Dimensions of biquadratic and bicubic spline spaces over hierarchical T-meshes

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ABSTRACT

This paper discusses the dimensions of biquadratic C^1 spline spaces and bicubic C^2 spline spaces over hierarchical T-meshes using the smoothing cofactor-conformality method. We obtain the dimension formula of biquadratic C^1 spline spaces over hierarchical T-meshes in a concise way. In addition, we provide a dimension formula for bicubic C^2 spline spaces over hierarchical T-meshes with fewer restrictions than that in the previous literature. A dimension formula for bicubic C^2 spline spaces over a new type of hierarchical T-meshes is also provided.

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1. Introduction

Splines are a useful tool for representing functions and surface models, and Non-uniform Rational B-Splines (NURBS), which are defined on the tensor product meshes, are the most popular splines in the industry. However, due to the tensor product structure, local refinement of NURBS is impossible; furthermore, NURBS models generally contain a large number of superfluous control points. Therefore, many splines defined on T-meshes are developed.

There are four main types of splines that are defined on T-meshes. Hierarchical B-splines [1] are polynomial splines defined on hierarchical T-meshes. The definition is improved in [2]. More papers [3–5] are published to discuss the completeness and the partition of unity. T-splines [6,7] are rational splines defined on T-meshes, and they are a piecewise rational polynomial on every cell of the meshes. The blending functions of T-splines may be linearly dependent, which is adverse to the application in IGA. Therefore, analysis-suitable T-splines are introduced in [8]. Polynomial splines over hierarchical T-meshes (PHT-splines) [9] are developed directly from the spline spaces. The bases of PHT-splines are linearly independent and form a partition of unity. The main drawback of PHT-splines is that they are only C^1 continuous. LR-splines [10], which are defined on a special type of T-meshes, are also polynomial splines. LR-splines are defined with the help of the dimension formulae, and they are complete. However, the linear independence cannot be guaranteed. Among all of these splines, one fundamental problem is to understand the spline space of the piecewise polynomials of a given smoothness on a T-mesh, which is called the spline space over a T-mesh.

The spline space over a T-mesh $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is first introduced in [11], which is a bi-degree (m, n) piecewise polynomial spline space over a T-mesh $\mathscr{T}$ with smoothness order α and β in two directions. When $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, a dimension formula has been given in [11], and the bases have been constructed in [9]. However, if we relax the constraints of $m \geqslant 2\alpha + 1$ and $n \geqslant 2\beta + 1$, there is not a general dimension formula. In 2011, [12] discovered that the dimension of the associated spline space has instability over some particular T-meshes, i.e., the dimension is associated not only with the topological information of the T-mesh but also with the geometric information of the T-mesh. In addition, in [13],
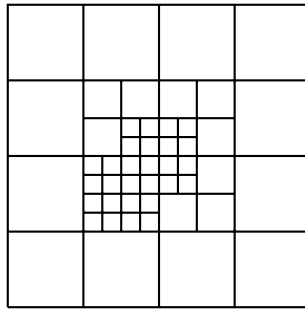

Fig. 1. A hierarchical T-mesh.

D. Berdinsky et al. give two more examples of $\mathbf{S}(5, 5, 3, 3, \mathscr{T})$ and $\mathbf{S}(4, 4, 2, 2, \mathscr{T})$ for the instability of dimensions. These results suggest that we should consider the dimension formula for the spline space over some special T-meshes. For this purpose, weighted T-meshes [14], diagonalizable T-meshes [15], and T-meshes for hierarchical B-spline [16], over which the dimensions are stable, are developed. The present paper is interested in hierarchical T-meshes, which have a nature tree structure and have existed in the finite element analysis community for a long time. For a hierarchical T-mesh, [17] derives a dimension formula for biquadratic C^1 spline spaces. And [18] provides the dimension formula for $\mathbf{S}(d, d, d-1, d-1, \mathscr{T})$ over a very special hierarchical T-mesh using the homological algebra technique.

In this paper, we present three results:

1. We provide a concise proof of the dimension formula of $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ over a hierarchical T-mesh.
2. We prove a dimension formula for $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$ over a hierarchical T-mesh with fewer restrictions than that in the previous literature.
3. We give a dimension formula for $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$ over a new type of hierarchical T-mesh that splits each cell into 3×3 parts during the local refinement.

[15,5,18] also discuss the dimensions of $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$. For [15,18], their conclusions are aimed at diagonalizable T-meshes or weighted T-meshes. In this paper, the T-meshes we considered may be not of these types. See Fig. 1 for an example. For [5], the authors present a new type of splines, which is called TDHB-splines. The idea of this paper is to construct the bases of TDHB-splines first, then to prove the completeness of the bases. That is to say, the dimension of the spline space has been obtained, which is the number of the bases. For $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$, their conclusion is our third result; moreover, they do not derive an explicit dimension formula.

The rest of the paper is organized as follows. In Section 2, we review the definitions and some results regarding T-meshes and spline spaces over T-meshes. In Section 3, the smoothing cofactor-conformality method is explored. In Section 4, we present some conclusions about homogeneous boundary conditions. We discuss the dimensions of $\mathbf{S}(2, 2, 1, 1, \mathscr{T})$ over hierarchical T-meshes in Section 5 and the dimensions of $\mathbf{S}(3, 3, 2, 2, \mathscr{T})$ over hierarchical T-meshes in Section 6. Section 7 ends the paper with conclusions and future work.

2. T-meshes and spline spaces

A T-mesh is a rectangular grid that allows T-junctions.

Definition 2.1 ([11,17])**.** Suppose $\mathscr{T}$ is a set of axis-aligned rectangles and the intersection of any two distinct rectangles in $\mathscr{T}$ either is empty or consists of points on the boundaries of the rectangles. Then, $\mathscr{T}$ is called a **T-mesh**. Furthermore, if the entire domain occupied by $\mathscr{T}$ is a rectangle, $\mathscr{T}$ is called a **regular T-mesh**. If some edges of $\mathscr{T}$ also form a T-mesh $\mathscr{T}'$, $\mathscr{T}'$ is called a **submesh** of $\mathscr{T}$.

In this paper, the T-meshes that we consider are regular, and we adopt the definitions of vertex, edge, and cell provided in [11]. T-meshes are allowed to have **T-junctions**, or **T-nodes**, which are the vertex of one rectangle that lies in the interior of an edge of another rectangle. A valence 4 vertex is called a **crossing-vertex**.

Vertices, edges and cells can be divided into two parts. If a vertex is on the boundary grid line of the T-mesh, it is called a **boundary vertex**. Otherwise, it is called an **interior vertex**. There are two types of interior vertices: T-junctions and crossing-vertices. If an edge is on the boundary of the T-mesh, it is called a **boundary edge**; otherwise, it is called an **interior edge**. A cell is called an **interior cell** if all its edges are interior edges; otherwise, it is called a **boundary cell**. An **l-edge** is a line segment that consists of several edges. It is the longest possible line segment, the interior edges of which are connected and the two end points are T-junctions or boundary vertices. If an l-edge is comprised of some boundary edges, the l-edge is called a **boundary l-edge**; otherwise, it is called an **interior l-edge**. There are three types of interior l-edges. If the two end points of an interior l-edge are both T-junctions, the l-edge is called a **T l-edge**. If the two end points of an interior l-edge are both boundary vertices, the l-edge is called a **cross-cut**. If one end point is a boundary vertex and the other one is an interior vertex, the l-edge is called a **ray**.
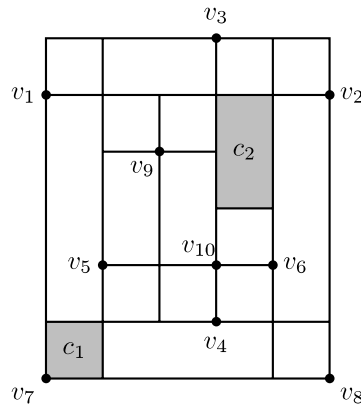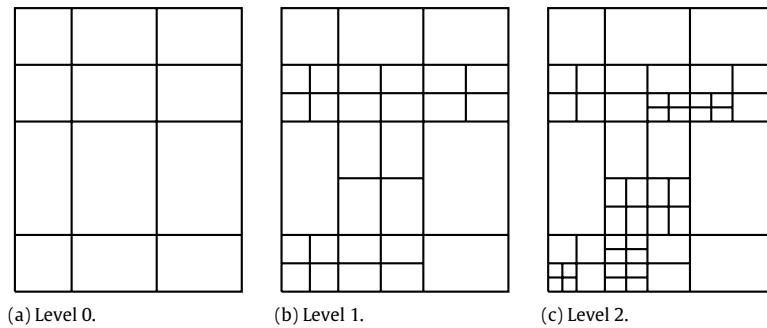
Fig. 2. A regular T-mesh.

(a) Level 0.　　　　　(b) Level 1.　　　　　(c) Level 2.

Fig. 3. A hierarchical T-mesh.

In Fig. 2, v_1, v_2, v_3, v_7 and v_8 are boundary vertices, v_9 and v_{10} are crossing-vertices, and v_4, v_5 and v_6 are T-junctions; c_1 is a boundary cell, and c_2 is an interior cell; the l-edge between v_1 and v_2 is a cross-cut, the l-edge between v_3 and v_4 is a ray, the l-edge between v_5 and v_6 is a T l-edge, and the l-edge between v_7 and v_8 is a boundary l-edge.

2.1. Hierarchical T-meshes

A hierarchical T-mesh [9] is a special type of T-mesh that has a natural level structure. It is defined recursively. Generally, we start from a tensor-product mesh (level 0), the elements (vertices, edges and cells) of which are called level 0 elements. Then, some cells at level k are each divided into $m \times n$ subcells equally, where the new vertices, the new edges and the new cells are of level $k + 1$. The resulting T-mesh is called **a hierarchical T-mesh of** $m \times n$ **division**. In the following, the default division of a hierarchical T-mesh is 2×2. Fig. 3 illustrates the process of generating a hierarchical T-mesh. To emphasize the level structure of a hierarchical T-mesh $\mathscr{T}$, in certain cases, we denote the T-mesh of level k as $\mathscr{T}^k$. The maximal level number that appears is called the **level** of the hierarchical T-mesh, and is denoted by lev($\mathscr{T}$).

