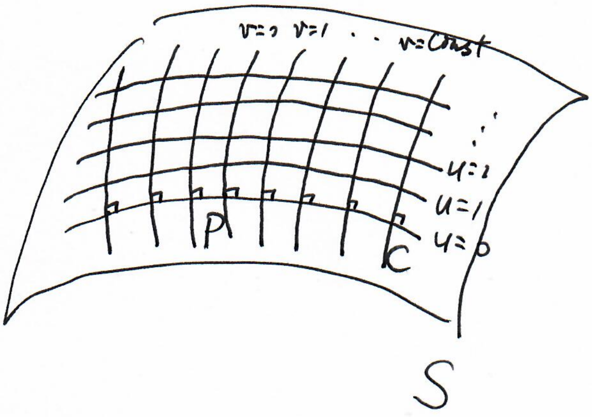
§4.6 测地坐标系

1. 测地平行坐标系

- S：正则曲面．取 $p\in S$．我们将在 p 附近建立一个方便的坐标系，它将类比于平面笛卡儿坐标系．
- 任取一条从 p 出发的、S 上的、以弧长为参数的测地线 C．

  C 的弧长参数记为 v．



- 过 C 上各点作与 C 正交的测地线，以弧长为参数，记为 u．

- (u,v)－曲线网：

  u－曲线：v 为常数的曲线．

  测地线，弧长参数：u

  v－曲线：u 为常数的曲线．

  $u=0$：曲线 C，测地线．

- $(ds)^2 = E(du)^2 + 2F\,du\,dv + G(dv)^2$．

$$E=\langle \vec{r}_u,\vec{r}_u\rangle\equiv 1. \quad \Rightarrow \langle \vec{r}_{uv},\vec{r}_u\rangle\equiv 0$$

$$F=\langle \vec{r}_u,\vec{r}_v\rangle \quad \Rightarrow \quad \frac{\partial F}{\partial u} = \langle \vec{r}_{uu},\vec{r}_v\rangle + \langle \vec{r}_u,\vec{r}_{uv}\rangle$$

u－曲线是测地线 $\Rightarrow \langle \vec{r}_{uu},\vec{r}_v\rangle\equiv 0$

$$\left.\begin{aligned} &\Rightarrow \frac{\partial F}{\partial u}\equiv 0 \\ &\text{而 } F(0,v)\equiv 0 \end{aligned}\right\} \Rightarrow F(u,v)\equiv F(0,v)\equiv 0.$$

$$\Rightarrow ds^2 = du^2 + G(u,v)\,dv^2$$

Q. $G(u,v)$ 满足什么初始条件？（$u=0$ 时）

$u=0$：测地线 C．v 是 C 的弧长参数 $\Rightarrow G(0,v)\equiv 1$．

测地线 C 的方程：$\begin{cases} u=0 \\ v=v. \end{cases}$ （v 是参数）

$$\Rightarrow \underbrace{\frac{d^2u^\alpha}{dv^2}}_{=0} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{dv}\frac{du^\gamma}{dv} = 0 \qquad (u=0\text{ 曲线})$$

$$\Rightarrow \Gamma^\alpha_{22}\cdot\frac{dv}{dv}\frac{dv}{dv} = 0 \qquad (u=0\text{ 曲线})$$

$$\Rightarrow \Gamma^\alpha_{22}\Big|_{u=0} = 0, \quad \alpha=1,2.$$

$$\Gamma^1_{22} = -\frac{1}{2E}\frac{\partial G}{\partial u} \Rightarrow \frac{\partial G}{\partial u}\Big|_{u=0} \equiv 0$$

$$\Gamma^2_{22} = \frac{1}{2}\frac{\partial \log G}{\partial v} \Rightarrow \frac{\partial G}{\partial v}\Big|_{u=0} \equiv 0 \quad ✓$$

Prop. 测地平行坐标系下，第一基本形式具有如下表达式：

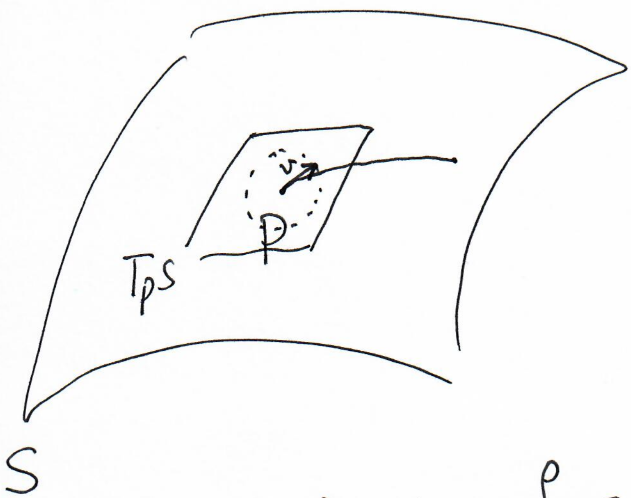
$$ds^2 = du^2 + G(u,v)dv^2,$$

其中，$G(0,v)\equiv 1$,
$G_u(0,v)\equiv 0$.

