4.3 变分原理

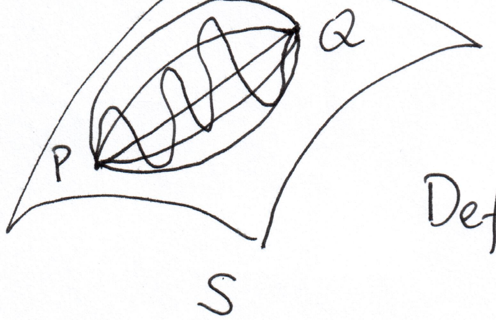
设 S 为一正则曲面，$\vec{r}=\vec{r}(u^1,u^2)$.

- 设 $L=L(\underline{u},\underline{\xi},t)$ 为 $TS\times\mathbb{R}$ 上的任一(给定的)光滑函数.

$\underline{u}=(u^1,u^2)$, $\underline{\xi}=(\xi^1,\xi^2)\in T_{\underline{u}}S$.

- 任取 $P,Q\in S$. 固定 P,Q. 考虑连接 P,Q 的 S 上的光滑曲线 γ: $u^\alpha=u^\alpha(t)$, $t\in[c,d]\subseteq(a,b)$

$$u^\alpha(c)=u^\alpha_P$$
$$u^\alpha(d)=u^\alpha_Q$$



Def. 定义泛函

$$S[\gamma]:=\int_c^d L(\underline{u}(t),\dot{\underline{u}}(t),t)\,dt,$$

称为 action (作用).

Question: $\min\limits_{\gamma} S[\gamma]=?$

定理：若 $S[\gamma]$ 在 γ^* 上取得驻值，则 γ^* 满足：

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right)-\frac{\partial L}{\partial u^\alpha}=0. \quad \text{(Euler-Lagrange 方程)}$$

这里，$\dfrac{\partial L}{\partial \dot{u}^\alpha}=\dfrac{\partial L}{\partial \xi^\alpha}\Big|_{\underline{u}=\underline{u}(t),\,\underline{\xi}=\dot{\underline{u}}(t)}$, $\dfrac{\partial L}{\partial u^\alpha}=\dfrac{\partial L}{\partial u^\alpha}\Big|_{\underline{u}=\underline{u}(t),\,\underline{\xi}=\dot{\underline{u}}(t)}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right)=\left(\frac{\partial^2 L}{\partial u^\beta\partial\xi^\alpha}\dot{u}^\beta+\frac{\partial^2 L}{\partial\xi^\alpha\partial\xi^\beta}\ddot{u}^\beta+\frac{\partial^2 L}{\partial\xi^\alpha\partial t}\right)\Bigg|_{\underline{u}=\underline{u}(t),\,\underline{\xi}=\dot{\underline{u}}(t)}$$

proof. 设 $\eta^\alpha=\eta^\alpha(t)$, $t\in(a,b)$ 为满足如下条件的任意光滑函数：

$$\eta^\alpha(c)=\eta^\alpha(d)=0.$$ 取定任何一个这样的 $\underline{\eta}(t)$.

γ^* 为驻值曲线 $\Rightarrow$ $$\lim_{\varepsilon\to 0}\frac{S[\gamma^*+\varepsilon\eta]-S[\gamma^*]}{\varepsilon}=0.$$

$$\begin{aligned}
LHS &= \frac{d}{d\varepsilon}\left(S[\gamma^*+\varepsilon\eta]\right)\Big|_{\varepsilon=0}\\
&= \frac{d}{d\varepsilon}\left(\int_c^d L(\underline{u}+\varepsilon\underline{\eta},\ \dot{\underline{u}}+\varepsilon\dot{\underline{\eta}},\ t)\,dt\right)\Big|_{\varepsilon=0}\\
&= \int_c^d\left[\frac{\partial L}{\partial u^\alpha}(\underline{u}+\varepsilon\underline{\eta},\dot{\underline{u}}+\varepsilon\dot{\underline{\eta}},t)\,\eta^\alpha+\frac{\partial L}{\partial \xi^\alpha}(\underline{u}+\varepsilon\underline{\eta},\dot{\underline{u}}+\varepsilon\dot{\underline{\eta}},t)\,\dot{\eta}^\alpha\right]dt\,\Big|_{\varepsilon=0}\\
&= \int_c^d\left(\frac{\partial L}{\partial u^\alpha}\eta^\alpha+\frac{\partial L}{\partial \dot{u}^\alpha}\dot{\eta}^\alpha\right)dt\\
&= \int_c^d\left(\frac{\partial L}{\partial u^\alpha}\eta^\alpha-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right)\eta^\alpha\right)dt+\underbrace{\frac{\partial L}{\partial \dot{u}^\alpha}\eta^\alpha\Big|_c^d}_{=0}\\
&= \int_c^d\left[\frac{\partial L}{\partial u^\alpha}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right)\right]\eta^\alpha\,dt\\
&\equiv 0\quad(\text{for any } \underline{\eta})
\end{aligned}$$

$$\Rightarrow\ \frac{\partial L}{\partial u^\alpha}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right)=0.\qquad \#$$

· Def. 称 L 为 Lagrange 量 (Lagrangian).

称 $\gamma+\varepsilon\eta$ 为 γ 的一个变分.

(其几何意义之后会看到。)

· 例子

例 1. $L=\sqrt{g_{\alpha\beta}(u)\,\xi^\alpha\xi^\beta}$

相应的作用 $S[\gamma]=\int_c^d L\,dt$ 称为 γ 的<u>长度</u>.
(长度泛函)

Euler-Lagrange 方程:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot u^\alpha}\right)=\frac{\partial L}{\partial u^\alpha}$$

$$\mathrm{LHS}=\frac{d}{dt}\left(\frac{1}{2}\left(g_{\alpha_1\beta_1}\dot u^{\alpha_1}\dot u^{\beta_1}\right)^{-\frac{1}{2}}\left(g_{\alpha\beta_2}\dot u^{\beta_2}+g_{\alpha_2\alpha}\dot u^{\alpha_2}\right)\right)$$

$$=\frac{d}{dt}\left(\frac{g_{\alpha\beta}\dot u^\beta}{\sqrt{g_{\alpha_1\beta_1}\dot u^{\alpha_1}\dot u^{\beta_1}}}\right)$$

$$\mathrm{RHS}=\frac{1}{2}\frac{\partial g_{\alpha_2\beta_2}}{\partial u^\alpha}\dot u^{\alpha_2}\dot u^{\beta_2}\frac{1}{\sqrt{g_{\alpha_1\beta_1}\dot u^{\alpha_1}\dot u^{\beta_1}}}$$

