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Several problems in extremal combinatorics and their applications in coding theory

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摘　要

作为组合数学近几十年来发展最为迅猛的分支之一, 极值组合不仅在数学理论上展现出了它蓬勃的生命力, 也在信息科学中展现出了广泛的应用. 在本学位论文中, 我们对几类极值组合学中的新型理论问题展开了探索性研究, 并从组合数学的观点出发, 对几类编码问题做出一定的推进.

在第一章绪论部分, 我们将简要介绍本文所涉及问题的背景并概述本文对这些问题所做的推进.

在第二章中, 针对著名的Erdős-Ko-Rado型定理, 我们提出了一类新型的反问题并对此类反问题做了系统性地思考与研究. 集族$\mathcal{F} \subseteq \binom{[n]}{k}$ 被称为是相交的, 如果它的任意两个成员都包含一个公共元素. 考虑相交族, 一个自然的问题是确定它最大可能的大小, 与之对应的反问题则是刻画它的极值构型及相应结构的稳定性. 通过解决相交集族的这两个问题, Erdős-Ko-Rado定理引领了集族上相交性问题研究. 从定量的观点出发, 我们提出了如下的Erdős-Ko-Rado型定理的反问题: 对于$\mathcal{F} \subseteq \binom{[n]}{k}$, 定义$\mathcal{I}(\mathcal{F}) = \sum_{F_1,F_2\in\mathcal{F}} |F_1 \cap F_2|$ 为$\mathcal{F}$ 的总相交数, 那么当$\mathcal{F}$ 在所有和它大小相同的集族中具有最大总相交数时, 它的结构是什么样的呢? 对于子空间族和置换族, 自然地, 我们可以提出同样的问题. 通过利用组合中的移位技术, 谱方法以及线性规划方法, 对于给定大小的具有极大总相交数的子集族, 子空间族以及置换族, 我们对相应的极值构型分别给出了结构性地刻画, 并且给出了$\mathcal{I}(\mathcal{F})$的上界. 这在很大程度上回答了这一新型的EKR问题的反问题.

在第三章中, 我们考虑了多部交叉相交族大小的上界. 给定正整数p及$n_1, \dots, n_p$, 对$1 \le i \le p$, 记$S_i = \{n_{(i-1)} + 1, \dots, n_i\}$(其中$n_0 = 0$)为集合$\bigsqcup_{i=1}^{p} S_i = [\sum_{i=1}^{p} n_i]$的第$i$部. 对于$1 \le k_i \le n_i$及$A_i \in \binom{S_i}{k_i}$, 称$\bigsqcup_{i\in[p]} A_i$为$\bigsqcup_{i\in[p]} S_i$的一个$p$部$(\sum_{i=1}^{p} k_i)$–子集. 那么, 形如 $\prod_{i\in[p]} \mathcal{F}_i = \{\bigsqcup_{i\in[p]} A_i :\ A_i \in \mathcal{F}_i \subseteq \binom{S_i}{k_i}\}$的集族可以看作是常规单部$k$一致集族的多部推广. 考虑两个$p$部集族$\prod_{i\in[p]} \mathcal{A}_i$和$\prod_{i\in[p]} \mathcal{B}_i$, 若对于任意$\bigsqcup_{i\in[p]} A_i \in \prod_{i\in[p]} \mathcal{A}_i$和$\bigsqcup_{i\in[p]} B_i \in \prod_{i\in[p]} \mathcal{B}_i$均存在至少一个$1 \le i \le p$使得$A_i \cap B_i \neq \emptyset$, 则称这两个$p$部集族为交叉相交的. 基于对顶点可迁图及其直积的独立集的刻画, 我们确定了最大的多部交叉相交族

的大小和相应的结构, 分别推广了Hilton和Frankl-Tohushige在单部情形下的结果.