2. 测地极坐标系

· S：正则曲面. 取 $p\in S$. 我们将在 p 附近建立另外两个方便的坐标系.

· 设 $\vec{v}\in T_pS$ 为 S 在 p 处的一个单位切向量.



记：$\underline{u}(\rho;\vec{v})$ 为从 p 出发、初速度是 $\vec{v}$ 的 S 上的测地线. 即：

$$\begin{cases} \dfrac{D\left(\frac{d\vec{r}}{d\rho}\right)}{d\rho} = 0 \\ \vec{r}\big|_{\rho=0} = p \\ \dfrac{d\vec{r}}{d\rho}\Big|_{\rho=0} = \vec{v} \end{cases}$$

容易看到：ρ 是弧长参数.

由ODE解对初值的连续依赖性及 S^1 的紧性

$\Rightarrow \exists\ \epsilon>0$, s.t. $\forall\ \vec{v}\in T_pS$, $|\vec{v}|=1$, $u(\rho;\vec{v})$ 在 $0\le\rho<\epsilon$ 上有定义.

· Def. (exponential map, 指数映射) $P\in S$. 定义映射:

$$\exp_p : T_pS \to S$$

$$\vec{w}\neq 0 \longmapsto \exp_p(\vec{w}) := \vec{r}\left(\gamma\left(|\vec{w}|;\frac{\vec{w}}{|\vec{w}|}\right)\right),$$

$$\vec{w}=0 \longmapsto P.$$

称为 P 点处的 exponential map. 这里, $|\vec{w}|<\epsilon$.

· T_pS 上的直线 $\rho\vec{v}$, $0\le\rho<\epsilon$, $|\vec{v}|=1$ (固定 $\vec{v}$) 映为测地线 $\exp_p(\rho\vec{v})$.

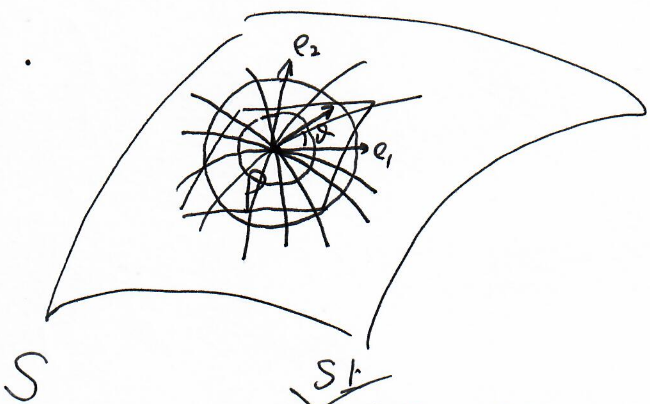
· Def. 取 P 处的, T_pS 的一个正交标架 e_1, e_2

$$\vec{w} = x^1e_1 + x^2e_2 \longmapsto \vec{r}(x^1,x^2) := \exp_p(\vec{w})$$

给出 S 在 p 附近的一个参数表示.

称 (x^1,x^2) 为 P 附近的测地法坐标系.

Consider $\begin{cases} x^1=\rho\cos\theta \\ x^2=\rho\sin\theta \end{cases}$, 称 (ρ,θ) 为 P 附近的测地极坐标系.

· 

$\theta=\theta_0$: ρ-曲线, 为从 P 出发的, 初始速度为 $v_0=\cos\theta_0 e_1+\sin\theta_0 e_2$ 的测地线, 记为 C_{θ_0}. 它以 ρ 为弧长参数.

$\rho=\rho_0$: θ-曲线, 是 T_pS 上以 P 为圆心, ρ_0 为半径的圆在 $\exp_p$ 下的像. 称为以 ρ_0 为半径的测地圆.

· 测地圆是到 P 点的测地距离是常数的点的全体. (注意: 常数要充分小.)

· Prop. (x^1,x^2) 是 S 在 P 附近的容许参数.

proof. 设 (u^1, u^2) 是 S 在 p 附近的容许参数且

$$\vec{r}_{u^1}(p) = e_1, \quad \vec{r}_{u^2}(p) = e_2$$

(其中, (e_1, e_2) 为 T_pS 的一个正交基.) 标准

记 C_θ: $\vec{r} = \vec{r}(u^1(\rho), u^2(\rho))$. 则:

$$\frac{d^2u^\alpha}{d\rho^2} + \Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho} = 0$$

$$\vec{v}(\theta) = \frac{d\vec{r}}{d\rho}\Big|_{\rho=0} = \frac{du^1}{d\rho}\Big|_{\rho=0}\vec{r}_{u^1}\Big|_{\rho=0} + \frac{du^2}{d\rho}\Big|_{\rho=0}\vec{r}_{u^2}\Big|_{\rho=0}$$