For a hierarchical T-mesh $\mathscr{T}$ of level n, we use T_k to denote the set of all of the level k l-edges and the level k vertices of $\mathscr{T}$, and T_k^o to denote the set of all of the level k l-edges and all of the vertices on these level k l-edges in $\mathscr{T}$. For any l-edge l of level j, we use $N(l)$ to denote the number of the cells of $\mathscr{T}^{j-1}$ that l crosses.

Fig. 4 is an example. $\mathscr{T}$ is a hierarchical T-mesh and lev($\mathscr{T}$) = 2. The level 1 l-edges are labeled with dashed lines, the level 1 vertices are labeled with "•", and the level 2 vertices are labeled with "○". Here, $N(l_1) = 2$ and $N(l_2) = 1$ for the two l-edges l_1 (between v_1 and v_2) and l_2 (between v_3 and v_4).

2.2. Spline spaces over T-meshes

Given a T-mesh $\mathscr{T}$, we use $\mathscr{F}$ to denote all of the cells in $\mathscr{T}$ and Ω to denote the region occupied by the cells in $\mathscr{T}$. In [11], the following spline space definition is proposed:

$$\mathbf{S}(m, n, \alpha, \beta, \mathscr{T}) := \{f(x, y) \in C^{\alpha,\beta}(\Omega) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}\},$$

where $\mathbb{P}_{mn}$ is the space of the polynomials with bi-degree (m, n) and $C^{\alpha,\beta}$ is the space consisting of all of the bivariate functions continuous in Ω with order α along the x-direction and with order β along the y-direction. It is obvious that $\mathbf{S}(m, n, \alpha, \beta, \mathscr{T})$ is a linear space. In this paper, we only discuss $\mathbf{S}(d, d, d - 1, d - 1, \mathscr{T})$, which is denoted as $\mathbf{S}^d(\mathscr{T})$ for convenience.
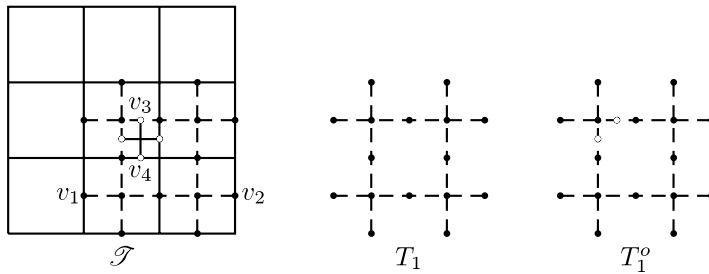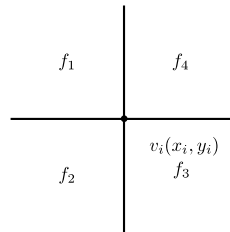
Fig. 4. T_k and T_k^o.

Fig. 5. Smoothing conditions in adjacent cells.

3. Smoothing cofactor-conformality method and conformality vector spaces

In this section, we review the smooth cofactor-conformality method introduced in [19,20] for computing the dimensions of spline spaces over T-meshes [21,15,18].

Referring to Fig. 5, let $f_j(x, y)$, $j = 1, 2, 3, 4$ be the bivariate polynomials surrounding the interior vertex $v_i(x_i, y_i)$ (if the vertex v_i is a T-junction, some of the polynomials are identical). Then there exist a constant $\gamma_i \in \mathbb{R}$ and two polynomials $a(y) \in \mathbb{P}_d[y]$, $b(x) \in \mathbb{P}_d[x]$, such that

$$f_1(x, y) - f_2(x, y) = b(x)(y - y_i)^d,$$
$$f_3(x, y) - f_2(x, y) = a(y)(x - x_i)^d,$$
$$f_4(x, y) - f_2(x, y) = a(y)(x - x_i)^d + b(x)(y - y_i)^d + \gamma_i(x - x_i)^d(y - y_i)^d,$$

where $a(y)$ and $b(x)$ are the edge co-factors associated with the corresponding edges, and γ_i is the vertex co-factor associated with vertex v_i.

There are other constraints for the T l-edges of the T-mesh. For example, given a horizontal T l-edge l_j with r vertices $v_{j_1}, v_{j_2}, \ldots, v_{j_r}$, let the x-coordinate of v_{j_i} be x_{j_i}, and the vertex co-factors of v_{j_i} be γ_{j_i}. Then we have

$$\sum_{i=1}^{r} \gamma_{j_i}(x - x_{j_i})^d = 0.$$

This equation is equivalent to a linear system denoted by $\mathscr{S}_{l_j} = 0$:

$$\begin{cases} \sum_{i=1}^{r} \gamma_{j_i} = 0, \\ \sum_{i=1}^{r} \gamma_{j_i} x_{j_i} = 0, \\ \ldots, \\ \sum_{i=1}^{r} \gamma_{j_i} x_{j_i}^d = 0. \end{cases} \tag{1}$$

Similarly, we can derive the linear system for a vertical T l-edge.

As in [18], we can define the **conformality vector space** for a set of T l-edges as follows.

Definition 3.1. Suppose L is a set of T l-edges: $L = \{l_1, l_2, \ldots, l_n : l_i$ is a T l-edge, $1 \leqslant i \leqslant n\}$, $v_1, v_2, \ldots, v_m$ are all of the vertices on $l_1, l_2, \ldots, l_n$, and γ_j is the co-factor of v_j. Then the **conformality vector space** $W[L]$ of L is defined by

$$W[L] := \{\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m)^T : \mathscr{S}_{l_i} = 0, \ 0 \leqslant i \leqslant n\},$$

where $\mathscr{S}_{l_i} = 0$ is the linear system as Eq. (1) associated with the l-edge l_i. For some predefined order of the vertex co-factors and the l-edges, the coefficient matrix for the homogeneous system of $W[L]$ is called the **conformality matrix** of L.

In particular, if L only contains one T l-edge l_1 with V_0 vertices, $\dim W[l_1] = (V_0 - d - 1)_+ := \max(0, V_0 - d - 1)$. In the following, we also call such an l-edge an l-edge of order $(V_0 - d - 1)_+$. When $(V_0 - d - 1)_+ = 0$, we say l_1 is a **trivial** l-edge.

For an l-edge of order 1, we have the following property.

Proposition 3.2. *In a given T-mesh $\mathscr{T}$, suppose l is a T l-edge with $d + 2$ vertices $v_0, v_1, \ldots, v_{d+1}$ (from one end point to the other end point) and the co-factor of v_i is γ_i. Then*

$$\gamma_{i+k} = (-1)^k \gamma_i \frac{\prod\limits_{j \neq i} d_{i,j}}{\prod\limits_{j \neq i+k} d_{i+k,j}}, \tag{2}$$

where $d_{i,j}$ is the distance between v_i and v_j, $k \geqslant 1$.

Proof. Suppose l is a horizontal T l-edge, and the x-coordinate of v_i is x_i. Then the linear system $\mathscr{S}_l = 0$ has the following form:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 \\ x_0 & x_1 & \cdots & x_i & \cdots & x_{i+k} & \cdots & x_{d+1} \\ \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ x_0^d & x_1^d & \cdots & x_i^d & \cdots & x_{i+k}^d & \cdots & x_{d+1}^d \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_{d+1} \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_0 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{d+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_0^d & \cdots & x_{i-1}^d & x_{i+1}^d & \cdots & x_{d+1}^d \end{pmatrix} \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{i-1} \\ \gamma_{i+1} \\ \vdots \\ \gamma_{d+1} \end{pmatrix} = -\gamma_i \begin{pmatrix} 1 \\ x_i \\ \vdots \\ x_i^d \end{pmatrix}.$$

According to Cramer's Rule of the linear equation system, we know that $\gamma_{i+k} = -\gamma_i \cdot \frac{\Delta'}{\Delta}$. Here

$$\Delta = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \\ x_0 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{d+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ x_0^d & \cdots & x_{i-1}^d & x_{i+1}^d & \cdots & x_{d+1}^d \end{vmatrix},$$

$$\Delta' = \begin{vmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ x_0 & \cdots & x_{i-1} & x_{i+1} & \cdots & x_{i+k-1} & x_i & x_{i+k+1} & \cdots & x_{d+1} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ x_0^d & \cdots & x_{i-1}^d & x_{i+1}^d & \cdots & x_{i+k-1}^d & x_i^d & x_{i+k+1}^d & \cdots & x_{d+1}^d \end{vmatrix}.$$

Δ and Δ' are both Vandermonde determinants. Hence, it is easy to verify that Eq. (2) holds. $\square$

Referring to [21], we have the following lemma for the dimensions of the spline spaces over T-meshes.

Lemma 3.3. *Given a T-mesh $\mathscr{T}$, suppose it has n_c cross-cuts and n_v interior vertices and M is the conformality matrix of all of the T l-edges. Then, it follows that*

$$\dim \mathbf{S}^d(\mathscr{T}) = (d + 1)^2 + n_c(d + 1) + n_v - \operatorname{rank} M.$$

Remark 3.4. For a given T-mesh $\mathscr{T}$, we use $L(\mathscr{T})$ to denote the set of all of the T l-edges in $\mathscr{T}$. Then, Lemma 3.3 states that

$$\dim \mathbf{S}^d(\mathscr{T}) = \dim W[L(\mathscr{T})] + \dim \mathbf{S}^d(\mathscr{T} \setminus L(\mathscr{T})). \tag{3}$$

Here, $\mathscr{T} \setminus L(\mathscr{T})$ is the mesh obtained by deleting $L(\mathscr{T})$ from $\mathscr{T}$. See Fig. 6 for an example.

