§1.2. 空间曲线 ($\mathbb{R}^3$) ①

作业: P28. 5, 7, 14, 15, 19.

· $n=3$.

· 设 $\vec{q}(t) = (q^1(t), q^2(t), q^3(t))$ 为 $\mathbb{R}^3$ 中的光滑参数曲线, $t \in (a,b)$.

$$\dot{\vec{q}}(t) \neq 0 \quad \forall t.$$

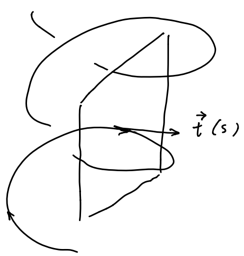
$$s(t) := \int_{t_0}^{t} |\dot{\vec{q}}(u)|\, du$$

弧长参数. $\frac{ds}{dt} > 0 \Rightarrow t = t(s)$

$$\vec{r}(s) = \vec{q}(t(s)) = (r^1(s), r^2(s), r^3(s)).$$

Recall that $\left|\frac{d\vec{r}}{ds}\right| \equiv 1$.

定义 $\vec{t}(s) := \frac{d\vec{r}(s)}{ds}$. 则 $\vec{t}(s)$ 为曲线 $\vec{r} = \vec{r}(s)$ 在 s 处的单位切向量.



过点 $\vec{r}(s)$ 与 $\vec{t}(s)$ 垂直的平面称为曲线在 $\vec{r}(s)$ 处的法平面.

$\langle \vec{t}(s), \vec{t}(s) \rangle \equiv 1$

$\Rightarrow \langle \frac{d\vec{t}(s)}{ds}, \vec{t}(s) \rangle \equiv 0.$

②

· 若 $\frac{d\vec{t}(s)}{ds} \neq 0$ 则 $\frac{d\vec{t}(s)}{ds}$ 为 $\vec{r}(s)$ 处的一个非零法向量. 称为 $\vec{r}(s)$ 的曲率向量.

· 记:

$$\kappa(s) = \left|\frac{d\vec{t}(s)}{ds}\right| = \sqrt{\ddot{r}^1(s)^2 + \ddot{r}^2(s)^2 + \ddot{r}^3(s)^2}.$$

称为曲线 $\vec{r}$ 在 s 处的曲率.

显然 $\kappa(s) \geq 0$.

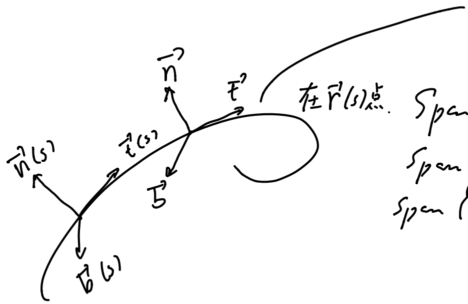
若 $\kappa(s) > 0$, 则可定义 $\vec{n}(s) := \frac{1}{\kappa(s)} \frac{d\vec{t}(s)}{ds}$

则 $\vec{n}(s)$ 为 s 处的一个单位法向量, 称为主法向量.

令: $\vec{b}(s) := \vec{t}(s) \times \vec{n}(s)$.

称 $\vec{b}(s)$ 为曲线 $\vec{r}$ 在 s 处的副法向量.

则 $(\vec{r}(s); \vec{t}(s), \vec{n}(s), \vec{b}(s))$ 为沿曲线 $\vec{r}$ 的正交标架. 称为 Frenet 标架.



在 $\vec{r}(s)$ 点. Span$\{\vec{n}, \vec{b}\}$: 法平面.

Span$\{\vec{n}, \vec{t}\}$: 密切平面.

Span$\{\vec{b}, \vec{t}\}$: 从切平面.

· Frenet 标架运动公式 (假定 $\kappa(s)>0$)

$$\frac{d\vec{t}(s)}{ds} = \kappa(s)\,\vec{n}(s).$$

$$\langle \vec{t}(s), \vec{n}(s)\rangle \equiv 0$$

$$\Rightarrow \left\langle \frac{d\vec{t}}{ds}, \vec{n}\right\rangle + \left\langle \vec{t}, \frac{d\vec{n}}{ds}\right\rangle \equiv 0$$

$$\Rightarrow \kappa(s) + \left\langle \vec{t}, \frac{d\vec{n}}{ds}\right\rangle \equiv 0 \qquad (1)$$

$$\langle \vec{n}, \vec{n}\rangle \equiv 1 \Rightarrow \left\langle \frac{d\vec{n}}{ds}, \vec{n}\right\rangle \equiv 0$$

$$\Rightarrow \frac{d\vec{n}(s)}{ds} = a(s)\,\vec{t}(s) + \tau(s)\,\vec{b}(s)$$

代入 (1) 知: $a(s) = -\kappa(s)$.

$$\langle \vec{b}, \vec{b}\rangle \equiv 1 \Rightarrow \left\langle \frac{d\vec{b}}{ds}, \vec{b}\right\rangle \equiv 0$$

$$\Rightarrow \frac{d\vec{b}}{ds} = c(s)\,\vec{t}(s) + d(s)\,\vec{n}(s)$$

$$\langle \vec{b}, \vec{t}\rangle \equiv 0 \Rightarrow \left\langle \frac{d\vec{b}}{ds}, \vec{t}\right\rangle + \left\langle \vec{b}, \frac{d\vec{t}}{ds}\right\rangle \equiv 0$$

$$\Rightarrow c(s) \equiv 0$$

$$\langle \vec{b}, \vec{n}\rangle \equiv 0 \Rightarrow \left\langle \frac{d\vec{b}}{ds}, \vec{n}\right\rangle + \left\langle \vec{b}, \frac{d\vec{n}}{ds}\right\rangle \equiv 0$$

$$\Rightarrow d(s) + \tau(s) \equiv 0$$

$$\Rightarrow d(s) \equiv -\tau(s)$$

综上:

$$\frac{d}{ds}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix} = \begin{pmatrix}0 & \kappa(s) & 0\\ -\kappa(s) & 0 & \tau(s)\\ 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{t}\\ \vec{n}\\ \vec{b}\end{pmatrix}$$

(空间曲线 Frenet 公式). That!

称 $\tau(s)$ 为曲线 $\vec{r}(s)$ 在 s 处的挠率.

· 挠率的几何意义:

设 $\vec{r}(s)$ 为平面曲线. i.e. $\exists$ 非零 常值向量 $\vec{a}$ s.t.

$$\langle \vec{r}(s) - \vec{r}(0), \vec{a}\rangle \equiv 0.$$

$$\xRightarrow{d/ds} \langle \vec{t}(s), \vec{a}\rangle \equiv 0 \xRightarrow{d/ds} \langle \kappa(s)\vec{n}(s), \vec{a}\rangle \equiv 0.$$

假定 $\kappa(s)>0$. 则 $\langle \vec{n}, \vec{a}\rangle \equiv 0$

$$\xRightarrow{d/ds} \langle -\kappa\vec{t} + \tau\vec{b}, \vec{a}\rangle \equiv 0 \Rightarrow \tau\langle \vec{b}, \vec{a}\rangle \equiv 0.$$

⑤

$\vec a \perp \vec t,\ \vec a \perp \vec n \Rightarrow \vec a \parallel \vec b \Rightarrow \tau \equiv 0.$

反之. 若 $\tau \equiv 0$. 则 $\frac{d\vec b}{ds} \equiv 0 \Rightarrow \vec b$ 为常值单位向量.

令: $f(s) = \langle \vec r(s) - \vec r(0), \vec b \rangle$. $f(0) = 0$.

$f'(s) = \langle \vec r'(s), \vec b \rangle = \langle \vec t(s), \vec b \rangle \equiv 0 \Rightarrow f(s) \equiv 0$.

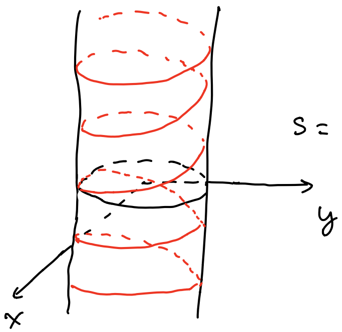
$\Rightarrow \vec r$ 为平面曲线.

Thm 2. 设空间曲线曲率 $k > 0$. 则 $\vec r$ 为平面曲线 iff $\tau \equiv 0$.

例.

求圆柱螺旋线 $\vec r(t) = (a\cos t, a\sin t, bt)$, $a>0$, $t>0$ 的曲率和挠率.

解:



$$\frac{ds}{dt} = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2}$$

$$s = \int_0^t \sqrt{a^2+b^2}\, dt = \sqrt{a^2+b^2}\, t$$

$$\Rightarrow \vec r(s) = \left(a\cos\left(\frac sc\right), a\sin\left(\frac sc\right), \frac bc s\right)$$

其中 $c := \sqrt{a^2+b^2}$.

$$\Rightarrow \vec t(s) = \frac{d\vec r(s)}{ds} = \left(-\frac ac \sin\left(\frac sc\right), \frac ac \cos\left(\frac sc\right), \frac bc\right)$$

⑥

$$\frac{d\vec t(s)}{ds} = \left(-\frac{a}{c^2}\cos\left(\frac sc\right), -\frac{a}{c^2}\sin\left(\frac sc\right), 0\right)$$

$$k(s) = \left|\frac{d\vec t(s)}{ds}\right| = \sqrt{\frac{a^2}{c^4}} = \frac{a}{c^2} = \frac{a}{a^2+b^2}.$$

$$\vec n = \left(-\cos\left(\frac sc\right), -\sin\left(\frac sc\right), 0\right).$$

$$\vec b = \vec t \times \vec n$$

$$= \begin{vmatrix} i & j & k \\ -\frac ac\sin\left(\frac sc\right) & \frac ac\cos\left(\frac sc\right) & \frac bc \\ -\cos\left(\frac sc\right) & -\sin\left(\frac sc\right) & 0 \end{vmatrix} = \left(\frac bc \sin\left(\frac sc\right), -b\cos\left(\frac sc\right), \frac ac\right)$$

$$\frac{d\vec b}{ds} = \left(\frac{b}{c^2}\cos\left(\frac sc\right), \frac{b}{c^2}\sin\left(\frac sc\right), 0\right) = -\frac{b}{c^2}\vec n(s).$$

$$\Rightarrow \quad \tau(s) = \frac{b}{a^2+b^2}.$$

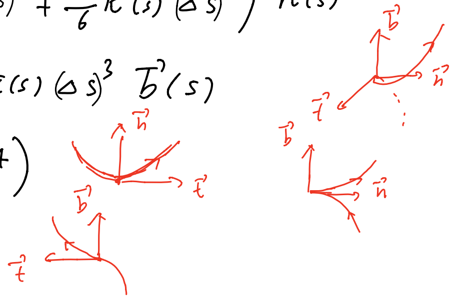
$$\vec r(s+\Delta s) = \vec r(s) + \vec r'(s)\Delta s + \frac{1}{2!}\vec r''(s)(\Delta s)^2 + \frac{1}{3!}\vec r'''(s)(\Delta s)^3 + O((\Delta s)^4)$$

$$= \vec r(s) + \vec t(s)\Delta s + \frac{1}{2!}\vec t'(s)(\Delta s)^2 + \frac{1}{3!}\vec t''(s)(\Delta s)^3 + O((\Delta s)^4)$$

$$= \vec{r}(s) + \vec{t}(s)\Delta s + \frac{1}{2!}\kappa(s)\vec{n}(s)(\Delta s)^2 + \frac{1}{3!}\left(\kappa'(s)\vec{n}(s) + \kappa(s)\vec{n}'(s)\right)(\Delta s)^3 + O((\Delta s)^4)$$

$$= \vec{r}(s) + \vec{t}(s)\Delta s + \frac{1}{2!}\kappa(s)\vec{n}(s)(\Delta s)^2 + \frac{1}{3!}\left(\kappa'(s)\vec{n}(s) - \kappa(s)^2\vec{t}(s) + \kappa(s)\tau(s)\vec{b}(s)\right)(\Delta s)^3 + O((\Delta s)^4)$$

$$= \vec{r}(s) + \left(\Delta s - \frac{\kappa(s)^2(\Delta s)^3}{6}\right)\vec{t}(s) + \left(\frac{1}{2}\kappa(s)(\Delta s)^2 + \frac{1}{6}\kappa'(s)(\Delta s)^3\right)\vec{n}(s) + \frac{1}{6}\kappa(s)\tau(s)(\Delta s)^3\,\vec{b}(s) + O((\Delta s)^4)$$



标架运动的观点：圆柱螺旋线逼近任一空间曲线.

$$\vec{t}(s+\Delta s) = \vec{t}(s) + \vec{t}'(s)\Delta s + \frac{1}{2!}\vec{t}''(s)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{t}(s) + \kappa(s)\vec{n}(s)\Delta s + \frac{1}{2}\left(\kappa'(s)\vec{n}(s) + \kappa(s)\vec{n}'(s)\right)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{t}(s) + \kappa(s)\vec{n}(s)\Delta s + \frac{(\Delta s)^2}{2}\left(\kappa'(s)\vec{n}(s) + \kappa(s)\left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right)\right) + O((\Delta s)^3)$$

$$= \left(1 - \frac{\kappa(s)^2}{2}(\Delta s)^2\right)\vec{t}(s) + \left(\kappa(s)\Delta s + \frac{\kappa'(s)}{2}(\Delta s)^2\right)\vec{n}(s) + \frac{\kappa(s)\tau(s)}{2}(\Delta s)^2\vec{b}(s) + O((\Delta s)^3)$$

$$\vec{n}(s+\Delta s) = \vec{n}(s) + \vec{n}'(s)\Delta s + \frac{1}{2!}\vec{n}''(s)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{n}(s) + \left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right)\Delta s + \frac{1}{2}\left(-\kappa'(s)\vec{t}(s) - \kappa(s)\vec{t}'(s) + \tau'(s)\vec{b}(s) + \tau(s)\vec{b}'(s)\right)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{n}(s) + \left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right)\Delta s + \frac{1}{2}\left(-\kappa'(s)\vec{t}(s) - \kappa(s)^2\vec{n}(s) + \tau'(s)\vec{b}(s) + \tau(s)\left(-\tau(s)\vec{n}(s)\right)\right)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \left(-\kappa(s)\Delta s - \frac{1}{2}\kappa'(s)(\Delta s)^2\right)\vec{t}(s) + \left(1 - \frac{1}{2}\kappa(s)^2(\Delta s)^2 - \frac{1}{2}\tau(s)^2(\Delta s)^2\right)\vec{n}(s) + \left(\tau(s)\Delta s + \frac{1}{2}\tau'(s)(\Delta s)^2\right)\vec{b}(s) + O((\Delta s)^3)$$

$$\vec{b}(s+\Delta s) = \vec{b}(s) + \vec{b}'(s)\Delta s + \frac{1}{2!}\vec{b}''(s)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{b}(s) + \left(-\tau(s)\vec{n}(s)\right)\Delta s$$

$$-\frac{1}{2}\left(\tau'(s)\vec{n}(s) + \tau(s)\vec{n}'(s)\right)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \vec{b}(s) - \tau(s)\vec{n}(s)\Delta s - \frac{(\Delta s)^2}{2}\tau'(s)\vec{n}(s) - \frac{1}{2}\tau(s)\left(-\kappa(s)\vec{t}(s) + \tau(s)\vec{b}(s)\right)(\Delta s)^2 + O((\Delta s)^3)$$

$$= \frac{(\Delta s)^2}{2}\tau(s)\kappa(s)\vec{t}(s) - \left(\tau(s)\Delta s + \tau'(s)\frac{(\Delta s)^2}{2}\right)\vec{n}(s)$$

$$\left(1 - \frac{1}{2}\tau(s)^2(\Delta s)^2\right)\vec{b}(s) + O((\Delta s)^3)$$

因此，

$$\left(\vec{t}(s+\Delta s), \vec{n}(s+\Delta s), \vec{b}(s+\Delta s)\right) \approx \left(\vec{t}(s), \vec{n}(s), \vec{b}(s)\right) \underbrace{\begin{pmatrix} \frac{\tau^2+\kappa^2\cos(\lambda\Delta s)}{\lambda^2} & -\frac{\kappa\sin(\lambda\Delta s)}{\lambda} & -\frac{\kappa\tau(-1+\cos(\lambda\Delta s))}{\lambda^2} \\ \frac{\kappa\sin(\lambda\Delta s)}{\lambda} & \cos(\lambda\Delta s) & -\frac{\tau\sin(\lambda\Delta s)}{\lambda} \\ -\frac{\kappa\tau(-1+\cos(\lambda\Delta s))}{\lambda^2} & \frac{\tau\sin(\lambda\Delta s)}{\lambda} & \frac{\kappa^2+\tau^2\cos(\lambda\Delta s)}{\lambda^2} \end{pmatrix}}_{=P}$$

这里"$\approx$"指一阶展开相等.

