gives the unique parametric locus of the order 2 self-intersection curve of $\mathbf{P}(s, t)$. The image of the rational ruled surface $\mathbf{P}(s, t)$ is shown in Fig. 1.

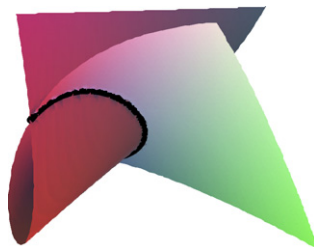


Fig. 1. Rational ruled surface with an order 2 self-intersection curve.

Example 4.2. Consider the rational ruled surface given by

$$\mathbf{P}(s, t) = \mathbf{P}_0(s) + t\mathbf{P}_1(s),$$

with

$$\mathbf{P}_0(s) = (s^4 - 6s^3 + 11s^2 - 6s, s^3 - 6s^2 + 11s - 7, s^3 - 6s^2 + 11s - 7, 1),$$

$$\mathbf{P}_1(s) = (3s^4 - 15s^3 + 15s^2 + 15s, 5s^4 - 25s^3 + 25s^2 + 25s - 29, 8, 0).$$

The first two elements of a μ -basis for $\mathbf{P}(s, t)$ are

$$\mathbf{p}(s) = (-40, 24, 87, 40s^4 - 351s^3 + 1106s^2 - 1461s + 777),$$

$$\mathbf{q}(s) = (-192000s^3 + 427200s^2 + 225480s - 512000, 115200s^3 - 256320s^2 - 135288s,$$

$$192000s^4 - 542400s^3 + 30840s^2 + 469581s, 369707s^4 - 1493442s^3 + 1729297s^2 - 731949s).$$

The principal subresultant coefficients of the μ -basis are

$$f_1(s, t) = -707788800000(s-1)^2(s-2)^2(s-3)^2H(s, t),$$

where $H(s, t)$ is a polynomial of total degree 12.

$$f_2(s, t) = 525335040000(s-1)^2(s-2)^2(s-3)^2(40s-111)^2,$$

$$f_3(s, t) = -282098523 + 599424000s + 231936000t - 7680000s^4 - 386304000s^2 + 96384000s^3.$$

So

$$\bar{f}_1(s, t) = f_1(s, t),$$

$$\bar{f}_2(s, t) = \gcd(f_1(s, t), f_2(s, t)) = (s-1)^2(s-2)^2(s-3)^2,$$

$$\bar{f}_3(s, t) = 1.$$

Hence

$$D_1(s, t) = H(s, t),$$

$$D_2(s, t) = (s-1)^2(s-2)^2(s-3)^2,$$

$$D_3(s, t) = 1.$$

Therefore, $L_1(s, t) \triangleq H(s, t) = 0$ is the unique parametric locus of an order 2 self-intersection curve, while $L_2(s) \triangleq s - 1 = 0$, $L_3(s) \triangleq s - 2 = 0$, $L_4(s) \triangleq s - 3 = 0$ are three different loci of the same self-intersection curve of order 3. Fig. 2 shows part of the order 3 self-intersection curve on this ruled surface.

Furthermore, if we implement Algorithm 2 by setting $i = 2$, we find that the parameter pair $(s_0, t_0) = (\frac{111}{40}, -\frac{7}{151})$ corresponds to a singular point \mathbf{Q} of order 4 lying on the order 2 self-intersection curve. The other three parameter pairs corresponding to the point \mathbf{Q} are $(s_1, t_1) = (1, -\frac{6575159}{77312000})$, $(s_2, t_2) = (2, -\frac{6575159}{77312000})$ and $(s_3, t_3) = (3, -\frac{6575159}{77312000})$. If we implement Algorithm 2 by setting $i = 3$, we also get the above three parameter pairs (s_1, t_1) , (s_2, t_2) , (s_3, t_3) , which shows that the point \mathbf{Q} is a point of order 4 lying on the order 3 self-intersection curve. Therefore, the point \mathbf{Q} is in fact an intersection point of the order 2 and order 3 self-intersection curves.