Lemma 3.5. *For a given T-mesh $\mathscr{T}$, suppose $L(\mathscr{T})$ is the set of all of the T l-edges in $\mathscr{T}$, and L is a subset of $L(\mathscr{T})$. Then, the following statements are equivalent:*

1. $\dim W[L(\mathscr{T})] = \dim W[L] + \dim W[L(\mathscr{T} \setminus L)]$.
2. *The projection mapping $\pi : W[L(\mathscr{T})] \to W[L]$ is surjective.*
3. *Suppose the conformality matrix of L is A, the conformality matrix of $L(\mathscr{T} \setminus L)$ is B, and the conformality matrix of $T(\mathscr{T})$ is M. Then, $\operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} B$.*
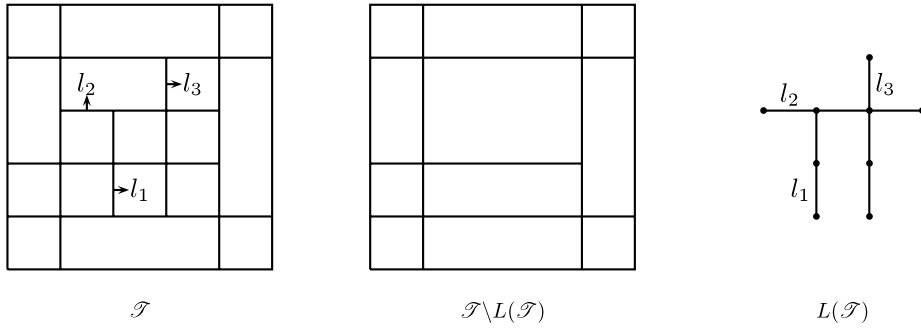
Fig. 6. Decompose a T-mesh into a semi-cross mesh and the set of all of the T l-edges.

Proof. Suppose $L(\mathcal{T})$ has V interior vertices and L has V' vertices. The matrix M has the form $\begin{pmatrix} A & \\ C & B \end{pmatrix}$ if we define the order of the vertex co-factors and the l-edges of $L(\mathcal{T})$ with that of L in front of the others.

1. $(1) \Leftrightarrow (2)$:
   For the projection mapping $\pi : W[L(\mathcal{T})] \to W[L]$, it is apparent that $\text{Ker}\pi = W[L(\mathcal{T} \setminus L)]$. Because $\dim W[L(\mathcal{T})] = \dim \text{Ker}\pi + \dim \text{Im}\pi$, (1) is equivalent to (2).

2. $(3) \Rightarrow (1)$:
   Because $\text{rank } M = \text{rank } A + \text{rank } B$, we have

   $$\dim W[L(\mathcal{T})] = V - \text{rank } M$$
   $$= (V - V') - \text{rank } B + (V' - \text{rank } A)$$
   $$= \dim W[L(\mathcal{T} \setminus L)] + \dim W[L].$$

3. $(1) \Rightarrow (3)$:
   Suppose $\text{rank } M \neq \text{rank } A + \text{rank } B$. Then, $\text{rank } M > \text{rank } A + \text{rank } B$. Thus,

   $$\dim W[L(\mathcal{T})] = V - \text{rank } M$$
   $$< (V - V') - \text{rank } B + (V' - \text{rank } A)$$
   $$= \dim W[L(\mathcal{T} \setminus L)] + \dim W[L],$$

   a contradiction. □

From the proof of Lemma 3.5, we know that

$$\dim W[L(\mathcal{T})] \leqslant \dim W[L(\mathcal{T} \setminus L)] + \dim W[L]. \tag{4}$$

Conversely, we have

$$\dim W[L(\mathcal{T})] \geqslant \dim W[L(\mathcal{T} \setminus L)]. \tag{5}$$

This is because $\text{rank } M \leqslant \text{rank } \begin{pmatrix} A \\ C \end{pmatrix} + \text{rank } (B) \leqslant V' + \text{rank } B$,

$$\dim W[L(\mathcal{T})] = V - \text{rank } M$$
$$\geqslant (V - V') - \text{rank } B$$
$$= \dim[L(\mathcal{T} \setminus L)].$$

Lemma 3.6. *For a given T-mesh $\mathcal{T}_1$, $L_1 = \{l_1, l_2, \ldots, l_n\}$ is a subset of $L(\mathcal{T}_1)$. We extend l_i to l_i' and get a new mesh $\mathcal{T}_2$, such that l_i' is still a T l-edge of $\mathcal{T}_2$. Thus, $L_2 = \{l_1', l_2', \ldots, l_n'\}$ is a subset of $L(\mathcal{T}_2)$. As in Fig. 7, $L_1 = \{l_1, l_3\}$, $L_2 = \{l_1', l_3'\}$. If $\dim W[L(\mathcal{T}_2)] = \dim W[L_2] + \dim W[L(\mathcal{T}_2 \setminus L_2)]$, then $\dim W[L(\mathcal{T}_1)] = \dim W[L_1] + \dim W[L(\mathcal{T}_1 \setminus L_1)]$.*

Proof. Suppose the conformality matrices of $L_1, L_2, L(\mathcal{T}_1)$ and $L(\mathcal{T}_2)$ are A_1, A_2, M_1 and M_2, respectively. We can arrange the order of the l-edges and the vertex co-factors such that $M_1 = \begin{pmatrix} A_1 & \\ C_1 & B \end{pmatrix}$, and $M_2 = \begin{pmatrix} A_2 & \\ C_2 & B \end{pmatrix}$. A_2 is obtained by adding some columns to A_1 on the vertices added from L_1 to L_2, and so is C_2.

Because $\dim W[L(\mathcal{T}_2)] = \dim W[L_2] + \dim W[L(\mathcal{T}_2 \setminus L_2)]$, it follows that the projection mapping $\pi' : W[L(\mathcal{T}_2)] \to W[L_2]$ is surjective, i.e., for any x that satisfies $A_2 x = 0$, the solution of $By = -C_2 x$ exists. Then, for any X_1 that satisfies $A_1 X_1 = 0$, we add 0 entries to X_1 on the vertices added from L_1 to L_2. Suppose we obtain X_2 after adding. It is easy to verify that $A_1 X_1 = A_2 X_2$ and $C_1 X_1 = C_2 X_2$. Thus, $A_1 X_1 = 0 \Rightarrow A_2 X_2 = 0 \Rightarrow$ the solution of $BY = -C_2 X_2 = -C_1 X_1$ exists. That is, the projection mapping $\pi : W[L(\mathcal{T}_1)] \to W[L_1]$ is surjective, which proves the lemma. □
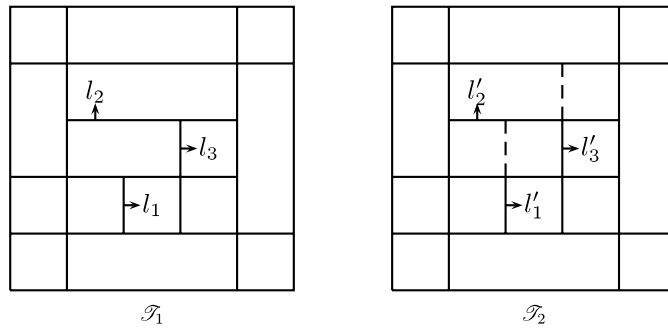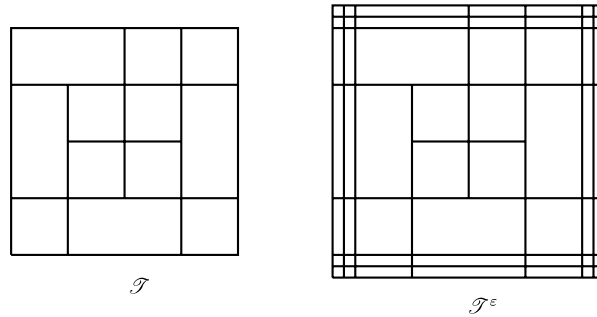
Fig. 7. Figure for Lemma 3.6.

Fig. 8. A T-mesh $\mathcal{T}$ and its extended T-mesh $\mathcal{T}^\varepsilon$ for $\mathbf{S}^2(\mathcal{T})$.

Lemma 3.7. *For a given T-mesh $\mathcal{T}$, L is a subset of $L(\mathcal{T})$. Suppose $L(\mathcal{T} \setminus L)$ has V vertices and E l-edges. If $\dim W[L(\mathcal{T} \setminus L)] = V - (d+1)E$, then $\dim W[L(\mathcal{T})] = \dim W[L] + \dim W[L(\mathcal{T} \setminus L)]$.*

Proof. Suppose the conformality matrix of L is A, the conformality matrix of $L(\mathcal{T} \setminus L)$ is B, and the conformality matrix of $L(\mathcal{T})$ is M. Then, we can arrange the order of the vertex co-factors and the l-edges such that $M = \begin{pmatrix} A & \\ C & B \end{pmatrix}$.

Because $\mathrm{rank}\, B = V - \dim W[L(\mathcal{T} \setminus L)] = V - (V - (d+1)E) = (d+1)E$, it follows that B is of full row rank. We have $\mathrm{rank}\, M \leqslant \mathrm{rank}\, A + \mathrm{rank}\, \begin{pmatrix} C & B \end{pmatrix} \leqslant \mathrm{rank}\, A + (d+1)E = \mathrm{rank}\, A + \mathrm{rank}\, B$. Conversely, $\mathrm{rank}\, M \geqslant \mathrm{rank}\, A + \mathrm{rank}\, B$. Thus, $\mathrm{rank}\, M = \mathrm{rank}\, A + \mathrm{rank}\, B$, which is equivalent to $\dim W[L(\mathcal{T})] = \dim W[L] + \dim W[L(\mathcal{T} \setminus L)]$ according to Lemma 3.5. □

Using Lemma 3.7, we can obtain the following corollary.