在第四章中, 我们研究了常重X-码. 作为对集成电路测试响应的结果进行压缩处理的一项重要技术, X-码被应用在具有未知逻辑变量(X)的测试响应的可靠性数据压缩中. 一个参数为(m,n,d,x)的X-码是一个大小为$m \times n$的二元矩阵, 其中参数d和x反映了集成电路中的测试质量. 通过利用极值组合, 概率论, 加法组合以及有限域中的工具, 针对重量为w参数为(m,n,d,x)的X-码, 我们得到了其最大码字个数的上界, 并且给出了具有最大或者近似最大码字个数的常重X-码的构造.

在第五章中, 我们研究了两类分布式存储码: 局部可修复码和极大可修复码. 作为现代分布式存储系统中两类重要的编码方案, 不论从实际应用的角度还是从理论研究的角度, 局部可修复码和极大可修复码都已经引起了众多学者和工程师的关注. 作为局部可修复码和极大可修复码研究中的一个重要主题, 码的各类参数的界及相应的最优构造, 尤其是具有较长码字长度的最优码的构造格外引人关注. 通过从校验矩阵入手, 对于具有全符号(r,δ)-局部性和信息(r,δ)-局部性的最优局部可修复码, 我们分别给出了一类新的构造并且利用稀疏超图的结果, 我们证明了这种构造方式所得到的最优局部可修复码的码长可以达到字母表的超线性阶. 除此之外, 对于实例化网格型拓扑$T_{m\times n}(1,b,0)$的极大可修复码, 我们还给出了几类存在该类型极大可修复码的最小有限域大小的新的上界.

在第六章中对其它成果及部分在研问题做了简要介绍.

关键词: 极值组合学, 总相交数, 交叉相交族, 常重X-码, 分布式存储编码.

Abstract

As the fastest growing branch of combinatorics in recent decades, extremal combinatorics shows not only its vigorous vitality in mathematical theory but also its wide applications in information science. In this dissertation, we investigate several new theoretical problems in extremal combinatorics and study several problems from coding theory via a combinatorial perspective.

In Chapter 1, we will briefly introduce the backgrounds of problems discussed in this dissertation and summarize our main contributions towards these problems.

In Chapter 2, we propose and investigate a new type of inverse problem of the prestigious Erdős-Ko-Rado theorem in extremal set theory. A family of subsets $\mathcal{F} \subseteq \binom{[n]}{k}$ is called intersecting if any two of its members share a common element. Consider an intersecting family, a direct problem is to determine its maximum size and the inverse problem is to characterize its extremal structure and corresponding stability. The Erdős-Ko-Rado theorem answered both direct and inverse problems and led the era of studying intersection problems for finite sets. From the quantitative persepctive, we consider the following inverse problem for Erdős-Ko-Rado type theorems: For $\mathcal{F} \subseteq \binom{[n]}{k}$, define its *total intersection number* as $\mathcal{I}(\mathcal{F}) = \sum_{F_1,F_2\in\mathcal{F}} |F_1 \cap F_2|$, then, what is the structure of $\mathcal{F}$ when it has the maximum total intersection number among all families in $\binom{[n]}{k}$ with the same family size? Similar problems can also be asked about families of subspaces and permutations. Using shifting techniques, spectra methods and linear programming, for families of subsets, subspaces and permutations, we provide structural characterizations of the optimal family of given size with maximum total intersection numbers and upper bounds on $\mathcal{I}(\mathcal{F})$. This answers these new inverse problems to a great extent.

In Chapter 3, we consider multi-part cross-intersecting families. For positive integers p and $n_1, \ldots, n_p$, write $S_i = \{n_{(i-1)} + 1, \ldots, n_i\}$ (set $n_0 = 0$) as the i_{th} part of $\bigsqcup_{i=1}^{p} S_i = [\sum_{i=1}^{p} n_i]$. For $1 \leq k_i \leq n_i$ and $A_i \in \binom{S_i}{k_i}$, let $\bigsqcup_{i\in[p]} A_i$ be a p-partite $(\sum_{i=1}^{p} k_i)$-subset of $\bigsqcup_{i\in[p]} S_i$. Then families of form $\prod_{i\in[p]} \mathcal{F}_i = \{\bigsqcup_{i\in[p]} A_i : A_i \in \mathcal{F}_i \subseteq \binom{S_i}{k_i}\}$ can be viewed as the multi-part generalization of traditional single-