$$= \cos\theta\, e_1 + \sin\theta\, e_2$$

$$\Rightarrow \frac{du^1}{d\rho}\Big|_{\rho=0} = \cos\theta, \quad \frac{du^2}{d\rho}\Big|_{\rho=0} = \sin\theta$$

$$u^\alpha(\rho) = u^\alpha(0) + \frac{du^\alpha}{d\rho}\Big|_{\rho=0}\rho + \frac{1}{2}\frac{d^2u^\alpha}{d\rho^2}\Big|_{\rho=0}\rho^2 + \cdots$$

$$= u^\alpha(0) + x^\alpha + \frac{1}{2}\left(-\Gamma^\alpha_{\beta\gamma}\frac{du^\beta}{d\rho}\frac{du^\gamma}{d\rho}\Big|_{\rho=0}\right)\rho^2 + \cdots$$

$$= u^\alpha(0) + x^\alpha - \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\Big|_{\rho=0} x^\beta x^\gamma + \cdots$$

$$\Rightarrow \frac{\partial u^\alpha}{\partial x^\beta}\Big|_p = \delta^\alpha_\beta + o(1) \Rightarrow \det\left(\frac{\partial u^\alpha}{\partial x^\beta}\right) \neq 0 \text{ for}$$

$|x|$ small. #

· 测地极坐标系 (ρ, θ).

ρ-曲线: $\begin{cases} x^1 = \rho\cos\theta_0 \\ x^2 = \rho\sin\theta_0 \end{cases}$ $0<\rho<\epsilon$, 为测地线.

$$\Rightarrow \frac{d^2x^\alpha}{d\rho^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\rho}\frac{dx^\gamma}{d\rho} = 0, \quad \alpha=1,2.$$

测地法坐标系下的 Christoffel 符号. 与之前的 $\Gamma^\alpha_{\beta\gamma}$ 不同!

$\Rightarrow \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\rho}\frac{dx^\gamma}{d\rho} = 0$, $\forall 0<\rho<\epsilon$.

令: $\rho\to 0$, 有: $\lim\limits_{\rho\to 0}\Gamma^\alpha_{\beta\gamma}(\rho\cos\theta_0, \rho\sin\theta_0)\frac{dx^\beta}{d\rho}\Big|_{\rho=0}\frac{dx^\gamma}{d\rho}\Big|_{\rho=0} = 0$

由 θ_0 的任意性 $\Rightarrow \lim\limits_{\rho\to 0}\Gamma^\alpha_{\beta\gamma} = 0$.

Thm. 测地法坐标系下，第一基本形式有如下形式：

$$\mathrm{I} = g_{\alpha\beta}\, dx^\alpha dx^\beta,$$

$$g_{\alpha\beta}\big|_p = \delta_{\alpha\beta}, \quad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\bigg|_p = 0.$$

测地极坐标系下，第一基本形式有如下形式：

$$\mathrm{I} = d\rho^2 + G(\rho,\theta)\, d\theta^2,$$

$$\lim_{\rho\to 0}\sqrt{G} \equiv 0, \quad \lim_{\rho\to 0}\left(\sqrt{G}\right)_\rho \equiv 1.$$

proof. 已经知道 $g_{\alpha\beta}|_p = \delta_{\alpha\beta}$ (see Prop. 中的证明)，

及 $\Gamma^\alpha_{\beta\gamma}\big|_p = 0$ (see 定理上面)。

由 $\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho}\left(\frac{\partial g_{\rho\beta}}{\partial u^\gamma} + \frac{\partial g_{\rho\gamma}}{\partial u^\beta} - \frac{\partial g_{\beta\gamma}}{\partial u^\rho}\right)$ 知：

$$\frac{\partial g_{\rho\beta}}{\partial x^\gamma} + \frac{\partial g_{\rho\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\rho}\bigg|_p = 0$$

$$\Rightarrow \quad \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\bigg|_p = 0.$$

在测地极坐标系下，设

$$\mathrm{I} = E\,d\rho^2 + 2F\,d\rho\, d\theta + G\,d\theta^2.$$

ρ-曲线是测地线，且 ρ 为弧长参数

$$\Rightarrow \begin{cases} \begin{cases}\rho=\rho, & 0<\rho<\epsilon \\ \theta=\theta_0\end{cases} \text{满足 测地线方程 } \dfrac{d^2u^\alpha}{d\rho^2} + \tilde{\Gamma}^\alpha_{\beta\gamma}\dfrac{du^\beta}{d\rho}\dfrac{du^\gamma}{d\rho} = 0 \\ E \equiv 1. \end{cases}$$

$u^1=\rho,\ u^2=\theta$，这里，$\tilde{\Gamma}^\alpha_{\beta\gamma}$ 为 (ρ,θ)-系下的 Christoffel 符号.