驻值曲线: $\displaystyle\frac{1}{2}\frac{\frac{\partial g_{\alpha_2\beta_2}}{\partial u^\alpha}\dot u^{\alpha_2}\dot u^{\beta_2}}{\sqrt{g_{\alpha_1\beta_1}\dot u^{\alpha_1}\dot u^{\beta_1}}}=\frac{d}{dt}\left(\frac{g_{\alpha\beta}\dot u^\beta}{\sqrt{g_{\alpha_1\beta_1}\dot u^{\alpha_1}\dot u^{\beta_1}}}\right)$

由于曲线的长度与其参数选取无关，取参数 t 满足:

$$g_{\alpha\beta}\dot u^\alpha\dot u^\beta=\mathrm{Const}=:l>0$$

则: $\displaystyle\frac{1}{2}\frac{1}{\sqrt{l}}\frac{\partial g_{\alpha_2\beta_2}}{\partial u^\alpha}\dot u^{\alpha_2}\dot u^{\beta_2}=\frac{1}{\sqrt{l}}\frac{d}{dt}\left(g_{\alpha\beta}\dot u^\beta\right)$

$$\Rightarrow\quad \frac{1}{2}\frac{\partial g_{\alpha_2\beta_2}}{\partial u^\alpha}\dot u^{\alpha_2}\dot u^{\beta_2}=\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\dot u^\gamma\dot u^\beta+g_{\alpha\beta}\ddot u^\beta$$

$$\Rightarrow\quad \ddot u^\beta=g^{\beta\alpha}\left(\frac{1}{2}\frac{\partial g_{\alpha_2\beta_2}}{\partial u^\alpha}\dot u^{\alpha_2}\dot u^{\beta_2}-\frac{\partial g_{\alpha\delta}}{\partial u^\gamma}\dot u^\gamma\dot u^\delta\right)$$

$$=g^{\beta\alpha}\left(\frac{1}{2}\frac{\partial g_{\gamma\delta}}{\partial u^\alpha}\dot u^\gamma\dot u^\delta-\frac{1}{2}\frac{\partial g_{\alpha\delta}}{\partial u^\gamma}\dot u^\gamma\dot u^\delta-\frac{1}{2}\frac{\partial g_{\alpha\gamma}}{\partial u^\delta}\dot u^\gamma\dot u^\delta\right),$$

i.e., $\ddot u^\beta+\Gamma^\beta_{\gamma\delta}\dot u^\gamma\dot u^\delta=0.$

· Def. 若曲面S上的曲线γ的参数t满足:

$$t = c_1 s\ ,\quad c_1 \neq 0 \text{ 为常数},$$

其中 s 是γ的弧长参数, 则称t为γ的自然参数.

· Thm. 曲面S上的长度泛函的驻值曲线在参数选为~~弧长~~自然参数时是测地线; 最短曲线一定是测地线的像.
测地线是长度泛函数的驻值曲线.

· 例2. $L = \frac{1}{2} g_{\alpha\beta} \xi^\alpha \xi^\beta = \frac{1}{2} g_{\gamma\delta} \xi^\gamma \xi^\delta$

$$S[\gamma] = \int_c^d L(\underline{u}, \dot{\underline{u}}, t)\, dt$$

Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right) = \frac{\partial L}{\partial u^\alpha}$$

$$\Longleftrightarrow \quad \frac{d}{dt}\left(\frac{1}{2} g_{\alpha\delta} \dot{u}^\delta + \frac{1}{2} g_{\gamma\alpha} \dot{u}^\gamma\right) = \frac{1}{2} \frac{\partial g_{\gamma\delta}}{\partial u^\alpha} \dot{u}^\gamma \dot{u}^\delta$$

$$\Longleftrightarrow \quad g_{\alpha\gamma} \ddot{u}^\gamma + \frac{1}{2} \frac{\partial g_{\alpha\delta}}{\partial u^\gamma} \dot{u}^\gamma \dot{u}^\delta + \frac{1}{2} \frac{\partial g_{\gamma\alpha}}{\partial u^\delta} \dot{u}^\gamma \dot{u}^\delta = \frac{1}{2} \frac{\partial g_{\gamma\delta}}{\partial u^\alpha} \dot{u}^\gamma \dot{u}^\delta$$

$$\Longleftrightarrow \quad \ddot{u}^\gamma + \Gamma^\gamma_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta = 0 .$$

· Thm. 曲面S上的曲线γ^*是测地线, iff γ^*是作用

$$S[\gamma] = \int_c^d L(\underline{u}, \dot{\underline{u}})\, dt,\quad L = \frac{1}{2} g_{\alpha\beta} \xi^\alpha \xi^\beta$$

的驻值曲线.

· Cor. (最小作用原理) 不受外力下, "作用"最小的(参数)曲线一定是测地线.

· 例3. $L = \frac{m}{2} g_{\alpha\beta}(\underline{u}) \xi^\alpha \xi^\beta - V(\underline{u})$

$$S[\gamma] = \int_c^d L(\underline{u}, \dot{\underline{u}})\, dt$$

Euler-Lagrange equation:

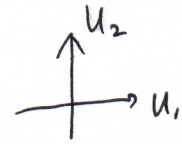
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot u^\alpha}\right) = \frac{\partial L}{\partial u^\alpha}$$

$$\Longleftrightarrow m\left(\ddot u^\gamma + \Gamma^\gamma_{\alpha\beta}\,\dot u^\alpha \dot u^\beta\right) = -g^{\gamma\alpha}\frac{\partial V}{\partial u^\alpha}$$

(曲面S上质点运动的 Newton 方程)

特别地，若S是平面，则：$\Gamma^\gamma_{\alpha\beta}\equiv 0$，$(g^{\alpha\beta}) = \begin{pmatrix}1&0\\0&1\end{pmatrix}$

$$\Rightarrow \quad m\ddot u_i = -\frac{\partial V}{\partial u_i}, \quad i=1,2.$$



Thm.（最小作用原理，有势外力情形）曲面上的质点在保守力作用下的运动曲线是使作用

$$S[\gamma] = \int_c^d L(\underline{u}, \dot{\underline{u}})\,dt, \quad L = \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta - V$$

取驻值的参数曲线.

$\cancel{L = \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta - V}$ · 设 $L = L(\underline{u}, \underline{\xi}, t) \in C^\infty(TS\times\mathbb{R})$.

