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Zeros of univariate interval polynomials

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Abstract

Polynomials with perturbed coefficients, which can be regarded as interval polynomials, are very common in the area of scientific computing due to floating point operations in a computer environment. In this paper, the zeros of interval polynomials are investigated. We show that, for a degree n interval polynomial, the number of interval zeros is at most n and the number of complex block zeros is exactly n if multiplicities are counted. The boundaries of complex block zeros on a complex plane are analyzed. Numeric algorithms to bound interval zeros and complex block zeros are presented.

Keywords: Interval polynomial; Interval zero; Complex block zero; Boundary polynomial

1. Introduction

Polynomials with perturbed coefficients are common in various areas of science and engineering applications due to floating point computing in computer environments. Manipulating such kind of polynomials, for example, computing the zeros of these polynomials is a very important problem in practical applications. Because the zeros of a polynomial are very sensitive to the coefficients of the polynomial, the stability of numerical methods is a very challenging problem. In this paper, we regard a polynomial with perturbed coefficients as an interval polynomial, i.e., a polynomial with interval coefficients, and study the properties and computation of the zeros of interval polynomials.

There have been a few literatures focusing on the zeros of interval polynomials. In [7], Hansen and Walster studied the bounds for the real zeros of univariate interval polynomials. The paper mainly focuses on the real zeros of interval polynomials of degree 2–4. In [5], Ferreira et al. discussed the distribution of the complex zeros of interval polynomials of degree 2–4. They further explored the real zeros of a special class of multivariate interval polynomials in [6]. In [8], the maximal modulus of the zeros of interval polynomials were investigated. In this paper, we will study the properties and computation of the real and complex zeros of an arbitrary interval polynomials.

The organization of the paper is as follows. In Section 2, some basic definitions and propositions are presented. In Section 3, we show that the number of real interval zeros of an interval polynomial is at most n, where n is the degree of the interval polynomial. In Section 4, we prove the *Fundamental theorem* for interval polynomials, i.e., the number of complex block zeros of an interval polynomial is exactly the degree of the interval polynomial if multiplicities are counted. A full description of the boundaries of the complex block zeros is also presented. In the last section, numerical methods to bound the real interval zeros and the complex block zeros are given. Some examples are also illustrated.

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2. Interval polynomials and zero sets

An *interval polynomial* of degree n is a polynomial whose coefficients are intervals:

$$[f](x) := \sum_{i=0}^{n}[a_i, b_i]x^i := \left\{\sum_{i=0}^{n} f_i\, x^i \;:\; f_i \in [a_i, b_i],\; i = 0, 1, \ldots, n\right\}, \tag{1}$$

where $[a_i, b_i]$, $i = 0, 1, \ldots, n$ are bounded closed intervals. $[a_n, b_n]$ is called the *leading interval coefficient* of $[f](x)$. In general, we require $[a_n, b_n]$ does not contain zero.

The upper/lower bound functions of $[f](x)$ are defined by

$$\mathscr{U}f(x) := \begin{cases} \mathscr{U}f^{+}(x) = \sum_{i=0}^{n} b_i x^i, & x \geqslant 0, \\ \mathscr{U}f^{-}(x) = \sum_{0\leqslant 2i\leqslant n} b_{2i}x^{2i} + \sum_{0\leqslant 2i+1\leqslant n} a_{2i+1}x^{2i+1}, & x < 0 \end{cases} \tag{2}$$

$$\mathscr{L}f(x) := \begin{cases} \mathscr{L}f^{+}(x) = \sum_{i=0}^{n} a_i x^i, & x \geqslant 0, \\ \mathscr{L}f^{-}(x) = \sum_{0\leqslant 2i\leqslant n} a_{2i}x^{2i} + \sum_{0\leqslant 2i+1\leqslant n} b_{2i+1}x^{2i+1}, & x < 0. \end{cases} \tag{3}$$

Note that the bound functions are piecewise polynomials with joints at $x = 0$. It follows that, for all $x_0 \in \mathbb{R}$, interval $[f](x_0)$ equals to $[\mathscr{L}f(x_0), \mathscr{U}f(x_0)]$. As an example, Fig. 1 shows the upper bound function and the lower bound function for the interval polynomial $x^2 + [-2, 2]x + [1/2, 2]$.

The *zero set* of an interval polynomial $[f](x)$ is defined as

$$Z([f]; k) := \{x_0 \in k : \exists f(x) \in [f](x) \text{ s.t.} f(x_0) = 0\}, \tag{4}$$

where k is a number field. In the following, we will consider two cases where k is the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. In each case, $Z([f](x); k)$ is a closed set.

When working over $\mathbb{R}$, we have

$$Z([f]; \mathbb{R}) = \{x_0 \in \mathbb{R} : \mathscr{L}f(x_0) \leqslant 0 \leqslant \mathscr{U}f(x_0)\}. \tag{5}$$

In this case, the zero set of $[f](x)$ is actually composed of several closed intervals (except the intervals containing $+\infty$ or $-\infty$). We call each of these intervals an *interval zero* of the interval polynomial. It follows that the endpoints (except $\pm\infty$) of the interval zeros must be zeros of the upper/lower bound functions $\mathscr{L}f(x)$ or $\mathscr{U}f(x)$. In Section 3, we will study the number of interval zeros of a given interval polynomial.

