· Last time:

· Gauss 方程:

$$\frac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\delta}_{\rho\gamma} - \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\delta}_{\rho\beta}$$

$$= b_{\alpha\beta} b^{\delta}_{\gamma} - b_{\alpha\gamma} b^{\delta}_{\beta}$$

· Codazzi 方程:

$$\frac{\partial b_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial b_{\alpha\gamma}}{\partial u^{\beta}} = \Gamma^{\delta}_{\alpha\gamma} b_{\delta\beta} - \Gamma^{\delta}_{\alpha\beta} b_{\delta\gamma}$$

记: $R^{\delta}_{\alpha\beta\gamma} :=$

$$\frac{\partial \Gamma^{\delta}_{\alpha\beta}}{\partial u^{\gamma}} - \frac{\partial \Gamma^{\delta}_{\alpha\gamma}}{\partial u^{\beta}} + \Gamma^{\rho}_{\alpha\beta}\Gamma^{\delta}_{\rho\gamma} - \Gamma^{\rho}_{\alpha\gamma}\Gamma^{\delta}_{\rho\beta}$$

Q1. $R^{\delta}_{\alpha\beta\gamma}$ 是什么量?

$\Gamma^{\alpha}_{\beta\gamma}$ 是什么量?

Q2. 对称性?

To answer these two questions we introduce tensor analysis.

· Brief introduction:

We call

$$T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}(u^1, u^2)$$

a tensor of type (r, s)

if under the change of

coords $(u^1, u^2) \longmapsto (\tilde{u}^1, \tilde{u}^2)$

$T^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}$ transform as follows:

$$\tilde{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}(u^1,u^2)$$

$$= T^{\gamma_1\cdots\gamma_r}_{\rho_1\cdots\rho_s}\frac{\partial u^{\rho_1}}{\partial \tilde{u}^{\beta_1}}\cdots\frac{\partial u^{\rho_s}}{\partial \tilde{u}^{\beta_s}}\cdot\frac{\partial \tilde{u}^{\alpha_1}}{\partial u^{\gamma_1}}\cdots\frac{\partial \tilde{u}^{\alpha_r}}{\partial u^{\gamma_r}}$$

Therefore:

$$\tilde{T}^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}(u^1,u^2)\,du^{\beta_1}\otimes\cdots\otimes du^{\beta_s}\,\frac{\partial}{\partial u^{\alpha_1}}\otimes\cdots\otimes\frac{\partial}{\partial u^{\alpha_r}}$$

is <u>invariant</u> under coords transf.

上 $\alpha_1,\cdots,\alpha_r$ 称为反变的

下 $\beta_1,\cdots,\beta_s$ ---协变的.

<u>Lemma.1</u>

$R^{\delta}_{\alpha\beta\gamma}$ 为具有一个反变分量和三个协变分量的(1,3)-型张量.

为证这个引理，我们先看 $g_{\alpha\beta}$，$g^{\alpha\beta}$，$\Gamma^{\gamma}_{\alpha\beta}$ 在坐标变换下如何改变.

Lemma.2 $\Gamma^{\alpha}_{\beta\gamma}$ 不是张量.

Pf. Recall that

$$\Gamma^{\delta}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial g_{\alpha\xi}}{\partial u^{\beta}} + \frac{\partial g_{\beta\xi}}{\partial u^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial u^{\xi}}\right)g^{\xi\delta}$$

$$\tilde{\Gamma}^{\delta}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \tilde{g}_{\alpha\xi}}{\partial \tilde{u}^{\beta}} + \frac{\partial \tilde{g}_{\beta\xi}}{\partial \tilde{u}^{\alpha}} - \frac{\partial \tilde{g}_{\alpha\beta}}{\partial \tilde{u}^{\xi}}\right)\tilde{g}^{\xi\delta}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial \tilde{u}^{\beta}}\left(g_{\rho\sigma}\frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}}\frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}}\right) + \frac{\partial}{\partial \tilde{u}^{\alpha}}\left(g_{\rho\sigma}\frac{\partial u^{\rho}}{\partial \tilde{u}^{\beta}}\frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}}\right) - \frac{\partial}{\partial \tilde{u}^{\xi}}\left(g_{\rho\sigma}\frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}}\frac{\partial u^{\sigma}}{\partial \tilde{u}^{\beta}}\right)\right)\tilde{g}^{\xi\delta}$$