· Noether's Theorem:

① 若 L 不显含 t，i.e., $\frac{\partial L}{\partial t}\equiv 0$，则

$$E := \left(\xi^\sigma \frac{\partial L}{\partial \xi^\sigma} - L\right)\Big|_\gamma$$ 是 Euler-Lagrange 方程的一个守恒量.

② 若 L 不显含 u^β，i.e., $\frac{\partial L}{\partial u^\beta}\equiv 0$，则

$$p_\beta := \left(\frac{\partial L}{\partial \xi^\beta}\right)\Big|_\gamma$$ 是 Euler-Lagrange 方程的一个守恒量.

（称 E 为 energy of a curve，p_β 为动量 momentum）.

proof. ① $\frac{dE}{dt} = \frac{d}{dt}\left(\dot{u}^\sigma \frac{\partial L}{\partial \dot{u}^\sigma} - L\right)$

$$= \cancel{\ddot{u}^\sigma \frac{\partial L}{\partial \dot{u}^\sigma}} + \dot{u}^\sigma \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\sigma}\right) - \frac{\partial L}{\partial u^\alpha}\dot{u}^\alpha - \cancel{\frac{\partial L}{\partial \dot{u}^\alpha}\ddot{u}^\alpha}$$

$$= \dot{u}^\sigma \frac{\partial L}{\partial u^\sigma} - \dot{u}^\alpha \frac{\partial L}{\partial u^\alpha} = 0 .$$

② $\frac{d p_\alpha}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}^\alpha}\right) = \frac{\partial L}{\partial u^\alpha} = 0$. #.

· 对于 $L = \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta - V$，有：

* $E = \xi^\sigma \frac{\partial L}{\partial \xi^\sigma} - L$

$$= \xi^\sigma \frac{1}{2} g_{\alpha\beta}\left(\delta^\alpha_\sigma \xi^\beta + \xi^\alpha \delta^\beta_\sigma\right) - \left(\frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta - V\right)$$

$$= \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta + V$$

* 显然，$\frac{\partial L}{\partial t} = 0$

$\Rightarrow E = \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta + V$ 为 Euler-Lagrange 方程的守恒量.

· 特别地，若 $V \equiv 0$，则这时驻值曲线是测地线.

$\Rightarrow E = \frac{1}{2} g_{\alpha\beta}\xi^\alpha\xi^\beta$ 是测地线的一个守恒量.

* $p_\alpha = \frac{\partial L}{\partial \xi^\alpha} = g_{\alpha\beta}\xi^\beta$ 为动量.

§4.4. More on covariant derivative

. S：正则曲面. $\vec{X} \in C^\infty(S, TS)$

· Recall that $\vec{X}$ 的协变微分是其全微分在 TS 上的正交投影, i.e.

$$D\vec{X} = \left(dX^\alpha + X^\rho \Gamma^\alpha_{\rho\gamma} du^\gamma\right)\vec{r}_{u^\alpha}.$$

这里，$\vec{X} = X^\alpha \vec{r}_{u^\alpha}$.

· C：S上光滑曲线．$\vec{X}$沿C的协变导数可定义为

$$\frac{D\vec{X}(u(t))}{dt}=\left(\frac{dX^\alpha(u(t))}{dt}+\Gamma^\alpha_{\rho\gamma}X^\rho\frac{du^\gamma}{dt}\right)\vec{r}_{u^\alpha}.$$

$\vec{X}$沿C的协变导数是$\vec{X}\circ c$沿C的协变导数．

（见§4.1）[注：在§4.1中，对沿曲线C的向量场定义了协变导数．]

· 我们把这个协变导数视为$\vec{X}$沿C的切向量场的协变导数．

· 引入如下的记号是方便的：对于$\vec{Y}\in C^\infty(S,TS)$，

$$\nabla_{\vec{Y}}\vec{X}:=\left(\frac{\partial X^\alpha}{\partial u^\gamma}Y^\gamma+\Gamma^\alpha_{\rho\gamma}X^\rho Y^\gamma\right)\vec{r}_{u^\alpha}\quad(*)$$

· 当$\vec{Y}$是C的切向量场时，记号$\nabla_{\vec{Y}}\vec{X}$也是有意义的，因为定义(*)中，只用到了$\vec{Y}$的分量而没有对这些分量求偏导；这时，$\vec{Y}=\frac{du^\gamma}{dt}\vec{r}_{u^\gamma}$，我们有：

$$\nabla_{\vec{Y}}\vec{X}:=\left(\frac{\partial X^\alpha}{\partial u^\gamma}\frac{du^\gamma}{dt}+\Gamma^\alpha_{\rho\gamma}X^\rho\frac{du^\gamma}{dt}\right)\vec{r}_{u^\alpha}=\frac{D\vec{X}(u(t))}{dt}.$$

[注：若$\vec{X}$是沿C定义的向量场，也可定义$\nabla_{\vec{Y}}\vec{X}$，i.e. $\nabla_{\vec{Y}}\vec{X}:=\frac{D\vec{X}}{dt}$.]

· 对于$\vec{X},\vec{Y}\in C^\infty(S,TS)$，我们来计算$\nabla_{\vec{X}}\vec{Y}-\nabla_{\vec{Y}}\vec{X}$．

$$\begin{aligned}&\nabla_{\vec{X}}\vec{Y}-\nabla_{\vec{Y}}\vec{X}\\&=\left(\frac{\partial Y^\alpha}{\partial u^\gamma}X^\gamma+\Gamma^\alpha_{\rho\gamma}Y^\rho X^\gamma\right)\vec{r}_{u^\alpha}\\&\quad-\left(\frac{\partial X^\alpha}{\partial u^\gamma}Y^\gamma+\Gamma^\alpha_{\rho\gamma}X^\rho Y^\gamma\right)\vec{r}_{u^\alpha}\\&=\left(X^\gamma\frac{\partial Y^\alpha}{\partial u^\gamma}-Y^\gamma\frac{\partial X^\alpha}{\partial u^\gamma}\right)\vec{r}_{u^\alpha}\end{aligned}$$

· Def. $\forall\vec{X},\vec{Y}\in C^\infty(S,TS)$，

$$[\vec{X},\vec{Y}]:=\left(X^\gamma\frac{\partial Y^\alpha}{\partial u^\gamma}-Y^\gamma\frac{\partial X^\alpha}{\partial u^\gamma}\right)\vec{r}_{u^\alpha}$$

称为$\vec{X},\vec{Y}$的Poisson括号．（$[\vec{X},\vec{Y}]\in C^\infty(S,TS)$）

· Prop. $\nabla_{\vec{X}}\vec{Y}-\nabla_{\vec{Y}}\vec{X}=[\vec{X},\vec{Y}]$.