part k-uniform families. Two p-partite families $\prod_{i\in[p]} \mathcal{A}_i$ and $\prod_{i\in[p]} \mathcal{B}_i$ are called cross-intersecting, if for any $\bigsqcup_{i\in[p]} A_i \in \prod_{i\in[p]} \mathcal{A}_i$ and $\bigsqcup_{i\in[p]} B_i \in \prod_{i\in[p]} \mathcal{B}_i$, there exists some $1 \leq i \leq p$ such that $A_i \cap B_i \neq \emptyset$. By characterizing the independent sets of vertex-transitive graphs and their direct products, we determine the sizes and structures of maximum-sized multi-part cross-intersecting families, which generalizes Hilton's and Frankl–Tohushige's results under the single-part setting, respectively.

In Chapter 4, we focus on constant weighted X-codes. As a crucial technique for integrated circuits (IC) test response compaction, X-codes are employed for reliable compressions of test responses in the presence of unknown logic values (Xs). An (m, n, d, x) X-code is an $m \times n$ binary matrix with parameters d, x corresponding to the test quality. Using tools from extremal combinatorics, probability theory, additive combinatorics and finite fields, we obtain several upper bounds on the maximum number of codewords for an (m, n, d, x) X-code of weight w and some new constructions for constant weighted X-codes for some specific ds and xs, which are optimal or nearly optimal with respect to known bounds.

In Chapter 5, we focus on two kinds of codes in distributed storage systems: locally repairable codes and maximally recoverable codes. As important coding schemes in modern distributed storage systems, locally repairable codes (LRCs) and maximally recoverable codes (MRCs) have attracted a lot of attentions from perspectives of both practical applications and theoretical research. As a major topic in the research of LRCs and MRCs, bounds and constructions of the corresponding optimal codes, especially longer codes are of particular concerns. Through parity check matrix approach, we provide constructions of both optimal LRCs with $(r, \delta)_a$-locality and optimal LRCs with $(r, \delta)_i$-locality, and with the help of constructions of large sparse hypergraphs, the length of LRCs constructed can be super-linear in the alphabet size. Besides, we also prove several new upper bounds on the field size required for the existence of MRCs instantiating grid-like topologies $T_{m\times n}(1, b, 0)$.

In Chapter 6, we briefly introduce several other results including topics still under investigation.

Key words: extremal combinatorics, total intersection number, cross-intersecting families, constant weighted X-codes, coding schemes in distributed storage system.

Contents

Chapter 1 Introduction

§ 1.1 New type of inverse problems of the Erdős-Ko-Rado type theorems

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and $\binom{[n]}{k}$ denote the collection of all k-element subsets of $[n]$. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called *intersecting* if any two of its members share at least one common element. The classic Erdős-Ko-Rado theorem [64] states that if $n \geq 2k + 1$, an intersecting family has size at most $\binom{n-1}{k-1}$; if the equality holds, the family must be consisted of all k-subsets of $[n]$ containing a fixed element. As one of the most fundamental results in extremal set theory, this theorem has inspired a great number of extensions and variations. Such as studies of cross-intersecting families (for examples, see [74, 76, 198, 199]), L-intersecting families (for examples, see [72, 75, 150, 180]), intersection problems on families of subspaces and permutations (for examples, see [41, 44, 54, 55, 58, 78]), etc. For readers interested in other extensions, we recommend Frankl and Tokushige's excellent survey [77] and references therein.

Following the path led by Erdős, Ko and Rado, most of these extensions and variations concerned problems of a same type of flavour: Consider a family (or families) of subsets, subspaces, or permutations with a certain kind of intersecting property, how large can this family (or these families) be? Since the intersecting property naturally leads to a clustering structure of the family, therefore, the size of the extremal family can not be very large and these kinds of questions are well asked.