$$= \frac{1}{2}\Bigg(\frac{\partial g_{\rho\sigma}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} + g_{\rho\sigma} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\alpha} \partial \tilde{u}^{\beta}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}}$$

$$+ \underline{g_{\rho\sigma} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial^2 u^{\sigma}}{\partial \tilde{u}^{\xi} \partial \tilde{u}^{\beta}}}$$

$$+ \frac{\partial g_{\rho\sigma}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} + g_{\rho\sigma} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\beta} \partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}}$$

$$+ g_{\rho\sigma} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\beta}} \frac{\partial^2 u^{\sigma}}{\partial \tilde{u}^{\xi} \partial \tilde{u}^{\alpha}}$$

$$- \frac{\partial g_{\rho\sigma}}{\partial \tilde{u}^{\xi}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\beta}} - g_{\rho\sigma} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\xi} \partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\beta}}$$

$$\underline{- g_{\rho\sigma} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial^2 u^{\sigma}}{\partial \tilde{u}^{\beta} \partial \tilde{u}^{\xi}}} \Bigg) \tilde{g}^{\xi\delta}$$

$$= \frac{1}{2}\Bigg(\frac{\partial g_{\rho\sigma}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} g^{\phi\psi} \frac{\partial \tilde{u}^{\xi}}{\partial u^{\phi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ \frac{\partial g_{\rho\sigma}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} g^{\phi\psi} \frac{\partial \tilde{u}^{\xi}}{\partial u^{\phi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$- \frac{\partial g_{\rho\sigma}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\xi}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\beta}} g^{\phi\psi} \frac{\partial \tilde{u}^{\xi}}{\partial u^{\phi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ 2 g_{\rho\sigma} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\alpha} \partial \tilde{u}^{\beta}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} \tilde{g}^{\xi\delta} \Bigg)$$

$$= \frac{1}{2}\Bigg(\frac{\partial g_{\rho\phi}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ \frac{\partial g_{\rho\phi}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\beta}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$- \frac{\partial g_{\rho\sigma}}{\partial u^{\phi}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\beta}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \Bigg)$$

$$+ g_{\rho\sigma} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\alpha} \partial \tilde{u}^{\beta}} \frac{\partial u^{\sigma}}{\partial \tilde{u}^{\xi}} g^{\phi\psi} \frac{\partial \tilde{u}^{\xi}}{\partial u^{\phi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$= \frac{1}{2}\Bigg(\frac{\partial g_{\rho\phi}}{\partial u^{\varepsilon}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ \frac{\partial g_{\varepsilon\phi}}{\partial u^{\rho}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$- \frac{\partial g_{\rho\varepsilon}}{\partial u^{\phi}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} g^{\phi\psi} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \Bigg)$$

(79)

$$+ g_{\rho\sigma} \frac{\partial^2 u^\rho}{\partial \tilde u^\alpha \partial \tilde u^\beta} g^{\sigma\psi} \frac{\partial \tilde u^\delta}{\partial u^\psi}$$

$$= \Gamma^\psi_{\rho\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\beta} \frac{\partial u^\rho}{\partial \tilde u^\alpha} \frac{\partial \tilde u^\delta}{\partial u^\psi} + \boxed{\frac{\partial^2 u^\psi}{\partial \tilde u^\alpha \partial \tilde u^\beta} \frac{\partial \tilde u^\delta}{\partial u^\psi}}$$

this term in general is not zero

Therefore,

$\Gamma^\gamma_{\alpha\beta}$ is not a tensor. #

Pf of Lemma 1.

$$R^\delta_{\alpha\beta\gamma} := \frac{\partial \Gamma^\delta_{\alpha\beta}}{\partial u^\gamma} - \frac{\partial \Gamma^\delta_{\alpha\gamma}}{\partial u^\beta} + \Gamma^\rho_{\alpha\beta}\Gamma^\delta_{\rho\gamma} - \Gamma^\rho_{\alpha\gamma}\Gamma^\delta_{\rho\beta}$$

Pf.