Examples. 1. $\nabla_{\vec{X}}\vec{X} = \left(\frac{\partial X^\alpha}{\partial u^\gamma} X^\gamma + \Gamma^\alpha_{\rho\gamma} X^\rho X^\gamma\right)\vec{r}_{u^\alpha}$.

$\forall \vec{X} \in C^\infty(S, TS)$

2. $\forall \vec{X}, \vec{Y} \in C^\infty(S, TS)$

$\nabla_{\vec{Y}}\vec{X} = \vec{X}$沿$\vec{Y}$的积分曲线的协变导数.

3. C : S上光滑曲线. $\underline{u} = \underline{u}(t)$. or $\vec{r} = \vec{r}(\underline{u}(t))$

$$\nabla_{\dot{\vec{r}}}\dot{\vec{r}} = \left(\ddot{u}^\alpha + \Gamma^\alpha_{\beta\gamma}\dot{u}^\beta\dot{u}^\gamma\right)\vec{r}_{u^\alpha}$$

- $$\nabla_{\vec{Y}}\vec{X} = \left(\frac{\partial X^\alpha}{\partial u^\gamma} Y^\gamma + \Gamma^\alpha_{\beta\gamma} X^\beta Y^\gamma\right)\vec{r}_{u^\alpha}$$
$$= \left(\frac{\partial X^\sigma}{\partial u^\rho} + \Gamma^\sigma_{\varphi\rho} X^\varphi\right) Y^\rho\, \vec{r}_{u^\sigma}$$

- $\nabla_{\vec{X}}\nabla_{\vec{Y}}(\vec{Z})$
$$= \nabla_{\vec{X}}\left[\left(\frac{\partial Z^\alpha}{\partial u^\gamma} + \Gamma^\alpha_{\beta\gamma} Z^\beta\right) Y^\gamma\, \vec{r}_{u^\alpha}\right]$$
$$= \left(\frac{\partial}{\partial u^\rho}\left[\left(\frac{\partial Z^\sigma}{\partial u^\gamma} + \Gamma^\sigma_{\beta\gamma} Z^\beta\right) Y^\gamma\right] + \Gamma^\sigma_{\varphi\rho}\left(\frac{\partial Z^\varphi}{\partial u^\gamma} + \Gamma^\varphi_{\beta\gamma} Z^\beta\right) Y^\gamma\right) X^\rho\, \vec{r}_{u^\sigma}$$
$$= \Big\{ \frac{\partial^2 Z^\sigma}{\partial u^\rho \partial u^\gamma} Y^\gamma + \frac{\partial \Gamma^\sigma_{\beta\gamma}}{\partial u^\rho} Z^\beta Y^\gamma + \Gamma^\sigma_{\beta\gamma}\frac{\partial Z^\beta}{\partial u^\rho} Y^\gamma + \left(\frac{\partial Z^\sigma}{\partial u^\gamma} + \Gamma^\sigma_{\beta\gamma} Z^\beta\right)\frac{\partial Y^\gamma}{\partial u^\rho}$$
$$+ \Gamma^\sigma_{\varphi\rho}\frac{\partial Z^\varphi}{\partial u^\gamma} Y^\gamma + \Gamma^\sigma_{\varphi\rho}\Gamma^\varphi_{\beta\gamma} Z^\beta Y^\gamma \Big\} X^\rho\, \vec{r}_{u^\sigma}$$

- $\nabla_{\vec{Y}}\nabla_{\vec{X}}(\vec{Z})$
$$= \Big\{ \frac{\partial^2 Z^\sigma}{\partial u^\rho \partial u^\gamma} X^\gamma + \frac{\partial \Gamma^\sigma_{\beta\gamma}}{\partial u^\rho} Z^\beta X^\gamma + \Gamma^\sigma_{\beta\gamma}\frac{\partial Z^\beta}{\partial u^\rho} X^\gamma + \left(\frac{\partial Z^\sigma}{\partial u^\gamma} + \Gamma^\sigma_{\beta\gamma} Z^\beta\right)\frac{\partial X^\gamma}{\partial u^\rho}$$
$$+ \Gamma^\sigma_{\varphi\rho}\frac{\partial Z^\varphi}{\partial u^\gamma} X^\gamma + \Gamma^\sigma_{\varphi\rho}\Gamma^\varphi_{\beta\gamma} Z^\beta X^\gamma \Big\} Y^\rho\, \vec{r}_{u^\sigma}$$

- $\nabla_{\vec{X}} \nabla_{\vec{Y}}(\vec{Z}) - \nabla_{\vec{Y}} \nabla_{\vec{X}}(\vec{Z})$