$\lambda = \sqrt{\tau^2+\kappa^2}$.

Ex. P为圆柱螺旋线的过渡矩阵.（作业）

§1.3. 曲线论基本定理　　　补充：$\mathbb{R}^3$中的刚体运动群

弧长参数的唯一性.

曲线的曲率、挠率与参数选取无关

Prop. 曲线的弧长、曲率、挠率在刚体运动下不变.

Pf. $\vec{r} = \vec{r}(s)$　参数曲线.　（假定曲率处处不为零）

$\sigma: \mathbb{R}^3 \to \mathbb{R}^3$　刚体运动, i.e.

$$\sigma: \vec{r} \longmapsto \sigma(\vec{r}) = A\vec{r} + \vec{r}_0$$

$$A^TA = I.\quad \det A = 1.$$

σ保距 $\Rightarrow$ s为 $\sigma\circ\vec{r}$ 的弧长参数.

设 $\vec{t}, \vec{n}, \vec{b}$ 为$\vec{r}$的 Frenet 标架.

$\tilde{\vec{t}}, \tilde{\vec{n}}, \tilde{\vec{b}}$ … $\sigma\circ\vec{r}$ … … … … …

则：$$\tilde{\vec{t}} = \frac{d\,\sigma\circ\vec{r}}{ds} = \sigma_*\left(\frac{d\vec{r}}{ds}\right) = \sigma_*\vec{t} = A\vec{t}$$

$$\frac{d\tilde{\vec{t}}}{ds} = A\frac{d\vec{t}}{ds} \Rightarrow \tilde{\kappa}(s)\tilde{\vec{n}}(s) = \kappa(s)A\vec{n}(s)$$

$$\Rightarrow \begin{cases} \tilde{\vec{n}}(s) = A\vec{n}(s) \\ \tilde{\kappa}(s) = \kappa(s) \end{cases}$$

$$\tilde{\vec{b}} = \tilde{\vec{t}} \times \tilde{\vec{n}} = (A\vec{t}) \times (A\vec{n}) = A(\vec{t}\times\vec{n}) = A\vec{b}$$

$$\tilde{\tau} = \frac{d\tilde{\vec{n}}}{ds}\cdot\tilde{\vec{b}} = A\frac{d\vec{n}}{ds}\cdot A\vec{b} = \tau.\ \#.$$

Thm. 设 $\vec{r}_1(s), \vec{r}_2(s)$ 为 $\mathbb{R}^3$ 中两条弧长参数曲线.

设 $\kappa_1(s) = \kappa_2(s) > 0$, $\forall s \in [0, s_1)$.

则存在一个刚体运动 $\sigma: \mathbb{R}^3 \to \mathbb{R}^3$, s.t.

唯一性　$$\vec{r}_1 = \sigma\circ\vec{r}_2.$$

Pf. 记 $(\vec{r}_1; \vec{t}_1, \vec{n}_1, \vec{b}_1)$ 与

$(\vec{r}_2; \vec{t}_2, \vec{n}_2, \vec{b}_2)$ 为两条曲线

的 Frenet 标架.

$s=0$时. 存在刚体运动 σ, s.t.

$$(\vec{r}_1(0); \vec{t}_1(0), \vec{n}_1(0), \vec{b}_1(0)) = \sigma(\vec{r}_2(0); \vec{t}_2(0), \vec{n}_2(0), \vec{b}_2(0))$$

下证明：$\forall s \in [0, s_1)$, 有：

$$(\vec{r}_1(s); \vec{t}_1(s), \vec{n}_1(s), \vec{b}_1(s)) = \sigma(\vec{r}_2(s); \vec{t}_2(s), \vec{n}_2(s), \vec{b}_2(s)).\quad (*)$$

Consider Frenet's formula:

$$\frac{d\vec{r}}{ds} = \vec{t}(s)$$

$$\frac{d\vec{t}}{ds} = \kappa(s)\, n(s)$$

$$\frac{d\vec{n}}{ds} = -\kappa(s)\vec{t}(s) \qquad + \tau(s)\vec{b}(s)$$

$$\frac{d\vec{b}}{ds} = \qquad -\tau(s)\vec{n}(s)$$

为关于 $\vec{r}, \vec{t}, \vec{n}, \vec{b}$ 的常微分方程组.
由解对初值的唯一性定理知：(*) 成立. #.

Thm. 设 $\kappa=\kappa(s)$, $\tau=\tau(s)$ 为 $[0, s_1)$ 上定义的两个连续可微函数 且 $\kappa(s)>0$, $\forall s\in[0,s_1)$. 则 ∃ $\mathbb{R}^3$ 的弧长参数曲线 $\vec{r}(s)$, $s\in[0,s_1)$, 以 $\kappa(s)$ 和 $\tau(s)$ 为曲率和挠率.

存在性

Pf. 考虑如下常微分方程组:

$$\frac{d}{ds}\begin{pmatrix}\vec{r}\\ \vec{e_1}\\ \vec{e_2}\\ \vec{e_3}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & \kappa(s) & 0\\ 0 & -\kappa(s) & 0 & \tau(s)\\ 0 & 0 & -\tau(s) & 0\end{pmatrix}\begin{pmatrix}\vec{r}\\ \vec{e_1}\\ \vec{e_2}\\ \vec{e_3}\end{pmatrix} \qquad (*)$$

给定初值
$\vec{r}_0, \vec{e}_{10}, \vec{e}_{20}, \vec{e}_{30}$. 其中 $(\vec{e}_{10}, \vec{e}_{20}, \vec{e}_{30})$ 为任一与 $\mathbb{R}^3$ 的自然定向同向的正交标架.

(*) 有唯一解满足: $(\vec{r}; \vec{e}_1, \vec{e}_2, \vec{e}_3)\big|_{s=0} = (\vec{r}_0; \vec{e}_{10}, \vec{e}_{20}, \vec{e}_{30})$

令: $g_{ij} = \langle \vec{e_i}, \vec{e_j}\rangle = g_{ji}$

则: $$\frac{dg_{ij}}{ds} = \left\langle \frac{d\vec{e_i}}{ds}, \vec{e_j}\right\rangle + \left\langle \vec{e_i}, \frac{d\vec{e_j}}{ds}\right\rangle$$

$$= \left\langle \sum_l \omega_{il}\vec{e_l}, \vec{e_j}\right\rangle + \left\langle \vec{e_i}, \sum_l \omega_{jl}\vec{e_l}\right\rangle$$

$$= \sum_l \left(\omega_{il} g_{lj} + \omega_{jl} g_{il}\right) \qquad (**)$$

$$\frac{dg_{ji}}{ds} = \sum_l \left(\omega_{jl} g_{li} + \omega_{il} g_{jl}\right) = \frac{dg_{ij}}{ds}$$

$\Rightarrow g_{ij}$, $i,j\geq 1$ 满足一个齐次线性常微分方程组.
初值 $g_{ij}\big|_{s=0} = \delta_{ij}$

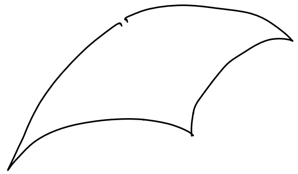
$g_{ij}\equiv\delta_{ij}$ 显然为 (**) 的一个解
由齐次线性常微分方程组解的唯一性定理
$\Rightarrow g_{ij}\equiv\delta_{ij}$.
$\Rightarrow e_1, e_2, e_3$ 在任意"时刻"s 是单位正交的.
且连续性 $\Rightarrow (e_1, e_2, e_3)\equiv 1$. $\Rightarrow$ 右手定向.

$\frac{d\vec{r}}{ds} = \vec{e}_1$ 且 $|\vec{e}_1| = 1 \Rightarrow \vec{r}$ 以 s 为弧长参数.
且 $\vec{e}_1$ 为单位切向量.

由 $\frac{d\vec{e_1}}{ds} = \kappa(s)\vec{e_2} \Rightarrow \kappa(s)$ 为 $\vec{r}$ 在 s 处的曲率.
$\vec{n} = \frac{1}{\kappa}\frac{d\vec{e_1}}{ds} = \vec{e_2}$ 为主法向量. $\vec{e_3} = \vec{e_1}\times\vec{e_2}$
$\Rightarrow \vec{e_3}$ 为副法向量. $\frac{d\vec{e_2}}{ds} = -\kappa\vec{e_1} + \tau\vec{e_3} \Rightarrow \tau$ 为挠率. #.

例: $\mathbb{R}^3$ 中曲率、挠率为常数的曲线的分类. ① $\kappa=0$. 直线.
② $\kappa>0$. $\tau=0$. 以 $|\frac{1}{\kappa}|$ 为半径的圆. ③ $\kappa>0$, $\tau\neq0$. 圆柱螺旋线.

§2. 曲面论



线动成面.

“参数曲面” 曲面

Def.1. 设 D 为 $\mathbb{R}^2$ 上的一个区域 (单连通开集).
称 映射 $\vec{q}: D \to \mathbb{R}^3$ 为光滑(参数)曲面, if
q^1, q^2, q^3 均为 D 上的光滑函数. 这里: $\vec{q}=(q^1,q^2,q^3)$.

Def.2. 设 D 为 $\mathbb{R}^2$ 上的一个区域. $\vec{q}: D \to \mathbb{R}^3$
为光滑曲面, 称 $\vec{q}$ 是正则的, if

$$\frac{\partial \vec{q}}{\partial u} \times \frac{\partial \vec{q}}{\partial v} \neq 0, \quad \forall (u,v) \in D.$$

这里: $\frac{\partial \vec{q}}{\partial u} = \left(\frac{\partial q^1}{\partial u}, \frac{\partial q^2}{\partial u}, \frac{\partial q^3}{\partial u}\right)$,

$\frac{\partial \vec{q}}{\partial v} = \left(\frac{\partial q^1}{\partial v}, \frac{\partial q^2}{\partial v}, \frac{\partial q^3}{\partial v}\right)$.

以后提到曲面常指 正则光滑参数曲面.

例. 函数的图.

$z=f(x,y)$. $\quad x,y \in D$.

$\vec{q}=(x, y, f(x,y))$.

$\vec{q}_x=(1,0,f_x)$, $\vec{q}_y=(0,1,f_y)$

$\vec{q}_x \times \vec{q}_y \neq 0$.

· $\vec{q}: D \to \mathbb{R}^3$. 改变参数:

$D \to \tilde{D}$

$(u,v) \longmapsto (\tilde{u},\tilde{v})$ $\quad \tilde{u}=\tilde{u}(u,v)$, $\tilde{v}=\tilde{v}(u,v)$

Jacobian

$$\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix}$$

若 $\frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} \neq 0$, 则由隐函数定理有: $\begin{cases} u=u(\tilde{u},\tilde{v}) \\ v=v(\tilde{u},\tilde{v}) \end{cases}$

$$\vec{q}(u,v)=\vec{q}(u(\tilde{u},\tilde{v}), v(\tilde{u},\tilde{v})) =: \vec{r}(\tilde{u},\tilde{v}).$$

(同一个“曲面”的不同参数表示.)

· 切平面.

参数曲面 S: $\vec{r}(u,v)=(r^1(u,v), r^2(u,v), r^3(u,v))$, $(u,v)\in D$.

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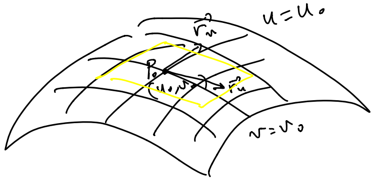


$\vec{r}_u := \frac{\partial \vec{r}}{\partial u}$

$= \left(\frac{\partial r^1}{\partial u}, \frac{\partial r^2}{\partial u}, \frac{\partial r^3}{\partial u}\right),$

$\vec{r}_v := \frac{\partial \vec{r}}{\partial v}$

$= \left(\frac{\partial r^1}{\partial v}, \frac{\partial r^2}{\partial v}, \frac{\partial r^3}{\partial v}\right).$

- 在点 $P_0 = \vec{r}(u_0, v_0)$ 处，$\vec{r}_u \times \vec{r}_v \neq 0$，
  由 $\vec{r}_u$，$\vec{r}_v$ 张成的平面称为曲面 S 在 P_0 处的切平面。记为 $T_{P_0}S$.

$\vec{r}_u$ 为参数曲线 $v=v_0$ 在 P 处的切向量，
$\vec{r}_v$ - - - - - $u=u_0$ - - - - - - - - -.

$\vec{r}_u \times \vec{r}_v$ 为 $T_{P_0}S$ 的一个法向量，记 $\vec{n} := \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.
单位法向量.

$\{\vec{r} = P;\ \vec{r}_u, \vec{r}_v, \vec{r}_u \times \vec{r}_v\}$ 构成 $\mathbb{R}^3$ 的一个自然定向的标架.

$T_{P_0}S = \{S$ 上过 P 点的参数曲线在 P 点的切向量$\}$
$\xleftrightarrow[\text{onto}]{1-1}$ $\{S$ 上过 P 点的参数曲线的等价类$\}$

16

设 $\vec{\gamma}: \mathbb{R} \to S \subseteq \mathbb{R}^3$
$t \mapsto (r^1(u(t), v(t)), r^2(u(t), v(t)), r^3(u(t), v(t))).$

$\dot{\vec{\gamma}} = \frac{d\vec{\gamma}}{dt} = \left(\frac{\partial r^1}{\partial u}\frac{du}{dt} + \frac{\partial r^1}{\partial v}\frac{dv}{dt}, \frac{\partial r^2}{\partial u}\dot{u} + \frac{\partial r^2}{\partial v}\dot{v}, \frac{\partial r^3}{\partial u}\dot{u} + \frac{\partial r^3}{\partial v}\dot{v}\right)$

$= \frac{\partial \vec{r}}{\partial u}\frac{du}{dt} + \frac{\partial \vec{r}}{\partial v}\frac{dv}{dt}$

$\dot{\gamma}|_{P_0} \in T_{P_0}S$.

反之，$\vec{r}_u|_{P_0} c_1 + \vec{r}_v|_{P_0} c_2$

$u = c_1 t$
$v = c_2 t$.

Prop. 1. 曲面的切平面与参数选取无关.

Proof. $\vec{q} = \vec{q}(u,v)$
$= (q^1(u,v), q^2(u,v))$

$\left.\begin{array}{l} u = u(\tilde{u}, \tilde{v}) \\ v = v(\tilde{u}, \tilde{v}) \end{array}\right\}$ 参数变换.