(80)

$$\tilde R^\delta_{\alpha\beta\gamma}$$

$$= \frac{\partial \tilde\Gamma^\delta_{\alpha\beta}}{\partial \tilde u^\gamma} - \frac{\partial \tilde\Gamma^\delta_{\alpha\gamma}}{\partial \tilde u^\beta} + \tilde\Gamma^\rho_{\alpha\beta}\tilde\Gamma^\delta_{\rho\gamma} - \tilde\Gamma^\rho_{\alpha\gamma}\tilde\Gamma^\delta_{\rho\beta}$$

$$= \frac{\partial}{\partial u^\phi}\left(\Gamma^\psi_{\rho\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\beta} \frac{\partial u^\rho}{\partial \tilde u^\alpha} \frac{\partial \tilde u^\delta}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\alpha \partial \tilde u^\beta} \frac{\partial \tilde u^\delta}{\partial u^\psi}\right) \frac{\partial u^\phi}{\partial \tilde u^\gamma}$$

$$- \frac{\partial}{\partial u^\phi}\left(\Gamma^\psi_{\rho\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\gamma} \frac{\partial u^\rho}{\partial \tilde u^\alpha} \frac{\partial \tilde u^\delta}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\alpha \partial \tilde u^\gamma} \frac{\partial \tilde u^\delta}{\partial u^\psi}\right) \frac{\partial u^\phi}{\partial \tilde u^\beta}$$

$$+ \left(\Gamma^\psi_{\mu\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\beta} \frac{\partial u^\mu}{\partial \tilde u^\alpha} \frac{\partial \tilde u^\rho}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\alpha \partial \tilde u^\beta} \frac{\partial \tilde u^\rho}{\partial u^\psi}\right)$$

$$\left(\Gamma^\psi_{\mu\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\gamma} \frac{\partial u^\mu}{\partial \tilde u^\rho} \frac{\partial \tilde u^\delta}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\rho \partial \tilde u^\gamma} \frac{\partial \tilde u^\delta}{\partial u^\psi}\right)$$

$$- \left(\Gamma^\psi_{\mu\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\gamma} \frac{\partial u^\mu}{\partial \tilde u^\alpha} \frac{\partial \tilde u^\rho}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\alpha \partial \tilde u^\gamma} \frac{\partial \tilde u^\rho}{\partial u^\psi}\right)$$

$$\left(\Gamma^\psi_{\mu\varepsilon} \frac{\partial u^\varepsilon}{\partial \tilde u^\beta} \frac{\partial u^\mu}{\partial \tilde u^\rho} \frac{\partial \tilde u^\delta}{\partial u^\psi} + \frac{\partial^2 u^\psi}{\partial \tilde u^\rho \partial \tilde u^\beta} \frac{\partial \tilde u^\delta}{\partial u^\psi}\right)$$

$$= \frac{\partial \Gamma^{\psi}_{\rho\varepsilon}}{\partial u^{\phi}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\gamma}}$$

$$+ \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial^2 u^{\varepsilon}}{\partial \tilde{u}^{\beta}\partial \tilde{u}^{\gamma}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\gamma}}$$

$$+ \frac{\partial^3 u^{\psi}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\beta}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \frac{\partial^2 u^{\psi}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\beta}} \frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\gamma}}$$

$$- \Bigg(\frac{\partial \Gamma^{\psi}_{\rho\varepsilon}}{\partial u^{\phi}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\beta}}$$

$$+ \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial^2 u^{\varepsilon}}{\partial \tilde{u}^{\beta}\partial \tilde{u}^{\gamma}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial^2 u^{\rho}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\beta}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}}$$

$$+ \Gamma^{\psi}_{\rho\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\rho}}{\partial \tilde{u}^{\alpha}} \frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\beta}}$$

$$+ \frac{\partial^3 u^{\psi}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\beta}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \frac{\partial^2 u^{\psi}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\gamma}} \frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\beta}} \Bigg)$$

$$+ \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}} \right) \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\rho}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \right)$$

$$- \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}} \right) \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\rho}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \right)$$

$$+ \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\beta}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi_1}} \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\rho}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \frac{\partial^2 u^{\psi}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \right)$$

$$+ \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi_1}} \Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}}$$

$$- \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\alpha}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi_1}} \left(\Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\rho}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} + \frac{\partial^2 u^{\psi}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\beta}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi}} \right)$$

$$- \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\beta}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi_1}} \Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}}$$