$$
\begin{aligned}
= & \left\{ \frac{\partial \Gamma^{\sigma}_{\beta\gamma}}{\partial u^\rho} Z^\beta Y^\gamma X^\rho + \Gamma^{\sigma}_{\varphi\rho} \Gamma^{\varphi}_{\beta\gamma} Z^\beta Y^\gamma X^\rho \right. \\
& \left. - \frac{\partial \Gamma^{\sigma}_{\beta\gamma}}{\partial u^\rho} Z^\beta X^\gamma Y^\rho - \Gamma^{\sigma}_{\varphi\rho} \Gamma^{\varphi}_{\beta\gamma} Z^\beta X^\gamma Y^\rho \right\} \vec{r}_{u^\sigma} \\
+ & \left\{ \Gamma^{\sigma}_{\beta\gamma} \frac{\partial Z^\beta}{\partial u^\rho} Y^\gamma X^\rho + \frac{\partial Z^\sigma}{\partial u^\gamma} \frac{\partial Y^\gamma}{\partial u^\rho} X^\rho + \Gamma^{\sigma}_{\beta\gamma} Z^\beta \frac{\partial Y^\gamma}{\partial u^\rho} X^\rho \right. \\
& + \Gamma^{\sigma}_{\varphi\rho} \frac{\partial Z^\varphi}{\partial u^\gamma} Y^\gamma X^\rho - \Gamma^{\sigma}_{\beta\gamma} \frac{\partial Z^\beta}{\partial u^\rho} X^\gamma Y^\rho - \frac{\partial Z^\sigma}{\partial u^\gamma} \frac{\partial X^\gamma}{\partial u^\rho} Y^\rho \\
& \left. - \Gamma^{\sigma}_{\beta\gamma} Z^\beta \frac{\partial X^\gamma}{\partial u^\rho} Y^\rho - \Gamma^{\sigma}_{\varphi\rho} \frac{\partial Z^\varphi}{\partial u^\gamma} X^\gamma Y^\rho \right\} \vec{r}_{u^\sigma} \\
= & \left\{ \Gamma^{\sigma}_{\varphi\rho} \Gamma^{\varphi}_{\beta\gamma} - \Gamma^{\sigma}_{\varphi\gamma} \Gamma^{\varphi}_{\beta\rho} + \frac{\partial \Gamma^{\sigma}_{\beta\gamma}}{\partial u^\rho} - \frac{\partial \Gamma^{\sigma}_{\beta\rho}}{\partial u^\gamma} \right\} Z^\beta Y^\gamma X^\rho \vec{r}_{u^\sigma} \\
+ & \left\{ \frac{\partial Z^\sigma}{\partial u^\gamma} \left(X^\rho \frac{\partial Y^\gamma}{\partial u^\rho} - Y^\rho \frac{\partial X^\gamma}{\partial u^\rho} \right) \right\} \vec{r}_{u^\sigma} \\
+ & \left\{ \Gamma^{\sigma}_{\beta\gamma} Z^\beta \left(X^\rho \frac{\partial Y^\gamma}{\partial u^\rho} - Y^\rho \frac{\partial X^\gamma}{\partial u^\rho} \right) \right\} \vec{r}_{u^\sigma} \\
= & R^{\sigma}_{\beta\gamma\rho} Z^\beta Y^\gamma X^\rho \vec{r}_{u^\sigma} + \nabla_{\left(X^\rho \frac{\partial Y^\gamma}{\partial u^\rho} - Y^\rho \frac{\partial X^\gamma}{\partial u^\rho}\right) \vec{r}_{u^\gamma}} \vec{Z} \\
= & R^{\sigma}_{\beta\gamma\rho} Z^\beta Y^\gamma X^\rho \vec{r}_{u^\sigma} + \nabla_{[\vec{X}, \vec{Y}]} \vec{Z}.
\end{aligned}
$$

- Lemma. $\nabla_{\vec{X}} \nabla_{\vec{Y}}(\vec{Z}) - \nabla_{\vec{Y}} \nabla_{\vec{X}}(\vec{Z}) = R^{\sigma}_{\beta\gamma\rho} Z^\beta Y^\gamma X^\rho \vec{r}_{u^\sigma} + \nabla_{[\vec{X},\vec{Y}]} \vec{Z}$.

- Example. 若 $\vec{X}$, $\vec{Y}$ 是坐标向量场, $\vec{X} = \vec{r}_{u^\alpha}$, $\vec{Y} = \vec{r}_{u^\beta}$

  则: $[\vec{X}, \vec{Y}] = 0$

  $\Rightarrow \nabla_{\vec{X}} \nabla_{\vec{Y}}(\vec{Z}) - \nabla_{\vec{Y}} \nabla_{\vec{X}}(\vec{Z}) = R^{\sigma}_{\beta\gamma\rho} Z^\beta Y^\gamma X^\rho \vec{r}_{u^\sigma}$

  i.e., 曲率张量刻画了沿坐标向量场协变导数可交换的 obstruction.

§4.5 二阶变分公式

- 设 S 为正则光滑曲面，令 $L = L(\underline{u}, \underline{\xi}) \in C^\infty(TS, \mathbb{R})$ 为切丛 TS 上的光滑函数。

$$S[\gamma] := \int_c^d L(\underline{u}(t), \dot{\underline{u}}(t))\, dt$$

- γ^* 为驻值曲线，即：

γ^* 满足 $\dfrac{\partial L}{\partial u^\alpha} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{u}^\alpha}\right)$，$\alpha = 1, 2$.（E-L 方程）

i.e., $\dfrac{d}{d\varepsilon}\left(S[\gamma^* + \varepsilon \eta]\right)\Big|_{\varepsilon=0} \equiv 0$ （$\forall \underline{\eta} = (\eta^1, \eta^2)$ $[c,d]$ 上的光滑函数 $\underline{\eta}(c) = \underline{\eta}(d) = 0$）

- 设 γ^* 为驻值曲线。

定义：$G_{\gamma^*}(\rho, \eta) = \dfrac{\partial^2 S[\gamma^* + \lambda\rho + \mu\eta]}{\partial\lambda\,\partial\mu}\Big|_{(\lambda,\mu)=0}$.

ρ, η 为 $[c,d]$ 上的光滑函数，满足 $\underline{\eta}(c) = \underline{\eta}(d) = 0$，$\underline{\rho}(c) = \underline{\rho}(d) = 0$.

下面我们计算 $\dfrac{\partial^2 S[\gamma^* + \lambda\rho + \mu\eta]}{\partial\lambda\,\partial\mu}$ ...

- $$\frac{\partial^2 S[\gamma^* + \lambda\rho + \mu\eta]}{\partial\lambda\,\partial\mu} = \frac{\partial}{\partial\lambda}\left(\frac{\partial}{\partial\mu}\int_c^d L\big(\overbrace{\underline{u} + \lambda\underline{\rho} + \mu\underline{\eta}, \dot{\underline{u}} + \lambda\dot{\underline{\rho}} + \mu\dot{\underline{\eta}}}^{\cdots}\big)\, dt\right)$$

$$= \frac{\partial}{\partial\lambda}\int_c^d \left(\frac{\partial L}{\partial u^\alpha}(\ldots)\eta^\alpha + \frac{\partial L}{\partial \xi^\alpha}(\ldots)\dot{\eta}^\alpha\right) dt$$

$$= \frac{\partial}{\partial\lambda}\int_c^d \eta^\alpha\left(\frac{\partial L}{\partial u^\alpha}(\ldots) - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^\alpha}(\ldots)\right)\right) dt$$

$$\Rightarrow \frac{\partial^2 S[\gamma^* + \lambda\rho + \mu\eta]}{\partial\lambda\,\partial\mu}\Big|_{\mu=0} = \frac{\partial}{\partial\lambda}\int_c^d \eta^\alpha\left(\frac{\partial L}{\partial u^\alpha}\big(\underbrace{u + \lambda\rho, \dot{u} + \lambda\dot{\rho}}\big) - \frac{d}{dt}\left(\frac{\partial L}{\partial \xi^\alpha}(u + \lambda\rho, \dot{u} + \lambda\dot{\rho})\right)\right) dt$$