When working over $\mathbb{C}$, the situation is much more complicated. $Z([f]; \mathbb{C})$ is separated into some connected parts in the complex plane. We call each connected part a *complex block zero* of the interval polynomial. In Section 4, the number and the boundaries of complex block zeros will be investigated.

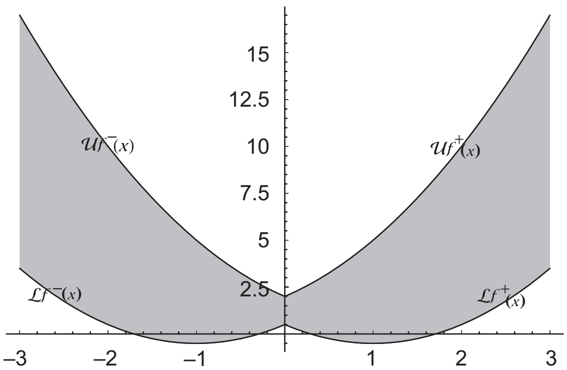

Fig. 1. Graph of $x^2 + [-2, 2]x + [1/2, 2]$.

3. Interval zeros

In this section, we will study the number of the interval zeros of an interval polynomial. We first make the following remarks for degenerate and infinite intervals for the convenience of description.

Remark 1. If $[a, a]$ is an interval zero of $[f](x)$, then the sum of the multiplicities of $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ at $x = a$ is at least 2. We consider a as a "multiple endpoint" of $[a, a]$. Hence every interval zero has two endpoints.

Here the multiplicity of $\mathscr{U}f(x)/\mathscr{L}f(x)$ at $x = 0$ is defined as the minimum multiplicity of $\mathscr{U}f^+(x)$ and $\mathscr{U}f^-(x)$ /($\mathscr{L}f^+(x)$ and $\mathscr{L}f^-(x)$) at $x = 0$.

Remark 2. If $(-\infty, a]$ is an interval zero of $[f](x)$, then the leading interval coefficient of $[f](x)$ must contain 0. Then there exists b such that $[b, +\infty)$ is also an interval zero of $[f](x)$. In this case, we count $(-\infty, a] \bigcup [b, +\infty)$ as one interval zero. In another word, if we require the leading interval coefficient does not contain 0, then $[f](x)$ cannot have infinite interval zeros.

Because the endpoints of interval zeros must be zeros of the upper or lower bound function, we can deduce the number of interval zeros of the interval polynomial if we know the number of zeros of its upper/lower bound functions. Since the upper/lower bound functions are piecewise polynomials, Descartes' rule of signs can be applied to determine the number of zeros of upper/lower bound functions. The following Definition 3 and Theorem 12 are well known in any textbooks of algebra.

Definition 3. Let $a_1, a_2, \ldots, a_n$ be a sequence of real numbers which does not contain 0. The number of sign changes of the sequence is defined as the number of negatives in $\{a_i a_{i+1} | 1 \leqslant i \leqslant n - 1\}$. If the sequence contains zeros, the number of sign changes is defined as that of the sequence by removing all the zeros from the original sequence.

Theorem 4 (*Descartes [1]*). *The number of positive zeros (counting multiplicities) of polynomial $f(x)$ does not exceed the number of the sign changes of its coefficients sequence.*

Lemma 5. *Let $[f](x) = \sum_{i=0}^{n} [a_i, b_i] x^i$ be an interval polynomial. Then the sum of the sign changes of the coefficients sequences of $\mathscr{L}f^+(x)$, $\mathscr{L}f^-(-x)$, $\mathscr{U}f^+(x)$ and $\mathscr{U}f^-(-x)$ does not exceed $2n$.*

Proof. The coefficients sequences of $\mathscr{L}f^+(x)$, $\mathscr{L}f^-(-x)$, $\mathscr{U}f^+(x)$ and $\mathscr{U}f^-(-x)$ are

$$\begin{aligned} &\mathscr{L}f^+(x) : a_0, a_1, a_2, a_3, \ldots, \\ &\mathscr{L}f^-(-x) : a_0, -b_1, a_2, -b_3, \ldots, \\ &\mathscr{U}f^+(x) : b_0, b_1, b_2, b_3, \ldots, \\ &\mathscr{U}f^-(-x) : b_0, -a_1, b_2, -a_3, \ldots . \end{aligned} \tag{6}$$

Firstly we consider the total number of the sign changes in the first two columns. We exchange all zero terms to nonzero real numbers because it does not decrease the number of sign changes. In this way we assume that all the coefficients are nonzero. The total number of the sign changes equals to the number of negatives of $a_0 a_1, -a_0 b_1, b_0 b_1, -b_0 a_1$. Because $a_0 a_1 \cdot (-a_0 b_1) \cdot b_0 b_1 \cdot (-b_0 a_1) = (a_0 a_1 b_0 b_1)^2 > 0$, the number of the sign changes is even. On the other hand, since $a_0 a_1 + (-a_0 b1) + b_0 b_1 + (-b_0 a_1) = (b_0 - a_0)(b_1 - a_1) > 0$, the four numbers cannot be all negative. So the total number of the sign changes is at most 2.