Corollary 3.8. *Given a T-mesh $\mathcal{T}$, if there is an order of all of the T l-edges, say $l_1, l_2, \ldots, l_n$, such that $n_{l_i} \geqslant d + 1$, where n_{l_i} is the number of the vertices on l_i but not on l_j, $j = 1, 2, \ldots, i - 1$, then $\dim W[L(\mathcal{T})] = V - (d+1)n$, where V is the number of the vertices on all of these l-edges.*

Proof. Let $L = \{l_1, l_2, \ldots, l_{n-1}\}$. Then, $L(\mathcal{T} \setminus L)$ has n_{l_n} vertices. Because $n_{l_n} \geqslant d + 1$, $\dim W[L(\mathcal{T} \setminus L)] = (n_{l_n} - (d+1))_+ = n_{l_n} - (d+1)$. According to Lemma 3.7, we have $\dim W[L(\mathcal{T})] = \dim W[L] + \dim W[L(\mathcal{T} \setminus L)] = \dim W[L] + n_{l_n} - (d+1)$. $\dim W[L]$ can be analyzed as $\dim W[L(\mathcal{T})]$. Continuing this process, at last we have

$$\dim W[L(\mathcal{T})] = \sum_{i=1}^{n} (n_{l_i} - (d+1)) = V - (d+1)n. \quad \square$$

For any subset L of $L(\mathcal{T})$, if it has at least one such order as Corollary 3.8, we say L has a **reasonable order**.

4. Extended T-meshes and homogeneous boundary conditions

For a T-mesh $\mathcal{T}$ of $\mathbf{S}(m, n, m-1, n-1, \mathcal{T})$, the extended T-mesh $\mathcal{T}^\varepsilon$ is an enlarged T-mesh by copying each horizontal boundary l-edge of $\mathcal{T}$ n times, and each vertical boundary l-edge of $\mathcal{T}$ m times, and by extending all of the l-edges with an end point on the boundary of $\mathcal{T}$. Fig. 8 is an example of an extended T-mesh for $\mathbf{S}^2(\mathcal{T})$.

A spline space over a given T-mesh $\mathcal{T}$ with homogeneous boundary conditions (HBC for short) is defined by [17]

$$\bar{\mathbf{S}}(m, n, \alpha, \beta, \mathcal{T}) := \{f(x, y) \in C^{\alpha, \beta}(\mathbb{R}^2) : f(x, y)|_\phi \in \mathbb{P}_{mn}, \forall \phi \in \mathscr{F}, \text{ and } f|_{\mathbb{R}^2 \setminus \Omega} \equiv 0\},$$

where $\mathbb{P}_{mn}$, $\mathscr{F}$, $C^{\alpha, \beta}$ are defined as before. One important observation in [17] is that the two spline spaces $\mathbf{S}(m, n, m-1, n-1, \mathcal{T})$ and $\bar{\mathbf{S}}(m, n, m-1, n-1, \mathcal{T}^\varepsilon)$ are closely related.
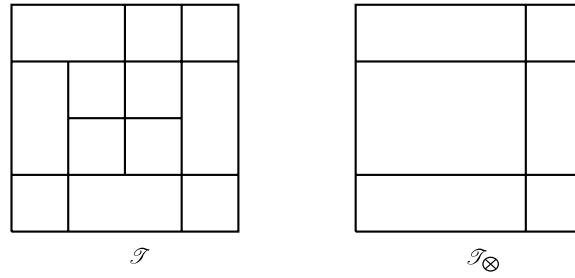
Fig. 9. An example T-mesh for Lemma 4.2.

Theorem 4.1 ([17]). *Given a T-mesh $\mathscr{T}$, let $\mathscr{T}^\varepsilon$ be the extended T-mesh associated with $\mathbf{S}(m, n, m-1, n-1, \mathscr{T})$. Then*

$$\mathbf{S}(m, n, m-1, n-1, \mathscr{T}) = \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon)|_\Omega,$$
$$\dim \mathbf{S}(m, n, m-1, n-1, \mathscr{T}) = \dim \bar{\mathbf{S}}(m, n, m-1, n-1, \mathscr{T}^\varepsilon).$$

In the T-mesh $\mathscr{T}$ of $\bar{\mathbf{S}}^d(\mathscr{T})$, for any l-edge l, the co-factors of all of the vertices on l satisfy the linear system $\mathscr{S}_l = 0$ as Eq. (1), which means we have the similar conformality vector spaces $W[\mathscr{T}]$ for all of the l-edges. HBC unifies all types of l-edges, which will bring convenience to the computation of dimensions. From [18], we know that

$$\bar{\mathbf{S}}^d(\mathscr{T}) \cong W[\mathscr{T}].$$

Lemma 4.2. *Given a T-mesh $\mathscr{T}$, we can construct a tensor-product mesh $\mathscr{T}_\otimes$ with all of the boundary l-edges and some of the cross-cuts. See Fig. 9 for an example. Let L be the set of all of the l-edges of $\mathscr{T}$ that are not contained in $\mathscr{T}_\otimes$. If $\mathscr{T}_\otimes$ has at least $d+1$ horizontal l-edges and $d+1$ vertical l-edges, then*

$$\dim \bar{\mathbf{S}}^d(\mathscr{T}) = \dim \bar{\mathbf{S}}^d(\mathscr{T}_\otimes) + \dim W[L].$$

Proof. Suppose $\mathscr{T}_\otimes$ is an $m \times n$ tensor-product mesh. Then $\dim \bar{\mathbf{S}}^d(\mathscr{T}_\otimes) = (m-d-1)(n-d-1)$. Now, we extend all of the interior l-edges in L to the boundary, where the new l-edges set is denoted as L_1, and we create a new $m_1 \times n_1$ tensor-product mesh $\mathscr{T}_1$. $\dim \bar{\mathbf{S}}^d(\mathscr{T}_1) = (m_1-d-1)(n_1-d-1)$.

Because $\mathscr{T}_\otimes$ has at least $d+1$ horizontal l-edges and $d+1$ vertical l-edges, it is easy to verify that L_1 has a reasonable order and we have $\dim W[L_1] = m_1 n_1 - mn - (d+1)((m_1-m) + (n_1-n))$, which means that

$$\dim \bar{\mathbf{S}}^d(\mathscr{T}_1) = \dim \bar{\mathbf{S}}^d(\mathscr{T}_\otimes) + \dim W[L_1].$$

Thus, the lemma is directly derived from Lemma 3.6.   □

5. Dimensions of $\mathbf{S}^2(\mathscr{T})$ over hierarchical T-meshes

In [17], the dimension formula of $\mathbf{S}^2(\mathscr{T})$ over Hierarchical T-meshes is provided, where the analysis is somewhat complex and the main strategy is to construct a mapping from $\mathbf{S}^2(\mathscr{T})$ to $\mathbf{S}^1(\mathscr{T})$ and decompose $\mathbf{S}^2(\mathscr{T})$ into the direct sum of some subspaces. In this section, we provide a concise proof for the dimension formula of $\mathbf{S}^2(\mathscr{T})$ over a hierarchical T-mesh $\mathscr{T}$.

First, we define some notations. Suppose $\text{lev}(\mathscr{T}) = n$. We use V, V^+, and E to denote the numbers of the vertices, the crossing-vertices, and the interior l-edges of $\mathscr{T}$, respectively. In addition, we use V_i, V_i^+, and E_i to represent the numbers of the vertices, the crossing-vertices, and the interior l-edges in T_i, respectively. Here, T_i is the set of all of the level i l-edges and the level i vertices of $\mathscr{T}$ (see Section 2.1). It follows that

$$V = \sum_{i=0}^n V_i, \qquad V^+ = \sum_{i=0}^n V_i^+, \qquad E = \sum_{i=0}^n E_i.$$

Then, the key procedure of the proof consists of the following components.

1. Compute $\dim W[T_i]$, $i = 1, \ldots, n$;
2. Prove that $\dim \bar{\mathbf{S}}^2(\mathscr{T}) = \dim \bar{\mathbf{S}}^2(\mathscr{T}^0) + \dim W[T_1] + \dim W[T_2] + \cdots + \dim W[T_n]$.

Lemma 5.1. *If $\mathscr{T}^0$ has at least 3 horizontal l-edges and 3 vertical l-edges, $\dim \bar{\mathbf{S}}^2(\mathscr{T}^0) = V_0^+ - E_0 + 1$.*

Proof. Suppose $\mathscr{T}^0$ is an $m \times n$ tensor-product mesh. It follows that $V_0^+ = (m-2)(n-2)$, $E_0 = m+n-4$ and $\dim \bar{\mathbf{S}}^2(\mathscr{T}^0) = (m-3)(n-3) = V_0^+ - E_0 + 1$.   □

Definition 5.2. For a hierarchical T-mesh $\mathscr{T}$, T_i ($i \geqslant 1$) can be divided into several parts, $T_{i_1}, T_{i_2}, \ldots, T_{i_{m_i}}$, such that the l-edges of T_{i_r} do not intersect with the l-edges of T_{i_s}, where $r, s \in \{1, 2, \ldots, m_i\}$ and $r \neq s$. T_{i_k} is called a **connected component** of T_i, $k \in \{1, 2, \ldots, m_i\}$.

For the hierarchical T-mesh in Fig. 3, T_1 has 1 connected component, and T_2 has 3 connected components.

Lemma 5.3. In a hierarchical T-mesh $\mathscr{T}$, suppose $T_{i_1}, T_{i_2}, \ldots, T_{i_{m_i}}$ are all of the connected components of T_i ($i \geqslant 1$), then we have

$$\dim W[T_i] = \sum_{k=1}^{m_i} \dim W[T_{i_k}]. \tag{6}$$

Proof. This can be directly derived from Lemma 3.5. □

Lemma 5.4. Suppose T_i ($i \geqslant 1$) has δ_{1i} connected components that divide only one cell of $\mathscr{T}^{i-1}$. Then, $\dim W[T_i] = V_i^+ - E_i + \delta_{1i}$.

Proof. According to Lemma 5.3, we can only consider $\dim W[T_{i_1}]$. Suppose T_{i_1} has V_{i_1} vertices, $V_{i_1}^+$ crossing-vertices and E_{i_1} l-edges. Then $V_{i_1} = V_{i_1}^+ + 2E_{i_1}$. Let $t_1, t_2, \ldots, t_k, t_{k+1}, \ldots, t_r$ be all of the l-edges in T_{i_1}, where $N(t_m) = 1$, $m \in \{1, 2, \ldots, k\}$ and $N(t_n) \geqslant 2$, $n \in \{k+1, \ldots, r\}$.