$$=$$

$$- \Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\gamma}} \left(\frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\beta}} + \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\beta}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi_1}} \right)$$

$$+ \Gamma^{\psi}_{\mu\varepsilon} \frac{\partial u^{\mu}}{\partial \tilde{u}^{\alpha}} \frac{\partial u^{\varepsilon}}{\partial \tilde{u}^{\beta}} \left(\frac{\partial^2 \tilde{u}^{\delta}}{\partial u^{\phi}\partial u^{\psi}} \frac{\partial u^{\phi}}{\partial \tilde{u}^{\gamma}} + \frac{\partial^2 u^{\psi_1}}{\partial \tilde{u}^{\rho}\partial \tilde{u}^{\gamma}} \frac{\partial \tilde{u}^{\rho}}{\partial u^{\psi}} \frac{\partial \tilde{u}^{\delta}}{\partial u^{\psi_1}} \right)$$

$$+\frac{\partial^2 u^{\varphi_1}}{\partial\tilde u^\alpha\partial\tilde u^\beta}\left(\frac{\partial^2\tilde u^\delta}{\partial u^\varphi\partial u^{\varphi_1}}\frac{\partial u^\varphi}{\partial\tilde u^\gamma}+\frac{\partial\tilde u^\rho}{\partial u^{\varphi_1}}\frac{\partial^2 u^\varphi}{\partial\tilde u^\rho\partial\tilde u^\gamma}\frac{\partial\tilde u^\delta}{\partial u^\varphi}\right)$$

$$-\frac{\partial^2 u^{\varphi_1}}{\partial\tilde u^\alpha\partial\tilde u^\gamma}\left(\frac{\partial^2\tilde u^\delta}{\partial u^\varphi\partial u^{\varphi_1}}\frac{\partial u^\varphi}{\partial\tilde u^\beta}+\frac{\partial\tilde u^\rho}{\partial u^{\varphi_1}}\frac{\partial^2 u^\varphi}{\partial\tilde u^\rho\partial\tilde u^\beta}\frac{\partial\tilde u^\delta}{\partial u^\varphi}\right)$$

$$=\cdots-\Gamma^\varphi_{\mu\varepsilon}\frac{\partial u^\mu}{\partial\tilde u^\alpha}\frac{\partial u^\varepsilon}{\partial\tilde u^\gamma}\frac{\partial}{\partial u^\varphi}\left(\frac{\partial\tilde u^\delta}{\partial u^{\varphi_1}}\frac{\partial u^{\varphi_1}}{\partial\tilde u^\beta}\right)$$

$$+\Gamma^\varphi_{\mu\varepsilon}\frac{\partial u^\mu}{\partial\tilde u^\alpha}\frac{\partial u^\varepsilon}{\partial\tilde u^\beta}\frac{\partial}{\partial u^\varphi}\left(\frac{\partial\tilde u^\delta}{\partial u^{\varphi_1}}\frac{\partial u^{\varphi_1}}{\partial\tilde u^\gamma}\right)$$

$$+\frac{\partial^2 u^{\varphi_1}}{\partial\tilde u^\alpha\partial\tilde u^\beta}\frac{\partial}{\partial u^{\varphi_1}}\left(\frac{\partial\tilde u^\delta}{\partial u^\varphi}\frac{\partial u^\varphi}{\partial\tilde u^\gamma}\right)$$

$$-\frac{\partial^2 u^{\varphi_1}}{\partial\tilde u^\alpha\partial\tilde u^\gamma}\frac{\partial}{\partial u^{\varphi_1}}\left(\frac{\partial\tilde u^\delta}{\partial u^\varphi}\frac{\partial u^\varphi}{\partial\tilde u^\beta}\right)$$

$$=0 .\qquad \#.$$

· 张量缩并.

例. 由 $g_{\alpha\beta}$ 与 $g^{\alpha\beta}$ 给出的张量缩并.

· 对称性

· $R_{\delta\alpha\beta\gamma}=R_{\gamma\beta\alpha\delta}=-R_{\alpha\delta\beta\gamma}=-R_{\delta\alpha\gamma\beta}$

§3.2. 唯一性定理

Thm. 设 S_1, S_2 为定义在 D 上的两个正则曲面 $\vec r(u^1,u^2)$, $\tilde{\vec r}(u^1,u^2)$. 若 $\forall (u^1,u^2)\in D$ S_1, S_2 在 (u^1,u^2) 点有相同的 I 和 II, 则存在 $\mathbb{R}^3$ 中的刚体运动 T s.t. $\tilde{\vec r}=T\circ\vec r$.