$\vec{q}(u,v) = \vec{q}(u(\tilde{u},\tilde{v}), v(\tilde{u},\tilde{v})) = \vec{r}(\tilde{u},\tilde{v})$

$P_0 \in S$. $T_{P_0}S \cong \mathrm{Span}\{\vec{q}_u, \vec{q}_v\}$

$\vec{q}_u = \frac{\partial \vec{r}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + \frac{\partial \vec{r}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u}$

$$\vec{q}_u = \vec{r}_{\tilde u}\frac{\partial \tilde u}{\partial u} + \vec{r}_{\tilde v}\frac{\partial \tilde v}{\partial v}$$

$$(\vec{q}_u, \vec{q}_v) = (\vec{r}_{\tilde u}, \vec{r}_{\tilde v})\begin{pmatrix} \frac{\partial \tilde u}{\partial u} & \frac{\partial \tilde u}{\partial v} \\ \frac{\partial \tilde v}{\partial u} & \frac{\partial \tilde v}{\partial v}\end{pmatrix}$$

$$\det\left(\quad\right) \neq 0$$

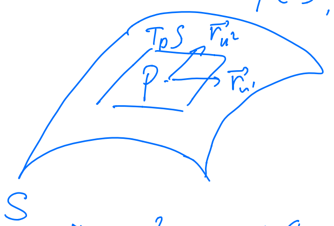
$$\Rightarrow T_{p_0}S \cong \mathrm{Span}\{\vec{r}_{\tilde u}, \vec{r}_{\tilde v}\}. \qquad \#.$$

推论. 单位法向量与曲面参数选取无关.

§2.1 第一基本形式

设 $\vec{r} = \vec{r}(u^1, u^2)$, $(u^1, u^2) \in D$ 为光滑曲面.

$p \in S$, $X, Y \in T_pS$.



$$X = X^1 \vec{r}_{u^1} + X^2 \vec{r}_{u^2}$$

$$Y = Y^1 \vec{r}_{u^1} + Y^2 \vec{r}_{u^2}$$

考虑 $\mathbb{R}^3$ 上的内积在 T_pS 上的限制. 仍为欧氏内积

$$\langle X, Y\rangle = \left\langle \sum_i X^i \vec{r}_{u^i}, \sum_j Y^j \vec{r}_{u^j}\right\rangle$$

$$= \sum_{i,j} X^i Y^j \langle \vec{r}_{u^i}, \vec{r}_{u^j}\rangle$$

$$= \sum_{i,j} X^i Y^j g_{ij}$$

$$g_{11} = \langle \vec{r}_{u^1}, \vec{r}_{u^1}\rangle$$

$$g_{12} = \langle \vec{r}_{u^1}, \vec{r}_{u^2}\rangle \qquad g_{ij} = \langle \vec{r}_{u^i}, \vec{r}_{u^j}\rangle$$

$$g_{21} = \langle \vec{r}_{u^2}, \vec{r}_{u^1}\rangle = g_{12}$$

$$g_{22} = \langle \vec{r}_{u^2}, \vec{r}_{u^2}\rangle$$

$$(g_{ij}) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22}\end{pmatrix} \qquad \det(g_{ij}) > 0, \ \forall p \in S.$$

称 g_{ij} 为曲面 S 上由 $\mathbb{R}^3$ 上欧氏内积诱导下来的度量. metric.

· g_{ij} 在不同参数下的变换关系:

(19)

$$\begin{cases} u^1 = u^1(\tilde u^1, \tilde u^2) \\ u^2 = u^2(\tilde u^1, \tilde u^2) \end{cases}$$

参数变换

$\tilde D \to D$

$(\tilde u^1, \tilde u^2) \mapsto (u^1, u^2)$

$$\vec r = \vec r(u^1, u^2) = \vec r\left(u^1(\tilde u^1, \tilde u^2), u^2(\tilde u^1, \tilde u^2)\right)$$

$$\tilde g_{ij} := \langle \vec r_{\tilde u^i}, \vec r_{\tilde u^j} \rangle$$

$$= \left\langle \frac{\partial \vec r}{\partial u^k}\frac{\partial u^k}{\partial \tilde u^i}, \frac{\partial \vec r}{\partial u^k}\frac{\partial u^k}{\partial \tilde u^j} \right\rangle$$

$$= \sum_{k,l} \frac{\partial u^k}{\partial \tilde u^i}\frac{\partial u^l}{\partial \tilde u^j} g_{kl}$$

Einstein summation convention

i.e., $$\boxed{\tilde g_{ij} = \frac{\partial u^k}{\partial \tilde u^i}\frac{\partial u^l}{\partial \tilde u^j} g_{kl}}$$

· 考虑 S上的光滑曲线：

$$\vec r = \vec r(u^1(t), u^2(t)), \quad t \in (a, b)$$

(20)

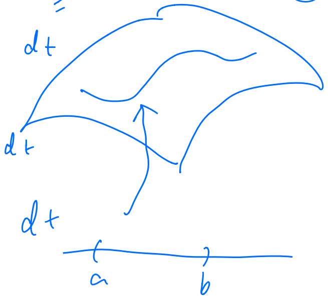
S 弧长参数.

$= \langle d\vec r, d\vec r \rangle$

$$ds = \sqrt{\left\langle \frac{d\vec r}{dt}, \frac{d\vec r}{dt} \right\rangle}\, dt$$

$$= \sqrt{\left\langle \sum \frac{\partial \vec r}{\partial u^i}\frac{du^i}{dt}, \sum \frac{\partial \vec r}{\partial u^j}\frac{du^j}{dt} \right\rangle}\, dt$$

$$= \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\, dt$$



$$(ds)^2 = g_{ij}\, du^i du^j \quad \text{(Einstein summation convention)}$$

$$= g_{11}(du^1)^2 + 2g_{12}\, du^1 du^2 + g_{22}(du^2)^2.$$

$$\begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} =: \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

称 $I := (ds)^2 = E(du^1)^2 + 2F\, du^1 du^2 + G(du^2)^2$ 为S的第一基本形式. 也称$(ds)^2$为S上的度量(metric).

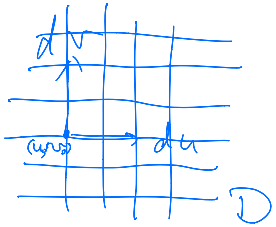
· 定理: 第一基本形式 $(ds)^2$ 与参数选取无关.

证: $\widetilde{(ds)}^2 = \tilde g_{ij}\, d\tilde u^i d\tilde u^j$

$$= \frac{\partial u^k}{\partial \tilde u^i}\frac{\partial u^l}{\partial \tilde u^j}\, g_{kl}\, \frac{\partial \tilde u^i}{\partial u^a} du^a \frac{\partial \tilde u^j}{\partial u^b} du^b$$

$$= \delta^k_a\, \delta^l_b\, g_{kl}\, du^a du^b$$

$$= g_{kl}\, du^k du^l = (ds)^2. \quad \#$$



$$(ds)^2 = E(du^1)^2 + 2F\, du^1 du^2 + G(du^2)^2$$

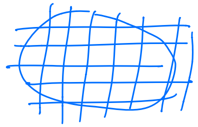
在每一点(u_0,v_0)，$(ds)^2$为二次形.

坐标变换关系: $\tilde g_{ij} = \frac{\partial u^k}{\partial \tilde u^i}\frac{\partial u^l}{\partial \tilde u^j}\, g_{kl}$

$$(\tilde g) = P^T \cdot (g) \cdot P$$

$P = \left(\frac{\partial u^k}{\partial \tilde u^i}\right) \qquad P^k_i = \frac{\partial u^k}{\partial \tilde u^i}$

· 面积: D: 平面区域 $\qquad \sigma(D) = \int_D |dx \wedge dy|$



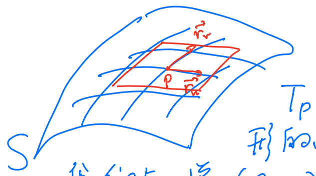
如何求光滑曲面的面积?

光滑曲面: $\vec r = \vec r(u,v)$, $(u,v) \in D$.

第I基本形式: $I = (ds)^2 = E(du^1)^2 + 2F\, du^1 du^2 + G(du^2)^2$

这里, $E = g_{11} = \langle \vec r_{u^1}, \vec r_{u^1} \rangle$,

$F = g_{12} = \langle \vec r_{u^1}, \vec r_{u^2} \rangle$, $G = g_{22} = \langle \vec r_{u^2}, \vec r_{u^2} \rangle$.



每点 $p = \vec r(u,v)$ 处, 固定p, 切平面 T_pS

T_pS上由 $\vec r_u, \vec r_v$ 张成的平行四边形的面积 = ?

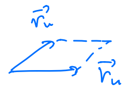
我们知道 (g_{ij}) 可由线性变换A化为标准形:

$(\tilde g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. $\qquad (\tilde g_{ij}) = A^T (g_{ij}) A$. (*)

$(e_1, e_2) = (\vec r_u, \vec r_v) A$, A为过渡矩阵.

i.e., $\vec r_u = p_{11} e_1 + p_{12} e_2$,

$\vec r_v = p_{21} e_1 + p_{22} e_2$,



$\Rightarrow$ $\vec r_u, \vec r_v$ 张成的平行四边形面积 $= \left| \begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{vmatrix} \right|$.

$= |\det(A^{-1})|$ 23

(*)两边同时取 det $\Rightarrow$ $1 = (\det A)^2 \det(g_{ij})$

$\Longrightarrow [\det(A^{-1})]^2 = g,$

这里, $g := \det(g_{ij}) = g_{11}g_{22} - g_{12}^2 > 0$

$\Rightarrow |\det(A^{-1})| = \sqrt{g}.$

Def. $\sigma(S) := \int_D \sqrt{g}\,|du\,dv|$

为曲面 $\vec{r} = \vec{r}(u,v)$ 在区域 D 上的面积,

Prop. $\sigma(D)$ 与参数(即曲面上的坐标)选取无关.

proof. $(u^1, u^2) \mapsto (\tilde{u}^1, \tilde{u}^2).$

$$\sigma(D) = \int_D \sqrt{g}\,|du^1 \wedge du^2| = \int_D \sqrt{g_{11}g_{22} - g_{12}^2}\,|du^1 \wedge du^2|$$

回顾到 $(\tilde{g}_{ij}) = P^T (g_{ij}) P,$

$P := \left(\frac{\partial u^k}{\partial \tilde{u}^i}\right)$, i.e., $P^k_i = \frac{\partial u^k}{\partial \tilde{u}^i}$

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Therefore,

$\tilde{g} = \det(\tilde{g}_{ij}) = \det(P^T)\, g\, \det(P)$

$= g \cdot (\det(P))^2$

$\Longrightarrow$

$$\sigma(D) = \int_{\tilde{D}} \sqrt{\frac{\tilde{g}}{(\det(P))^2}}\left|\left(\frac{\partial u^1}{\partial \tilde{u}^1} d\tilde{u}^1 + \frac{\partial u^1}{\partial \tilde{u}^2} d\tilde{u}^2\right) \wedge \left(\frac{\partial u^2}{\partial \tilde{u}^1} d\tilde{u}^1 + \frac{\partial u^2}{\partial \tilde{u}^2} d\tilde{u}^2\right)\right|$$

$$= \int_{\tilde{D}} \sqrt{\tilde{g}}\, \frac{1}{|\det(P)|} \left(\frac{\partial u^1}{\partial \tilde{u}^1}\frac{\partial u^2}{\partial \tilde{u}^2} - \frac{\partial u^1}{\partial \tilde{u}^2}\frac{\partial u^2}{\partial \tilde{u}^1}\right) d\tilde{u}^1 \wedge d\tilde{u}^2$$

$$= \int_{\tilde{D}} \sqrt{\tilde{g}}\, |d\tilde{u}^1 \wedge d\tilde{u}^2|. \quad \#$$

定理: $(ds)^2$ 在刚体运动下不变. (25)

pf. 曲面 S: $\vec r = \vec r(u,v)$, $(u,v)\in D$.

$\sigma: \mathbb{R}^3 \to \mathbb{R}^3$

$\vec r \mapsto A\vec r + \vec r_0$. $A \in SO(3)$.

$\tilde{\vec r}_u = A\vec r_u$, $\tilde{\vec r}_v = A\vec r_v$

$$(d\tilde s)^2 = \langle \tilde{\vec r}_u, \tilde{\vec r}_u\rangle (du)^2 + 2\langle \tilde{\vec r}_u, \tilde{\vec r}_v\rangle du\,dv + \langle \tilde{\vec r}_v, \tilde{\vec r}_v\rangle (dv)^2$$
$$= \langle \vec r_u, \vec r_u\rangle (du)^2 + 2\langle \vec r_u, \vec r_v\rangle du\,dv + \langle \vec r_v, \vec r_v\rangle (dv)^2$$
$$= (ds)^2. \quad \#$$

Example 1. 平面.

$$\vec r(u,v) = (u, v, \text{const})^T \quad \vec r_u = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \vec r_v = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

$$(ds)^2 = (du)^2 + (dv)^2.$$

Example 2. 柱面.

$$\vec r(u,v) = (x(u), y(u), v)$$
$$\vec r_u = (x', y', 0)$$
$$\vec r_v = (0, 0, 1)$$
$$\Rightarrow \mathrm{I} = (ds)^2 = ((x')^2 + (y')^2)(du)^2 + (dv)^2$$

Example 3. 球面.

$$\vec r(\theta,\varphi) = (a\cos\theta\cos\varphi, a\cos\theta\sin\varphi, a\sin\theta)$$
$$\vec r_\theta = (-a\sin\theta\cos\varphi, -a\sin\theta\sin\varphi, a\cos\theta)$$
$$\vec r_\varphi = (-a\cos\theta\sin\varphi, a\cos\theta\cos\varphi, 0)$$

$$\mathrm{I}(\theta,\varphi) = a^2(d\theta)^2 + a^2\cos^2\theta\,(d\varphi)^2$$

(26)

§2.2. 第二基本形式

S: $\vec r = \vec r(u,v)$, $(u,v)\in D$.

$$\vec n := \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|} \quad \text{单位法向量.}$$

$$\mathrm{II} := -\langle d\vec r, d\vec n\rangle$$

$$\langle \vec r_u, \vec n\rangle \equiv 0, \quad \langle \vec r_v, \vec n\rangle \equiv 0$$
$$\Rightarrow \langle \vec r_{uu}, \vec n\rangle + \langle \vec r_u, \vec n_u\rangle \equiv 0,$$
$$\langle \vec r_{uv}, \vec n\rangle + \langle \vec r_u, \vec n_v\rangle \equiv 0,$$

$$\langle \vec r_{vu}, \vec n\rangle + \langle \vec r_v, \vec n_u\rangle \equiv 0,$$
$$\langle \vec r_{vv}, \vec n\rangle + \langle \vec r_v, \vec n_v\rangle = 0$$

定义

$$L := \langle \vec r_{uu}, \vec n\rangle = -\langle \vec r_u, \vec n_u\rangle$$
$$M := \langle \vec r_{uv}, \vec n\rangle = -\langle \vec r_u, \vec n_v\rangle = -\langle \vec r_v, \vec n_u\rangle$$
$$N := \langle \vec r_{vv}, \vec n\rangle = -\langle \vec r_v, \vec n_v\rangle.$$

$$\mathrm{II} = -\langle d\vec r, d\vec n\rangle$$
$$= -\langle \vec r_u du + \vec r_v dv, \vec n_u du + \vec n_v dv\rangle$$
$$= L(du)^2 + 2M\,du\,dv + N(dv)^2.$$

几何意义:　　　　　　　　　　　　　　　　　　　　27

$$\vec r(u_0+\Delta u, v_0+\Delta v) - \vec r(u_0,v_0)$$
$$= \vec r_u(u_0,v_0)\Delta u + \vec r_v(u_0,v_0)\Delta v$$
$$+ \frac{1}{2}\Big(\vec r_{uu}(u_0,v_0)(\Delta u)^2 + 2\vec r_{uv}(u_0,v_0)\Delta u\Delta v$$
$$+ \vec r_{vv}(u_0,v_0)(\Delta v)^2\Big) + o\big((\Delta u)^2+(\Delta v)^2\big)$$

点 $\vec r(u_0+\Delta u, v_0+\Delta v)$ 到 $T_{\vec r(u_0,v_0)}S$ 的有向距离为

$$\langle \vec r(u_0+\Delta u, v_0+\Delta v) - \vec r(u_0,v_0), \vec n\rangle$$
$$= \frac{1}{2}\big(L(\Delta u)^2 + 2M\Delta u\Delta v + N(\Delta v)^2\big)$$
$$+ o\big((\Delta u)^2+(\Delta v)^2\big)$$

- (u_0,v_0) 附近.