If $t_1, t_2, \ldots, t_k$ do not intersect with each other, $t_1, t_2, \ldots, t_k, \ldots, t_r$ form a reasonable order of T_{i_1}. Actually, for any j, if $j \leqslant k$, the number of the vertices on t_j but not on t_l, $l < j$, is exactly 3 and for $j > k$, because t_j crosses at least two cells in the upper level, the number of the vertices on t_j but not on t_l, $l < j$, is at least 3. Hence, we have $\dim W[T_{i_1}] = V_{i_1} - 3E_{i_1} = V_{i_1}^+ - E_{i_1}$.

Otherwise, T_{i_1} only divides one cell of $\mathscr{T}^{i-1}$, and it is easy to determine that $\dim W[T_{i_1}] = 0 = V_{i_1}^+ - E_{i_1} + 1$. Therefore,

$$\dim W[T_{i_1}] = \begin{cases} V_{i_1}^+ - E_{i_1} + 1, & T_{i_1} \text{ only divides one cell of } \mathscr{T}^{i-1}, \\ V_{i_1}^+ - E_{i_1}, & \text{otherwise.} \end{cases}$$

Suppose there are δ_{1i} connected components of T_i that divide only one cell of $\mathscr{T}^{i-1}$. Then,

$$\dim W[T_i] = V_i^+ - E_i + \delta_{1i}. \quad \square$$

Let $T = T_1^o \cup T_2^o \cup \cdots \cup T_n^o$, $T' = T_2^o \cup \cdots \cup T_n^o$, where T_i^o has the same meaning as in Section 2.1. For the second step of the proof, we only have to prove that $\dim \overline{\mathbf{S}}^2(\mathscr{T}^\varepsilon) = \dim \overline{\mathbf{S}}^2(\mathscr{T}^0) + \dim W[T]$ and $\dim W[T] = \dim W[T_1] + \dim W[T']$. First, we have the following lemma.

Lemma 5.5. For $\overline{\mathbf{S}}^2(\mathscr{T})$, with the definition above, we have

$$\dim W[T] = \sum_{i=1}^{n} \dim W[T_i].$$

Proof. We only need to prove that $\dim W[T] = \dim W[T_1] + \dim W[T']$. According to Lemma 5.4, we suppose T_1 has only two connected components $\overline{T_1}$ and $\widetilde{T_1}$. $\overline{T_1}$ divides only one cell of $\mathscr{T}^0$, while $\widetilde{T_1}$ divides more. Correspondingly, T also has two connected components; say they are $\overline{T}$ and $\widetilde{T}$. T' is divided into two parts; say they are $\overline{T'}$ and $\widetilde{T'}$, where $\overline{T'} \subseteq \overline{T}$, $\widetilde{T'} \subseteq \widetilde{T}$.

Because $\dim W[\widetilde{T_1}] = \widetilde{V}_1^+ - \widetilde{E}_1$, where $\widetilde{V}_1^+$ and $\widetilde{E}_1$ are the numbers of the crossing-vertices and the l-edges of $\widetilde{T_1}$, respectively, according to Lemma 3.7, we have $\dim W[\widetilde{T}] = \dim W[\widetilde{T_1}] + \dim W[\widetilde{T'}]$. For $\overline{T_1}$, we consider the mesh $\mathscr{T}'$ generated by $\overline{T}$ and the edges of the cell of $\mathscr{T}^0$ divided by $\overline{T_1}$. Then, we have

$$\dim W[\overline{T'}] = \dim \overline{\mathbf{S}}^2(\mathscr{T}') \leqslant \dim W[\overline{T}] \leqslant \dim W[\overline{T'}] + \dim W[\overline{T_1}] = \dim W[\overline{T'}].$$

That is, $\dim W[\overline{T}] = \dim W[\overline{T'}] = \dim W[\overline{T'}] + \dim W[\overline{T_1}]$, which proves the lemma. □

Now, we can give a new proof for the dimension formula of $\overline{\mathbf{S}}^2(\mathscr{T})$ provided in [17].

Theorem 5.6 ([17]). Suppose $\mathscr{T}$ is a hierarchical T-mesh with V^+ crossing-vertices, E interior l-edges and δ_{1i} connected components in T_i ($i \geqslant 1$) that divide only one cell of $\mathscr{T}^{i-1}$. Denote $\delta_1 = \sum_{i=1}^{n} \delta_{1i}$. If $\mathscr{T}^0$ has at least 3 horizontal l-edges and 3 vertical l-edges, then

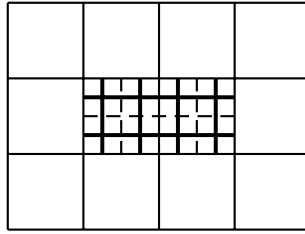$$\dim \overline{\mathbf{S}}^2(\mathscr{T}) = V^+ - E + \delta_1 + 1.$$

Fig. 10. An example of a T-mesh.

Proof. According to Lemma 5.1, Lemma 5.4, Lemma 4.2 and Lemma 5.5, we have

$$\dim \overline{\mathbf{S}}^2(\mathcal{T}) = \dim \overline{\mathbf{S}}^2(\mathcal{T}^0) + \dim W[T]$$

$$= V_0^+ - E_0 + 1 + \sum_{i=1}^n \dim W[T_i]$$

$$= V_0^+ - E_0 + 1 + \sum_{i=1}^n (V_i^+ - E_i + \delta_{1i})$$

$$= V^+ - E + \delta_1 + 1. \quad \square$$

Theorem 5.7. *Suppose $\mathcal{T}$ is a hierarchical T-mesh with V^+ crossing-vertices, V^b boundary vertices, E interior l-edges and δ_{1i} connected components in T_i ($i \geqslant 1$) that divide only one interior cell of $\mathcal{T}^{i-1}$. Denote $\delta_1 = \sum_{i=1}^n \delta_{1i}$. Then*

$$\mathbf{S}^2(\mathcal{T}) = 2V^b + V^+ - E + \delta_1 + 1.$$

Proof. According to Theorem 4.1, we have only to compute $\dim \overline{\mathbf{S}}^2(\mathcal{T}^\varepsilon)$. It is easy to check that $\mathcal{T}^\varepsilon$ has $2V^b + V^+ + 8$ crossing-vertices, $E + 8$ interior l-edges and δ_{1i} connected components in T_i^ε ($i \geqslant 1$) that divide only one cell of $\mathcal{T}^{\varepsilon^{i-1}}$.

We should notice that $\mathcal{T}^\varepsilon$ is not a hierarchical T-mesh of 2×2 division if any boundary cell of $\mathcal{T}$ is divided. However, after extending, every l-edge of $\mathcal{T}^{\varepsilon^i}(i \geqslant 1)$ has more vertices than the original l-edge. Therefore, Theorem 5.6 holds for $\mathcal{T}^\varepsilon$. Using the conclusion of Theorem 5.6 on $\mathcal{T}^\varepsilon$, we complete the proof. $\quad \square$

6. Dimensions of $\mathbf{S}^3(\mathcal{T})$ over hierarchical T-meshes

The difficulty in generalizing the above method to $\mathbf{S}^3(\mathcal{T})$ over a hierarchical T-mesh is the fact that the equation $\dim W[T] = \dim W[T_1] + \dim W[T']$ is not always true. For example, as indicated in Fig. 10, it is easy to compute that $\dim W[T_1] = 0$, $\dim W[T_2] = 6$, but $\dim W[T] \leqslant 5$. This tells us that we cannot compute the dimension of $\mathbf{S}^3(\mathcal{T})$ level by level if we consider general hierarchical T-meshes of 2×2 division. Thus, we consider hierarchical T-meshes with some restrictions. In Section 6.1, we consider hierarchical T-meshes with $N(t) \geqslant 2$ for any l-edge t of T, and in Section 6.2, we consider hierarchical T-meshes of 3×3 division. The symbols in this section have the same meanings as in Section 5.

6.1. Hierarchical T-meshes with $N(t) \geqslant 2$

Lemma 6.1. *If $\mathcal{T}^0$ has at least 4 horizontal l-edges and 4 vertical l-edges, then $\dim \overline{\mathbf{S}}^3(\mathcal{T}^0) = V_0^+ - 2E_0 + 4$.*

Proof. Suppose $\mathcal{T}^0$ is an $m \times n$ tensor-product mesh. Then, $V_0^+ = (m - 2)(n - 2)$, $E_0 = m + n - 4$, and $\dim \overline{\mathbf{S}}^3(\mathcal{T}^0) = (m - 4)(n - 4) = V_0^+ - 2E_0 + 4$. $\quad \square$

Lemma 6.2. *Suppose T_i ($i \geqslant 1$) has δ_{4i} connected components that only divide the 2×2 neighbor cells in $\mathcal{T}^{i-1}$ (see Fig. 11(a)). Then, $\dim W[T_i] = V_i^+ - 2E_i + \delta_{4i}$.*

Proof. Similar to the discussion in the last section, we only consider $\dim W[T_{i_1}]$. Suppose T_{i_1} has V_{i_1} vertices, $V_{i_1}^+$ crossing-vertices and E_{i_1} l-edges. Then $V_{i_1} = V_{i_1}^+ + 2E_{i_1}$. We also assume $t_1, t_2, \ldots, t_k, \ldots, t_r$ are all of the l-edges of T_{i1}, where $N(t_i) = 2$ when $i \leqslant k$; $N(t_i) > 2$ when $i \geqslant k + 1$. For $i = 1, 2 \ldots, k$, if there is an order, say $t_{j_1}, t_{j_2}, \ldots, t_{j_k}$, such that t_{j_p} only intersects with at most one l-edge of $t_{j_q}, q < p$, then $t_{j_1}, t_{j_2}, \ldots, t_{j_k}, t_{k+1}, \ldots, t_r$ form a reasonable order. Hence, $\dim W[T_{i_1}] = V_{i_1} - 4E_{i_1} = V_{i_1}^+ - 2E_{i_1}$.