$$\Rightarrow \tilde{\Gamma}^1_{11} = 0, \quad \tilde{\Gamma}^2_{11} = 0 \quad \text{沿 } \rho\text{-曲线}.$$

$$\tilde{\Gamma}^1_{11} = \frac{1}{EG-F^2}\left(-F\frac{\partial F}{\partial \rho}\right), \quad \tilde{\Gamma}^2_{11} = \frac{1}{EG-F^2}\left(E\frac{\partial F}{\partial\rho}\right) = 0 \Rightarrow \frac{\partial F}{\partial\rho} = 0$$

$$\vec r_\theta = -\frac{\partial \vec r}{\partial x^1}\rho\sin\theta + \frac{\partial \vec r}{\partial x^2}\rho\cos\theta$$

$$\Rightarrow \lim_{\rho\to 0}\vec r_\theta = 0$$

$$F = \lim_{\rho\to 0} F = \lim_{\rho\to 0}\langle \vec r_\rho, \vec r_\theta\rangle = 0.$$

$$\Rightarrow \sqrt{G} = \sqrt{EG-F^2} = \frac{\partial(x^1,x^2)}{\partial(\rho,\theta)}\sqrt{\det(g_{\alpha\beta})} = \rho\sqrt{\det(g_{\alpha\beta})}$$

这里, $g_{\alpha\beta}$是(x^1,x^2)-系下的 Riemannian metric.

$$\Rightarrow \lim_{\rho\to 0}\sqrt{G} = 0.$$

又 $$(\sqrt{G})_\rho = \sqrt{\det(g_{\alpha\beta})} + \rho\frac{\partial\sqrt{\det(g_{\alpha\beta})}}{\partial\rho}$$

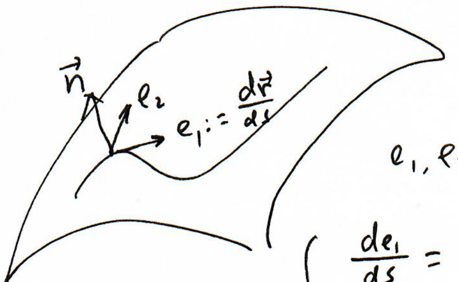
$$= \sqrt{\det(g_{\alpha\beta})} + \rho\,\frac{1}{2}\,\frac{1}{\sqrt{\det(g_{\alpha\beta})}}\underbrace{\left(\cos\theta\frac{\partial\det(g_{\alpha\beta})}{\partial x^1} + \sin\theta\frac{\partial\det(g_{\alpha\beta})}{\partial x^2}\right)}_{\to 0\ (\rho\to 0)}$$

$$\Rightarrow \lim_{\rho\to 0}(\sqrt{G})_\rho = \sqrt{\det(g_{\alpha\beta})}\Big|_{\rho=0} = 1. \quad \#$$

§4.7 测地曲率与 Darboux-Serret-Frenet 公式

S: 正则曲面. C: $u^\alpha = u^\alpha(s)$, s: 弧长参数.

定义: $e_1 = \frac{d\vec r}{ds}$, $\vec n = \frac{\vec r_{u^1}\times\vec r_{u^2}}{|\vec r_{u^1}\times\vec r_{u^2}|}$, $e_2 = \vec n\times\vec e_1$



$e_1, e_2, \vec n$ 构成沿C的一个正交标架场. 有:

$$\begin{cases} \frac{de_1}{ds} = K_g e_2 + K_n\vec n & ①\\ \frac{de_2}{ds} = -K_g e_1 + \tau\vec n & ②\\ \frac{de_3}{ds} = -K_n e_1 - \tau e_2 & ③\end{cases}$$

K_g: 测地曲率

K_n: 法曲率

τ: 测地挠率

① $\Rightarrow \frac{De_1}{ds} = K_g e_2$

$\Downarrow$

$K_g = \langle\frac{De_1}{ds}, e_2\rangle$

$\Downarrow$

对给定的曲线C, K_g只与曲面第I基本形式有关. 内蕴量

. remark. $K = \left|\frac{d^2\vec r}{ds^2}\right| = \left|\frac{de_1}{ds}\right| = \sqrt{K_g^2 + K_n^2}$

. 命题: 对于S上以弧长为参数的光滑曲线 C. 有:

C是测地线 iff 其测地曲率恒为零.