椭圆点: $LN-M^2\big|_{(u_0,v_0)} > 0$　　$\mathrm{II}(u_0,v_0)$ 正定, 凸(凹)

双曲点: $LN-M^2\big|_{(u_0,v_0)} < 0$　　------ 不定, 马鞍型

抛物点: $LN-M^2\big|_{(u_0,v_0)} = 0$　　---- 退化.

- 定理: 改变参数, 同向 II 不变
  反向 II 反号.

Pf. 同向 $\vec n = \tilde{\vec n}$.
反向 $\vec n = -\tilde{\vec n}$.　　#.

- 定理: II 在刚体运动下保持不变.　　　　　　　　28

Pf. $\sigma: \mathbb{R}^3 \to \mathbb{R}^3$

$\vec r \mapsto A\vec r + \vec r_0 = \tilde{\vec r}$　　$\tilde{\vec r}_u = A\vec r_u$, $\tilde{\vec r}_v = A\vec r_v$

$$\tilde{\vec r}_u \times \tilde{\vec r}_v = A(\vec r_u \times \vec r_v)$$
$$\Rightarrow \tilde{\vec n} = A\vec n.$$　　#.

国庆放假补充作业思考题. 证明如下欧拉公式:

$$\sum_{m\in\mathbb{Z}} (-1)^m q^{3m^2+m} = \prod_{k=1}^{\infty}(1-q^{2k})$$

(提示1. 两边均等于 $1 - q^2 - q^4 + q^{10} + q^{14} - q^{24} - q^{30} + q^{44} + q^{52} - q^{70} - q^{80} + \cdots$

提示2. 请未必按提示1作答.)

29

A: Geometry is number theory?

B: Number theory is geometry??

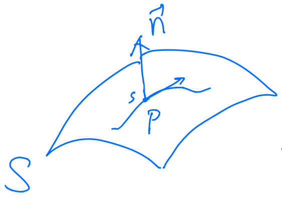
C: Just 写作业 ~ ☺

30

· 法曲率.

$\vec{r} = \vec{r}(u,v) \subset \mathbb{R}^3$ 中光滑曲面 S

$(u,v) \in D.$



设 $\vec{r} = \vec{r}(u(s), v(s))$, $s > 0$ 为 S 上弧长参数曲线.

$$\vec{t} = \frac{d\vec{r}}{ds} = \vec{r}_u \frac{du}{ds} + \vec{r}_v \frac{dv}{ds}$$

$$\frac{d^2\vec{r}}{ds^2} = \vec{r}_u \frac{d^2u}{ds^2} + \vec{r}_v \frac{d^2v}{ds^2} + \vec{r}_{uu}\left(\frac{du}{ds}\right)^2 + \vec{r}_{vu}\frac{du}{ds}\frac{dv}{ds} + \vec{r}_{uv}\frac{dv}{ds}\frac{du}{ds} + \vec{r}_{vv}\left(\frac{dv}{ds}\right)^2$$

$$\begin{aligned} k_n &:= \left(\frac{d^2\vec{r}}{ds^2}, \vec{n}\right) \\ &= (\vec{r}_{uu}, \vec{n})\left(\frac{du}{ds}\right)^2 + 2(\vec{r}_{uv}, \vec{n})\frac{du}{ds}\frac{dv}{ds} + (\vec{r}_{vv}, \vec{n})\left(\frac{dv}{ds}\right)^2 \\ &= L\left(\frac{du}{ds}\right)^2 + 2M\frac{du}{ds}\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2. \end{aligned}$$

称 k_n 为曲面 S 在 P 点沿曲线 $\vec r=\vec r(u(s),v(s))$ 的法曲率. (31)

$$k_n=\frac{L(du)^2+2M\,du\,dv+N(dv)^2}{(ds)^2}$$
$$=\frac{L(du)^2+2M\,du\,dv+N(dv)^2}{E(du)^2+2F\,du\,dv+G(dv)^2}$$
$$=\frac{II(s)}{I(s)}$$

· $L=\langle \vec r_{uu},\vec n\rangle=-\langle \vec r_u,\vec n_u\rangle$

$M=\langle \vec r_{uv},\vec n\rangle=-\langle \vec r_u,\vec n_v\rangle=-\langle \vec r_v,\vec n_u\rangle$

$N=\langle \vec r_{vv},\vec n\rangle=-\langle \vec r_v,\vec n_v\rangle$

$$II:=-\langle d\vec r,d\vec n\rangle=L(du)^2+2M\,du\,dv+N(dv)^2.$$

Examples.

1. 平面.

$$\vec r(u,v)=(u,v,Const)^T$$

$$\vec r_u=\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \vec r_v=\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \vec n=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$I=(du)^2+(dv)^2 \qquad II=0.$

2. 柱面. (32)

$$\vec r(u,v)=(x(u),y(u),v)^T$$
$$\vec r_u=(x'(u),y'(u),0)^T$$
$$\vec r_v=(0,0,1)^T$$
$$I=(x'(u)^2+y'(u)^2)(du)^2+(dv)^2$$
$$\vec r_{uu}=(x''(u),y''(u),0)^T$$
$$\vec r_{uv}=0$$
$$\vec r_{vv}=0.$$
$$\begin{vmatrix} i & j & k\\ x' & y' & 0\\ 0 & 0 & 1\end{vmatrix}$$
$$\vec n=(y'(u),-x'(u),0)^T$$
$$\Rightarrow II=(x''(u)y'(u)-y''(u)x'(u))(du)^2$$

若u是平面曲线 $\begin{cases}x=x(u)\\ y=y(u)\\ z=Const\end{cases}$ 的弧长参数,

则有: $I=(du)^2+(dv)^2$

$II=-\kappa(du)^2$

当平面曲线是 a 为半径的圆时, $\kappa=\frac{1}{a}$. 此时 $II=-\frac{1}{a}(du)^2$.

2.3. Pair of quadratic forms: $I=g_{ij}\,du^i du^j$

$II=b_{ij}\,du^i du^j$

S

记: $G:=(g_{ij})=\begin{pmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}$, $B:=\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}$

$G = G^T, \quad B = B^T. \quad G > 0. \quad$ real 33

Consider the pencil $B - \lambda G$:

$$\det(B - \lambda G)$$

$$= \begin{vmatrix} b_{11} - \lambda g_{11} & b_{12} - \lambda g_{12} \\ b_{21} - \lambda g_{21} & b_{22} - \lambda g_{22} \end{vmatrix}$$

$$= (b_{11} - \lambda g_{11})(b_{22} - \lambda g_{22}) - (b_{12} - \lambda g_{12})^2$$

$$= \lambda^2 (g_{11} g_{22} - g_{12}^2) - (g_{11} b_{22} + b_{11} g_{22} - 2 b_{12} g_{12})\lambda$$

$$+ b_{11} b_{22} - b_{12}^2 = 0 \quad (*)$$

· Lemma: 此二次方程有两个实根 λ_1, λ_2.

称 λ_1, λ_2 为 pair (G, B) 的特征值(或广义特征值).

Pf. of Lemma. G正定 $\Rightarrow$ 可取 T_pS 的一组标准正交基 d_1, d_2.

$$\langle d_i, d_j \rangle = \delta_{ij}.$$

记: $(r_{u^1}, r_{u^2}) = (d_1, d_2) A$, $\quad A$ real, invertible.

则: $G = A^T I A = A^T A$.

$$((A^{-1})^T B A^{-1})^T = (A^{-1})^T B^T A^{-1} = (A^{-1})^T B A^{-1}$$

i.e. real symmetric

$\Rightarrow$ ∃ 实正交阵 Q s.t. $Q^{-1}(A^{-1})^T B A^{-1} Q = \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}$.

记: $C = A^{-1} Q$ 则: $C^T = Q^T (A^{-1})^T = Q^{-1} (A^{-1})^T$ 34

$$\begin{cases} C^T G C = Q^{-1} (A^{-1})^T A^T A Q = I \\ C^T B C = \begin{pmatrix} \mu_1 & \\ & \mu_2 \end{pmatrix}. \end{cases}$$

$\underbrace{\text{Columns}}_{v_1, v_2}$ of Q are eigenvectors of $(A^{-1})^T B A^{-1}$ forming a basis of $\mathbb{R}^2$.

$$(A^{-1})^T B A^{-1} v_j = \mu_j v_j$$

$$\Rightarrow B A^{-1} v_j = \mu_j A^T v_j$$

$$\Rightarrow B A^{-1} A A^{-1} v_j = \mu_j A^T A A^{-1} v_j$$

$$\Rightarrow B A^{-1} v_j = \mu_j G A^{-1} v_j$$

$\xRightarrow{A可逆}$ μ_1, μ_2 are eigenvalues of (G, B).

$A^{-1} v_1$, $A^{-1} v_2$ are corresp. 广义 eigenvectors.

in particular λ_1, λ_2 are real. #.

In the above pf, we also showed that

Lemma. ∃ 可逆实矩阵 C, s.t.

$$C^T G C = I$$

$$C^T B C = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}.$$

Def. 称 λ_1, λ_2 为曲面 S 在点 P 的主曲率.
称 $H := \frac{1}{2}(\lambda_1+\lambda_2)$ 为曲面 S 在点 P 的平均曲率 (中曲率).
称 $K := \lambda_1\lambda_2$ 为曲面 S ------ Gauss 曲率.

(*)(⇒)
$$\lambda^2 - \frac{b_{22}g_{11} + b_{11}g_{22} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2}\lambda + \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = 0$$

由于 λ_1, λ_2 为根 $\Rightarrow (\lambda-\lambda_1)(\lambda-\lambda_2) = 0$

$$\Rightarrow \lambda_1+\lambda_2 = \frac{b_{22}g_{11} - 2b_{12}g_{12} + b_{11}g_{22}}{g} = 2H$$

$$\lambda_1\lambda_2 = \frac{b_{11}b_{22} - b_{12}^2}{g} = K$$

$$\lambda^2 - 2H\lambda + K = 0$$

$$\lambda_{1,2} = H \pm \sqrt{H^2 - 4K}$$

Lemma. λ_1, λ_2 与参数选取无关.

proof. $b_{ij} = \langle \vec{r}_{u^iu^j}, \vec{n} \rangle$

$$\vec{r}_{\tilde{u}^i} = \vec{r}_{u^k}\frac{\partial u^k}{\partial \tilde{u}^i}$$

$$\tilde{b}_{ij} = \langle \vec{r}_{\tilde{u}^i\tilde{u}^j}, \vec{n} \rangle = \left\langle \vec{r}_{u^ku^l}\frac{\partial u^k}{\partial \tilde{u}^i}\frac{\partial u^l}{\partial \tilde{u}^j}, \vec{n} \right\rangle$$

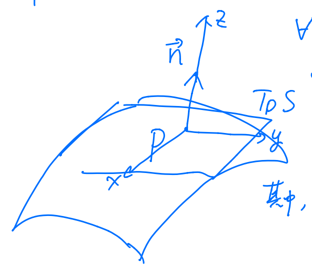
$$= \frac{\partial u^k}{\partial \tilde{u}^i} b_{kl} \frac{\partial u^l}{\partial \tilde{u}^j}$$

i.e., $(\tilde{b}_{ij}) = P^T (b_{ij}) P \qquad P = \left(\frac{\partial u^k}{\partial \tilde{u}^i}\right)$

recall that $(\tilde{g}_{ij}) = P^T (g_{ij}) P \qquad \det P \neq 0.$

$$\det(B - \lambda G) = 0 \Leftrightarrow \det(P^T(B-\lambda G)P) = 0$$
$$\Leftrightarrow \det(\tilde{B} - \lambda\tilde{G}) = 0. \quad \#.$$

* 一个好的坐标系. S: 正则光滑曲面, 参数 $(\beta_1, \beta_2) \in D \subseteq \mathbb{R}^2$.



$\forall P \in S$. Fix P. 以 P 为原点, $\vec{n}$ 为 z 轴单位向量
建 (任一) P-xyz 右手直角坐标系. 则 S 可有
如下参数表示:
$\vec{r} = \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v))^T$.

其中, $\begin{cases} x(u,v) = u = u^1 \\ y(u,v) = v = u^2 \\ z(u,v) = f(u,v) \end{cases}$

(Exercise: 证明 $(\beta_1, \beta_2) \mapsto (u,v)$ 为容许参数变换.)

$\vec{r}_u = (1, 0, f_u)$, $\vec{r}_v = (0, 1, f_v)$

$$g_{11} = 1 + f_u^2, \quad g_{12} = f_uf_v, \quad g_{22} = 1 + f_v^2$$

在 P 点处: $f_u = f_v = 0$. 以下计算仅在 P 处.

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1.$$

$$\mathrm{I}|_P = (du)^2 + (dv)^2 \qquad (\text{仅在 } P \text{ 一点处})$$

$\vec{r}_{uu} = (0, 0, f_{uu})$

$\vec{r}_{uv} = (0, 0, f_{uv})$ $\qquad$ $\vec{n}$ 为 z 轴上单位向量.

$\vec{r}_{vv} = (0, 0, f_{vv})$

$$b_{11} = f_{uu}|_p,\quad b_{12} = f_{uv}|_p,\quad b_{22} = f_{vv}|_p.$$

$$\mathrm{II}|_p = b_{ij}\,du^i du^j = f_{uu}(du)^2 + (f_{uv})\,du\,dv + f_{vv}(dv)^2$$

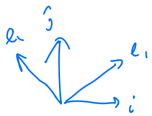
$\det(B - \lambda G) = 0$ $\quad$ 在P处: $\det(B-\lambda) = 0$

$\iff (\lambda-\lambda_1)(\lambda-\lambda_2) = 0$

λ_1, λ_2: 主曲率.

由于 B 是实对称阵 $\Rightarrow \exists$ 正交变换 $\alpha: T_pS \to T_pS$, s.t.

$$\alpha^T B \alpha = \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}.$$



$$(e_1, e_2) = (i, j)\,\alpha \Rightarrow \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \alpha^{-1}\begin{pmatrix} u \\ v \end{pmatrix}.$$

$$\begin{cases} x = \tilde{u} \\ y = \tilde{v} \\ z = f(u(\tilde{u},\tilde{v}), v(\tilde{u},\tilde{v})) = \frac{1}{2}\lambda_1\tilde{u}^2 + \frac{1}{2}\lambda_2\tilde{v}^2 + o(\tilde{u}^2+\tilde{v}^2). \end{cases}$$

$K = \lambda_1\lambda_2 > 0$ $\quad$ 椭圆点

$K = \lambda_1\lambda_2 < 0$ $\quad$ 双曲点.

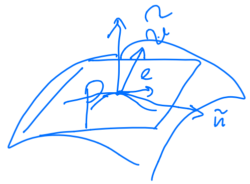
$K = \lambda_1\lambda_2 = 0$ $\quad$ 抛物点.