For such problems, once we determine the maximum size of the family with given intersecting property, an immediate inverse problem is to characterize the structure of the extremal family. Starting from this, the stability and supersaturation for extremal families are then well worth studying. In recent years, there have been a lot of works concerning this kind of inverse problems, for examples, see [18, 19, 47, 48, 53, 73, 89, 121, 167].

In this thesis, with the same spirit, we consider a new type of inverse problems for families of subsets, subspaces and permutations from another point of view. Instead of being intersecting, we assume that these families possess a certain property that "maximizes" the intersections quantitatively: Let X be the underlying set with finite members, X can be $\binom{[n]}{k}$, or $\begin{bmatrix} V \\ k \end{bmatrix}$ for an n-dimensional space V over $\mathbb{F}_q$, or S_n. Consider a family $\mathcal{F} \subseteq X$, the *total intersection number* of $\mathcal{F}$ is defined by

$$\mathcal{I}(\mathcal{F}) = \sum_{A\in\mathcal{F}}\sum_{B\in\mathcal{F}} int(A,B), \tag{1.1}$$

where $int(A,B)$ has different meanings for different Xs. When $X = \binom{[n]}{k}$, $int(A,B) = |A\cap B|$; when $X = \begin{bmatrix} V \\ k \end{bmatrix}$, $int(A,B) = \dim(A\cap B)$; when $X = S_n$, $int(A,B) = |\{i \in [n] : A(i) = B(i)\}|$. Moreover, we denote

$$\mathcal{MI}(X,\mathcal{F}) = \max_{\mathcal{G}\subseteq X, |\mathcal{G}|=|\mathcal{F}|} \mathcal{I}(\mathcal{G}) \tag{1.2}$$

as the maximum total intersection number among all families in X with the same size of $\mathcal{F}$ and we denote it as $\mathcal{MI}(\mathcal{F})$ for short if X is clear.

Certainly, the value of $\mathcal{I}(\mathcal{F})$ reveals the level of intersections among the members of $\mathcal{F}$: the larger $\mathcal{I}(\mathcal{F})$ is, the more intersections there will be in $\mathcal{F}$. For an integer $t \geq 1$, note that being t-intersecting also indicates that $\mathcal{F}$ possesses a large amount of intersections, therefore, it is natural to ask the relationship between being t-intersecting and having large $\mathcal{I}(\mathcal{F})$:

Question 1.1.1. *For $t \geq 1$ and n large enough, denote $M(X,t)$ as the maximum size of the t-intersecting family in X. Let $\mathcal{F} \subseteq X$ with size $M(X,t)$, if $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$, is $\mathcal{F}$ a t-intersecting family? Or, if $\mathcal{F}$ is a maximal t-intersecting family in X, do we have $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$?*

In this thesis, by taking $X = \binom{[n]}{k}$, we show that when $|\mathcal{F}| = \binom{n-t}{k-t}$ and $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$, the full t-star (the family consisting of all k-sets in $\binom{[n]}{k}$ containing t fixed elements) is the only structure of $\mathcal{F}$, which answers the Question 1.1.1 for the case $X = \binom{[n]}{k}$. When $X = \begin{bmatrix} V \\ k \end{bmatrix}$ and $\dim(V) = n$ is large enough, we obtain similar results for general $t \geq 1$; when $X = S_n$, we answer the Question 1.1.1 for the case $t = 1$. Noticed that the property of having maximum total intersection number can be considered for families of any size. Actually, we can ask the following more general question:

Question 1.1.2. *For a family $\mathcal{F} \subseteq X$, if $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$, what can we say about its structure?*

Aiming to answer these questions, for $X = \binom{[n]}{k}$, $\begin{bmatrix} \mathbb{F}_q^n \\ k \end{bmatrix}$ and S_n, we provide structural characterizations of the optimal family satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$ for several ranges of size of $\mathcal{F}$. Moreover, we also obtain some upper bounds on $\mathcal{MI}(\mathcal{F})$ for several ranges of $|\mathcal{F}|$ for all three cases. The detailed descriptions of our results will be shown in Chapter 2 .