Pf. 任取一点 $u_0=(u_0^1,u_0^2)\in D$

$$\left.\begin{array}{l} \mathrm{I}_1|_{u_0}=\mathrm{I}_2|_{u_0}\\ \mathrm{II}_1|_{u_0}=\mathrm{II}_2|_{u_0}\\ \vec r_{u^1},\vec r_{u^2},\vec n\text{ 右手},\ \tilde{\vec r}_{u^1},\tilde{\vec r}_{u^2},\tilde{\vec n}\text{ 右手}\end{array}\right\}\Rightarrow \text{标架 }\{\vec r(u_0);\vec r_{u^\alpha}(u_0),\vec n(u_0)\}$$

经过一个适当的刚体运动后, 与标架

$$\{\tilde{\vec r}(u_0);\tilde{\vec r}_{u^\alpha}(u_0),\tilde{\vec n}(u_0)\}\text{ 重合.}$$

i.e.,
$$\left.\begin{array}{l}\tilde{\vec r}(u_0)=T\circ\vec r(u_0)\\ \tilde{\vec r}_{u^\alpha}(u_0)=T\circ\vec r_{u^\alpha}(u_0)\\ \tilde{\vec n}(u_0)=T\circ\vec n(u_0)\end{array}\right\}\quad ①$$

由于自然标架运动方程的系数由 I 与 II 唯一确定.

而 $\mathrm{I}_1 \equiv \mathrm{I}_2$, $\mathrm{II}_1 \equiv \mathrm{II}_2$，由一阶线性 PDE 方程组初值问题

解的唯一性及 ① 知：$\tilde{\vec{r}} \equiv T \circ \vec{r}$. #

§ 存在性定理

Thm. 对于 $\mathrm{I} = g_{\alpha\beta}(u)\, du^\alpha\, du^\beta$

$\mathrm{II} = b_{\alpha\beta}(u)\, du^\alpha\, du^\beta$, $\quad u = (u^1, u^2) \in D$

其中 D 为 $\mathbb{R}^2$ 上的单连通区域，$g_{11}, g_{12} = g_{21}, g_{22}$,

$b_{11}, b_{12} = b_{21}, b_{22}$ 为 D 上给定的光滑函数，且 $g_{11}g_{22} - g_{12}^2 > 0$.
(恒成立)

定义：

$$\Gamma_{\alpha\beta}^{\gamma} = \frac{1}{2} g^{\gamma\xi}\left(\frac{\partial g_{\alpha\xi}}{\partial u^\beta} + \frac{\partial g_{\beta\xi}}{\partial u^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial u^\xi}\right).$$

若 $\Gamma_{\alpha\beta}^{\gamma}$, $b_{\alpha\beta}$, b_β^α 满足高斯方程和 Codazzi 方程，

则 $\forall\, u_0 \in D$，存在 u_0 的一个邻域 $U \subseteq D$，以及

定义在 U 上的曲面 $\vec{r}: U \to \mathbb{R}^3$, s.t.

I、II 为其第 I、第 II 基本形式.

Pf. Consider the following linear PDE system:

$$\begin{cases} \dfrac{\partial \vec{r}}{\partial u^\alpha} = \vec{r}_\alpha, \\[2mm] \dfrac{\partial \vec{r}_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^{\gamma} \vec{r}_\gamma + b_{\alpha\beta} \vec{n}, \\[2mm] \dfrac{\partial \vec{n}}{\partial u^\alpha} = -b_\alpha^\gamma \vec{r}_\gamma \end{cases} \qquad (*)$$

这个一阶变系数线性偏微分方程组的相容性条件为：

$$\partial_{u^\beta}(\partial_{u^\alpha} \vec{r}) = \alpha \leftrightarrow \beta$$

$$\partial_{u^\gamma}(\partial_{u^\beta} \vec{r}_\alpha) = \beta \leftrightarrow \gamma$$

$$\partial_{u^\beta}(\partial_{u^\alpha} \vec{n}) = \alpha \leftrightarrow \beta$$

$$\Longleftrightarrow \begin{cases} \Gamma_{\alpha\beta}^{\gamma} = \Gamma_{\beta\alpha}^{\gamma}, \\ b_{\alpha\beta} = b_{\beta\alpha}, \\ \text{Gauss 方程以及 Codazzi 方程.} \end{cases} \qquad (**)$$