在新参数下,

$$\mathrm{I}|_p = (d\tilde{u})^2 + (d\tilde{v})^2$$

$$\mathrm{II}|_p = \lambda_1(d\tilde{u})^2 + \lambda_2(d\tilde{v})^2$$

法曲率 $\quad k_n = \dfrac{\mathrm{II}|_p}{\mathrm{I}|_p} = \dfrac{\lambda_1(d\tilde{u})^2 + \lambda_2(d\tilde{v})^2}{(d\tilde{u})^2 + (d\tilde{v})^2}$



$$e = (\dot{\tilde{u}}, \dot{\tilde{v}})$$

$$d\tilde{u} = \dot{\tilde{u}}\,dt,\quad d\tilde{v} = \dot{\tilde{v}}\,dt$$

$$k_n = \frac{\lambda_1\dot{\tilde{u}}^2 + \lambda_2\dot{\tilde{v}}^2}{\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2} = \lambda_1\frac{\dot{\tilde{u}}^2}{\dot{\tilde{u}}^2+\dot{\tilde{v}}^2} + \lambda_2\frac{\dot{\tilde{v}}^2}{\dot{\tilde{u}}^2+\dot{\tilde{v}}^2}$$

$= \lambda_1\cos^2\theta + \lambda_2\sin^2\theta$ $\quad$ (θ是e与principal direction夹角)

(Euler 公式).

定理: $\quad k_n = \lambda_1\cos^2\theta + \lambda_2\sin^2\theta \quad (*)$

Pf. $\quad \lambda_1 \neq \lambda_2$. $\quad P = \left(\dfrac{\partial u^k}{\partial \tilde{u}^l}\right)$

$$(\vec{r}_{\tilde{u}}, \vec{r}_{\tilde{v}}) = (\vec{r}_u, \vec{r}_v)\,P$$

$$B\xi_m = \lambda_m G\xi_m,\quad m = 1, 2.$$

$$\Rightarrow P^T B P P^{-1}\xi_m = \lambda_m P^T G P P^{-1}\xi_m$$

由于 $\tilde{B} = P^T B P$, $\tilde{G} = P^T G P$

有 $\tilde{B} P^{-1}\xi_m = \lambda_m \tilde{G} P^{-1}\xi_m \overset{(引理)}{\Rightarrow} \tilde{\xi}_m = P^{-1}\xi_m$

$\Rightarrow$ 主方向与参数选取无关.

$\Rightarrow$ RHS of (*) 与参数选取无关.

LHS of (*) 也与 - - — — .

$\lambda_1 = \lambda_2 = \lambda_0 \Rightarrow k_n = \lambda_0(\cos^2\theta + \sin^2\theta) = \lambda_0$ #.

Remark. 如下引理也可说明"好"参数的存在性.

<u>Lemma.</u> $A^{-1}v_1$, $A^{-1}v_2$ 构成 T_pS 的标准正交基.

Pf. 记 $\xi_1 = A^{-1}v_1$, $\xi_2 = A^{-1}v_2$

$$\xi_i^T G \xi_j = v_i^T (A^{-1})^T A^T A A^{-1} v_j = v_i^T v_j = \delta_{ij} \quad \#.$$

· 若 $\lambda_1 \neq \lambda_2$, 有:

设 $\lambda_1 < \lambda_2$. 则:

$$\lambda_1 \le k_n \le \lambda_2$$

· 若 $\lambda_1 = \lambda_2$, 有:

$$k_n = \lambda_1 = \lambda_2 =: k$$

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2} \equiv k$$

取 $\dot{u} = 0$, $\dot{v} = 1$. $\Rightarrow N = kG$

取 $\dot{u} = 1$, $\dot{v} = 0$ $\Rightarrow L = kE$

取 $\dot{u} = \dot{v} = 1$. $\Rightarrow M = kF$.

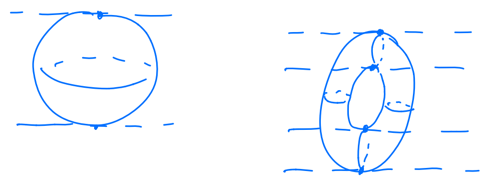
$\Big\}\ \mathrm{II} = k\mathrm{I}$.

· $\lambda_1 = \lambda_2$ 的点称为曲面的脐点.

· Global geometry.

Morse theory

$$\chi(M) = \sum_{p \in \{\text{critical pts of a Morse fct on } M\}} (-1)^{\sigma(p)}$$



2.4 Examples.

1. $\begin{cases} x = u \\ y = v \\ z = f(u,v) \end{cases}$

$\vec{r}_u = (1, 0, f_u)^T$

$\vec{r}_v = (0, 1, f_v)^T$

$E = \vec{r}_u \cdot \vec{r}_u = 1 + f_u^2$

$F = \vec{r}_u \cdot \vec{r}_v = f_u f_v$

$G = \vec{r}_v \cdot \vec{r}_v = 1 + f_v^2$

$$\vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} = \frac{(-f_u, -f_v, 1)^T}{|(-f_u, -f_v, 1)|} = \frac{(-f_u, -f_v, 1)^T}{\sqrt{f_u^2 + f_v^2 + 1}}$$

$$L = \vec{r}_{uu} \cdot \vec{n} = (0,0,f_{uu})^T \cdot \vec{n} = \frac{f_{uu}}{\sqrt{f_u^2+f_v^2+1}}$$

$$M = \vec{r}_{uv} \cdot \vec{n} = \frac{f_{uv}}{\sqrt{f_u^2+f_v^2+1}}$$

$$N = \vec{r}_{vv} \cdot \vec{n} = \frac{f_{vv}}{\sqrt{f_u^2+f_v^2+1}}$$

$$(g_{ij}) = \begin{pmatrix} E & F \\ F & G \end{pmatrix}, \quad (b_{ij}) = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

$$K = \frac{\det(b_{ij})}{\det(g_{ij})} = \frac{LN-M^2}{EG-F^2} = \frac{\frac{f_{uu}f_{vv}}{f_u^2+f_v^2+1} - \frac{f_{uv}^2}{f_u^2+f_v^2+1}}{(1+f_u^2)(1+f_v^2) - f_u^2 f_v^2}$$

$$= \frac{f_{uu}f_{vv} - f_{uv}^2}{(1+f_u^2+f_v^2)^2}$$

$$\det\begin{pmatrix} L-\lambda E & M-\lambda F \\ M-\lambda F & N-\lambda G \end{pmatrix} = (L-\lambda E)(N-\lambda G) - (M-\lambda F)^2$$

$$= \lambda^2(EG-F^2) - \lambda(EN+LG-2MF) + LN - M^2$$

$$2H = \lambda_1+\lambda_2 = \frac{EN+LG-2MF}{EG-F^2} = \frac{(1+f_u^2)\frac{f_{vv}}{\sqrt{g}} + \frac{f_{uu}}{\sqrt{g}}(1+f_v^2) - 2\frac{f_{uv}f_u f_v}{\sqrt{g}}}{g}$$

$$= \frac{(1+f_u^2)f_{vv} + (1+f_v^2)f_{uu} - 2f_u f_v f_{uv}}{(1+f_u^2+f_v^2)^{\frac{3}{2}}}$$

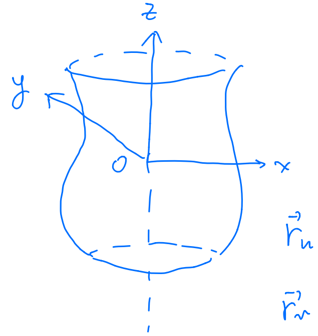
Def. 中曲率处处为零的曲面称为极小曲面.

· 从例1中, 我们看到曲面
$\vec{r}(u,v) = (u, v, f(u,v))^T$ 为极小曲面, iff.

$$(1+f_u^2)f_{vv} + (1+f_v^2)f_{uu} - 2f_u f_v f_{uv} = 0.$$

2. 旋转曲面.

$$\vec{r}(u,v) = \big(f(u)\cos v,\ f(u)\sin v,\ g(u)\big)^T, \quad \text{且 } f(u) > 0.$$
$(u,v) \in D$ $\qquad \forall (u,v)\in D$



xz 平面上与 z 轴不交的曲线
$x = f(u),\ z = g(u)$
绕 z 轴旋转得到.

$$\vec{r}_u = \big(f'(u)\cos v,\ f'(u)\sin v,\ g'(u)\big)^T$$

$$\vec{r}_v = \big(-f(u)\sin v,\ f(u)\cos v,\ 0\big)^T$$

$$E = \vec{r}_u \cdot \vec{r}_u = f'(u)^2 + g'(u)^2$$

$$F = \vec{r}_u \cdot \vec{r}_v = 0, \quad G = \vec{r}_v \cdot \vec{r}_v = f(u)^2$$

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$$\vec r_{uu} = \left(f''(u)\cos v,\ f''(u)\sin v,\ g''(u) \right)^T$$
$$\vec r_{uv} = \left(-f'(u)\sin v,\ f'(u)\cos v,\ 0 \right)^T$$
$$\vec r_{vv} = \left(-f(u)\cos v,\ -f(u)\sin v,\ 0 \right)^T$$

$$\vec n = \frac{\vec r_u \times \vec r_v}{|\vec r_u \times \vec r_v|} = \frac{\left(-fg'\cos v,\ -fg'\sin v,\ ff' \right)^T}{\sqrt{f^2 g'^2 + f^2 f'^2}}$$

$$= \frac{\left(-g'\cos v,\ -g'\sin v,\ f' \right)^T}{\sqrt{g'^2 + f'^2}}$$

$$L = \vec r_{uu}\cdot \vec n = \frac{f'g'' - g'f''}{\sqrt{f'^2 + g'^2}}$$

$$M = \vec r_{uv}\cdot \vec n = 0$$

$$N = \vec r_{vv}\cdot \vec n = \frac{fg'}{\sqrt{f'^2+g'^2}}$$

$$K = \frac{\det B}{\det G} = \frac{LN - M^2}{EF - G^2} = \frac{\dfrac{(f'g''-g'f'')fg'}{f'^2+g'^2}}{(f'^2+g'^2)f^2}$$

$$= \frac{g'(f'g''-g'f'')}{f\cdot(f'^2+g'^2)^2}$$

$$\det(B - \lambda G) = \begin{vmatrix} \dfrac{f'g''-g'f''}{\sqrt{f'^2+g'^2}} - \lambda(f'^2+g'^2) & 0 \\ 0 & \dfrac{fg'}{\sqrt{f'^2+g'^2}} - \lambda f^2 \end{vmatrix}$$

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$$\Rightarrow k_1 = \frac{f'g''-g'f''}{(f'^2+g'^2)^{\frac{3}{2}}}, \quad k_2 = \frac{fg'}{f^2\sqrt{f'^2+g'^2}} = \frac{g'}{f\sqrt{f'^2+g'^2}}$$

$$H = \frac{1}{2}\left(\frac{f'g''-g'f''}{(f'^2+g'^2)^{3/2}} + \frac{g'}{f\sqrt{f'^2+g'^2}} \right).$$

若 u 为 xz 平面上 曲线 $(f(u), g(u))$ 的弧长参数, 则:

$$f'(u)^2 + g'(u)^2 = 1. \qquad (*)$$

$$k_1 = f'g'' - g'f'', \quad k_2 = \frac{g'}{f}$$

$$K = \frac{g'}{f}\left(f'g'' - g'f'' \right)$$

$$H = \frac{1}{2}\left(f'g'' - g'f'' + \frac{g'}{f} \right).$$

Or,

$$\xRightarrow{\frac{\partial}{\partial u}(*)} 2f'f'' + 2g'g'' = 0$$

$$\Rightarrow g'(f'g'' - g'f'') = f'g'g'' - g'^2 f''$$
$$= -f'^2 f'' - f''(1 - f'^2)$$

$$= -f''$$

$$\Rightarrow \quad k_1 = -\frac{f''}{g'}, \quad k_2 = \frac{g'}{f}, \quad K = -\frac{f''}{f},$$

$$H = \frac{1}{2}\left(\frac{g'}{f} - \frac{f''}{g'}\right)$$

In particular,

1) 常 Gauss 曲率旋转曲面.

(1a). $K = c^2 > 0$.

$$\Rightarrow f''(u) + c^2 f(u) = 0$$

$$f = A\cos(cu) + B\sin(cu)$$

$$g(u) = \pm \int_0^u \sqrt{1 - f'(u)^2}\, du$$

$$= \pm \int_0^u \sqrt{1 - c^2(-A\sin(cu) + B\cos(cu))^2}\, du$$

(e.g. $B = 0$. $A = \frac{1}{c}$ 时, $f = \frac{1}{c}\cos(cu)$, $g = \pm\frac{1}{c}\sin cu$
半径为 $1/c$ 的球面.)

(1b) $K = 0$. $f'' = 0$

$f = Au + b$.

$$g(u) = \pm\sqrt{1-A^2}\, u + C$$

$0 \le A \le 1$

$A = 0$, 圆柱面.

$A = 1$, 平面.

$0 < A < 1$, 圆锥面.

(1c) $K = -c^2 < 0$. $f'' - c^2 f = 0$

$$f = A\cosh(cu) + B\sinh(cu)$$

$$g(u) = \pm \int \sqrt{1 - c^2(A\sinh(cu) + B\cosh(cu))^2}\, du$$

$$\text{e.g.} \left(\begin{array}{l} f(u) = \frac{1}{c} e^{-cu} \\ g(u) = \pm \int_0^u \sqrt{1 - e^{-2ct}}\, dt \end{array} \right)$$

pseudo-sphere

2) 常平均曲率旋转曲面.

(2a) $H = 0$. 极小曲面.

$$f'^2 + g'^2 = 1.$$

$$\frac{g'}{f} - \frac{f''}{g'} = 0$$

$$\Rightarrow ff'' = g'^2 = 1 - f'^2$$

$$\Rightarrow (ff')' = 1$$

$\Rightarrow f f' = u + A$

$\Rightarrow f^2 = u^2 + 2Au + B$

$f = \sqrt{u^2 + 2Au + B}$.

$g' = \pm\sqrt{1 - f'^2} = \pm\sqrt{\dfrac{B - A}{u^2 + 2Au + B}}$

e.g. $A = 0$, $B = a^2$.

(对u作平移变换总可有$A=0$).

xz平面上:

$x = f(u) = \sqrt{u^2 + a^2}$

$z = g(u) = \pm\int_0^u \dfrac{a\,dt}{\sqrt{t^2 + a^2}} = \pm a \operatorname{arcsinh}\left(\dfrac{u}{a}\right)$.

$\Rightarrow \quad x = a\cosh\left(\dfrac{z}{a}\right)$

悬链线. (Catenary)

悬链面 (Catenoid)

2b). $\dfrac{g'}{f} - \dfrac{f''}{g'} = 2H_0 \neq 0$.

$g' = \sqrt{1 - f'^2}$

$f'^2 + g'^2 = 1 \Rightarrow f'f'' + g'g'' = 0$

$$\Rightarrow \frac{d(g'f)}{df} = g' + f g'' \frac{1}{f'} = g' - \frac{f f''}{g'}$$

$$= f\left(\frac{g'}{f} - \frac{f''}{g'}\right) = 2H_0 f$$

$\Rightarrow \quad g'f = H_0 f^2 - A^2$.

$\Rightarrow \quad \sqrt{1 - f'^2} = \dfrac{H_0 f^2 - A^2}{f}$

$$\frac{df}{du} = f' = \frac{\sqrt{f^2 - (H_0 f^2 - A^2)^2}}{f}$$

令 $B^2 = 1 + 4A^2 H_0$ $\Rightarrow \dfrac{4H_0^2\, d(f^2)}{\sqrt{4B^2 - (4H_0^2 f^2 - (1+B^2))^2}} = 2H_0\, du$

$$\Rightarrow \arcsin \frac{(2Hf)^2 - (1+B^2)}{2B} = 2H_0 u + C$$

考虑关于u的平移变换，可不妨设 $C = 0$.

$$\Rightarrow f = \frac{1}{2H_0}\sqrt{1 + B^2 + 2B\sin(2H_0 u)}$$

$$g(u) = \int_0^u \sqrt{1 - f'^2}\, du = \int_0^u \frac{1 + B\sin(2Hu)}{\sqrt{1 + B^2 + 2B\sin(2Hu)}}\, du.$$

(Delaunay surface).

plane, cylinder, sphere,

catenoid, unduloid, nodoid.