The first part of this work (the work concerning about families of subsets) has been submitted to the journal *Journal of Combinatorial Theory, Series A*, the second part of this work (the work concerning about families of subspaces and permutations) has been published on the journal *SCIENCE CHINA Mathematics*.

§ 1.2 Multi-part cross-intersecting families

Because of its fundamental status in extremal set theory, the Erdős-Ko-Rado theorem has numerous extensions in different ways. One of the major extensions is to study *cross-t-intersecting* families. Unlike the original theorem, this extension concerns the intersection relationship among a group of different families: Denote $2^{[n]}$ as the *power set* of $[n]$; let $\mathcal{A}_i \subseteq 2^{[n]}$ for each $1 \leq i \leq m$; $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be cross-t-intersecting, if $|A \cap B| \geq t$ for any $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \neq j$. Especially, we say $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are cross-intersecting if $t = 1$.

Over the decades, a lot of works have been done about cross-intersecting families. In 1967, Hilton and Milner [105] first dealt with pair of non-empty cross-intersecting families $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ and obtain an upper bound on $|A| + |B|$. Then in 1977, Hilton [104] investigated general cross-intersecting families in $\binom{[n]}{k}$ and obtained an upper bound on $\sum\limits_{i=1}^{m} |\mathcal{A}_i|$. Since then, there have been many extensions about these two results, for examples, see [74, 76, 199] for extensions of Hilton and Milner's result for pair of non-empty families and see [30–33, 198] for extensions of Hilton's result for general cross-intersecting families.

In this thesis, we extend these two results to the *multi-part* version. The concept of multi-part family was first introduced by Frankl [70] in connection with a similar result of Sali [168]. For positive integers p and $n_1, \ldots, n_p$, take $[\sum_{i\in[p]} n_i]$ as the ground set. Then it can be viewed as the disjoint union of p parts

$$[\sum_{i\in[p]} n_i] = \bigsqcup_{i=1}^{p} S_i = [n_1] \sqcup [n_1 + 1, n_2] \sqcup \cdots \sqcup [n_{p-1} + 1, n_p].$$

For integers $1 \leq k_i \leq n_i$ and $A_i \in \binom{S_i}{k_i}$, let $\bigsqcup_{i\in[p]} A_i$ be a $(\sum_{i=1}^{p} k_i)$-subset of $\bigsqcup_{i\in[p]} S_i$ with A_i in the i-th part and denote $\prod_{i\in[p]} \binom{[n_i]}{k_i}$ as the family of all subsets of $\bigsqcup_{i\in[p]} S_i$ which have exactly k_i elements in the i-th part. Then families of form $\prod_{i\in[p]} \mathcal{F}_i = \{\bigsqcup_{i\in[p]} A_i : \ A_i \in \mathcal{F}_i \subseteq \binom{S_i}{k_i}\}$ can be viewed as the natural generalization of k-uniform families to the multi-part setting. Similarly, a multi-part family is intersecting if any two sets of this family intersect in at least one of the p parts. Let n_i, t_i, s_i be positive integers satisfying $n_i \geq s_i + t_i + 1$, $2 \leq s_i, t_i \leq \frac{n_i}{2}$ for every $i \in [p]$. Our first extension consider cross-intersecting families $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ over $\prod_{i\in[p]} \binom{[n_i]}{s_i}$ with $\mathcal{A}_1 \neq \emptyset$ and prove an upper bound on $\sum\limits_{i=1}^{m} |\mathcal{A}_i|$. Our second extension consider cross-intersecting families $\mathcal{A} \subseteq \prod_{i\in[p]} \binom{[n_i]}{t_i}$ and $\mathcal{B} \subseteq \prod_{i\in[p]} \binom{[n_i]}{s_i}$ and prove an upper bound on $|\mathcal{A}| + |\mathcal{B}|$. The detailed description of these results will be shown in Chapter 3 .