对 $u_0 \in D$，任取向量 $\vec{r}_{0\alpha}$, $\vec{n}_0$ 满足：

$\langle \vec{n}_0, \vec{n}_0 \rangle = 1$,

$\langle \vec{r}_{0\alpha}, \vec{n}_0 \rangle = 0$,

$\langle \vec{r}_{0\alpha}, \vec{r}_{0\beta} \rangle = g_{\alpha\beta}(u_0)$.

$(\vec{r}_{01}, \vec{r}_{02}, \vec{n}_0) > 0$

Consider (*) 的初值问题:

$$\vec{p}(u_0) = \vec{p}_0 \quad (\text{任取 } \mathbb{R}^3 \text{ 中一点})$$

$$\vec{p}_\alpha(u_0) = \vec{p}_{0\alpha}$$

$$\vec{f}(u_0) = \vec{f}_0$$

由于(**)成立 $\Rightarrow$ 存在 u_0 的一个邻域 $U \subseteq D$, s.t.
(*) 在 U 中有解:

$\vec{p}$, $\vec{p}_\alpha$, $\vec{f}$ $\Rightarrow$ $\vec{p}$ 给出 $U \to \mathbb{R}^3$ 的一个参数曲面.

(*) $\Rightarrow$
$$\frac{\partial\langle \vec{p}_\alpha, \vec{p}_\beta\rangle}{\partial u^\gamma} = \Gamma^\xi_{\alpha\gamma}\langle \vec{p}_\xi, \vec{p}_\beta\rangle + \Gamma^\xi_{\beta\gamma}\langle \vec{p}_\xi, \vec{p}_\alpha\rangle + b_{\alpha\gamma}\langle \vec{p}_\beta, \vec{f}\rangle + b_{\beta\gamma}\langle \vec{p}_\alpha, \vec{f}\rangle$$

$$\frac{\partial\langle \vec{p}_\alpha, \vec{f}\rangle}{\partial u^\beta} = \Gamma^\xi_{\alpha\beta}\langle \vec{p}_\xi, \vec{f}\rangle + b_{\alpha\beta}\langle \vec{f}, \vec{f}\rangle - b^\gamma_\beta\langle \vec{p}_\alpha, \vec{p}_\gamma\rangle$$

$$\frac{\partial\langle \vec{f}, \vec{f}\rangle}{\partial u^\alpha} = -2 b^\beta_\alpha \langle \vec{p}_\beta, \vec{f}\rangle$$

视为关于
$\langle \vec{p}_\alpha, \vec{p}_\beta\rangle$, $\langle \vec{p}_\alpha, \vec{f}\rangle$, $\langle \vec{f}, \vec{f}\rangle$
的一阶线性偏微分方程组.

显然 $g_{\alpha\beta}$, 0, 1 为一组解.

给定初值 $g_{\alpha\beta}(u_0)$, 0, 1. 解唯一.

由于 $\langle \vec{p}_\alpha, \vec{p}_\beta\rangle(u_0) = g_{\alpha\beta}(u_0)$,
$\langle \vec{p}_\alpha, \vec{f}\rangle(u_0) = 0$,
$\langle \vec{f}, \vec{f}\rangle(u_0) = 1$

$$\Rightarrow \langle \vec{p}_\alpha, \vec{p}_\beta\rangle \equiv g_{\alpha\beta}$$

$$\langle \vec{p}_\alpha, \vec{f}\rangle \equiv 0$$

$$\langle \vec{f}, \vec{f}\rangle \equiv 1.$$

$\Rightarrow$ $\vec{p}_1$, $\vec{p}_2$ 在 U 上处处线性无关.
且由于 $\vec{p}_\alpha = \partial_{u^\alpha}\vec{p}$ 知: $\vec{p}$ 以 I 为其第一基本形式.
$\vec{p}_1$, $\vec{p}_2$, $\vec{f}$ 为右手, $\langle \vec{f}, \vec{f}\rangle \equiv 1$ 及

(89)

$$\frac{\partial \vec P_\alpha}{\partial u^\beta} = \Gamma_{\alpha\beta}^{\gamma}\vec P_\gamma + b_{\alpha\beta}\vec f$$

知: $\vec P$ 以 II 为其第二基本形式. 定义 $\vec r = \vec P$, $u\in U$. #.