2.5. 等温参数系.

$$\vec{r}(p,q) = (x(p,q), y(p,q), z(p,q))^T, \quad (p,q) \in D \subseteq \mathbb{R}^2$$

$$dx^2 + dy^2 + dz^2 = E(dp)^2 + 2F\,dp\,dq + G(dq)^2 = I$$

$$g = EG - F^2 > 0.$$

Thm. 设 E, F, G 是 p, q 的实解析函数. 则:

$\exists\, u, v$ s.t.

$$I = f(u,v)\left((du)^2 + (dv)^2\right)$$

Pf.

$$I = E(dp)^2 + 2F\,dp\,dq + G(dq)^2$$

$$= \left(\sqrt{E}\,dp + \frac{F + i\sqrt{g}}{\sqrt{E}}\,dq\right)\left(\sqrt{E}\,dp + \frac{F - i\sqrt{g}}{\sqrt{E}}\,dq\right)$$

$$= f(u,v)\left((du)^2 + (dv)^2\right)$$

To find $\lambda = \lambda(p,q)$ s.t.

$$\lambda(p,q)\left(\sqrt{E}\,dp + \frac{F + i\sqrt{g}}{\sqrt{E}}\,dq\right) = du + i\,dv$$

$$= \frac{\partial u}{\partial p}dp + \frac{\partial u}{\partial q}dq + i\left(\frac{\partial v}{\partial p}dp + \frac{\partial v}{\partial q}dq\right).$$

$$\bar{\lambda}(p,q)\left(\sqrt{E}\,dp + \frac{F - i\sqrt{g}}{\sqrt{E}}\,dq\right) = du - i\,dv.$$

$$\Longleftrightarrow \quad \lambda\sqrt{E} = \frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p}$$

$$\lambda\frac{F + i\sqrt{g}}{\sqrt{E}} = \frac{\partial u}{\partial q} + i\frac{\partial v}{\partial q}$$

$$\Longrightarrow \quad (F + i\sqrt{g})\left(\frac{\partial u}{\partial p} + i\frac{\partial v}{\partial p}\right) = \left(\frac{\partial u}{\partial q} + i\frac{\partial v}{\partial q}\right)\cdot E$$

$$\Longleftrightarrow \quad \begin{cases} F\dfrac{\partial u}{\partial p} - \sqrt{g}\dfrac{\partial v}{\partial p} = \dfrac{\partial u}{\partial q}\cdot E \\[2ex] \sqrt{g}\dfrac{\partial u}{\partial p} + F\dfrac{\partial v}{\partial p} = \dfrac{\partial v}{\partial q}\cdot E \end{cases}$$

$$\Longleftrightarrow \quad \frac{\partial v}{\partial p} = \frac{F\frac{\partial u}{\partial p} - E\frac{\partial u}{\partial q}}{\sqrt{g}}, \quad \frac{\partial v}{\partial q} = \frac{G\frac{\partial u}{\partial p} - F\frac{\partial u}{\partial q}}{\sqrt{g}}$$

$$\frac{\partial u}{\partial p} = \frac{E\frac{\partial v}{\partial q} - F\frac{\partial v}{\partial p}}{\sqrt{g}}, \quad \frac{\partial u}{\partial q} = \frac{F\frac{\partial v}{\partial q} - G\frac{\partial v}{\partial p}}{\sqrt{g}}$$

(51)

$\partial_p \partial_q v = \partial_p \partial_q v$,
$\partial_p \partial_q u = \partial_q \partial_p u$, $\Rightarrow$ $Lu = 0$ and $Lv = 0$

$$L = \frac{\partial}{\partial q}\left(\frac{F\frac{\partial}{\partial p} - E\frac{\partial}{\partial q}}{\sqrt{g}}\right) + \frac{\partial}{\partial p}\left(\frac{F\partial_q - G\partial_p}{\sqrt{g}}\right)$$

(Beltrami 方程).

$$E, F, G \xrightarrow[\text{real analytic}]{} \text{有解} \Rightarrow u, v, \lambda. \quad \#$$

令: $w = u + iv$
$\bar{w} = u - iv$

则: $I = g(w, \bar{w})\, dw\, d\bar{w}$

· Another pf of the above Thm.

Lemma 1. 设 $f(u,v)$, $g(u,v)$ 为单连通区域 D 上的实解析函数. 则存在 D 上的非零解析函数 $\lambda(u,v)$ 及 $k(u,v)$

$$\lambda(f\,du + g\,dv) = dk(u,v)$$

$$\begin{cases} \dfrac{\partial k}{\partial u} = \lambda f \\ \dfrac{\partial k}{\partial v} = \lambda g \end{cases} \qquad \frac{\partial(\lambda f)}{\partial v} = \frac{\partial(\lambda g)}{\partial u}$$

$$\lambda_v f + \lambda f_v = \lambda_u g + \lambda g_u$$

(52)

$$\lambda_u g - \lambda_v f + \lambda(f_v - g_u) = 0. \quad \#$$

Lemma 2. 正则参数曲面 S: $\vec{r} = \vec{r}(u,v)$ 上给定两个处处线性无关的光滑切向量场 $\vec{a}(u,v)$, $\vec{b}(u,v)$. 则 $\forall$ 点 $p \in S$, 存在 U_p 及 $\tilde{u}, \tilde{v}$ s.t. $\vec{r}_{\tilde{u}} \parallel \vec{a}$, $\vec{r}_{\tilde{v}} \parallel \vec{b}$.

pf. $\vec{a} = a_1 \vec{r}_u + a_2 \vec{r}_v$
$\vec{b} = b_1 \vec{r}_u + b_2 \vec{r}_v$

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0.$$

分析: 若存在 $\tilde{u}, \tilde{v}$ s.t. $\vec{r}_{\tilde{u}} \parallel \vec{a}$, $\vec{r}_{\tilde{v}} \parallel \vec{b}$.

$$\Rightarrow \vec{r}_{\tilde{u}} = \lambda \vec{a}, \quad \vec{r}_{\tilde{v}} = \mu \vec{b}$$

$$\Rightarrow \vec{r}_{\tilde{u}} = \lambda a_1 \vec{r}_u + \lambda a_2 \vec{r}_v, \quad \vec{r}_{\tilde{v}} = \mu b_1 \vec{r}_u + \mu b_2 \vec{r}_v$$

$$\Rightarrow S = \begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix} = \begin{pmatrix} \lambda a_1 & \lambda a_2 \\ \mu b_1 & \mu b_2 \end{pmatrix}$$

$$\Rightarrow S^{-1} = \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} = \frac{1}{\lambda\mu(a_1b_2-a_2b_1)} \begin{pmatrix} \mu b_2 & -\lambda a_2 \\ -\mu b_1 & \lambda a_1 \end{pmatrix}$$ 53

$$d\tilde{u} = \frac{\partial \tilde{u}}{\partial u} du + \frac{\partial \tilde{u}}{\partial v} dv$$

$$= \frac{1}{\lambda(a_1b_2-a_2b_1)} (b_2 du - b_1 dv)$$

$$d\tilde{v} = \frac{\partial \tilde{v}}{\partial u} du + \frac{\partial \tilde{v}}{\partial v} dv$$

$$= \frac{1}{\mu(a_1b_2-a_2b_1)} (-a_2 du + a_1 dv)$$

由引理1： $\forall p \in S$，$\exists$ p 的邻域 U 及 U 上处处非零的连续可微函数 ξ, η s.t. U 上存在容许参数 $\tilde{u}, \tilde{v}$ 满足：

$$d\tilde{u} = \xi(b_2 du - b_1 dv)$$

$$d\tilde{v} = \eta(-a_2 du + a_1 dv)$$

$$\Rightarrow \begin{pmatrix} \frac{\partial \tilde{u}}{\partial u} & \frac{\partial \tilde{v}}{\partial u} \\ \frac{\partial \tilde{u}}{\partial v} & \frac{\partial \tilde{v}}{\partial v} \end{pmatrix} = \begin{pmatrix} \xi b_2 & -\eta a_2 \\ -\xi b_1 & \eta a_1 \end{pmatrix}$$

$$|\ | = \xi\eta(a_1b_2-a_2b_1) \neq 0.$$

$\Rightarrow$ $(\tilde{u}, \tilde{v})$ 是曲面 S 在 U 内的容许参数.

且 $$\begin{pmatrix} \frac{\partial u}{\partial \tilde{u}} & \frac{\partial v}{\partial \tilde{u}} \\ \frac{\partial u}{\partial \tilde{v}} & \frac{\partial v}{\partial \tilde{v}} \end{pmatrix} = \frac{1}{\xi\eta(a_1b_2-a_2b_1)} \begin{pmatrix} \eta a_1 & \eta a_2 \\ \xi b_1 & \xi b_2 \end{pmatrix}$$ 54

$$\Rightarrow \vec{r}_{\tilde{u}} = \frac{1}{\xi(a_1b_2-a_2b_1)} a$$

$$\vec{r}_{\tilde{v}} = \frac{1}{\eta(a_1b_2-a_2b_1)} b \quad . \quad \#$$

Lemma 3. 设 $S: \vec{r} = \vec{r}(u,v)$ 为正则曲面. 则 $\forall p \in S$，存在 p 的一个邻域 U 及 U 上的参数 $(\tilde{u}, \tilde{v})$，s.t. $\vec{r}_{\tilde{u}}, \vec{r}_{\tilde{v}}$ 是正交的.

Pf. 在每一点，对 $(\vec{r}_u, \vec{r}_v)$ 作 Schmidt 正交化.

令： $e_1 = \frac{\vec{r}_u}{|\vec{r}_u|} = \frac{r_u}{\sqrt{E}}$

$$e_2 = \frac{1}{\sqrt{EG-F^2}} \left(-\frac{F}{\sqrt{E}} \vec{r}_u + \sqrt{E} \vec{r}_v \right)$$

则 e_1, e_2 为 S 上的单位正交标架场.（光滑）

由 Lemma 2 知：U 上存在 $(\tilde{u}, \tilde{v})$ s.t. $\vec{r}_{\tilde{u}} \parallel e_1$，$\vec{r}_{\tilde{v}} \parallel e_2$. #

Pf. $I = E(du)^2 + G(dv)^2$

$$= (\sqrt{E}\,du + i\sqrt{G}\,dv)(\sqrt{E}\,du - i\sqrt{G}\,dv)$$

令 $\omega = \sqrt{E}\,du + i\sqrt{G}\,dv$

$\Rightarrow$ 在小邻域U及U上 $\exists\, \lambda(u,v)$ 处处非零复解析函数 s.t.

$$\lambda(u,v)\,\omega = d\mu(u,v)$$

$\mu = \mu_1 + \sqrt{-1}\,\mu_2$

则

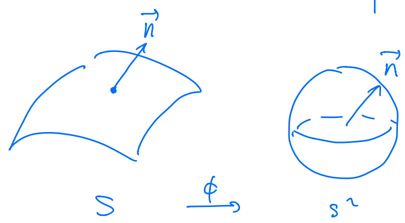
$$I = \omega\bar{\omega} = \frac{1}{|\lambda|^2}\left(d\mu_1^2 + d\mu_2^2\right). \qquad \#$$

2.6 Weingarten 变换

S: 光滑正则曲面. $\vec{r} = \vec{r}(u,v)$, $u,v \in D \subseteq \mathbb{R}^2$.

Gauss 映射: $\phi: S \longrightarrow S^2$ (unit sphere)

$$p \longmapsto \vec{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$



考虑 $d\phi$. $\quad d\vec{n} = \vec{n}_u\,du + \vec{n}_v\,dv$

$(\vec{n},\vec{n}) \equiv 1 \Rightarrow (\vec{n}_u, \vec{n}) \equiv 0,\ (\vec{n}_v, \vec{n}) \equiv 0$

$\Rightarrow \vec{n}_u, \vec{n}_v \in T_pS$.

定义 $W: T_pS \to T_pS$

$$\vec{r}_u\,du + \vec{r}_v\,dv \longmapsto -(\vec{n}_u\,du + \vec{n}_v\,dv)$$

称 W 为曲面的 Weingarten 变换.

Lemma. Weingarten 变换与曲面参数选取无关.

Pf. $(u,v) \mapsto (\tilde{u},\tilde{v})$

$$W(\vec{r}_{\tilde{u}}\,d\tilde{u} + \vec{r}_{\tilde{v}}\,d\tilde{v})$$
$$= W\left(\left(\vec{r}_u\frac{\partial u}{\partial\tilde{u}} + \vec{r}_v\frac{\partial v}{\partial\tilde{u}}\right)\left(\frac{\partial\tilde{u}}{\partial u}du + \frac{\partial\tilde{u}}{\partial v}dv\right) + \left(\vec{r}_u\frac{\partial u}{\partial\tilde{v}} + \vec{r}_v\frac{\partial v}{\partial\tilde{v}}\right)\left(\frac{\partial\tilde{v}}{\partial u}du + \frac{\partial\tilde{v}}{\partial v}dv\right)\right)$$
$$= W(\vec{r}_u\,du + \vec{r}_v\,dv)$$
$$= -(\vec{n}_u\,du + \vec{n}_v\,dv) = -(\vec{n}_{\tilde{u}}\,d\tilde{u} + \vec{n}_{\tilde{v}}\,d\tilde{v}). \quad \#$$

Prop. 设 $\vec{v} \in T_pS$. $|v| = 1$. 则 S 沿 v 方向的法曲率

$$k_n(v) = \langle W(v), v\rangle.$$

Pf. $\vec{v} = \vec{r}_u\,du + \vec{r}_v\,dv$

$$\langle W(\vec{v}), \vec{v}\rangle = -\langle \vec{n}_u\,du + \vec{n}_v\,dv,\ \vec{r}_u\,du + \vec{r}_v\,dv\rangle$$
$$= -\langle \vec{n}_u, \vec{r}_u\rangle (du)^2 - 2M\,du\,dv - \langle \vec{n}_v, \vec{r}_v\rangle (dv)^2$$
$$= k_n(\vec{v}). \quad \#$$

Thm. $W: T_pS \to T_pS$ 是 self-adjoint 的, i.e.,

$$\langle W(v), w\rangle = \langle v, W(w)\rangle.$$

Pf. $v = \vec{r}_u\,du^{(1)} + \vec{r}_v\,dv^{(1)}$

$w = \vec{r}_u\,du^{(2)} + \vec{r}_v\,dv^{(2)}$

$$LHS = -\langle \vec{n}_u\,du^{(1)} + \vec{n}_v\,dv^{(1)},\ \vec{r}_u\,du^{(2)} + \vec{r}_v\,dv^{(2)}\rangle$$
$$= -L\,du^{(1)}du^{(2)} - M\,du^{(1)}dv^{(2)} - M\,dv^{(1)}du^{(2)} - N\,dv^{(1)}dv^{(2)}$$
$$= -\langle \vec{r}_u\,du^{(1)} + \vec{r}_v\,dv^{(1)},\ \vec{n}_u\,du^{(2)} + \vec{n}_v\,dv^{(2)}\rangle$$
$$= RHS. \quad \#$$

下面我们来看 W 在基 $\{\vec{r}_u, \vec{r}_v\}$ 下的矩阵.

$$W(\vec{r}_u, \vec{r}_v) = (\vec{r}_u, \vec{r}_v)\, W = (\vec{r}_u, \vec{r}_v)\begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

$$W(\vec{r}_u) = a\vec{r}_u + b\vec{r}_v = -\vec{n}_u, \quad (1)$$
$$W(\vec{r}_v) = c\vec{r}_u + d\vec{r}_v = -\vec{n}_v. \quad (2)$$

$$(1) \Rightarrow \begin{cases} aE + bF = L \\ aF + bG = M \end{cases}$$

$$\Rightarrow a = \frac{LG - MF}{EG - F^2}, \quad b = \frac{ME - LF}{EG - F^2}$$

$$(2) \Rightarrow \begin{cases} cE + dF = M \\ cF + dG = N \end{cases}$$

$$\Rightarrow c = \frac{MG - NF}{EG - F^2}, \quad d = \frac{NE - MF}{EG - F^2}$$

i.e. $$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

(Indeed, $$\frac{\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}}{EG - F^2} \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \frac{\begin{pmatrix} GL - FM & GM - FN \\ ME - LF & EN - MF \end{pmatrix}}{EG - F^2}).$$

i.e. $W = G^{-1}B$. (过渡矩阵).