Similarly the total number of the sign changes in any two consecutive columns is at most 2, so the total sum of the sign changes is at most $2n$. □

Lemma 6. *Let $[f](x)$ be a degree n interval polynomial and $\mathscr{U}f(x)/\mathscr{L}f(x)$ its upper/lower bound functions. Then $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ in total have at most $2n$ zeros (counting multiplicities).*

Proof. When $\mathscr{L}f(0) \neq 0$ and $\mathscr{U}f(0) \neq 0$, the total number of the zeros of $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ equals to the total number of the positive zeros of $\mathscr{L}f^{+}(x)$, $\mathscr{L}f^{-}(-x)$, $\mathscr{U}f^{+}(x)$ and $\mathscr{U}f^{-}(-x)$. According to Lemma 5 and Theorem 4, the number is at most $2n$.

If $\mathscr{L}f(0)=0$, the coefficients sequences 6 change into

$$\begin{aligned}
&\mathscr{L}f^{+}(x): 0, a_1, a_2, a_3, \ldots,\\
&\mathscr{L}f^{-}(-x): 0, -b_1, a_2, -b_3, \ldots,\\
&\mathscr{U}f^{+}(x): b_0, b_1, b_2, b_3, \ldots,\\
&\mathscr{U}f^{-}(-x): b_0, -a_1, b_2, -a_3, \ldots.
\end{aligned} \tag{7}$$

According to the Proof of Lemma 5, it follows that the total number of the sign changes in the last n columns in (7) is at most $2(n-1)$. Because $b_0 \geqslant a_0 = 0$, the number of the sign changes in the first two columns equals to the number of negatives of b_1 and $-a_1$. Since $a_1 \leqslant b_1$, b_1 and $-a_1$ cannot be both negative. So the number of the sign changes in the first two columns is at most 1.

Similarly, we can prove that if $x=0$ is a m-multiple zero of $\mathscr{U}f(x)$ and a l-multiple zero of $\mathscr{L}f(x)$, respectively (without loss of generality, we assume $m \geqslant l$), the total number of sign changes in the first $m+1$ columns is at most $m-l$, and that in the last $n-m+1$ columns is at most $2(n-m)$. So the total number of non-zero roots of $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ is at most $2(n-m)+m-l=2n-m-l$. Thus, $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ in total have at most $2n-m-l+m+l=2n$ zeros. □

Theorem 7. *A degree n interval polynomial $[f](x)$ has at most n interval zeros.*

Proof. Assume $[f](x)$ has at least $n+1$ interval zeros. These intervals have at least $2(n+1)$ endpoints and every endpoint is a zero of $\mathscr{U}f(x)$ or $\mathscr{L}f(x)$. But $\mathscr{U}f(x)$ and $\mathscr{L}f(x)$ have in total at most $2n$ zeros by Lemma 6. A contradiction. □

Let's look at an example.

Example 8. Let $[f](x)=x^2+[-2,2]x+[1/2,2]$, whose figure is shown in Fig. 1. The coefficients sequences of the upper/lower functions are

$\mathscr{L}f^{+}(x)$:	1/2,	−2,	1,
$\mathscr{L}f^{-}(-x)$:	1/2,	−2,	1,
$\mathscr{U}f^{+}(x)$:	2,	2,	1,
$\mathscr{U}f^{-}(-x)$:	2,	2,	1.

Since the sum of the sign changes is 4, $[f](x)$ has at most two interval zeros.

4. Complex block zeros

In order to determine if a real number x_0 is in the zero set of $[f](x)$, we merely need to see if 0 is in the interval $[f](x_0)$. It is much harder to decide if a complex number z is a zero of $[f](x)$ or not. Suppose $z=r\,\mathrm{e}^{\mathrm{i}\theta}$, $[f](x)=\sum_{j=0}^{n}[a_j,b_j]x^j$ and $f(x)=\sum_{j=0}^{n} f_j x^j \in [f](x)$. Then

$$\begin{aligned}
\operatorname{Re}(f(z)) &= \sum_{j=0}^{n} r^j \cos(j\theta) f_j, \quad f_j \in [a_j,b_j],\\
\operatorname{Im}(f(z)) &= \sum_{j=0}^{n} r^j \sin(j\theta) f_j, \quad f_j \in [a_j,b_j].
\end{aligned} \tag{8}$$

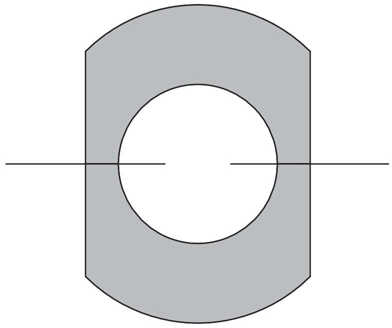

Fig. 2. The zero set of $z^2 + [-2, 2]z + [1/2, 2]$.