This work has been submitted to the journal *Journal of Algebraic Combinatorics.*

§ 1.3 Constant weighted X-codes

Typical digital circuit testing applies test patterns to the circuit and observes the circuit's responses to the applied patterns. The observed response to a test pattern is compared with the expected response, and a chip in the circuit is determined to be defective if the comparison mismatches. With the development of the large scale integrated circuits (IC), although the comparison for each testing output is simple, the ever increasing amount of testing data costs much more time and space for processing. This leads to the requirement of more advanced test compression techniques [145]. Since then, various related techniques have been studied such as automatic test pattern generation (ATPG) (see [46, 106, 131, 173] and the reference therein) and compression-based approaches (e.g., [148, 157]).

Usually, voltages on signal lines in digital circuit system are interpreted as logic values 0 or 1. However, due to timing constraints, uninitialized memory elements and other uncertainties in practical scenarios, some simulated responses cannot be uniquely determined as states 0 or 1. These unknown states are modeled as "X" states. In the presence of Xs, the technique of X-compact was proposed in [147] as one of the compression-based approaches that have high reliability and error detection ability in actual digital systems.

X-codes are used as linear maps to compress test responses during the processing of X-compact. An (m, n, d, x) X-code is a set of m-dimensional $\{0, 1\}$-vectors of size n. The parameters d, x correspond to the test quality of the code. The value of $\frac{n}{m}$ is called the *compaction ratio* and X-codes with large *compaction ratios* are desirable for actual IC testing. The weight of a codeword corresponds to the required fan-out of the X-compactor. For an X-compactor, smaller fan-out reduces power requirements, area, and delay [147, 203]. From this point of view, codewords in X-codes are expected to have small weights. Therefore, X-codes of constant weight can be a good starting point for the study.

Let $M_w(m, d, x)$ be the maximum number n of codewords for which there exists an (m, n, d, x) X-code of constant weight w. In this thesis, based on results from superimposed codes, we derive general lower and upper bounds on $M_w(m,d,x)$. The lower bounds for $M_w(m,3,2)$ and $M_w(m,7,2)$ are further improved through explicit constructions of corresponding X-codes based on 3-AP-free subsets from additive combinatorics. These constructions provide a nearly optimal lower bound for $M_3(m,3,2)$ and an optimal lower bound for $M_4(m,3,2)$, when m is large enough. And the lower bound for $M_3(m,d,2)$ is further improved through a probabilistic approach. Moreover, we also improve the best known lower bound on the maximum number of codewords for the special class of $(m,n,1,2)$ X-codes of constant weight 3 introduced in [79]. The detailed descriptions of these results will be shown in Chapter 4 .

This work has been published on the journal *IEEE Transactions on Information Theory*.

§ 1.4 Two kinds of codes in distributed storage systems: locally repairable codes and maximally recoverable codes

With rapidly increasing amounts of data created and processed in internet s-cale companies such as Google, Facebook, and Amazon, the efficient storage of such copious amounts of data has thus become a fundamental and acute problem in modern computing. This resulted in distributed storage systems relying on distinct storage nodes. Traditional large scale distributed storage systems entails large s-torage overhead and is nonadaptive for modern systems supporting the "Big Data" environment.

To ensure the reliability with better storage efficiency, erasure coding based schemes are employed to provide efficient repair for failed storage nodes, such as in Windows Azure [107] and in Facebook's Hadoop cluster [185]. Among all these storage codes, maximal distance separable (MDS) codes are favored for their high repair efficiency and reliability. However, due to the large bandwidth and disk

I/O during repair process (see [171]), schemes based on MDS codes can be costly when only a few nodes fail in the system. This greatly affects the practicability of MDS codes in storage systems, especially in large-scale distributed file systems. To address this efficiency problem, a lot of works have emerged in two aspects: local regeneration and local reconstruction.