Prop. κ_1, κ_2 为 W 的特征值, 主方向为 W 的特征向量.

pf. $$\det(\lambda I - W) = \det(\lambda I - G^{-1}B) = \det G \cdot \det(\lambda G - B). \quad \#.$$

self-adjoint 也可在自然坐标系中看:

pf. 设 ξ, η: 列向量, v, w 的坐标.
$$\langle W(v), w \rangle = (G^{-1}B\xi)^T G\eta = \xi^T B G^{-1} G \eta = \xi^T B \eta$$
$$\langle v, Ww \rangle = \xi^T G G^{-1} B \eta = \xi^T B \eta. \quad \#.$$

· 关于自然记号:

$$\vec{n} = \vec{n}(u, v) \qquad p = (u_0, v_0)$$

$$\vec{n} = \vec{n}(u_0, v_0) + \vec{n}_u(u_0, v_0)\, du + \vec{n}_v(u_0, v_0)\, dv + O\left((du)^2 + (dv)^2\right)$$

$$d\phi(\vec{r}_u\, du + \vec{r}_v\, dv) = \vec{n}_u\, du + \vec{n}_v\, dv.$$

(push-forward:
$$\phi_*\left(\frac{\partial}{\partial u}\right) = \frac{\partial}{\partial u}, \quad \phi_*\left(\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial v}.$$).

2.7. More examples

Digression:

椭圆点： II 正定或负定 $\Leftrightarrow LN-M^2>0 \Leftrightarrow K>0$

双曲点： II 不定 $\Leftrightarrow LN-M^2<0 \Leftrightarrow K<0$

抛物点： II 退化 $\Leftrightarrow LN-M^2=0 \Leftrightarrow K=0.$

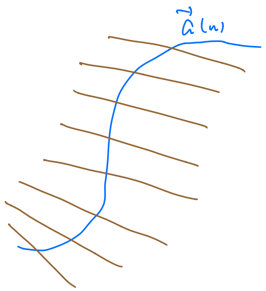
$$\left(K=\frac{LN-M^2}{EG-F^2}\right).$$

---

1. 直纹面.

S: $\vec{r}(u,v)=\vec{a}(u)+v\vec{b}(u)$, $\vec{b}(u)\neq 0$

其中, $\vec{a}(u)$ 为一条空间曲线. 固定 u,
$\vec{r}(u,v)$ 为过点$\vec{a}(u)$ 且方向为$\vec{b}(u)$ 的直线,
称为 S 的直母线. 假定S是正则的.



· 直纹面的 Gauss 曲率:

$\vec{r}_u=\vec{a}'(u)+v\vec{b}'(u)$

$\vec{r}_v=\vec{b}(u)$

$\vec{r}_{uu}=\vec{a}''(u)+v\vec{b}''(u)$

$\vec{r}_{uv}=\vec{b}'(u)$

$\vec{r}_{vv}=0$

$\vec{n}=\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$. $N=\langle\vec{r}_{vv},\vec{n}\rangle=0$

$\Rightarrow K=\dfrac{LN-M^2}{EG-F^2}=\dfrac{-M^2}{EG-F^2}\leq 0.$

· Def. Gauss 曲率恒为零的直纹面称为可展曲面.

· Prop. ① S是可展曲面 iff $(\vec{a}',\vec{b},\vec{b}')\equiv 0.$

② - - - - - - - - 沿直母线, S的单位法向量不变.

proof. ① $\Leftrightarrow M\equiv 0. \Leftrightarrow \langle\vec{r}_{uv},\vec{n}\rangle\equiv 0$

$\Leftrightarrow \vec{b}'(u)\in \mathrm{Span}_{\mathbb{R}}\{\vec{r}_u,\vec{r}_v\}=T_{\vec{r}(u,v)}S.$

$\Leftrightarrow (\vec{a}',\vec{b},\vec{b}')\equiv 0.$

② $\vec{n}=\dfrac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}=\dfrac{(\vec{a}'(u)+v\vec{b}'(u))\times\vec{b}(u)}{|(\vec{a}'(u)+v\vec{b}'(u))\times\vec{b}(u)|}$

$\forall v_1,v_2,\ \vec{n}(u,v_1)\equiv\vec{n}(u,v_2)$

$\Leftrightarrow \forall v_1,v_2,\ (\vec{a}'(u)+v_1\vec{b}'(u))\times\vec{b}(u)\,/\!/\,(\vec{a}'(u)+v_2\vec{b}'(u))\times\vec{b}(u)$

$\Leftrightarrow \forall v_1,v_2,\ \left[(\vec{a}'(u)+v_1\vec{b}'(u))\times\vec{b}(u)\right]\times\left[(\vec{a}'(u)+v_2\vec{b}'(u))\times\vec{b}(u)\right]\equiv 0$

$\left((\vec{a}\times\vec{b})\times\vec{c}=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{b}\cdot\vec{c})\vec{a}\right)$

$\Leftrightarrow \forall v_1,v_2,\ (\vec{a}'(u)+v_1\vec{b}'(u))\cdot\left((\vec{a}'(u)+v_2\vec{b}'(u))\times\vec{b}(u)\right)\vec{b}(u)$
$-\ \vec{b}(u)\cdot\left((\vec{a}'(u)+v_2\vec{b}'(u))\times\vec{b}(u)\right)(\vec{a}'(u)+v_1\vec{b}'(u))\equiv 0$

$\Leftrightarrow \forall v_1,v_2,\ v_2\,\vec{a}'(u)\cdot(\vec{b}'(u)\times\vec{b}(u))\,\vec{b}(u)$
$+\ v_1\,\vec{b}'(u)\cdot(\vec{a}'(u)\times\vec{b}(u))\,\vec{b}(u)\equiv 0$

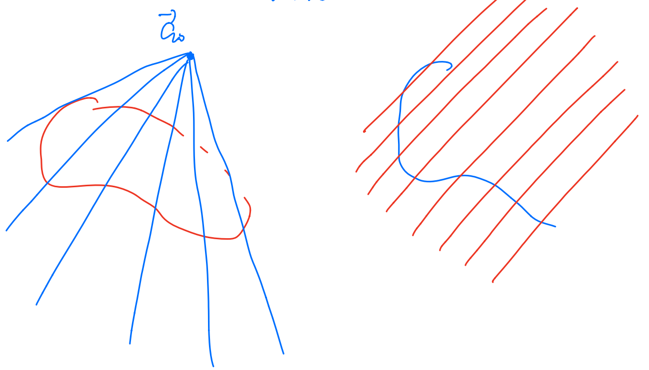
$\Leftrightarrow \forall v_1,v_2,\ (v_1-v_2)(\vec{a}'(u),\vec{b}(u),\vec{b}'(u))\,\vec{b}(u)\equiv 0$

$\Leftrightarrow (\vec{a}'(u),\vec{b}(u),\vec{b}'(u))\equiv 0.$ #.

e.g. ① $\vec{r}(u,v)=\vec{a}_0+v\vec{b}(u)$. 以 a_0 为顶点的锥面.
S 的直母线均过定点 $\vec{a}_0$.

② $\vec{r}(u,v)=\vec{a}(u)+v\vec{b}_0$，柱面.

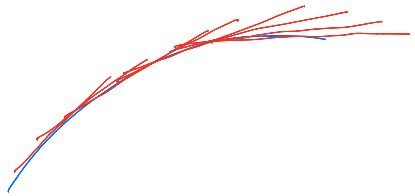
S的直母线均平行.



③ $\vec{\rho}=\vec{\rho}(t)$ 正则曲线.

$\vec{r}(v,t):=\vec{\rho}(t)+v\vec{\rho}'(t)$

切线全体构成空间的一张曲面. 称为曲线的切线面.



作业：验证：锥面、柱面、切线面均为可展曲面.

可展曲面分类：

$\vec{r}(u,v)=\vec{a}(u)+v\vec{b}(u)$，且

$\vec{b}(u)\neq 0$，$(\vec{a}',\vec{b},\vec{b}')\equiv 0$.

1. $\vec{b}(u)\times\vec{b}'(u)\equiv 0$.

$\vec{c}(u):=\dfrac{\vec{b}(u)}{|\vec{b}(u)|}$ $\Rightarrow |\vec{c}(u)|\equiv 1$.

$\Rightarrow \vec{c}(u)\cdot\vec{c}'(u)\equiv 0$

$\Rightarrow \vec{c}'(u)\cdot\vec{b}(u)\equiv 0$

$\vec{b}(u)=|\vec{b}(u)|\,\vec{c}(u)$

$\Rightarrow \vec{b}'(u)=\frac{d}{du}(|\vec{b}(u)|)\,\vec{c}(u)+|\vec{b}(u)|\,\vec{c}'(u)$

$\Rightarrow \vec{b}'(u)\times\vec{b}(u)=|\vec{b}(u)|\,\vec{c}'(u)\times\vec{b}(u)\equiv 0$

$\Rightarrow \vec{c}'(u)=\lambda(u)\vec{b}(u)$

$\vec{c}'(u)\cdot\vec{b}(u)=\lambda(u)|\vec{b}(u)|^2\equiv 0 \Rightarrow \lambda(u)\equiv 0$

$\Rightarrow \vec{c}(u)\equiv\vec{c}_0$.

$\Rightarrow$ $\vec{b}(u)$ 方向与u无关. $\Rightarrow$ 柱面.

2. $\vec{b}(u)\times\vec{b}'(u)\neq 0$.

$\Rightarrow \vec{a}'(u)=\lambda(u)\vec{b}(u)+\mu(u)\vec{b}'(u)$

令，$\tilde{\vec{a}}(u):=\vec{a}(u)-\mu(u)\vec{b}(u)$.

则 $\tilde{\vec{a}}'(u)=\vec{a}'(u)-\frac{d}{du}\big(\mu(u)\vec{b}(u)\big)$

$=\lambda(u)\vec{b}(u)+\mu(u)\vec{b}'(u)-\mu(u)\vec{b}'(u)-\mu'(u)\vec{b}(u)$

$=\big(\lambda(u)-\mu'(u)\big)\vec{b}(u)$.

2a). $\tilde{\vec{a}}'(u)\equiv 0$. 则 $\tilde{\vec{a}}(u)\equiv\vec{a}_0$.

$\vec{r}(u,v)=\vec{a}_0+(\mu(u)+v)\vec{b}(u)$

$\Rightarrow$ 所有直母线均经过 $\vec{c}$ $\Rightarrow$ 锥面.

2b). $\tilde{\vec{a}}'(u) \neq 0$. $\Rightarrow$ $\tilde{\vec{a}}$ 为正则曲线.

$$\vec{r}(u,v) = a(u) + v\,\vec{b}(u)$$
$$= \tilde{\vec{a}}(u) + (\mu(u) + v)\,\vec{b}(u)$$
$$= \tilde{\vec{a}}(u) + \frac{\mu(u)+v}{\lambda(u) - \mu'(u)}\,\tilde{\vec{a}}'(u)$$

$\Rightarrow$ 直母线为 $\tilde{\vec{a}}$ 的切线. $\Rightarrow$ S 是切线面.

2. 全脐点曲面.

S: 正则曲面.

回顾: $p \in S$ 为脐点, iff $\mathrm{II} = k(p)\,\mathrm{I}$, $k(p) \in \mathbb{R}$.

称 S 为全脐点曲面, if $\forall\, p \in S$, p 为脐点.

Thm. S 是全脐点曲面, iff S 是平面或球面.

pf. "$\Leftarrow$". 平面 $\mathrm{II} = 0$. $\checkmark$.

球面: $\vec{r}(u,v) - \vec{r}_0 = \pm R\,\vec{n}$, R半径.

$$\Rightarrow d\vec{r} = \pm R\, d\vec{n}.$$

$$\mathrm{II} = -d\vec{r}\cdot d\vec{n} = \mp R\, d\vec{n}\cdot d\vec{n}$$
$$\mathrm{I} = d\vec{r}\cdot d\vec{r} = R^2\, d\vec{n}\cdot d\vec{n}$$
$$\frac{\mathrm{II}}{\mathrm{I}} = \frac{\mp R}{R^2} = \mp\frac{1}{R}.$$

"$\Rightarrow$". $\mathrm{II} = k\,\mathrm{I}$

$$\Rightarrow L = kE$$
$$M = kF$$
$$N = kG.$$

$$\langle \vec{n}_u + k\vec{r}_u,\ \vec{r}_u \rangle = -L + kE = 0$$
$$\langle \vec{n}_u + k\vec{r}_u,\ \vec{r}_v \rangle = -M + kF = 0$$
$$\langle \vec{n}_u + k\vec{r}_u,\ \vec{n} \rangle = 0.$$
$$\Rightarrow \vec{n}_u + k\vec{r}_u = 0$$

同理: $\vec{n}_v + k\vec{r}_v = 0$

$$\Rightarrow \begin{cases} \vec{n}_{uv} + k_v\vec{r}_u + k\vec{r}_{uv} = 0 \\ \vec{n}_{uv} + k_u\vec{r}_v + k\vec{r}_{uv} = 0 \end{cases}$$
$$\Rightarrow k_v\vec{r}_u - k_u\vec{r}_v = 0$$
$$\Rightarrow k_u = k_v = 0 \Rightarrow k \equiv k_0.$$

1. $k_0 = 0$ $\Rightarrow \mathrm{II} \equiv 0 \Rightarrow \vec{n} \equiv \vec{n}_0 \Rightarrow S$ 是平面.

2. $k_0 \neq 0$.

$$\Rightarrow \vec{n} + k_0\vec{r} = \mathrm{Const} =: \vec{c}_0$$

$\Rightarrow$ S 为半径为 $\frac{1}{|k_0|}$ 的球面. #.

§3. 曲面论基本定理

§3.1 结构方程

S: $\vec{r} = \vec{r}(u^1, u^2)$ 光滑参数曲面. $(u^1, u^2) \in D \subseteq \mathbb{R}^2$

正则. i.e., $\vec{r}_{u^1} \times \vec{r}_{u^2}$ 处处非零.

$$\vec{n} = \frac{\vec{r}_{u^1} \times \vec{r}_{u^2}}{|\vec{r}_{u^1} \times \vec{r}_{u^2}|}$$ 单位法向量场.