Otherwise, T_{i_1} only divides the 2×2 neighbor cells in $\mathcal{T}^{i-1}$. Then, we can obtain $\dim W[T_{i_1}] = 1$ according to [18]. In this case, $\dim W[T_{i_1}] = 1 = V_{i_1}^+ - 2E_{i_1} + 1$. Thus, if δ_{4i} is the number of the connected components that only divide the 2×2 neighbor cells in $\mathcal{T}^{i-1}$, then $\dim W[T_i] = V_i^+ - 2E_i + \delta_{4i}$. $\quad \square$
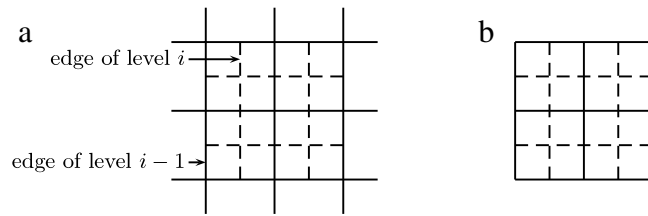
Fig. 11. The special connected component T_i (a) and the new mesh $\mathscr{T}'$ (b).

Lemma 6.3. *For* $\overline{\mathbf{S}}^3(\mathscr{T})$, *let* $T = T_1^o \cup T_2^o \cup \cdots \cup T_n^o$. *If* $N(t) \geqslant 2$ *for any l-edge t, we have*

$$\dim W[T] = \sum_{i=1}^{n} \dim W[T_i].$$

Proof. Let $T' = T_2^o \cup \cdots \cup T_n^o$. Then, we have only to prove that $\dim W[T] = \dim W[T_1] + \dim W[T']$. According to Lemma 6.2, we suppose T_1 has only two connected components $\overline{T_1}$ and $\widetilde{T_1}$. $\overline{T_1}$ only divides the 2×2 neighbor cells in $\mathscr{T}^0$, while $\widetilde{T_1}$ does not. Correspondingly, T also has two connected components; say they are $\overline{T}$ and $\widetilde{T}$. T' is divided into two parts; say they are $\overline{T'}$ and $\widetilde{T'}$, where $\overline{T'} \subseteq \overline{T}$, $\widetilde{T'} \subseteq \widetilde{T}$.

Because $\dim W[\widetilde{T_1}] = \widetilde{V}_1^+ - 2\widetilde{E}_1$, where $\widetilde{V}_1^+$, $\widetilde{E}_1$ are the numbers of the crossing-vertices and the l-edges of $\widetilde{T_1}$, respectively; according to Lemma 3.7, we have $\dim W[\widetilde{T}] = \dim W[\widetilde{T_1}] + \dim W[\widetilde{T'}]$. For $\overline{T_1}$, we consider the mesh $\mathscr{T}'$ (Fig. 11(b) is the mesh excluding $\overline{T'}$) generated by $\overline{T}$ and the edges of the cells of $\mathscr{T}^0$ divided by $\overline{T_1}$. Then, we have

$$1 + \dim W[\overline{T'}] = \dim \overline{\mathbf{S}}^3(\mathscr{T}') \leqslant \dim W[\overline{T}] \leqslant \dim W[\overline{T'}] + \dim W[\overline{T_1}] = \dim W[\overline{T'}] + 1.$$

That is, $\dim W[\overline{T}] = \dim W[\overline{T'}] + 1 = \dim W[\overline{T'}] + \dim W[\overline{T_1}]$, which proves the lemma. $\square$

Theorem 6.4. *Suppose* $\mathscr{T}$ *is a hierarchical T-mesh with* V^+ *crossing-vertices,* E *interior l-edges, and any l-edge t of T satisfies* $N(t) \geqslant 2$. *There are* δ_{4i} *connected components in* T_i *that only divide the* 2×2 *neighbor cells in* $\mathscr{T}^{i-1}$. *Let* $\delta_4 = \sum_{i=1}^{n} \delta_{4i}$. *If* $\mathscr{T}^0$ *has at least 4 horizontal l-edges and 4 vertical l-edges, then*

$$\dim \overline{\mathbf{S}}^3(\mathscr{T}) = V^+ - 2E + 4 + \delta_4.$$

Proof. According to Lemma 6.1, Lemma 6.2, Lemma 4.2 and Lemma 6.3, we have

$$\dim \overline{\mathbf{S}}^3(\mathscr{T}) = \dim \overline{\mathbf{S}}^3(\mathscr{T}^0) + \dim W[T]$$

$$= V_0^+ - 2E_0 + 4 + \sum_{i=1}^{n} \dim W[T_i]$$

$$= V_0^+ - 2E_0 + 4 + \sum_{i=1}^{n} (V_i^+ - 2E_i + \delta_{4i})$$

$$= V^+ - 2E + \delta_4 + 4. \quad \square$$

Similar to Theorem 5.7, we have the following theorem.

Theorem 6.5. *Suppose* $\mathscr{T}$ *is a hierarchical T-mesh with* V^+ *crossing-vertices,* V^b *boundary vertices,* E *interior l-edges, and any l-edge t of T satisfies* $N(t) \geqslant 2$. *There are* δ_{4i} *connected components in* T_i *that only divide the* 2×2 *interior neighbor cells in* $\mathscr{T}^{i-1}$. *Let* $\delta_4 = \sum_{i=1}^{n} \delta_{4i}$. *Then*

$$\dim \mathbf{S}^3(\mathscr{T}) = 3V^b + V^+ - 2E + 4 + \delta_4.$$

Now, we give an example for the dimension computation.

Example 6.6. In Fig. 12, there are 20 crossing-vertices (labeled with "•"), 16 boundary vertices (labeled with "○"), and 12 interior l-edges (labeled with dashed lines). $\delta_{42} = 1$, $\delta_{41} = 0$. Hence, $\dim \mathbf{S}^3(\mathscr{T}) = 3 \times 16 + 20 - 2 \times 12 + 4 + 1 = 49$.

6.1.1. The topological explanation

In [17], the following structure is introduced to propose a topological explanation to the dimension formula.

Definition 6.7 ([17]). Given a hierarchical T-mesh $\mathscr{T}$, we can construct a graph $\mathscr{G}$ by retaining the crossing-vertices and the edges with two end points that are crossing-vertices and removing the other vertices and the edges in $\mathscr{T}$. $\mathscr{G}$ is called the crossing–vertex-relationship graph (CVR graph for short) of $\mathscr{T}$.
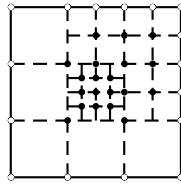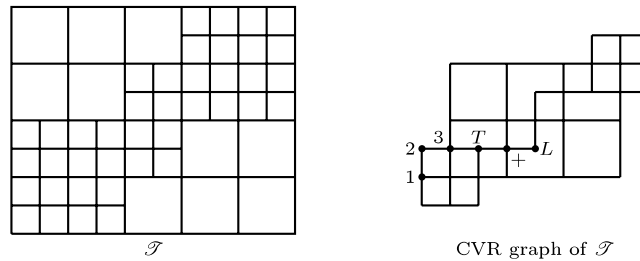
Fig. 12. A hierarchical T-mesh $\mathscr{T}$.

Fig. 13. A hierarchical T-mesh and its CVR graph.

It should be noted that, when $N(t) \geqslant 2$ for any l-edge of t in a hierarchical T-mesh $\mathscr{T}$, the CVR graph of $\mathscr{T}$ is connected. See Fig. 13 for an example. In addition, in [17], the following conjecture is stated:

Conjecture 6.8. *Suppose $\mathscr{T}$ is a hierarchical T-mesh whose CVR graph is $\mathscr{G}$. When $m, n \geqslant 2$, it follows that*

$$\dim \overline{\mathbf{S}}(m, n, m - 1, n - 1, \mathscr{T}) = \dim \overline{\mathbf{S}}(m - 2, n - 2, m - 3, n - 3, \mathscr{G}),$$

where $\overline{\mathbf{S}}(m, n, \alpha, \beta, \mathscr{G})$ is defined in a similar way to the spline space over a T-mesh.

In [17], the conjecture is proved to hold as $m = n = 2$. Now, we will prove that it also holds as $m = n = 3$ over a hierarchical T-mesh specified in Theorem 6.4.

In $\mathscr{G}$, we can define boundary vertex, interior vertex, crossing-vertex, T-junction, edge, l-edge, and so on in the same way as in $\mathscr{T}$. We use $V_{\mathscr{G}}$ to denote the number of the vertices of the CVR graph. Therefore, $V_{\mathscr{G}} = V^+$. In $\mathscr{G}$, a vertex is the intersection of two l-edges. We distinguish the vertices of $\mathscr{G}$ into six types as follows:

1. Type 1: a boundary vertex of $\mathscr{G}$, which is an end point of an l-edge and an interior vertex of another l-edge,
2. Type 2: a boundary vertex of $\mathscr{G}$, which is the end points of two l-edges,
3. Type 3: a boundary vertex of $\mathscr{G}$, which is the interior vertices of two l-edges,
4. Type T: an interior vertex of $\mathscr{G}$, which is a T-junction of $\mathscr{G}$,
5. Type +: an interior vertex of $\mathscr{G}$, which is a crossing-vertex of $\mathscr{G}$,
6. Type L: an interior vertex of $\mathscr{G}$, which is the end points of two l-edges.

The six types of vertices have been marked in Fig. 13. We use $V_{\mathscr{G}}^1, V_{\mathscr{G}}^2, V_{\mathscr{G}}^3, V_{\mathscr{G}}^T, V_{\mathscr{G}}^+, V_{\mathscr{G}}^L$ to denote the numbers of the six types of vertices, respectively.

Lemma 6.9. $V_{\mathscr{G}}^2 = V_{\mathscr{G}}^3 + 4$.