The concept of local regeneration was introduced by Dimakis et al. [50]. They established a tradeoff between the repair bandwidth and the storage capacity of a node, and introduced a new family of codes, called regenerating codes, which attain this tradeoff. The concept of local reconstruction was introduced by Gopalan et al. [92], and they initiated the study of locally repairable codes (LRCs). A block code is called a locally repairable code with locality r if any failed code symbol can be recovered by accessing at most r survived ones. Recent years, the theory of regenerating codes and LRCs has developed rapidly. There have been a lot of related works focusing on the bounds and the constructions of optimal codes, see [108, 143, 160, 161, 163, 174, 181, 182, 186, 187, 200, 204, 205] and the reference therein.

Over the past few years, the concept of LRCs has been generalized in many different aspects. As one major generalization, the notion of locally repairable codes with (r, δ)-locality ((r, δ)-LRCs) was introduced by Prakash et al. [160], which extends the capability of repairing one erasure within each repair set to $\delta - 1$ erasures. Like original LRCs, a Singleton-type upper bound on the minimum distance of (r, δ)-LRCs was given in [160]. Recently, finding constructions of the optimal LRCs and optimal (r, δ)-LRCs with respect to such bounds has become an interesting and challenging work, which attracted lots of researchers. Usually, longer codes over smaller fields are favored for their efficient transmission performances and fast implementations in practical applications. Therefore, given the size q of the underlying field and other parameters, it is natural to ask how long a code with such parameters can be. For optimal (r, δ)-LRCs, this question was recently asked by Guruswami et al. [98].

In this thesis, through parity-check matrix approach, we provide general constructions for both optimal (r, δ)-LRCs with all symbol locality and optimal (r, δ)-LRCs with information locality and extra global recoverability. We obtain optimal $(r, \delta)_a$-LRCs (codes with all symbol (r, δ)-locality) and optimal $(r, \delta)_i$-LRCs (codes with information (r, δ)-locality) with length super-linear in alphabet size. Compared to the results in [39] and [40], our results provide longer codes for $d \geq 3\delta + 1$. Furthermore, as two applications of our constructions, we construct optimal H-LRCs with super-linear length, which improves the results given by [213]; and we also provide a construction of generalized sector-disk codes with unbounded length.

Along with locality, *maximally recoverable property* was introduced by Chen et al. [43] for multi-protection group codes, and then extended by Gopalan et al. [91] to general settings. In [91], the authors introduced the *topology* of the code and obtained a general upper bound on the minimal size of the field over which maximally recoverable codes (MRCs) exist. Different from the parity check matrix, the topology of the code only specifies the number of redundant symbols and the data symbols on which the redundant ones depend. This makes the topology a crucial characterization of the structures of codes used under distributed storage settings.

With the purpose of deploying longer codes in storage, Gopalan et al. [93] proposed the notion of *grid-like topologies*, which unified a number of topologies considered both in theory and practice. Consider an $m \times n$ matrix, each entry storing a data from a finite field $\mathbb{F}$. Every row satisfies a parity constraints, every column satisfies b parity constraints and in addition, h global parity constraints are involved among all mn entries. This grid-like topology is denoted by $T_{m\times n}(a, b, h)$. In this thesis, we focus on MRCs instantiating topologies of the form $T_{m\times n}(1, b, 0)$ and obtain an upper bound on the size of the field required for the existence of MRCs instantiating topology $T_{m\times n}(1, b, 0)$. For topologies $T_{4\times n}(1, 2, 0)$ and $T_{3\times n}(1, 3, 0)$, this upper bound is further improved. Moreover, we also obtain a polynomial lower bound on the size of the field required for MRCs instantiating $T_{m\times n}(1, 2, 0)$. The detailed description of our results will be shown in Chapter 5 .