Obviously, z is a zero of $[f](x)$ if and only if the following linear system (9) (with f_j being unknowns) has a solution:

$$\begin{cases} \sum_{j=0}^{n} r^j \cos(j\theta) f_j = 0, \\ \sum_{j=0}^{n} r^j \sin(j\theta) f_j = 0, \end{cases} \quad a_j \leqslant f_j \leqslant b_j, \quad j = 0, 1, \ldots, n. \tag{9}$$

Let

$$\begin{aligned} S_m(z) &= \sin(m\theta)\mathrm{Re}([f](z)) - \cos(m\theta)\mathrm{Im}([f](z)) \\ &= \sum_{j \neq m} r^j \sin((m-j)\theta)[a_j, b_j]. \end{aligned} \tag{10}$$

Suppose that $\sin\theta \neq 0$ (i.e., z is not a real number), then z is a zero of $[f](x)$ if and only if for all $0 \leqslant m \leqslant n$, $0 \in S_m(z)$.

Remark 9. In Eq. (10) we assume that $\sin\theta \neq 0$, that is, z is not a real number. The real zeroes of $[f](z)$ can be calculated by the methods of previous sections. So in the following sections that discuss the complex zeroes we will always suppose that z is not a real number for the convenient to describe.

The set of the complex zeros of an interval polynomial could be very complicated. Fig. 2 shows the complex zero set of $z^2 + [-2, 2]z + [1/2, 2]$ in the complex plane. The points of the shadowed part and the solid curves are the complex zeros of the interval polynomial.

The two horizontal line segments are on the real axis, that is, they are real interval zeros, see Fig. 1. The boundaries of the shadowed part are enclosed by $x+1=0$, $x-1=0$, $x^2+y^2-1/2=0$, and $x^2+y^2-2=0$, where $z=x+\mathrm{i}\,y$, $x, y \in \mathbb{R}$.

The complex zero set is usually composed of several disconnected regions in the complex plane. Each region is called a *complex block zero*. In the following context, we will discuss the number and the boundaries of the complex block zeros of an interval polynomial.

4.1. Number of complex block zeros

In this subsection, we will prove that a degree n interval polynomial has exactly n complex block zeros (counting with multiplicities). The proof relies on the following basic result about the continuity of the zeros with respect to the coefficients of a univariate polynomial [2].

Theorem 10 (*Bhatia [2]*). *Suppose $a_j(t)(1 \leqslant j \leqslant n)$ are continuous complex-valued functions defined on an interval I. Then there exist continuous complex-valued functions $\alpha_1(t), \ldots, \alpha_n(t)$ which, for each $t \in I$, constitute the zeros of the univariate polynomial $x^n + a_1(t)x^{n-1} + \cdots + a_n(t)$.*

The following lemmas follows directly from the above theorem.

Lemma 11. *Suppose the leading interval coefficient of* $[f](z)$ *does not contain* 0 *and* Ω *is a complex block zero of* $[f](z)$. *Then for all* $f(z) \in [f](z)$, $f(z)$ *has a zero in* Ω.

Proof. Let $z_0 \in \Omega$. Then there exists $f_0(z) \in [f](z)$ such that $f_0(z_0) = 0$. Suppose the leading coefficients of $f(z)$ and $f_0(z)$ are a and b respectively, where a and b have the same sign. Consider

$$g(z; t) := \frac{tf(z) + (1-t)f_0(z)}{at + b(1-t)}, \quad t \in [0, 1]. \tag{11}$$

For all $t \in [0, 1]$, the denominator of $g(z, t)$ cannot be zero and the numerator is a polynomial which belongs to $[f](z)$. So any zero of $g(z; t)$ is a zero of $[f](z)$.

It follows that $g(z; 0) = f_0(z)/b$, $g(z; 1) = f(z)/a$. According to Theorem 10, there exists a continuous function $\alpha(t)$ such that $\alpha(t)$ is a zero of $g(z; t)$ and $\alpha(0) = z_0$. Namely $\alpha(t)$ is a zero of $[f](z)$ for any $t \in [0, 1]$. Since $\alpha(t)$ is continuous, $\alpha(t) \in \Omega$, $t \in [0, 1]$. Hence $f(z)$ has a zero $\alpha(1)$ which is in Ω. □

Similarly, we can prove that, for each complex block zero Ω, all the polynomials $f(z)$ in $[f](z)$ have the same number of zeros in Ω if multiplicities are counted. We define the number as the *multiplicity of the complex block zero* Ω of $[f](z)$. The following theorem follows immediately from the above argument.

Theorem 12. *Suppose* $[f](z)$ *is a degree n interval polynomial whose leading interval coefficient does not contain* 0. *Then* $[f](z)$ *has n complex block zeros* (*counting with multiplicities*).