Proof. The CVR graph $\mathscr{G}$ is connected. We choose a vertex v as the starting point on the boundary of $\mathscr{G}$, and run through the boundary of $\mathscr{G}$ in the clockwise direction. When we meet a vertex of type 2, the direction rotates 90° clockwise; and when we meet a vertex of type 3, the direction rotates 90° anti-clockwise. Finally, we return to v, and the direction rotates 360° clockwise. Therefore,

$$90V_{\mathscr{G}}^2 - 90V_{\mathscr{G}}^3 = 360.$$

That is $V_{\mathscr{G}}^2 = V_{\mathscr{G}}^3 + 4$. □

Theorem 6.10. *Suppose $\mathscr{T}$ is a hierarchical T-mesh with $N(t) \geqslant 2$ for any l-edge t, and its CVR graph is $\mathscr{G}$. There are δ_{4i} connected components as in Fig. 11(a) in T_i. $\delta_4 = \sum_{i=1}^{n} \delta_{4i}$. Then*

$$\dim \overline{\mathbf{S}}^3(\mathscr{T}) = V_{\mathscr{G}}^+ - V_{\mathscr{G}}^L + \delta_4.$$
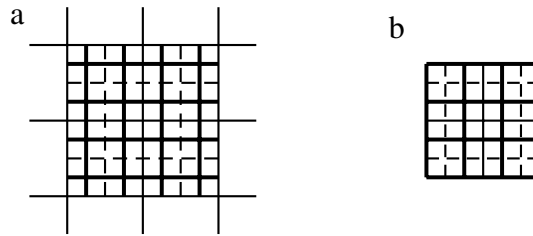
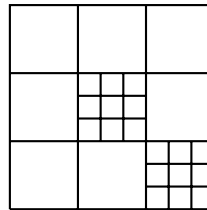**Fig. 14.** Figure for the proof of Theorem 6.11.

Fig. 15. A hierarchical T-mesh of 3×3 division.

Proof. The end points of every l-edge in $\mathscr{G}$ could be the types of $1, 2, T, L$. Running through all of the l-edges of the CVR graph, we get $2E = V_{\mathscr{G}}^1 + 2V_{\mathscr{G}}^2 + V_{\mathscr{G}}^T + 2V_{\mathscr{G}}^L$. Then,

$$V^+ - 2E + 4 = V_{\mathscr{G}} - V_{\mathscr{G}}^1 - 2V_{\mathscr{G}}^2 - V_{\mathscr{G}}^T - 2V_{\mathscr{G}}^L$$
$$= V_{\mathscr{G}}^+ - V_{\mathscr{G}}^L + V_{\mathscr{G}}^3 - V_{\mathscr{G}}^2 + 4.$$

According to Lemma 6.9, we know the theorem holds. □

Now, we prove Conjecture 6.8 holds for $\overline{\mathbf{S}}^3(\mathscr{T})$ over a hierarchical T-mesh with $N(t) \geqslant 2$ for any l-edge t.

Theorem 6.11. *Suppose $\mathscr{T}$ is a hierarchical T-mesh with $N(t) \geqslant 2$ for any l-edge t. If the CVR graph of $\mathscr{T}$ is $\mathscr{G}$, we have*

$$\dim \overline{\mathbf{S}}^3(\mathscr{T}) = \dim \overline{\mathbf{S}}^1(\mathscr{G}).$$

Proof. $\mathscr{G}$ has a natural hierarchical structure, and we use the same notations in Section 2.1.

First, we have $\dim W[G_i] = \dim W[T_i]$. Suppose all of the l-edges of T_i are $t_1, t_2, \ldots, t_m$. Every vertex of t_j, $j = 1, 2, \ldots, m$, excluding the two end points, is a crossing-vertex. Because $N(t_j) \geqslant 2$, t_j has at least 5 vertices. Therefore, G_i also has m l-edges $t_1', t_2', \ldots, t_m'$, where t_j' is derived from t_j and has two less vertices than t_j. t_l' intersects with t_k' when t_l intersects t_k, where $k \neq l$, $k, l \in \{1, 2, \ldots, m\}$. Using the method similar to the computation of $\dim W[T_i]$, we can get $\dim W[G_i]$ easily, and we have $\dim W[G_i] = \dim W[T_i]$.

Second, we have $\dim W[G] = \sum_{i=1}^n \dim W[G_i]$, where $G = G_1^o \cup G_2^o \cup \cdots \cup G_n^o$. We can suppose that $n = 2$. When T_1 only divides the 2×2 neighbor cells in $\mathscr{T}^0$, according to Lemma 3.6, we should only prove this conclusion when all of the cells of level 1 are divided (see Fig. 14(a)). Fig. 14(b) is the corresponding CVR graph, in which G_2 consists of the thick lines and G consists of the thick lines and the dashed lines. From the first part, we know $\dim W[G_1] = 1$, $\dim W[G_2] = 24$. To compute $\dim W[G]$, we consider the mesh $\mathscr{G}'$ of Fig. 14(b). It is easy to determine that $\dim \overline{\mathbf{S}}^1(\mathscr{G}') = 25$. We have $\dim W[G] = \dim \overline{\mathbf{S}}^1(\mathscr{G}') = 25$. Therefore, $\dim W[G] = \dim W[G_1] + \dim W[G_2]$. When T_1 divides more cells, G_1 has a reasonable order, we also have $\dim W[G] = \dim W[G_1] + \dim W[G_2]$.

Third, we have $\dim \overline{\mathbf{S}}^1(\mathscr{G}) = \dim \overline{\mathbf{S}}^1(\mathscr{G}^0) + \dim W[G]$. This is a generalization of Lemma 4.2. If any boundary cell of $\mathscr{T}^0$ is divided, the bound of $\mathscr{G}$ may be not a rectangle. In this situation, we can use the same method as in the second part to change $\mathscr{G}$ into a regular mesh. We omit the proof process here.

Combining the three parts, we have proved this theorem. □

6.2. Hierarchical T-meshes of 3×3 division

In this section we consider hierarchical T-meshes of 3×3 division. See Fig. 15 for an example.

Before discussing the dimension of $W[T_i]$, we first introduce some notations. Suppose T_i only has one connected component, and all of the l-edges are $t_{11}, \ldots, t_{1r}, t_{21}, \ldots, t_{2s}, t_1, \ldots, t_m$, where $N(t_{1j}) = 1$, $N(t_{2k}) = 2$, $N(t_l) \geqslant 3$, $j \in \{1, \ldots, r\}$, $k \in \{1, \ldots, s\}$, $l \in \{1, \ldots, m\}$. Then, T_i can be the following several types:
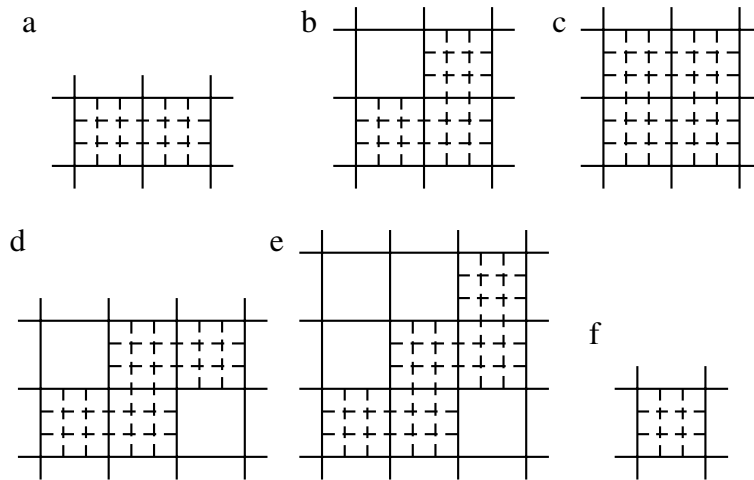
Fig. 16. Special cases of T_i.

1. $m > 0$,
2. $m = 0, s > 0$. This case includes five possible sub-cases:
    (a) The connected component refines two cells in the $i - 1$th level, see Fig. 16(a).
    (b) The connected component refines three cells in the $i - 1$th level, see Fig. 16(b).
    (c) The connected component refines four cells in the $i - 1$th level which are the 2×2 neighbor cells, see Fig. 16(c).
    (d) The connected component refines four cells in the $i - 1$th level which are not the 2×2 neighbor cells, see Fig. 16(d).
    (e) The connected component refines more than four cells in the $i - 1$th level, see Fig. 16(e).
3. $m = s = 0$, see Fig. 16(f).

Lemma 6.12. *Suppose T_i ($i \geqslant 1$) has δ_{1i} connected components of case 3, δ_{2i} connected components of case 2(a), δ_{3i} connected components of case 2(b) and δ_{4i} connected components of case 2(c). Then* $\dim W[T_i] = V_i^+ - 2E_i + 4\delta_{1i} + 2\delta_{2i} + \delta_{3i} + \delta_{4i}$.

Proof. Suppose T_i only has one connected component. We discuss every case of T_i discussed above.