$$= \int_c^d \eta^\alpha\left(\frac{\partial^2 L}{\partial u^\alpha \partial u^\beta}(\text{——})\rho^\beta + \frac{\partial^2 L}{\partial u^\alpha \partial \xi^\beta}(\text{——})\dot{\rho}^\beta - \frac{\partial}{\partial\lambda}\left(\frac{\partial^2 L}{\partial \xi^\alpha \partial u^\beta}(\text{——})(\dot{u} + \lambda\dot{\rho}) + \frac{\partial^2 L}{\partial \xi^\alpha \partial \xi^\beta}(-)(\ddot{u} + \lambda\ddot{\rho})\right)\right) dt$$

$$= \int_c^d \eta^\alpha\Big(\frac{\partial^2 L}{\partial u^\alpha \partial u^\beta}(-)\rho^\beta + \frac{\partial^2 L}{\partial u^\alpha \partial \xi^\beta}(-)\dot{\rho}^\beta - \frac{\partial^3 L}{\partial \xi^\alpha \partial u^\beta \partial u^\gamma}(-)\rho^\gamma(\dot{u}^\beta + \lambda\dot{\rho}^\beta) - \frac{\partial^3 L}{\partial \xi^\alpha \partial u^\beta \partial \xi^\gamma}(-)\dot{\rho}^\gamma(\dot{u}^\beta + \lambda\dot{\rho}^\beta)$$
$$- \frac{\partial^2 L}{\partial \xi^\alpha \partial u^\beta}(-)\dot{\rho}^\beta - \frac{\partial^2 L}{\partial \xi^\alpha \partial \xi^\beta \partial u^\gamma}(-)\rho^\gamma(\ddot{u}^\beta + \lambda\ddot{\rho}^\beta) - \frac{\partial^3 L}{\partial \xi^\alpha \partial \xi^\beta \partial \xi^\gamma}(-)\dot{\rho}^\gamma(\ddot{u}^\beta + \lambda\ddot{\rho}^\beta) - \frac{\partial^2 L}{\partial \xi^\alpha \partial \xi^\beta}(-)\ddot{\rho}^\beta\Big) dt$$

$\overset{\lambda=0}{\Longrightarrow}$ $$G_{\gamma^*}(\rho,\eta)=\int_c^d \eta^\alpha\Big(\frac{\partial^2 L}{\partial u^\alpha\partial u^\beta}\rho^\beta+\frac{\partial^2 L}{\partial u^\alpha\partial\xi^\beta}\dot\rho^\beta-\frac{\partial^3 L}{\partial\xi^\alpha\partial u^\beta\partial u^\gamma}\rho^\gamma\dot u^\beta-\frac{\partial^3 L}{\partial\xi^\beta\partial u^\beta\partial\xi^\gamma}\rho^\gamma\dot u^\beta$$
$$-\frac{\partial^2 L}{\partial\xi^\alpha\partial u^\beta}\dot\rho^\beta-\frac{\partial^3 L}{\partial\xi^\alpha\partial\xi^\beta\partial u^\gamma}\rho^\gamma\ddot u^\beta-\frac{\partial^3 L}{\partial\xi^\alpha\partial\xi^\beta\partial\xi^\gamma}\dot\rho^\gamma\ddot u^\beta$$
$$-\frac{\partial^2 L}{\partial\xi^\alpha\partial\xi^\beta}\ddot\rho^\beta\Big)dt$$
$$=\int_c^d \eta^\alpha\Big(-\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial\xi^\alpha\partial u^\beta}\Big)\rho^\beta+\frac{\partial^2 L}{\partial u^\alpha\partial u^\beta}\rho^\beta+\frac{\partial^2 L}{\partial u^\alpha\partial\xi^\beta}\dot\rho^\beta$$
$$-\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial\xi^\alpha\partial\xi^\beta}\dot\rho^\beta\Big)\Big)dt$$
$$=-\int_c^d\eta^\alpha\big(J_{\alpha\beta}(\rho^\beta)\big)dt$$
$$J_{\alpha\beta}(\rho^\beta):=\frac{d}{dt}\Big(\frac{\partial^2 L}{\partial\xi^\alpha\partial u^\beta}\rho^\beta+\frac{\partial^2 L}{\partial\xi^\alpha\partial\xi^\beta}\dot\rho^\beta-\frac{\partial^2 L}{\partial u^\alpha\partial u^\beta}\rho^\beta-\frac{\partial^2 L}{\partial u^\alpha\partial\xi^\beta}\dot\rho^\beta\Big)$$

称 $J=(J_{\alpha\beta})$ 为 Jacobi 算子.（作用在沿曲线的向量场上）

（注：上面的记号如 $\frac{\partial^2 L}{\partial\xi^\alpha\partial u^\beta}$ 中，省略了 $\frac{\partial^2 L}{\partial\xi^\alpha\partial u^\beta}(u,\dot u)$ 中的 $(u,\dot u)$，用之前给出的记号应是 $\frac{\partial^2 L}{\partial\dot u^\alpha\partial u^\beta}$ 代表此省略.）

下面我们看一个具体例子.

. Example. $L=\frac{1}{2}g_{\alpha\beta}\xi^\alpha\xi^\beta$. $\quad S[\gamma]=\int_c^d L\,dt$

我们知道驻值曲线是测地线.

设 γ^* 为一测地线. 下计算 $G_{\gamma^*}(\rho,\eta)$. 对任一光滑曲线 γ: $u^\alpha=u^\alpha(t)$.

$$\frac{\partial}{\partial\lambda}S[\gamma+\lambda\rho]\Big|_{\lambda=0}\overset{\text{page 4}}{=\!=}-\int_c^d(\ddot u^\alpha+\Gamma^\alpha_{\gamma\delta}\dot u^\gamma\dot u^\delta)g_{\alpha\beta}\rho^\beta dt$$
$$=-\int_c^d(\ddot u^\alpha+\Gamma^\alpha_{\gamma\delta}\dot u^\gamma\dot u^\delta)\langle\vec r_{u^\alpha},\vec r_{u^\beta}\rangle\rho^\beta dt$$
$$=-\int_c^d\langle\nabla_{\dot{\vec r}}\dot{\vec r},\vec V_0\rangle dt\ ,\quad \vec V_0:=\rho^\beta\vec r_{u^\beta}$$

我们称 $\vec{V}_0 = \rho^\beta(t)\,\vec{r}_{u^\beta}$ 为沿 γ 的一个变分切向量场.