As an example, the interval polynomial $x^2 + [-2, 2]x + [1/2, 2]$ has just one complex block zero (see Fig. 2) with multiplicity 2.

4.2. Boundaries of complex block zeros

In this subsection, we will give a full description of the boundaries of complex block zeros, and prove that the boundaries are consisted of some algebraic curves.

Definition 13. Suppose $[f](z) = \sum_{i=0}^{n}[a_i, b_i]z^i$. If $f(z) \in [f](z)$ and at least n coefficients of $f(z)$ come from either a_i or b_i, $i = 0, 1, \ldots, n$, then we call $f(z)$ a boundary polynomial of $[f](z)$.

Theorem 14. *Any boundary point of a complex block zero of* $[f](z)$ *must be a zero of some boundary polynomial.*

Proof. Suppose z_0 is on the boundary of a complex block zero of $[f](z)$ and $f_0(z) \in [f](z)$, $f_0(z_0) = 0$. Assume the coefficients of $f_0(z)$ have the maximum number of the endpoints of the interval coefficients $[a_i, b_i]$, $i = 0, 1, \ldots, n$ among $\{f(z) \in [f](z) : f(z_0) = 0\}$.

Assume z_0 is not a zero of any boundary polynomial. Then at least two of the coefficients of $f_0(z)$ come from the interior points of the intervals $[a_i, b_i]$, $i = 0, 1, \ldots, n$. Without loss of generality, assume the coefficients of z^m and z^k in $f_0(z)$ are not the endpoints of the interval coefficients. If $z_0^k = \alpha z_0^m$, $\alpha \in \mathbb{R}$, then z_0 is also a root of $f_t(z) := f_0(z) + t(z^k - \alpha z^m) = 0$. We can choose t such that $f_t(z) \in [f](z)$ and $f_t(z)$ has more coefficients coming from a_i, b_i than $f_0(z)$. This contradicts with our assumption. Thus, the ratio of z_0^k and z_0^m cannot be real.

Now we will show that, any complex number sufficiently close to z_0 is also a zero of $[f](x)$. This conflicts with the assumption that z_0 is a boundary point, and the theorem is thus proved.

Notice that, for any two real numbers ε_m and ε_k whose absolute values are sufficiently small, $f_0(z) + \varepsilon_k z^k + \varepsilon_m z^m \in [f](z)$. Let $z = z_0 + z_\varepsilon$, where z_ε is a complex number whose absolute values is sufficiently small. Then

$$\begin{aligned} f_0(z) + \varepsilon_k z^k + \varepsilon_m z^m &= f_0(z_0 + z_\varepsilon) + \varepsilon_k (z_0 + z_\varepsilon)^k + \varepsilon_m (z_0 + z_\varepsilon)^m \\ &= \varepsilon_k (z_0^k + \mathrm{O}(z_\varepsilon)) + \varepsilon_m (z_0^m + \mathrm{O}(z_\varepsilon)) + \mathrm{O}(z_\varepsilon). \end{aligned} \tag{12}$$

Since the ratio of z_0^k and z_0^m are not real, for any z_ε whose absolute values is sufficiently small, we can find real numbers ε_k and ε_m such that the above expression equals to zero, that is, any point in a neighborhood of z_0 is also a zero of $[f](z)$. This completes the proof of the theorem. □

Suppose $f(z)$ is a boundary polynomial whose coefficient of z^m is an interior point of the corresponding interval coefficient and the other coefficients are endpoints of the interval coefficients. Write $f(z)=\sum_{i=0}^n f_j z^j$, where f_j, $j\neq m$ are from $a_i, b_i, i=0,1,\ldots,n$. Suppose $z=r\,\mathrm{e}^{\mathrm{i}\theta}(\sin\,\theta\neq 0)$ is a zero of $f(z)$. Eliminate f_m from (9), we have

$$r^m\ \sin(m\theta)\,\mathrm{Re}(f(z)) - r^m\ \cos(m\theta)\,\mathrm{Im}(f(z)) = \sum_{j\neq m} f_j r^{m+j}\ \sin((m-j)\theta). \tag{13}$$

Substituting $\cos\ \theta = x/r$ and $\sin\ \theta = y/r$ into the right-hand side of the above expression, we can get a polynomial in x and y which represents an algebraic curve. Since m is an integer ranging from 0 to n and every $f_j(0\leqslant j\leqslant n,\ j\neq =m)$ has two choices (a_j or b_j), there are in total $(n+1)2^n$ boundary polynomials which correspond $(n+1)2^n$ algebraic curves. All the boundary points of the complex block zeros are on these algebraic curves. These curves divide the complex plane into disconnected parts. Each part is either a block zero of $[f](z)$ or does not contain any zero of $[f](z)$. Applying the method of cylindrical algebraic decomposition [4], we can compute an algebraic point in every part. We can detect if these points are the zeros of the interval polynomial or not from the linear system (9). If the algebraic point in some part is a zero of the interval polynomial, then the corresponding part is a block zero of $[f](z)$. Otherwise, it is not.