回顾: 黎曼度量 $g_{\alpha\beta} := \vec{r}_{u^\alpha} \cdot \vec{r}_{u^\beta}$ $\quad u = u^1, v = u^2$.

$$\mathrm{I} = g_{\alpha\beta}\, du^\alpha du^\beta \quad \left(= E(du)^2 + 2F\,du\,dv + G(dv)^2\right)$$

记: $b_{\alpha\beta} = \vec{r}_{u^\alpha u^\beta} \cdot \vec{n}$

$$= -\vec{r}_{u^\alpha} \cdot \vec{n}_{u^\beta} = -\vec{r}_{u^\beta} \cdot \vec{n}_{u^\alpha}$$

$$\mathrm{II} = b_{\alpha\beta}\, du^\alpha du^\beta \quad \left(= L(du)^2 + 2M\,du\,dv + N(dv)^2\right).$$

$$g = \det((g_{\alpha\beta})) = g_{11}g_{22} - g_{12}^2 \quad \left(= EG - F^2\right)$$

$$b = \det((b_{\alpha\beta})) = b_{11}b_{22} - b_{12}^2 \quad \left(= LN - M^2\right)$$

$\{\vec{r}; \vec{r}_{u^\alpha}, \vec{r}_{u^\beta}, \vec{n}\}$ 构成 S 上一个标架场, 称为自然标架场.

考虑自然标架场的运动.

$$\frac{\partial \vec{r}}{\partial u^\alpha} = \vec{r}_{u^\alpha}$$

常用记号

$$\frac{\partial \vec{r}_{u^\alpha}}{\partial u^\beta} = \vec{r}_{u^\alpha u^\beta} = \Gamma^\gamma_{\alpha\beta}\, \vec{r}_\gamma + C_{\alpha\beta}\, \vec{n}, \quad (*1)$$

常用记号

$$\frac{\partial \vec{n}}{\partial u^\beta} = \vec{n}_{u^\beta} = D^\gamma_\beta\, \vec{r}_\gamma + D_\beta \vec{n} \quad (*2)$$

$\vec{n} \cdot \vec{n} \equiv 0 \Rightarrow D_\beta = 0$,

$(*1) \cdot \vec{n} \Rightarrow C_{\alpha\beta} = \vec{r}_{u^\alpha u^\beta} \cdot \vec{n} = b_{\alpha\beta}$,

$(*2) \cdot \vec{r}_{u^\xi} \Rightarrow D^\gamma_\beta\, g_{\gamma\xi} = \vec{n}_{u^\beta} \cdot \vec{r}_{u^\xi} = -b_{\beta\xi}$,

or $D^\gamma_\beta = -b_{\beta\xi}\, g^{\xi\gamma}$.

记: $b^\gamma_\beta = b_{\beta\xi}\, g^{\xi\gamma}$ 则: $D^\gamma_\beta = -b^\gamma_\beta$.

$$\Rightarrow \begin{cases} \vec{r}_{u^\alpha u^\beta} = \Gamma^\gamma_{\alpha\beta}\, \vec{r}_\gamma + b_{\alpha\beta}\, \vec{n}, & (*3) \\ \vec{n}_{u^\beta} = -b^\gamma_\beta\, \vec{r}_\gamma & (*4) \end{cases}$$

$(*3)\cdot \vec r_{u^\xi} \implies \vec r_{u^\alpha u^\beta}\cdot \vec r_{u^\xi} = \Gamma^\gamma_{\alpha\beta}\, g_{\gamma\xi}$

由定义 $\vec r_{u^\alpha}\cdot \vec r_{u^\xi} = g_{\alpha\xi}$

$$\vec r_{u^\alpha u^\beta}\cdot \vec r_{u^\xi} + \vec r_{u^\alpha}\cdot \vec r_{u^\xi u^\beta} = \frac{\partial g_{\alpha\xi}}{\partial u^\beta}$$

$$\vec r_{u^\beta u^\alpha}\cdot \vec r_{u^\xi} + \vec r_{u^\beta}\cdot \vec r_{u^\xi u^\alpha} = \frac{\partial g_{\beta\xi}}{\partial u^\alpha}$$

$$\vec r_{u^\alpha u^\xi}\cdot \vec r_{u^\beta} + \vec r_{u^\alpha}\cdot \vec r_{u^\beta u^\xi} = \frac{\partial g_{\alpha\beta}}{\partial u^\xi}$$

$$\Rightarrow\quad 2\,\vec r_{u^\alpha u^\beta}\cdot \vec r_{u^\xi} = \frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}$$

$$\Rightarrow\quad \underbrace{\vec r_{u^\alpha u^\beta}\cdot \vec r_{u^\xi}}_{\Gamma^\gamma_{\alpha\beta}\cdot g_{\gamma\xi}=} = \frac{1}{2}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}\right)$$

$$\Gamma^\gamma_{\alpha\beta}\, g_{\gamma\xi}\, g^{\xi\delta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}\right) g^{\xi\delta}$$

Hence, $$\Gamma^\delta_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}\right) g^{\xi\delta}$$

称 $\Gamma^\delta_{\alpha\beta}$ 为 S 的度量 $g_{\alpha\beta}$ 给出的Christoffel符号.

显然 $\Gamma^\delta_{\alpha\beta} = \Gamma^\delta_{\beta\alpha}$.

$$\begin{cases} \vec r_{u^\alpha u^\beta} = \Gamma^\gamma_{\alpha\beta}\,\vec r_\gamma + b_{\alpha\beta}\,\vec n \\ \vec n_{u^\beta} = -\,b^\gamma_\beta\,\vec r_\gamma \end{cases} \qquad b^\gamma_\beta := g^{\gamma\xi} b_{\xi\beta}.$$

- $\partial_{u^\gamma}(\vec r_{u^\alpha u^\beta})$ 关于 α,β,γ 全对称.

$$\partial_{u^\gamma}(\vec r_{u^\alpha u^\beta}) = \partial_{u^\gamma}\left(\Gamma^\delta_{\alpha\beta}\,\vec r_{u^\delta} + b_{\alpha\beta}\,\vec n\right)$$

$$= \frac{\partial \Gamma^\delta_{\alpha\beta}}{\partial u^\gamma}\vec r_{u^\delta} + \Gamma^\delta_{\alpha\beta}\,\vec r_{u^\delta u^\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma}\vec n + b_{\alpha\beta}\vec n_{u^\gamma}$$

$$= \frac{\partial \Gamma^\delta_{\alpha\beta}}{\partial u^\gamma}\vec r_\delta + \Gamma^\delta_{\alpha\beta}\left(\Gamma^\varphi_{\delta\gamma}\,\vec r_\varphi + b_{\delta\gamma}\vec n\right)$$

$$+ \frac{\partial b_{\alpha\beta}}{\partial u^\gamma}\vec n + b_{\alpha\beta}\left(-b^\delta_\gamma\,\vec r_\delta\right)$$

$$= \left(\frac{\partial \Gamma^\delta_{\alpha\beta}}{\partial u^\gamma} + \Gamma^\rho_{\alpha\beta}\Gamma^\delta_{\rho\gamma} - b_{\alpha\beta}\,b^\delta_\gamma\right)\vec r_\delta$$

$$+ \left(\Gamma^\delta_{\alpha\beta}\,b_{\delta\gamma} + \frac{\partial b_{\alpha\beta}}{\partial u^\gamma}\right)\vec n$$

交换 β,γ

$$\partial_{u^\beta}(\vec r_{u^\alpha u^\gamma}) = \left(\frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\rho_{\alpha\gamma}\Gamma^\delta_{\rho\beta} - b_{\alpha\gamma}b^\delta_\beta\right)\vec r_\delta + \left(\Gamma^\delta_{\alpha\gamma}\,b_{\delta\beta} + \frac{\partial b_{\alpha\gamma}}{\partial u^\beta}\right)\vec n$$

$$\Rightarrow \begin{cases} \dfrac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\delta}_{\rho\gamma} - b_{\alpha\beta} b^{\delta}_{\gamma} = \dfrac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\delta}_{\rho\beta} - b_{\alpha\gamma} b^{\delta}_{\beta} \\ \Gamma^{\delta}_{\alpha\beta} b_{\delta\gamma} + \dfrac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} = \Gamma^{\delta}_{\alpha\gamma} b_{\delta\beta} + \dfrac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} \end{cases}$$

i.e.,
$$\begin{cases} \dfrac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} - \dfrac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\delta}_{\rho\gamma} - \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\delta}_{\rho\beta} = b_{\alpha\beta} b^{\delta}_{\gamma} - b_{\alpha\gamma} b^{\delta}_{\beta} \quad (\text{Gauss方程}) \\ \dfrac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \dfrac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} = \Gamma^{\delta}_{\alpha\gamma} b_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} b_{\delta\gamma} \quad (\text{Codazzi方程}) \end{cases}$$

记: $R^{\delta}_{\alpha\beta\gamma} :=$
$$\frac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\delta}_{\rho\gamma} - \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\delta}_{\rho\beta}$$

称 $R^{\delta}_{\alpha\beta\gamma}$ 为 S 的 $g_{\alpha\beta}$ 对应的曲率张量.

记: $R_{\delta\alpha\beta\gamma} := g_{\delta\eta} R^{\eta}_{\alpha\beta\gamma}$. 则:

- $R_{\delta\alpha\beta\gamma} = b_{\alpha\beta} b_{\gamma\delta} - b_{\alpha\gamma} b_{\beta\delta}$

Thm (Gauss' Egregium Thm). "绝妙定理"

Gauss 曲率由第一基本形式唯一确定.

Pf. 取: $\delta=1$, $\alpha=2$, $\beta=1$, $\gamma=2$, 有:

$$R_{1212} = b_{21}b_{21} - b_{22}b_{11} = -(LN - M^2)$$

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-R_{1212}}{g}. \quad \#.$$

- $\vec{n}_{u^\beta u^\gamma}$ 关于 β, γ 全对称. 下证明此对称性条件可由 Gauss-Codazzi方程推出.

$$\partial_{u^\gamma}(\vec{n}_{u^\beta}) = -\partial_{u^\gamma}(b^{\rho}_{\beta} \vec{r}_{u^\rho})$$
$$= -\frac{\partial b^{\rho}_{\beta}}{\partial u^{\gamma}} \vec{r}_{u^\rho} - b^{\rho}_{\beta} \vec{r}_{u^\rho u^\gamma}$$
$$= -\frac{\partial b^{\rho}_{\beta}}{\partial u^{\gamma}} \vec{r}_{u^\rho} - b^{\rho}_{\beta}\left(\Gamma^{\delta}_{\rho\gamma}\vec{r}_{\delta} + b_{\rho\gamma}\vec{n}\right)$$
$$= -\left(\frac{\partial b^{\sigma}_{\beta}}{\partial u^{\gamma}} + b^{\rho}_{\beta}\Gamma^{\sigma}_{\rho\gamma}\right)\vec{r}_{\sigma} - b^{\rho}_{\beta} b_{\rho\gamma}\vec{n}$$

Z1

$$\partial_{u^\beta}(\vec n_{u^\gamma}) = -\left(\frac{\partial b_\gamma^\sigma}{\partial u^\beta} + b_\gamma^\rho\, \Gamma^\sigma_{\rho\beta}\right)\vec r_\sigma - b_\gamma^\rho b_{\rho\beta}\,\vec n$$

$$\Rightarrow \frac{\partial b_\beta^\sigma}{\partial u^\gamma} + b_\beta^\rho\, \Gamma^\sigma_{\rho\gamma} = \frac{\partial b_\gamma^\sigma}{\partial u^\beta} + b_\gamma^\rho\, \Gamma^\sigma_{\rho\beta}$$

$$b_\beta^\rho\, b_{\rho\gamma} = b_\gamma^\rho\, b_{\rho\beta} \quad (\text{trivial})$$

$$b_\beta^\rho = g^{\rho\sigma} b_{\sigma\beta} \qquad g^{\rho\sigma} b_{\sigma\beta} b_{\rho\gamma} = g^{\rho\sigma} b_{\sigma\gamma} b_{\rho\beta}$$

$$\Vert$$

$$b_{\gamma\rho}\, g^{\rho\sigma}\, b_{\sigma\beta} = b_{\gamma\sigma}\, g^{\sigma\rho}\, b_{\rho\beta}$$

$$\frac{\partial(b_{\beta\rho}\, g^{\rho\sigma})}{\partial u^\gamma} + b_{\beta\xi}\, g^{\xi\rho}\, \Gamma^\sigma_{\rho\gamma} = \frac{\partial(b_{\gamma\rho}\, g^{\rho\sigma})}{\partial u^\beta} + b_{\gamma\xi}\, g^{\xi\rho}\, \Gamma^\sigma_{\rho\beta}$$

$$\Leftrightarrow \frac{\partial b_{\beta\rho}}{\partial u^\gamma} g^{\rho\sigma} + b_{\beta\rho}\frac{\partial g^{\rho\sigma}}{\partial u^\gamma} + b_{\beta\xi}\, g^{\xi\rho}\, \Gamma^\sigma_{\rho\gamma} = \frac{\partial b_{\gamma\rho}}{\partial u^\beta} g^{\rho\sigma} + b_{\gamma\rho}\frac{\partial g^{\rho\sigma}}{\partial u^\beta} + b_{\gamma\xi}\, g^{\xi\rho}\, \Gamma^\sigma_{\rho\beta}$$

Z2

$$\Leftrightarrow \frac{\partial b_{\beta\varphi}}{\partial u^\gamma} + b_{\beta\rho}\frac{\partial g^{\rho\sigma}}{\partial u^\gamma} g_{\sigma\varphi} + b_{\beta\xi}\, g^{\xi\rho}\, \Gamma^\sigma_{\rho\gamma}\, g_{\sigma\varphi} = (\beta\leftrightarrow\gamma)$$

$$\Leftrightarrow \frac{\partial b_{\beta\varphi}}{\partial u^\gamma} \underline{- b_{\beta\rho}\, g^{\rho\sigma}\frac{\partial g_{\sigma\varphi}}{\partial u^\gamma}} + \underline{\frac12 b_{\beta\xi}\, g^{\xi\rho}}\left(\frac{\partial g_{\rho\varphi}}{\partial u^\gamma} + \frac{\partial g_{\gamma\varphi}}{\partial u^\rho} - \frac{\partial g_{\rho\gamma}}{\partial u^\varphi}\right) = (\beta\leftrightarrow\gamma) \quad (***)$$

Codazzi 方程:

$$\frac{\partial b_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial b_{\alpha\gamma}}{\partial u^\beta} = \Gamma^\delta_{\alpha\gamma}\, b_{\delta\beta} - \Gamma^\delta_{\alpha\beta}\, b_{\delta\gamma}$$

$$\Rightarrow \frac{\partial b_{\varphi\beta}}{\partial u^\gamma} - \frac{\partial b_{\varphi\gamma}}{\partial u^\beta} = \Gamma^\delta_{\varphi\gamma}\, b_{\delta\beta} - \Gamma^\delta_{\varphi\beta}\, b_{\delta\gamma}$$

$$= \frac12 g^{\delta\xi}\left(\frac{\partial g_{\varphi\xi}}{\partial u^\gamma} + \frac{\partial g_{\gamma\xi}}{\partial u^\varphi} - \frac{\partial g_{\varphi\gamma}}{\partial u^\xi}\right) b_{\delta\beta} - \frac12 g^{\delta\xi}\left(\frac{\partial g_{\varphi\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\varphi} - \frac{\partial g_{\varphi\beta}}{\partial u^\xi}\right) b_{\delta\gamma}$$

$$\Rightarrow \frac{\partial b_{\varphi\beta}}{\partial u^\gamma} \underline{- \frac12 b_{\beta\delta}\, g^{\delta\xi}\frac{\partial g_{\xi\varphi}}{\partial u^\gamma}} + \frac12 b_{\beta\delta}\, g^{\delta\xi}\frac{\partial g_{\gamma\varphi}}{\partial u^\xi} - \frac12 b_{\beta\delta}\, g^{\delta\xi}\frac{\partial g_{\xi\gamma}}{\partial u^\varphi}$$

$$= (\beta\leftrightarrow\gamma). \Rightarrow ***. \ \#$$