. proof. $|K_g| = \left|\frac{De_1}{ds}\right| \Rightarrow K_g = 0$ iff $\left|\frac{De_1}{ds}\right| \equiv 0$ iff $\frac{De_1}{ds}\equiv 0$. #

· Last times:

· 测地极坐标系:

$$I = (d\rho)^2 + G(\rho, \vartheta)(d\vartheta)^2$$

· $\lim_{\rho \to 0} \sqrt{G} = 0$

· $\lim_{\rho \to 0} (\sqrt{G})_\rho = 1$

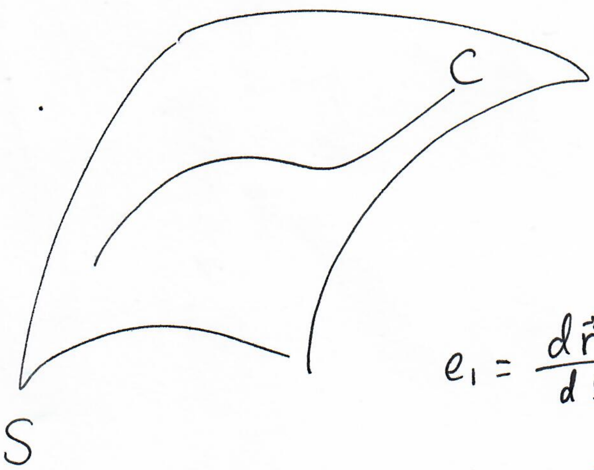
· 测地平行坐标系:

$$I = (du)^2 + G(u, v)(dv)^2$$

· $G(0, v) \equiv 1$,

· $G_u(0, v) \equiv 0$.

· 测地法坐标系: 可推广到高维 Riemannian manifold.

· 

S: 正则曲面. $\vec{r} = \vec{r}(u^1, u^2)$

C: $u^\alpha = u^\alpha(s)$, s: 弧长参数

$$e_1 = \frac{d\vec{r}}{ds}, \quad \vec{n} = \frac{\vec{r}_{u^1} \times \vec{r}_{u^2}}{|\vec{r}_{u^1} \times \vec{r}_{u^2}|}, \quad e_2 = \vec{n} \times e_1$$

$e_1, e_2, \vec{n}$: 沿 C 正交标架场.

$$\text{D-F-S:} \begin{cases} \dfrac{de_1}{ds} = \kappa_g e_2 + \kappa_n \vec{n} \\ \dfrac{de_2}{ds} = -\kappa_g e_1 + \tau \vec{n} \\ \dfrac{d\vec{n}}{ds} = -\kappa_n e_1 - \tau e_2 \end{cases}$$

κ_g: 测地曲率

κ_n: 法曲率

τ: 测地挠率

$$k_g = \left\langle \frac{de_1}{ds}, e_2 \right\rangle = \left\langle \frac{De_1}{ds}, e_2 \right\rangle,$$

or $$\frac{De_1}{ds} = k_g e_2 .$$

· k_g 的计算： $C: u^\alpha = u^\alpha(s)$. s: arc-parameter

$$e_1 = \frac{d\vec{r}}{ds} = \frac{du^\alpha}{ds} \vec{r}_{u^\alpha}$$

$$\frac{De_1}{ds} = \left(\frac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right) \vec{r}_{u^\alpha}$$

$$k_g = \left\langle \frac{De_1}{ds}, e_2 \right\rangle$$

$$= \left\langle \left(\frac{d^2u^\alpha}{ds^2} + \Gamma^\alpha_{\beta\gamma} \frac{du^\beta}{ds} \frac{du^\gamma}{ds} \right) \vec{r}_{u^\alpha}, e_2 \right\rangle$$

代入 $e_2 = \vec{n} \times \dfrac{d\vec{r}}{ds}$

$$= \sqrt{g_{11}g_{22} - g_{12}^2} \begin{vmatrix} u^{1\prime} & u^{1\prime\prime} + \Gamma^1_{\alpha\beta} u^{\alpha\prime} u^{\beta\prime} \\ u^{2\prime} & u^{2\prime\prime} + \Gamma^2_{\alpha\beta} u^{\alpha\prime} u^{\beta\prime} \end{vmatrix}.$$

· Prop. (Liouville). 设 (u,v) 是 S 上的正交参数系，$I = E(du)^2 + G(dv)^2$

s: arc-parameter.

$C: u = u(s), v = v(s)$ 是 S 上的光滑曲线．

设 C 与 u-曲线的夹角是 $\theta(s)$．则 C 的测地曲率 $k_g(s)$ 有公式：

$$k_g(s) = \frac{d\theta}{ds} - \frac{1}{2\sqrt{G}} \frac{\partial \log E}{\partial v} \cos\theta + \frac{1}{2\sqrt{E}} \frac{\partial \log G}{\partial u} \sin\theta.$$