Example 4.3. Consider the rational ruled surface

$$\mathbf{P}(s, t) = \mathbf{P}_0(s) + t\mathbf{P}_1(s),$$

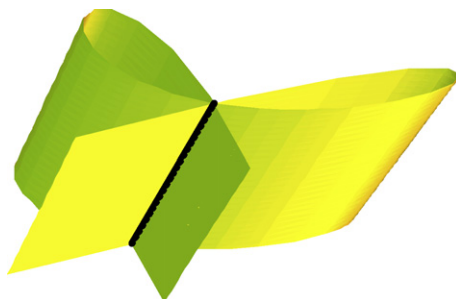


Fig. 2. Rational ruled surface with an order 3 self-intersection curve.



Fig. 3. Rational ruled surface with an order 3 singular point as the intersection of two order 2 self-intersection curves.

with

$$\mathbf{P}_0(s) = (2s, s, s - 1, 1),$$

$$\mathbf{P}_1(s) = (s^4 - 6s^3 + 11s^2 - 8s, s^3 - 6s^2 + 10s - 7, s^3 - 6s^2 + 10s - 6, 0).$$

The first two elements of a μ -basis for $\mathbf{P}(s, t)$ are

$$\mathbf{p}(s) = (-1, -s^2 + 2s, s^2 - s, s),$$

$$\mathbf{q}(s) = (-s, -4s^2 + 10s - 6, 5s^2 + 7 - 10s, -s^3 + 7s^2 - 11s + 7).$$

Computing the principal subresultant coefficients yields

$$f_1(s, t) = -(s^2t - 4st + 2t + 1)(s^4t^2 - 8s^3t^2 + 23s^2t^2 - 28st^2 + 13t^2 - 2t + 1),$$

$$f_2(s, t) = 1 - t.$$

Thus

$$D_1(s, t) = \bar{f}_1(s, t) = f_1(s, t),$$

$$D_2(s, t) = \bar{f}_2(s, t) = 1.$$

Therefore, both $L_1(s, t) = s^2t - 4st + 2t + 1 = 0$ and $L_2(s, t) = s^4t^2 - 8s^3t^2 + 23s^2t^2 - 28st^2 + 13t^2 - 2t + 1 = 0$ are loci for order 2 self-intersection curves of $\mathbf{P}(s, t)$.

Note that $\mathbf{p}(s)$ has an axis point $\mathbf{Q} = (0, -1, -1, 1)$. Thus according to Theorem 3.3, the point \mathbf{Q} is an order 3 singular point. The parameter pairs corresponding to \mathbf{Q} are $(s_0, t_0) = (1, 1)$, $(s_1, t_1) = (2, 1)$ and $(s_2, t_2) = (3, 1)$.

Fig. 3 shows the image of the self-intersection curve of $\mathbf{P}(s, t)$. We find that the order 3 singular point \mathbf{Q} is, in fact, the intersection point of two different order 2 self-intersection curves.

Example 4.4. Consider the rational ruled surface

$$\mathbf{P}(s, t) = \mathbf{P}_0(s) + t\mathbf{P}_1(s),$$

with

$$\mathbf{P}_0(s) = (21s^2 - 175s^3 + 385s^4 - 315s^5 + 77s^6 + 7s^7, -21s^2 + 35s^3 + 21s^5 - 84s^6 + 49s^7, -35s^4 + 126s^5 - 147s^6 + 56s^7, 1),$$

$$\mathbf{P}_1(s) = (1, 1, 1, 0).$$

The first two elements of a μ -basis for $\mathbf{P}(s, t)$ are

$$\mathbf{p}(s) = (196984s^3 - 1388498s^2 + 37537s + 565487, -1378888s^3 + 3612982s^2 - 3904529s + 307258, 1181904s^3 - 2224484s^2 + 3866992s - 872745, 5422809s^3 - 5422809s^2),$$

$$\mathbf{q}(s) = (37974s^3 - 263587s^2 - 19697s + 98492, -265818s^3 + 667915s^2 - 690762s, 227844s^3 - 404328s^2 + 710459s - 98492, -1075403s^4 + 3143735s^3 - 2068332s^2).$$



Fig. 4. Rational ruled surface with order 2 and order 3 self-intersection curves.