作业: ① 给出 $R_{\alpha\beta\gamma\delta}$ 对称性的内蕴证明.

② P102 : 1

P103 : 3, 5, 9

§4. 联络与测地线

§4.1 平行移动

S: 光滑曲面. 正则. $\vec r = \vec r(u^1,u^2)$ $\quad \vec r: D \to \mathbb{R}^3$, $(u^1,u^2)\mapsto \vec r(u^1,u^2)$.

设 $\vec X(u^1,u^2)$ 为 S上的一个切向量场.

$\vec X = X^\alpha(u^1,u^2)\,\vec r_{u^\alpha}(u^1,u^2)$.

称 $\vec X$ 是光滑的, if X^1, X^2 均是光滑的. (on D)

$$d\vec X = dX^\alpha \vec r_{u^\alpha} + X^\alpha d\vec r_{u^\alpha}$$

$$= dX^\alpha \vec r_{u^\alpha} + X^\alpha\left(\Gamma_{\alpha\beta}^\rho du^\beta \vec r_{u^\rho} + b_{\alpha\beta}du^\beta \vec n\right)$$

$$= \left(dX^\alpha + X^\rho\Gamma_{\rho\gamma}^\alpha du^\gamma\right)\vec r_{u^\alpha} + X^\alpha b_{\alpha\beta}du^\beta\vec n.$$

定义: $D\vec X(u^1,u^2) := \left(dX^\alpha + X^\rho\Gamma_{\rho\gamma}^\alpha du^\gamma\right)\vec r_{u^\alpha}$.

称为曲面S上的切向量场 $\vec X$ 的协变微分.

(附注, $\forall p\in S$, $D\vec X(p)$ 为 $d\vec X(p)$ 向 T_pS 的正交投影).

(90)

记: $DX^\alpha = dX^\alpha + X^\beta\Gamma_{\beta\gamma}^\alpha du^\gamma$.

则: $D\vec X = DX^\alpha\, \vec r_{u^\alpha}$.

称 DX^α 为分量 X^α 的协变微分. $DX^\alpha = \frac{\partial X^\alpha}{\partial u^\gamma}du^\gamma + X^\beta\Gamma^\alpha_{\beta\gamma}du^\gamma = \left(\frac{\partial X^\alpha}{\partial u^\gamma} + X^\beta\Gamma^\alpha_{\beta\gamma}\right)du^\gamma$

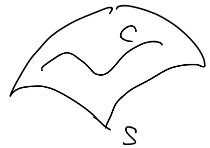
Thm. $\forall\, \vec X,\vec Y\in\mathfrak{X}(S)$, 有:

① $D(\vec X+\vec Y) = D(\vec X)+D(\vec Y)$

② $D(f\cdot\vec X) = df\cdot\vec X + f\cdot D\vec X$, $\forall f\in C^\infty(D)$

③ $d\langle\vec X,\vec Y\rangle = \langle D\vec X,\vec Y\rangle + \langle\vec X, D\vec Y\rangle$

Pf. exercise. #



设 C: $u^\alpha = u^\alpha(t)$ 为 S上的一条曲线.

设 $\vec X(t)$ 为 S上 沿 C 的一个切向量场. (S上的)

$\frac{d\vec X(t)}{dt}$ 为 $\mathbb{R}^3$ 中沿 C 的一个向量场.

一般来说, $\frac{d\vec X(t)}{dt}$ 不是 S上的切向量场.

定义: $\frac{D\vec X(t)}{dt} := \frac{d\vec X(t)}{dt}$ 在 $T_{(u^1(t),\,u^2(t))}S$ 上的投影.