Proof. 记 $\alpha_1 = \dfrac{\vec{r}_u}{\sqrt{E}}$, $\alpha_2 = \dfrac{\vec{r}_v}{\sqrt{G}}$，则：

$$\vec{r}' = \cos\theta\, \alpha_1 + \sin\theta\, \alpha_2 = \frac{\cos\theta}{\sqrt{E}} \vec{r}_u + \frac{\sin\theta}{\sqrt{G}} \vec{r}_v = u'\vec{r}_u + v'\vec{r}_v$$

$$\Rightarrow \cos\theta = \sqrt{E}\, u' , \quad \sin\theta = \sqrt{G}\, v'$$

$$e_1 = \vec{r}' . \quad e_2 = \vec{n}\times e_1 \overset{\alpha_1,\alpha_2,\vec{n}\text{右手}}{\Longrightarrow} e_2 = -\sin\theta\, \alpha_1 + \cos\theta\, \alpha_2$$

$$e_1' = \vec{r}'' = -\sin\theta\, \theta'\, \alpha_1 + \cos\theta\, \alpha_1' + \cos\theta\, \theta'\, \alpha_2 + \sin\theta\, \alpha_2'$$

$$= \underbrace{(-\sin\theta\, \alpha_1 + \cos\theta\, \alpha_2)}_{e_2}\theta' + \cos\theta\, \alpha_1' + \sin\theta\, \alpha_2'$$

$$K_g = \langle \vec{r}'', e_2\rangle = \theta' \langle e_2, e_2\rangle + \cos\theta \langle \alpha_1', e_2\rangle + \sin\theta \langle \alpha_2', e_2\rangle$$

$$= \theta' + \cos\theta \langle \alpha_1', -\sin\theta\, \alpha_1 + \cos\theta\, \alpha_2\rangle + \sin\theta \langle \alpha_2', -\sin\theta\, \alpha_1 + \cos\theta\, \alpha_2\rangle$$

$$= \theta' + \cos^2\theta \langle \alpha_1', \alpha_2\rangle - \sin^2\theta \langle \alpha_2', \alpha_1\rangle$$

$$\langle \alpha_1, \alpha_2\rangle \equiv 0$$

$$= \theta' + \langle \alpha_1', \alpha_2\rangle$$

$$\alpha_1' = \frac{d}{ds}\left(\frac{1}{\sqrt{E}}\right)\vec{r}_u + \frac{1}{\sqrt{E}}\left(\vec{r}_{uu}\, u' + \vec{r}_{uv}\, v'\right)$$

$$\langle \alpha_1', \alpha_2\rangle = \frac{1}{\sqrt{EG}}\left(u' \langle \vec{r}_{uu}, \vec{r}_v\rangle + v' \langle \vec{r}_{uv}, \vec{r}_v\rangle\right)$$

$$\langle \vec{r}_{uu}, \vec{r}_v\rangle = \langle \vec{r}_u, \vec{r}_v\rangle_u - \langle \vec{r}_u, \vec{r}_{uv}\rangle = -\langle \vec{r}_u, \vec{r}_{uv}\rangle$$

$$= -\frac{1}{2}\frac{\partial}{\partial v}\langle \vec{r}_u, \vec{r}_u\rangle = -\frac{1}{2}\frac{\partial E}{\partial v}$$

$$\langle \vec{r}_{uv}, \vec{r}_v\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle \vec{r}_v, \vec{r}_v\rangle = \frac{1}{2}\frac{\partial G}{\partial u}$$

$$\Rightarrow \langle \alpha_1', \alpha_2\rangle = -\frac{1}{\sqrt{EG}}\frac{1}{2}\frac{\cos\theta}{\sqrt{E}}\frac{\partial E}{\partial v} + \frac{1}{\sqrt{EG}}\frac{1}{2}\frac{\sin\theta}{\sqrt{G}}\frac{\partial G}{\partial u}$$

$$= -\frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta . \quad \#.$$

在正交参数系下：

- 例1. u-曲线. $\theta \equiv 0$.

$$K_g(u\text{-曲线}) \overset{(\text{弧长参数}s)}{=} -\frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}$$

v-曲线. $\theta \equiv \pi/2$.

$$K_g(v\text{-曲线}) \overset{(\text{弧长参数}s)}{=} \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}$$

- 正交参数系下，一条 u-曲线 (弧长参数 s) 是测地线 iff $\frac{\partial E}{\partial v} \equiv 0$ (沿 u-曲线).
一条 v-…… (弧长参数 s) …… $\partial G/\partial u \equiv 0$ (.. v-……).

· 例2. $D=\{(u,v)\mid v>0\}$ $\quad u^1=u,\ u^2=v$

$$(ds)^2 = v\left((du)^2+(dv)^2\right)$$

Consider $L=\frac{1}{2}g_{\alpha\beta}\xi^\alpha\xi^\beta \in C^\infty(TD)$, $(g_{\alpha\beta})=\begin{pmatrix} v & 0 \\ 0 & v\end{pmatrix}$

γ: $u=u(t), v=v(t)$ smooth curve

沿γ的动能: $L=\frac{1}{2}v(t)\,\dot{u}(t)^2+\frac{1}{2}v(t)\,\dot{v}(t)^2$

γ是测地线 iff γ满足关于L的 Euler-Lagrange方程.

$$\frac{\partial L}{\partial t}=0 \Rightarrow \frac{1}{2}v(t)\,\dot{u}(t)^2+\frac{1}{2}v(t)\,\dot{v}(t)^2 = \text{Const} =: C_1$$

思考为什么.