Remark 15. If $\sin\theta = 0$, the former analysis will fail. This means the method here cannot determine the real zeros of an interval polynomial. The specified methods for real zeros must be applied to determine real zeros.

5. Numerical method to bound the zero sets

The previous sections studied the properties of the zeros of interval polynomials. In this section, we present numerical methods to bound the zero sets of an interval polynomial, which is an essential step to compute the interval/block zeros robustly and accurately.

5.1. Bound interval zeros

In this subsection, we present a numerical algorithm to find a set of intervals which bound the interval zeros of a given interval polynomial $[f](x)$. Furthermore, the intervals converge to the exact interval zeros when the computing accuracy tends to infinity.

The following proposition is obvious and is the foundation of our algorithm.

Proposition 16. *Suppose* $[f](x)=\sum_{i=0}^n [a_i, b_i]\,x^i$, *and* $\mathscr{U}f(x)$ *and* $\mathscr{L}f(x)$ *are the upper/lower bound functions of* $[f](x)$. *Then*

(1) *Any real number in* $I := [a, b]$ *is not a zero of* $[f](x)$ *if* $0\notin [f](I)$.
(2) *Any real number in* $I := [a, b]$ *is a zero of* $[f](x)$ *if* $0\in [f]((a+b)/2), 0\notin \mathscr{U}f(I)$ *and* $0\notin \mathscr{L}f(I)$.

For a given interval I, there are three cases to consider. In case (1), we discard the interval. In case (2), we save the interval. Otherwise we have to bisection the interval and test each subinterval until the width of the interval is small enough (we call these intervals *undecided intervals*). In order not to lose any zeros, we can add the undecided intervals into the set of interval zeros. The initial interval which contains all the real zeros can be computed from the so-called Cauchy bound.

Theorem 17 (*Cauchy* [3], *Kurosh* [9]). *Suppose* $f(x)=x^m+a_{m-1}x^{m-1}+\cdots+a_0\in\mathbb{C}[x]$. $z_1,\ldots,z_m$ *are all the complex zeros of* $f(x)$. *Let*

$$r_0 = \max\{|z_1|,\ldots,|z_m|\}.$$

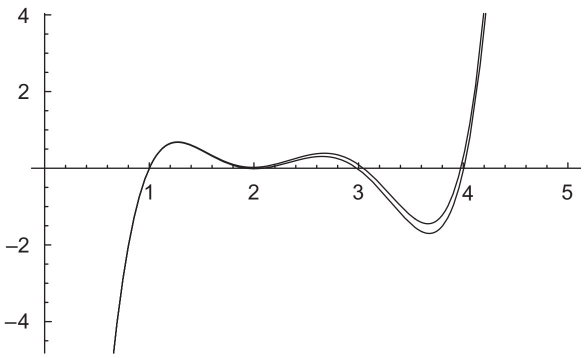

Fig. 3. Graph of the interval polynomial in Example 19.

Then

$$r_0 \leqslant 1 + \max\{|a_0|, \ldots, |a_{m-1}|\}.$$

Notice that we can take different forms of $\mathscr{L}f(x)$ and $\mathscr{U}f(x)$ to calculate $\mathscr{L}f(I)$ and $\mathscr{U}f(I)$. The center form is recommended. For more information about this topic, the reader is referred to [10]. The idea of center form also works to calculate $[f](I)$.

The algorithm is outlined below.

Input: An interval polynomial $[f](x) = \sum_{0\leqslant i\leqslant n}[a_i, b_i]x^i$.

Output: A set of intervals which contain all the interval zeros of $[f](x)$.

Step 1: Set the initial interval $I = [-r_0, r_0]$. Here $r_0 = 1 + \max\{|a_0|, |b_0|, \ldots, |a_n|, |b_n|\}$. Let S be an empty set.

Step 2: For the given interval I, calculate $[f](I)$. If $0 \notin [f](I)$, discard the interval and process the next interval. Otherwise go to Step 3.

Step 3: If $0 \in [f]((a+b)/2)$, $0 \notin \mathscr{U}f(I)$ and $0 \notin \mathscr{L}f(I)$, or if the width of I is less than a given tolerance ε, append I to the set S. Otherwise bisection I into two intervals at midpoint and for each subinterval, go to Step 2.

Step 4: Union all the neighboring intervals in S. That is, if $[a, b] \in S$ and $[b, c] \in S$, replace $[a, b]$, $[b, c]$ with $[a, c]$.

Remark 18. In the above algorithm, let $I_1^{\mathrm{U}}, I_2^{\mathrm{U}}, \ldots, I_m^{\mathrm{U}}$ be the set of output intervals, $I_1^{\mathrm{L}}, I_2^{\mathrm{L}}, \ldots, I_l^{\mathrm{L}}$ be the set of intervals produced by the algorithm without adding the undecided intervals (whose widths are less than ε), and $I_1, I_2, \ldots, I_k$ be the interval zeros of $[f](x)$. Then $m \leqslant k \leqslant l$. Thus if $l = m$, then the number of interval zeros is $m(=l)$. Furthermore, if we set $I^{\mathrm{U}} = \bigcup_{i=1}^{m} I_i^{\mathrm{U}}$, $I^{\mathrm{L}} = \bigcup_{i=1}^{l} I_i^{\mathrm{L}}$, and $Z([f], \mathbb{R}) = \bigcup_{i=1}^{k} I_i$, then $I^{\mathrm{L}} \subset Z([f], \mathbb{R}) \subset I^{\mathrm{U}}$.