$\vec X(t) = X^\alpha(t)\,\vec r_{u^\alpha}(u^1(t),u^2(t))$.

$$\frac{d\vec X(t)}{dt} = \frac{dX^\alpha(t)}{dt}\vec r_{u^\alpha} + X^\alpha(t)\frac{\partial \vec r_{u^\alpha}}{\partial u^\gamma}\frac{du^\gamma(t)}{dt}$$

$$= \underbrace{\left(\frac{dX^\alpha(t)}{dt} + \Gamma^\alpha_{\beta\gamma}X^\beta(t)\frac{du^\gamma(t)}{dt}\right)\vec r_{u^\alpha}} + b_{\alpha\gamma}X^\alpha(t)\frac{du^\gamma}{dt}\vec n$$

$$\Rightarrow \frac{D\vec{X}(t)}{dt} = \left(\frac{dX^\alpha(t)}{dt} + \Gamma^\alpha_{\beta\gamma} X^\beta(t)\frac{du^\gamma(t)}{dt}\right)\vec{r}_{u^\alpha}$$

称 $\frac{D\vec{X}(t)}{dt}$ 为 S上沿 C 定义的 S上的切向量场 X 沿C在S上的协变导数.

$$\frac{dX^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} X^\beta(t)\frac{du^\gamma(t)}{dt} =: \frac{DX^\alpha}{dt}$$ 为分量 X^α 沿 C 的协变导数.

Recall that $DX^\alpha = \left(\frac{\partial X^\alpha}{\partial u^\gamma} + X^\beta \Gamma^\alpha_{\beta\gamma}\right)du^\gamma$

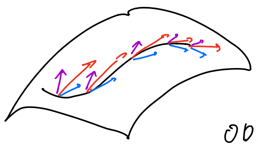
$\frac{\partial X^\alpha}{\partial u^\gamma} + X^\beta \Gamma^\alpha_{\beta\gamma}$ 为沿参数曲线的协变导数.

记为 $X^\alpha_{,\gamma}$. 因而, $DX^\alpha = X^\alpha_{,\gamma}\, du^\gamma$.

定义: 设 $\vec{X}(t)$ 是 S上沿 $C: u^\gamma = u^\gamma(t)$ 定义的 S上的光滑向量场.

if $\frac{D\vec{X}(t)}{dt} = 0$ 则称 $X(t)$ 沿C在S上是平行的.

· $\frac{D\vec{X}(t)}{dt} = 0 \iff \frac{dX^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} X^\beta \frac{du^\gamma}{dt} = 0.$ (*)



$C: u^\gamma = u^\gamma(t)$, $t \in [c,d] \subseteq (a,b)$

$X^\alpha|_{t=t_0} = X^\alpha_0$

ODE 理论 $\Rightarrow$ $X^\alpha(t)$ 为 (*) 的唯一解.

称 $\vec{X}(t) := X^\alpha(t)\vec{r}_{u^\alpha}$ 为 $X^\alpha_0 \vec{r}_{u^\alpha}(u^1(t_0), u^2(t_0))$ 沿C的平行移动.

· 若 $\vec{X}(t)$ 沿 C 在S上平行,

$\vec{Y}(t)$ . . . .

则 $\frac{d}{dt}\langle \vec{X}(t), \vec{Y}(t)\rangle = \langle \frac{D\vec{X}(t)}{dt}, \vec{Y}(t)\rangle + \langle \vec{X}(t), \frac{D\vec{Y}(t)}{dt}\rangle$

$= 0$

$\overset{(沿C)}{\Rightarrow}$ $\langle \vec{X}(t), \vec{Y}(t)\rangle$ 内积保持不变.

特别, (沿C) $\vec{X}(t)$ 长度 . . . . . .

§4.2. 测地线.

S: $\vec{r} = \vec{r}(u^1, u^2)$ 正则曲面.

C: $u^\alpha = u^\alpha(t)$ 为 S上的光滑曲线.

定义: 若 C 的切向量场沿 C(在S上) 是平行的, 则称 C 为 S上的测地线.

$$\vec{X}(t) := \frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u^\alpha}(u^1(t), u^2(t))\frac{du^\alpha(t)}{dt}$$

$$\Rightarrow X^\alpha(t) = \frac{du^\alpha(t)}{dt}$$

$\Rightarrow$ 测地线方程: $$\frac{d^2u^\alpha}{dt^2} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{dt}\frac{du^\gamma}{dt} = 0.$$

Next time: 变分法!!!

作业: 求 $x^2+y^2+z^2=1$ 上的测地线.