(沿测地线)

$$\frac{\partial L}{\partial u}=0 \Rightarrow \frac{\partial L}{\partial \dot{u}} = \text{Const} = C_2, \text{ i.e. } v(t)\,\dot{u}(t)=C_2$$

(沿测地线).

§4.8 Gauss-Bonnet 公式.

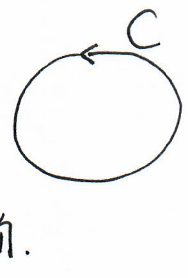
· Thm. 设C是有向正则曲面S上的一条由n段光滑曲线$\widehat{s_0\cdots s_{n-1}s_0}$组成的分段光滑的简单闭曲线. 设其包围的区域D是单连通的. 则:

$$\oint_C k_g\,ds + \int_D K\,d\sigma = 2\pi - \sum_{j=1}^{n}\alpha_j$$

这里, k_g是C的测地曲率, K: Gauss曲率.

α_j为C在$s=s_j$处的外角, $s_n=s_0$, s: 弧长参数.

proof. 先证明C是连续可微情形, 且D落在某个坐标邻域$(D_0;(u,v))$内, (u,v)为正交参数系. 此时,



$$\mathrm{I} = E(du)^2+G(dv)^2.$$

C: $u=u(s)$, $v=v(s)$. $\theta(s)$: C与u-曲线在s处夹角.

Liouville :

$$K_g = \theta' - \frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta$$

$$\oint K_g\, ds = \oint_C \theta'\, ds + \oint_C \left(-\frac{1}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\cos\theta + \frac{1}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\sin\theta\right) ds$$

$$= \oint_C d\theta + \oint_C \left(-\frac{\sqrt{E}}{2\sqrt{G}}\frac{\partial \log E}{\partial v}\, du + \frac{\sqrt{G}}{2\sqrt{E}}\frac{\partial \log G}{\partial u}\, dv\right)$$

$$= \oint_C d\theta + \oint_C \left(-\frac{(\sqrt{E})_v}{\sqrt{G}}\, du + \frac{(\sqrt{G})_u}{\sqrt{E}}\, dv\right)$$

$$\overset{\text{Green 公式}}{=} \oint_C d\theta + \int_D \left[\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right] du \wedge dv$$

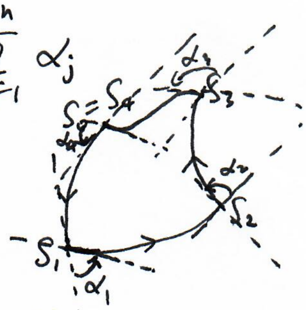
$$= \oint_C d\theta - \int_D K\sqrt{EG}\, du \wedge dv \qquad K = -\frac{1}{\sqrt{EG}}\left(\left(\frac{(\sqrt{E})_v}{\sqrt{G}}\right)_v + \left(\frac{(\sqrt{G})_u}{\sqrt{E}}\right)_u\right).$$

$$\overset{\text{旋转指标定理}}{=} 2\pi - \int_D K\, d\sigma .$$

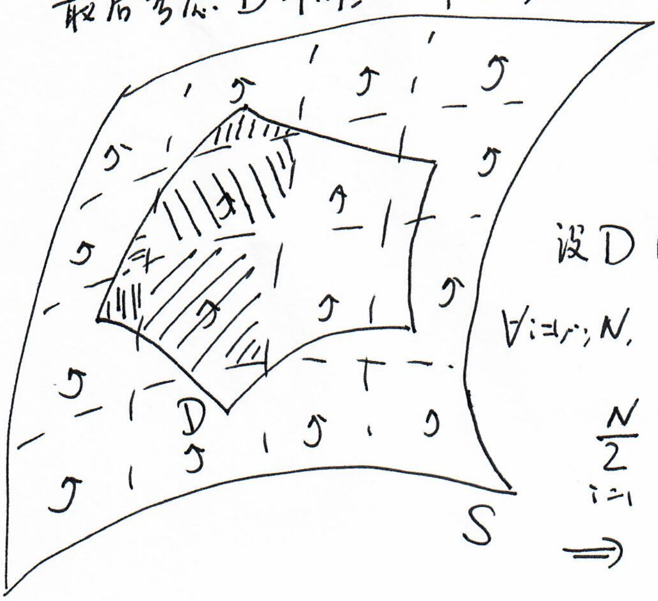
再看C是分段光滑的简单闭曲线情形. 仍假定 D 在一个正交系内.

$$\oint_C d\theta + \sum_{j=1}^{n} \alpha_j = 2\pi \quad (\text{旋转指标定理})$$

$$\Rightarrow \oint_C K_g\, ds + \int_D K\, d\sigma = 2\pi - \sum_{j=1}^{n} \alpha_j$$



最后考虑 D不在一个正交参数系内情形.