We illustrate the algorithm with an example.

Example 19. Given an interval polynomial

$$[f](x) = [-48.001, -47.999] + [123.999, 124.001]x + [-120.000, -119.999]x^2$$
$$+ [54.999, 55.000]x^3 + [-12.000, -11.999]x^4 + [1, 1]x^5,$$

which is obtained by perturbing the coefficients of the polynomial $(x-1)(x-2)^2(x-3)(x-4)$ with a tolerance $\varepsilon = 10^{-5}$. Our algorithm produces four intervals [0.99933, 1.00051], [1.92990, 2.07955], [2.98434, 3.04656], [3.97613, 4.00571] which bound the four interval zeros of $[f](x)$, respectively. For each endpoint of the interval zeros, it corresponds to an undecided interval whose width is less than ε (so we have eight undecided intervals in total). Thus the difference between the width of each interval and the width of the corresponding interval zero is less than 2ε. Fig. 3 depicts the interval polynomial.

Our program is designed under Mathematica 5.0. We design the program without paying much little attention to the efficiency of the codes. The time consumed by the example is about 0.25 s in a computer with CPU 1.6 GHz and Memory 512M.

5.2. Complex block zeros

Section 4.2 gives a full description about the boundaries of the complex block zeros. However, it is computationally expensive to compute the boundaries accurately. In this subsection, we will provide a numerical algorithm to bound the complex block zeros by a set of sectors (that is, $[r_1, r_2] \cdot e^{i[\theta_1,\theta_2]}$) on the complex plane. Similar to interval zero case, we need conditions for which any complex number in a sector is (is not) a zero of $[f](x)$. It is not hard to tell that a sector does not contain any zero, but it is not easy to give conditions for which any complex number in a sector is a zero of the interval polynomial. So we will just give sufficient conditions for which any point on the four edges of a sector is a zero of $[f](x)$.

Proposition 20. *Suppose* $[f](z) = \sum_{i=0}^{n}[a_j, b_j]\, z^j$, $R = [r_1, r_2]$, $\Theta = [\theta_1, \theta_2]$ *and* $\sin\theta \neq 0$ *for any* $\theta \in [\theta_1, \theta_2]$. *Let* $z = r \cdot e^{i\theta}$ *and*

$$\mathrm{Re}(f(r, \theta)) = \sum_{j=0}^{n} r^j \cos(j\theta)[a_j, b_j],$$

$$\mathrm{Im}(f(r, \theta)) = \sum_{j=0}^{n} r^j \sin(j\theta)[a_j, b_j],$$

$$S_m(r, \theta) = \sum_{j\neq m} r^j \sin((m - j)\theta)[a_j, b_j], \quad 0 \leqslant m \leqslant n. \tag{14}$$

Then

(1) *Any complex number in* $[r_1, r_2]\cdot e^{i[\theta_1,\theta_2]}$ *is not a zero of* $[f](z)$ *if* $0 \notin \mathrm{Re}(f([r_1, r_2], [\theta_1, \theta_2]))$ *or* $0 \notin \mathrm{Im}(f([r_1, r_2], [\theta_1, \theta_2]))$.
(2) *Any complex number in* $[r_1, r_2] \cdot e^{i\theta_1}$ *is a zero of* $[f](z)$ *if any real number in* $[r_1, r_2]$ *is a zero of* $S_m(r, \theta_1)$ $(\forall\ 0 \leqslant m \leqslant n)$.
(3) *Any complex number in* $r_1 \cdot e^{i[\theta_1,\theta_2]}$ *is a zero of* $[f](z)$ *if any real number* $[\theta_1, \theta_2]$ *is a zero of* $S_m(r_1, \theta)$ $(\forall\ 0 \leqslant m \leqslant n)$.

$S_m(r, \theta_1)$ in (2) is an interval polynomial. So item (2) can be tested by Proposition 16. For (3), we have to employ a new method to determine if 0 is in $S_m(r_1, \theta)$ for any $\theta \in [\theta_1, \theta_2]$. Let

$$\begin{aligned}\mathscr{U}S_m(\theta) = &\sum_{2k\pi \leqslant (m-j)[\theta_1,\theta_2] \leqslant (2k+1)\pi} \sin((m - j)\theta) b_j r_1^j \\ &+ \sum_{(2k-1)\pi \leqslant (m-j)[\theta_1,\theta_2] \leqslant 2k\pi} \sin((m - j)\theta) a_j r_1^j \\ &+ \sum_{\text{other}\, j} \min \sin((m - j)[\theta_1, \theta_2]) r_1^j [a_j, b_j].\end{aligned} \tag{15}$$

$$\begin{aligned}\mathscr{L}S_m(\theta) = &\sum_{2k\pi \leqslant (m-j)[\theta_1,\theta_2] \leqslant (2k+1)\pi} \sin((m - j)\theta) a_j r_1^j \\ &+ \sum_{(2k-1)\pi \leqslant (m-j)[\theta_1,\theta_2] \leqslant 2k\pi} \sin((m - j)\theta) b_j r_1^j \\ &+ \sum_{\text{other}\, j} \max \sin((m - j)[\theta_1, \theta_2]) r_1^j [a_j, b_j].\end{aligned}$$