事实上, consider $\vec{r} = \vec{r}(u(t) + \lambda\rho(t))$

$$\vec{r}_t = \vec{r}_{u^\alpha}(\underline{u}(t) + \lambda\underline{\rho}(t)) \cdot (\dot{u}^\alpha(t) + \lambda\dot{\rho}^\alpha(t))$$

$$\vec{r}_\lambda = \vec{r}_{u^\alpha}(\underline{u}(t) + \lambda\underline{\rho}(t)) \cdot \rho^\alpha(t)$$

可见 $\vec{V}_0 = \vec{r}_\lambda \big|_{\lambda=0}$.

· 观察: $\vec{r}_t \times \vec{r}_\lambda \big|_{\lambda=0} = (\dot{u}^1\rho^2 - \dot{u}^2\rho^1)\,\vec{r}_{u^1}\times\vec{r}_{u^2}$

$\dot{\underline{u}} \neq 0 \Rightarrow$ 对 generic 的 $\underline{\rho}$, 以及 $|\lambda|$ small

t, λ 为容许参数.

· $\dfrac{\partial^2}{\partial\lambda\,\partial\mu} S[\gamma^* + \lambda\rho + \mu\eta]\Big|_{\lambda=\mu=0}$

$$= -\frac{\partial}{\partial\lambda}\int_c^d \left\langle \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho}),\ \eta^\beta\vec{r}_{u^\beta}(\underline{u}+\lambda\underline{\rho})\right\rangle dt\Big|_{\lambda=0}$$

$$= -\int_c^d \left\langle \nabla_{\rho^\alpha\vec{r}_{u^\alpha}(\underline{u}+\lambda\underline{\rho})}\left(\nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})\right),\ \eta^\beta\vec{r}_{u^\beta}(\underline{u}+\lambda\underline{\rho})\right\rangle dt\Big|_{\lambda=0}$$

$$-\int_c^d \left\langle \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho}),\ \nabla_{\rho^\alpha\vec{r}_{u^\alpha}(\underline{u}+\lambda\underline{\rho})}\left(\eta^\beta\vec{r}_{u^\beta}(\underline{u}+\lambda\underline{\rho})\right)\right\rangle dt\Big|_{\lambda=0}$$

$\underline{\underline{\text{记}\ \vec{V} = \rho^\beta\vec{r}_{u^\beta}(\underline{u}+\lambda\underline{\rho})}}$

$\vec{W} = \eta^\beta\vec{r}_{u^\beta}(\underline{u}+\lambda\underline{\rho})$

$$-\int_c^d \left\langle \nabla_{\vec{V}}\left(\nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})\right),\ \vec{W}\right\rangle dt\Big|_{\lambda=0}$$

$$-\int_c^d \left\langle \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho}),\ \nabla_{\vec{V}}\vec{W}\right\rangle dt\Big|_{\lambda=0}$$

$$= -\int_c^d \left\langle \nabla_{\vec{V}}\left(\nabla_{\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})}\dot{\vec{r}}(\underline{u}+\lambda\underline{\rho})\right),\ \vec{W}\right\rangle dt\Big|_{\lambda=0}$$

$$-\int_c^d \Big\langle \underbrace{\nabla_{\dot{\vec{r}}(\underline{u})}\dot{\vec{r}}(\underline{u})}_{=\,0},\ \nabla_{\vec{V}}\vec{W}\Big\rangle dt$$

$$\nabla_{\vec{V}} \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\rho)} \dot{\vec{r}}(\underline{u}+\lambda\rho) - \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\rho)} \nabla_{\vec{V}} \dot{\vec{r}}(\underline{u}+\lambda\rho)$$

$$= R_{\beta\gamma\rho}{}^{\sigma} z^\beta z^\gamma V^\rho \vec{r}_{u\sigma} + \nabla_{\underbrace{[\vec{V},\ \dot{\vec{r}}(\underline{u}+\lambda\rho)]}_{=0}} \vec{z}, \quad \vec{z} := \dot{\vec{r}}(\underline{u}+\lambda\rho)$$

$$\Rightarrow G_{\gamma^*}(\rho,\eta) = -\int_c^d \left\langle \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\rho)} \nabla_{\vec{V}} \dot{\vec{r}}(\underline{u}+\lambda\rho) + R_{\beta\gamma\rho}{}^{\sigma} z^\beta z^\gamma V^\rho \vec{r}_{u\sigma}, \vec{W} \right\rangle dt \Big|_{\lambda=0}$$

$\nabla_{\vec{V}} \dot{\vec{r}} - \nabla_{\dot{\vec{r}}} \vec{V} = [\vec{V}, \dot{\vec{r}}] = 0$

$$= -\int_c^d \left\langle \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\rho)} \nabla_{\dot{\vec{r}}(\underline{u}+\lambda\rho)} \vec{V} + R_{\beta\gamma\rho}{}^{\sigma} z^\beta z^\gamma V^\rho \vec{r}_{u\sigma}, \vec{W} \right\rangle dt \Big|_{\lambda=0}$$

$$= -\int_c^d \left\langle \nabla_{\dot{\vec{r}}} \nabla_{\dot{\vec{r}}} \vec{V}_0 + R_{\beta\gamma\rho}{}^{\sigma} \dot{u}^\beta \dot{u}^\gamma \rho^\rho \vec{r}_{u\sigma}, \vec{W}_0 \right\rangle dt$$

$$\vec{W}_0 := \eta^\beta \vec{r}_{u\beta}(\underline{u})$$

$$= -\int_c^d \langle J\vec{V}_0, \vec{W}_0 \rangle dt$$

$$(J\vec{V}_0)^\alpha := \nabla_{\dot{\vec{r}}}^2 \rho^\alpha + R_{\beta\gamma\rho}{}^{\alpha} \dot{u}^\beta \dot{u}^\gamma \rho^\rho .$$

~~Def. 极~~

Thm. 在充分小的测地线长度范围内，所有连接 P, Q 的光滑曲线中 $S[\gamma^*]$ 为极小.

proof. $G_{\gamma^*}(\rho,\rho) = -\int_c^d \langle \nabla_{\dot{\vec{r}}}^2 \vec{V}_0 + \dot{u}^\beta \dot{u}^\gamma \rho^\delta R_{\beta\gamma\delta}{}^{\alpha} \vec{r}_{u\alpha}, \vec{V}_0 \rangle dt$

$$= -\int_c^d \langle \nabla_{\dot{\vec{r}}} \vec{V}_0, \nabla_{\dot{\vec{r}}} \vec{V}_0 \rangle dt - \int_c^d \langle \dot{u}^\beta \dot{u}^\gamma \rho^\delta R_{\beta\gamma\delta}{}^{\alpha} \vec{r}_{u\alpha}, \vec{V}_0 \rangle dt$$