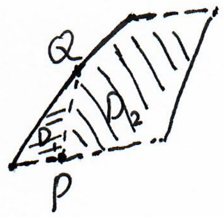
对 S ~~?~~ 进行剖分.

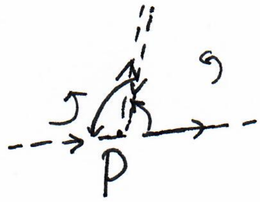
剖分的每一个四边形区域都在某正交系内.

设D由此分为 N 份: $D_1, \cdots, D_N$.

$$\forall i=1,\cdots,N,\ \oint_{C_i} K_g\, ds + \int_{D_i} K\, d\sigma = 2\pi - \sum_{j=1}^{n_i} \alpha_{i,j}$$

$$\overset{\sum_{i=1}^{N}}{\Longrightarrow} \oint_C K_g\, ds + \int_D K\, d\sigma = 2\pi N - \sum_{i=1}^{N}\sum_{j=1}^{n_i} \alpha_{i,j}$$



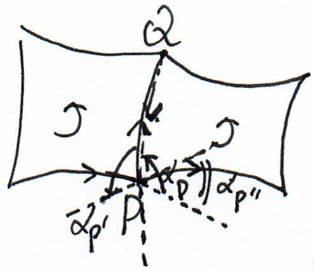


$$\alpha_P + \alpha_{P'} = \pi$$
$$\alpha_Q + \alpha_{Q'} = \pi$$

$$\alpha_P + \alpha_Q + \alpha_{P'} + \alpha_{Q'} = 2\pi \Longrightarrow 2\pi N - \sum \alpha_{ij} = 2\pi - \sum_{j=1}^{n} \alpha_j .$$

如果剖分是由分段光滑曲线完成的，

也有类似的结果.



$$\alpha_P + \alpha_{P'} - \alpha_{P''} = \pi$$
$$\alpha_Q + \alpha_{Q'} - \alpha_{Q''} = \pi$$

$$\alpha_P + \alpha_Q + \alpha_{P'} + \alpha_{Q'} = 2\pi + \alpha_{P''} + \alpha_{Q''}$$

#.

Application I.

· 测地三角形.

C 测地线组成. 三段.

$$\text{Gauss-Bonnet} \Rightarrow \int_D K\,d\sigma = 2\pi - \sum_{j=1}^{3} \alpha_j$$

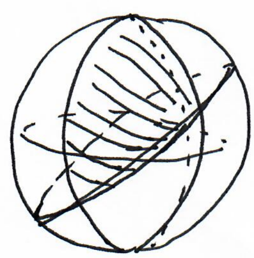
or $\beta_j := \pi - \alpha_j$

$$\sum_{j=1}^{3} \beta_j = \pi + \int_D K\,d\sigma$$

[illegible]

$K \equiv 1$. $$\sum_{j=1}^{3} \beta_j = \pi + \int_D d\sigma = \pi + \text{Area}(D).$$

$K \equiv 0$. $$\sum_{j=1}^{3} \beta_j = \pi$$



$K \equiv -1$ $$\sum_{j=1}^{3} \beta_j = \pi - \text{Area}(D).$$

Application II.

Cor. 设 S 为 $\mathbb{R}^3$ 中的紧致无边的可定向曲面. 则:

$$\int_S K \, d\sigma = 2\pi \chi(S).$$

proof. perform triangulation on S.

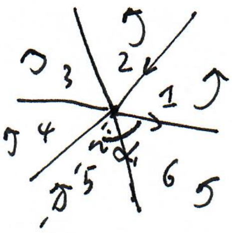
E: 棱数

V: 顶点数, 顶点: $1, \dots, V$. Valencies: $V_1, \dots, V_V$.

F: 面数

$$3F = 2E, \quad \sum_{i=1}^{V} V_i = 2E$$

α_{ij}: 第 i 个顶点的第 j 个三角形的外角



$$\sum_{j=1}^{V_i} \alpha_{ij} = \sum_{j=1}^{V_i} (\pi - \beta_{ij})$$

$$= V_i \pi - 2\pi = (V_i - 2)\pi$$

$$\int_S K \, d\sigma = 2\pi F - \sum_{i=1}^{V} \sum_{j=1}^{V_i} \alpha_{ij}$$

$$= 2\pi F - \sum_{i=1}^{V} (V_i - 2)\pi$$

$$= 2\pi F - \pi \sum_{i=1}^{V} V_i + 2\pi V$$

$$= 2\pi F - 2\pi E + 2\pi V$$

$$= 2\pi (V - E + F) = 2\pi \chi(S). \quad \#$$

Remark. 定理最后的证明也可用此计数方式, $V = E$, $F = 1$, 但要保留 C 上顶点处的外角.