1. $m > 0$:
   First, we should mention that we can remove all of $t_{11}, \ldots, t_{1r}$ without changing the dimension. For the other l-edges, if $s = 0$, it is obvious that $t_1, \ldots, t_m$ have a reasonable order. If $s > 0$, we define the order of the l-edges such that it is a reasonable order. First, select any l-edge as t_{21}. If there is an l-edge t with $N(t) = 2$ intersecting with t_{21}, we choose t as t_{22}. If there is an l-edge t' with $N(t') = 2$ intersecting with t_{22}, we choose t' as t_{23}. Repeat this process until all of the l-edges that cross two cells on the upper level have been sorted. In this case, $t_{11}, \ldots, t_{1r}, t_{21}, \ldots, t_{2s}, t_1, \ldots, t_m$ is a reasonable order.
2. $m = 0, s > 0$. This case includes five possible sub-cases:
    (a) The connected component refines two cells in the $i - 1$th level.
       It is evident that $\dim W[T_i] = 0 = V_i^+ - 2E_i + 2$.
    (b) The connected component refines three cells in the $i - 1$th level.
       It is evident that $\dim W[T_i] = 1 = V_i^+ - 2E_i + 1$.
    (c) The connected component refines four cells in the $i - 1$th level that are the 2×2 neighbor cells.
       It is evident that $\dim W[T_i] = 9 = V_i^+ - 2E_i + 1$.
    (d) The connected component refines four cells in the $i - 1$th level that are not the 2×2 neighbor cells.
       We delete t_{1r} and get the mesh S_i in Fig. 17.
       First, we prove that the projection mapping $\pi : W[S_i] \to W[t]$ is surjective. Because t is an l-edge of order one, we only need to verify that the co-factors of the vertices on t could be nonzero in $W[S_i]$. Let $W[t_1] = W[t_2] = 0$. Then we change S_i to S_i' (see Fig. 17) as with the mesh in case 2(b). Because $\dim W[S_i'] = 1$, the co-factor of every vertex on S_i' is nonzero. Thus, the mapping π is surjective. We have $\dim W[S_i] = \dim W[S_i''] + 1$, where S_i'' is the mesh $S_i \setminus t$ (see Fig. 17). According to Eq. (5), we have $\dim W[S_i''] \geqslant \dim W[S_i'] = 1$. Therefore, $\dim W[S_i] \geqslant 2$.
       For the upper bound, we let $\gamma(v_0) = \gamma(v_0') = 0$, where $\gamma(v_0), \gamma(v_0')$ are the co-factors of v_0 and v_0', respectively. Then, S_i is changed into S_i''' (see Fig. 17). Suppose the lengths of the four edges of t_1 are l, d, d, d, respectively. We will prove that $\dim W[S_i'''] = 0$. Let $\gamma(v_1) = x$; according to Eq. (2), we can obtain that

$$\gamma(v_2) = ax, \quad a = -\frac{d+l}{2d+l}, \quad -1 < a < 0.$$

   Similarly, we have

$$\gamma(v_3) = abx, \qquad \gamma(v_4) = abcx, \quad b \in \mathbb{R}, -1 < c < 0.$$

   We can get $\gamma(v_1) = acx$ from $\gamma(v_4)$. Thus, we have $x = acx$. Because $0 < ac < 1$, we get $x = 0$, which means that $\dim W[S_i'''] = 0$. Therefore, $\{\gamma(v_0), \gamma(v_0')\}$ is the determining set of $W[S_i]$. Then, $\dim W[S_i] \leqslant 2$.
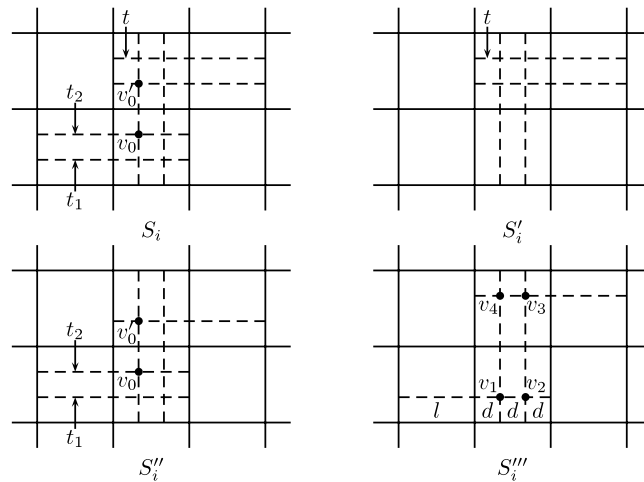   Combining the former two parts, we have $\dim W[S_i] = 2$; and it is easy to check that $\dim W[S_i] = V_i^+ - 2E_i$.
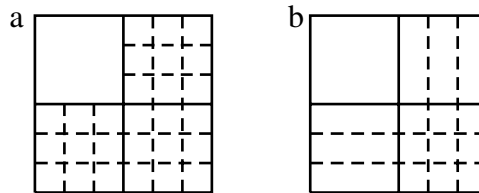
Fig. 17. Figure for the case 2(d).

Fig. 18. Figure for the proof of Lemma 6.13.

(e) The connected component refines more than four cells in the $i - 1$th level.
   This case is illustrated in Fig. 16(e). Suppose it refines p connected cells in the $i - 1$th level. As $m = 0$, so after removing all of $t_{11}, \ldots, t_{1r}$, T_i can be regarded as a mesh that is generated by adding two l-edges of order one to the mesh refining $p - 1$ connected cells in the $i - 1$th level. For example, Fig. 16(e) is the mesh generated by adding two l-edges of order one to the mesh S_i in Fig. 17. Continuing this process until the T-mesh follows the case 2(d), and according to Lemma 3.7, we know that dim $W[T_i] = V_i^+ - 2E_i$.

3. $m = s = 0$.
   It is evident that dim $W[T_i] = 0 = V_i^+ - 2E_i + 4$.

Combining the above discussion, we obtain this lemma. □

Lemma 6.13. *For* $\overline{\mathbf{S}}^3(\mathscr{T})$ *over a hierarchical T-mesh of* 3×3 *division, we have*

$$\dim W[T] = \sum_{i=1}^{n} \dim W[T_i].$$

Proof. Suppose T_1 has only one connected component. We will prove this lemma for all of the possible cases of T_1. For case 1, case 2(d), and case 2(e), the conclusion is right according to Lemma 3.7. For case 2(a), case 2(b), case 2(c), and case 3, we can use the same method in the proof of Lemma 6.2. Here, we only prove this conclusion for case 2(b). In the same way as in the proof of Lemmas 5.5 and 6.3, we construct a new mesh $\mathscr{T}'$ (Fig. 18(a) is the mesh excluding T') generated by T and the edges of the 2×2 neighbor cells. We choose the mesh in Fig. 18(b) as $\mathscr{T}'_\otimes$, and use L to denote the set of all of the l-edges of T not contained in $\mathscr{T}'_\otimes$. T' is a subset of L. According to Lemma 4.2 and Eq. (5), we have

$$\dim(\mathscr{T}') = 1 + \dim W[L] \geqslant 1 + \dim W[T'].$$

Conversely,

$$\dim(\mathscr{T}') \leqslant \dim W[T] \leqslant \dim W[T_1] + \dim W[T'] = 1 + \dim W[T'].$$

Therefore, dim $W[T] = 1 + \dim W[T'] = \dim W[T_1] + \dim W[T']$. □

Theorem 6.14. *Suppose* $\mathscr{T}$ *is a hierarchical T-mesh of* 3×3 *division with* V^+ *crossing-vertices,* E *l-edges, and there are* δ_{1i}, δ_{2i}, δ_{3i}, δ_{4i} *connected components of case 2(a), case 2(b), case 2(c), and case 3 in* T_i, *respectively.* $\delta_k = \sum_{i=1}^{n} \delta_{ki}$, $k = 1, 2, 3, 4$.
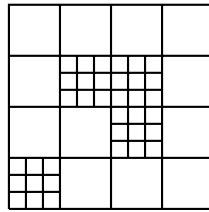
Fig. 19. A hierarchical T-mesh $\mathcal{T}$.

Then, we have

$$\dim \overline{\mathbf{S}}^3(\mathcal{T}) = V^+ - 2E + 4 + 4\delta_1 + 2\delta_2 + \delta_3 + \delta_4.$$

Proof. This is directly derived from the above lemmas. □

Similar to Theorem 5.7, we have the following theorem.

Theorem 6.15. *Suppose $\mathcal{T}$ is a hierarchical T-mesh of 3×3 division with V^+ crossing-vertices, V^b boundary vertices and E interior l-edges. There are $\delta_{1i}, \delta_{2i}, \delta_{3i}, \delta_{4i}$ connected components of case 2(a), case 2(b), case 2(c), and case 3 in T_i^ε, respectively, where T_i^ε is the ith level of the extended mesh $\mathcal{T}^\varepsilon$. $\delta_k = \sum_{i=1}^n \delta_{ki}, \ k = 1, 2, 3, 4$. Then, we have*

$$\dim \mathbf{S}^3(\mathcal{T}) = 3V^b + V^+ - 2E + 4 + 4\delta_1 + 2\delta_2 + \delta_3 + \delta_4.$$

Now, we give an example for the dimension computation.

Example 6.16. In Fig. 19, $V^+ = 29, V^b = 20, E = 18, \delta_{31} = 1$. Hence, $\dim \mathbf{S}^3(\mathcal{T}) = 3 \times 20 + 29 - 2 \times 18 + 4 + 1 = 58$.

7. Conclusions and future work

In this paper, the smoothing cofactor-conformality method is explored. We present a new proof of the dimension formula of $\mathbf{S}(2, 2, 1, 1, \mathcal{T})$ over hierarchical T-meshes and obtain the dimension formula of $\mathbf{S}(3, 3, 2, 2, \mathcal{T})$ over hierarchical T-meshes under the condition that $N(t) \geqslant 2$. As an application of this method, we give the dimension formula of $\mathbf{S}(3, 3, 2, 2, \mathcal{T})$ when $\mathcal{T}$ is a hierarchical T-mesh of 3×3 division.

Although most contents in this paper are discussing hierarchical T-meshes, many conclusions also apply to the general T-meshes. For a general T-mesh, it is not very easy to determine whether all the l-edges have a reasonable order. Therefore, these conclusions only make sense for some special T-meshes. For a hierarchical T-mesh, this method is effective when we can compute the dimensions level by level. Although we only discuss $\mathbf{S}(2, 2, 1, 1, \mathcal{T})$ and $\mathbf{S}(3, 3, 2, 2, \mathcal{T})$, it is not very difficult to compute $\dim \mathbf{S}(m, n, m-1, n-1, \mathcal{T})$ over hierarchical T-meshes with some restrictions. However, if we cannot compute the dimensions level by level, this method is invalid. We have discussed this phenomenon at the beginning of Section 6.

For the dimension of $\mathbf{S}(3, 3, 2, 2, \mathcal{T})$ over hierarchical T-meshes without any restrictions, the proposed smoothing cofactor-conformality method may be invalid. The solution to this problem will be left to future study.

Acknowledgments

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