注意到 $\nabla_{\dot{\vec{r}}} \vec{V}_0 = \frac{D\vec{V}_0}{dt}$

$$\langle R_{\beta\gamma\delta}{}^{\alpha} \dot{u}^\beta \dot{u}^\gamma \rho^\delta \vec{r}_{u\alpha}, \vec{V}_0 \rangle = g(t) \langle \vec{V}_0(t), \vec{V}_0(t) \rangle \leq C_1 \langle \vec{V}_0(t), \vec{V}_0(t) \rangle$$

$$\int_c^d \langle R_{\beta\gamma\delta}{}^{\alpha} \dot{u}^\beta \dot{u}^\gamma \rho^\delta \vec{r}_{u\alpha}, \vec{V}_0 \rangle$$

$$\leq C_1 \int_c^d \langle \vec{V}_0(t), \vec{V}_0(t) \rangle dt = C_1 \int_c^d \left(\sqrt{\langle \vec{V}_0(t), \vec{V}_0(t) \rangle} \right)^2 dt$$

$$= C_1 \int_c^d \left(\int_c^t \frac{d}{d\tilde{t}} \sqrt{\langle \vec{V}_0(\tilde{t}), \vec{V}_0(\tilde{t}) \rangle}\, d\tilde{t} \right)^2 dt = C_1 \int_c^d \left(\int_c^t \frac{\langle \frac{D\vec{V}_0}{dt}, \vec{V}_0 \rangle}{\sqrt{\langle \vec{V}_0, \vec{V}_0 \rangle}} d\tilde{t} \right)^2 dt$$

$$\leq C_1 \int_c^d \left(\int_c^t \frac{1 \left|\frac{D\vec{V}_0}{dt}\right| |\vec{V}_0|}{|\vec{V}_0|} d\tilde{t} \right)^2 dt \overset{\text{Hölder}}{\leq} C_1 \int_c^d \left(\left(\int_c^t 1 d\tilde{t}\right)^{\frac{1}{2}} \left(\int_c^t \left|\frac{D\vec{V}}{dt}\right|^2 d\tilde{t} \right)^{\frac{1}{2}} \right)^2 dt$$

$$= C_1 \int_c^d (t-c) \int_c^t \left< \frac{D\vec{V}_0}{d\tilde{t}}, \frac{D\vec{V}_0}{d\tilde{t}} \right> d\tilde{t} \; dt$$

$$\leq C_1 \int_c^d (d-c) \int_c^d \left< \frac{D\vec{V}_0}{d\tilde{t}}, \frac{D\vec{V}_0}{d\tilde{t}} \right> d\tilde{t} \; dt$$

$$= C_1 (d-c)^2 \int_c^d \left< \frac{D\vec{V}_0}{dt}, \frac{D\vec{V}_0}{dt} \right> dt$$

$$\Rightarrow G_{\gamma^*}(\rho, \rho) \geq \int_c^d \left(1 - C_1 (d-c)^2\right) \left< \frac{D\vec{V}_0}{dt}, \frac{D\vec{V}_0}{dt} \right> dt$$

$$> 0 \qquad (d \to c). \quad \#.$$

· Def. 称沿从 P 到 Q 的驻值曲线 γ 的向量场 $\vec{V}$ 是 Jacobi 场

if $\quad J\vec{V} = 0$ 且 $\vec{V}|_P = \vec{V}|_Q = 0$.

对于 $L = \frac{1}{2} g_{\alpha\beta} \xi^\alpha \xi^\beta$ 的情形，即：$\nabla_{\dot{\gamma}}^2 \rho^\alpha + u^\beta u^\gamma \rho^\delta R_{\beta\gamma\delta}{}^\alpha = 0$.

$\rho^\alpha|_P = 0,\ \rho^\alpha|_Q = 0$.

· Def. 称 P, Q 沿从 P 到 Q 的测地线 γ^* 是共轭点，if

存在沿 γ^* 的非零 Jacobi 场.

· Lemma. $G_{\gamma^*}(\xi, \eta)$ 非退化 iff γ^* 的端点 P, Q 不是沿

γ^* 的共轭点.

proof. "$\Rightarrow$" 若不然，是共轭点，$\Rightarrow \exists$ 非零 Jacobi 场 ξ,

s.t. $\quad J\xi = 0 \quad \Rightarrow \quad G(\xi, \eta) \equiv 0,\ \forall \eta$.

"$\Leftarrow$". 若不然，设 $G_{\gamma^*}(\xi, \eta)$ 在 ξ 处退化 $\Rightarrow G(\xi, \eta) \equiv 0 \quad \forall \eta$.

令 $\eta = a(t) J(\xi)$, $\quad a(c) = a(d) = 0. \quad a(t) > 0$

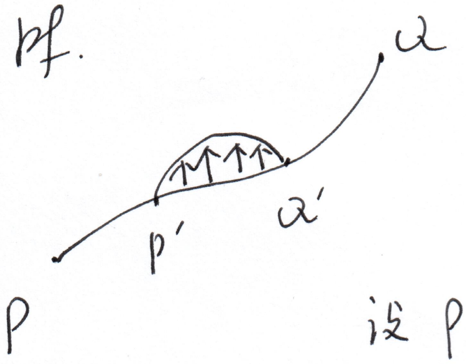
$$G(\xi, \eta) = -\int_c^d a(t) \left< J\xi, J\xi \right> dt \equiv 0 \qquad \forall a.$$

$\Rightarrow J\xi = 0. \qquad \#.$

Thm. 若 P 到 Q 的测地线 γ^* 包含了一对共轭点 P'、Q'，

则 γ 不是最小.

Pf.



若 P、Q 不共轭.

$\Rightarrow G_{\gamma^*}(\rho, \eta)$ 非退化.

若 γ^* 最小 $\Rightarrow G_{\gamma^*}(\rho, \rho)$ 正定.

设 ρ_1 为 P'、Q' 之间的 Jacobi 场

构造 P、Q 之间的间断场 ρ

$$\rho = \begin{cases} \rho_1 & P', Q' \text{之间} \\ 0 & \text{之外} \end{cases}$$

则 $G_{\gamma^*}(\rho, \rho) = 0$. 与 G_{γ^*} 正定矛盾. #