Computing the principal subresultant coefficients yields

$$D_1(s, t) = H(s, t), \quad D_2(s, t) = s^2(s - 1),$$

where $H(s, t) = 0$ is a high degree polynomial in s and t .

Therefore, $H(s, t) = 0$ is the locus for an order 2 self-intersection curve. The loci $s - 1 = 0$ and $s = 0$ corresponds to the same self-intersection curve of order 3. Fig. 4 shows both the order 2 and order 3 self-intersection curves of $\mathbf{P}(s, t)$. Notice that one part of the ruled surface intersects another part of the ruled surface tangentially along the order 3 self-intersection curve.

Moreover, the μ -basis elements $\mathbf{p}(s)$ and $\mathbf{q}(s)$ are both axial, and their axes are both the point $(1, 1, 1, 0)$ at infinity. Thus according to Theorem 3.3, the point $(1, 1, 1, 0)$ is a singular point of order 7. In fact, the ruled surface $\mathbf{P}(s, t)$ is a cone with vertex at $(1, 1, 1, 0)$.

5. Implicit equations of the self-intersection curves

Let $L(s, t) = 0$ be a parametric locus of the self-intersection curve C of $\mathbf{P}(s, t)$. Denote the corresponding ideal of C by $\mathbf{I}(C)$. Then $\mathbf{I}(C)$ is the radical ideal of

$$\mathbf{I}' = \langle d(s, t)x - a(s, t), d(s, t)y - b(s, t), d(s, t)z - c(s, t), L(s, t) \rangle \cap \mathbf{R}[x, y, z].$$

Now compute the Gröbner basis of the ideal $\sqrt{\mathbf{I}'}$. The elements in the Gröbner basis give the implicit equations of the self-intersection curve.

Example 5.1. Consider again Example 4.1. Let \mathbf{I}' be defined as above. Compute the Gröbner basis of $\sqrt{\mathbf{I}'}$ with respect to the order $t > s > x > y > z$. The polynomials involving x, y, z in the Gröbner basis are

$$\begin{aligned} F_1(x, y, z) = & 24416583y^2z^3 + 5439513z^4y + 77699340z^2y^3 + 2179460z^5 \\ & + 109796803y^4z + 58235817y^5 - 149004551y^4 - 289590703y^3z \\ & - 84276941z^2y^2 - 28086775yz^3 - 7325794z^4 + 159753123y^3 \\ & + 211689640zy^2 + 32805127yz^2 + 15353408z^3 - 79397830y^2 \\ & - 52968724yz - 10730042z^2 + 16487682y + 6115380z - 1355372. \end{aligned}$$

$$\begin{aligned} F_2(x, y, z) = & 24416583y^2z^3 + 5439513z^4y + 77699340z^2y^3 + 2179460z^5 \\ & + 109796803y^4z + 58235817y^5 - 149004551y^4 - 289590703y^3z \\ & - 84276941z^2y^2 - 28086775yz^3 - 7325794z^4 + 159753123y^3 \\ & + 211689640zy^2 + 32805127yz^2 + 15353408z^3 - 79397830y^2 \\ & - 52968724yz - 10730042z^2 + 16487682y + 6115380z - 1355372. \end{aligned}$$

The implicit representation for the self-intersection curve is given by

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases}$$

6. Conclusion

This paper presents an approach to computing the singular locus (i.e., self-intersection curves and isolated singular points) of rational ruled surfaces. By computing the principal subresultant coefficients of the μ -basis for a rational surface, we can generate the parametric loci of the self-intersection curves as well as isolated singular points. Examples are provided to illustrate the method.

It would be interesting to generalize this method to compute the singular locus of general rational surfaces or algebraic surfaces. However, currently there are no complete results about μ -bases for these general surfaces. So at present it is not possible for us to use μ -basis techniques to compute the singularities of these surfaces. Moreover, it is only because the μ -basis elements of a rational ruled surface are polynomials that subresultant technique can be applied. Thus even generalizing our current method to compute singularities of general (non-rational) ruled surface is also a problem worthy of further study.

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