Here an interval $[a, b] \geqslant c$ means $a \geqslant c$ and $[a, b] \leqslant c$ means $b \leqslant c$. Then $[\mathscr{L}S_m(\theta), \mathscr{U}S_m(\theta)] \subset S_m(r_1, \theta)$ for any $\theta \in [\theta_1, \theta_2]$. So a sufficient condition for (3) is

$$\mathscr{U}S_m([\theta_1, \theta_2]) \geqslant 0 \quad \text{and} \quad \mathscr{L}S_m([\theta_1, \theta_2]) \leqslant 0. \tag{16}$$

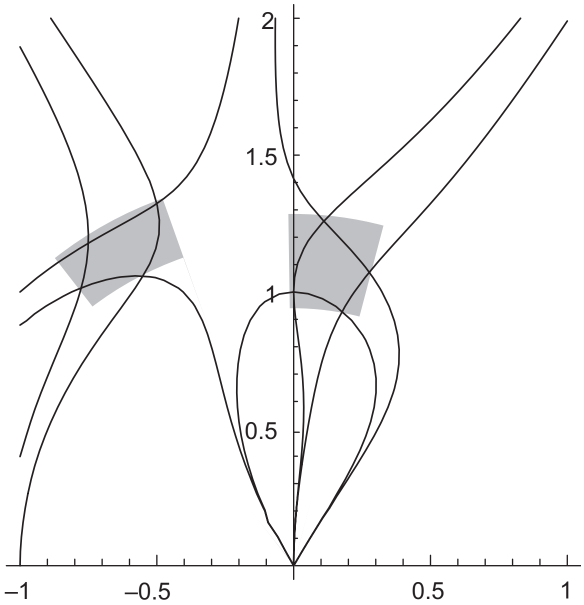

Fig. 4. Complex block zeros of $-2 + z + [-2, -1]z^2 + [1, 2]z^3 + z^5$.

According to Proposition 20, the algorithm to bound complex block zeros is similar to the algorithm to bound interval zeros in Section 5.1. For a given sector S, there are three cases to consider.

(1) If S satisfies the condition (1) in Proposition 20, then we discard the sector.
(2) If S satisfies the condition (2) or (3) in Proposition 20, then we save the sector.
(3) Otherwise we subdivide S, and for each sub-sector, repeat the previous processes until the sector is small enough.

The complex roots of real coefficient polynomials are conjugate, so we merely need to find the block zeros on the upper half plane. The initial sector can be choose as $[0, r]e^{i(0,\pi)}$, where r is obtained by Theorem 17. Here is an illustrating example.

Example 21. Let $[f](z) = -2 + z + [-2, -1]z^2 + [1, 2]z^3 + z^5$ be an interval polynomial. According to Section 4.2, the boundaries of its complex block zeros are on the following four algebraic curves:

$$-4x + x^2 - 2x^4 - 3x^6 + y^2 - 4x^2y^2 - 5x^4y^2 - 2y^4 - x^2y^4 + y^6 = 0,$$
$$-4x + x^2 - x^4 - 3x^6 + y^2 - 2x^2y^2 - 5x^4y^2 - y^4 - x^2y^4 + y^6 = 0,$$
$$-6x^2 + 2x^3 - x^4 - 2x^7 + 2y^2 + 2xy^2 - 2x^2y^2 - 6x^5y^2 - y^4 - 6x^3y^4 - 2xy^6 = 0,$$
$$-6x^2 + 2x^3 - 2x^4 - 2x^7 + 2y^2 + 2xy^2 - 4x^2y^2 - 6x^5y^2 - 2y^4 - 6x^3y^4 - 2xy^6 = 0.$$

It has four complex block zeros. Since they are symmetric about the real axis, we just show the half plane above the real axis in Fig. 4. The gray sectors are the outputs of our algorithm which bound the complex block zeros tightly. The example is implemented in the same computer as Example 20.

6. Conclusions

In this paper, the zeros of univariate interval polynomials are investigated. We prove that a univariate interval polynomial of degree n has at most n interval zeros, and has exactly n complex block zeros if multiplicities are counted. The boundaries of complex block zeros are explicitly described. Efficient numerical algorithms are developed to bound interval zeros and complex block zeros. Examples show that the proposed numerical algorithm generally produce good results.

It is interesting to generalize the results in this paper to multivariate interval polynomials. And how to improve the efficiency of the methods proposed in Section 5 is our task in the near future.

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