The first part of this work (the work concerning about locally repairable codes) has been submitted to the journal *IEEE Transactions on Information Theory*, the second part of this work (the work concerning about maximal recoverable codes) has been published on the journal *Journal of Algebraic Combinatorics*.

Chapter 2 New type of inverse problems of the Erdős-Ko-Rado type theorems

§ 2.1 Introduction

For a positive integer n, let $[n]$ denote the set of the first n positive integers, $[n] = \{1, 2, \ldots, n\}$. Let $2^{[n]}$ and $\binom{[n]}{k}$ denote the power set and the collection of all k-element subsets of $[n]$, respectively. $\mathcal{F} \subseteq 2^{[n]}$ is called a family of subsets, and moreover k-uniform, if $\mathcal{F} \subseteq \binom{[n]}{k}$. A family is called *intersecting* if any two of its members share at least one common element. In 1961, Erdős, Ko and Rado published the following classic result.

Theorem 2.1. *(Erdős-Ko-Rado [64]) Let $n > k > t > 0$ be integers and let $\mathcal{F} \subseteq \binom{[n]}{k}$ satisfy $|F \cap F'| \geq t$ for all $F, F' \in \mathcal{F}$. Then the following holds:*

- *(i) If $t = 1$ and $n \geq 2k$, then*

$$|\mathcal{F}| \leq \binom{n-1}{k-1}. \tag{2.1}$$

- *(ii) If $t \geq 2$ and $n > n_0(k, t)$, then*

$$|\mathcal{F}| \leq \binom{n-t}{k-t}. \tag{2.2}$$

As one of the most fundamental results in extremal set theory, this theorem has inspired a great number of extensions and variations. As two major extensions, intersection problems for families of permutations and families of subspaces over a given finite field have drawn lots of attentions in these years (for examples, see [41, 44, 54, 55, 58, 77, 78] etc).

Let $\mathbb{F}_q$ be the finite field with q elements and V be an n-dimensional vector space over $\mathbb{F}_q$. Denote $\begin{bmatrix} V \\ k \end{bmatrix}$ as the collection of all k-dimensional subspaces of V and for $t \geq 1$, $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ is called *t-intersecting* if $\dim(F \cap F') \geq t$ holds for all $F, F' \in \mathcal{F}$. In 1986, using spectra method, Frankl and Wilson [78] proved the following analogous result of Erdős-Ko-Rado theorem for t-intersecting family of subspaces of V. Since then, many other kinds of intersection problems for families of subspaces have been studied, for examples, see [27, 44, 189].

Theorem 2.2. *([78]) Let $n \geq 2k$ and $k \geq t > 0$ be integers and let $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ be a t-intersecting family, then $|\mathcal{F}| \leq \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$. Moreover, when $n \geq 2k+1$, the equality holds if and only if $\mathcal{F}$ is the family of k-dim subspaces containing a fixed t-dim subspace.*

Let S_n be the symmetric group of all permutations of $[n]$ and for $t \geq 1$, a subset $\mathcal{F} \subseteq S_n$ is called *t-intersecting* if there exist t distinct integers $i_1, i_2, \ldots, i_t \in [n]$ such that $\sigma(i_j) = \tau(i_j)$ for $j = 1, 2, \ldots, t$ and $\sigma, \tau \in \mathcal{F}$. Let $\mathcal{C}_{i_1 \to j_1, \ldots, i_t \to j_t} = \{\sigma \in S_n : \sigma(i_s) = j_s, \text{ for } s = 1, \ldots, t\}$, if $i_1, \ldots, i_t$ are distinct and $j_1, \ldots, j_t$ are distinct, then $\mathcal{C}_{i_1 \to j_1, \ldots, i_t \to j_t}$ is a coset of the stabilizer of t points, which is referred as a *t-coset*. In [71], Deza and Frankl proved the following theorem for 1-intersecting family of permutations.

Theorem 2.3. *([71]) For any positive integer n, if $\mathcal{F} \subseteq S_n$ is 1-intersecting, then $|\mathcal{F}| \leq (n-1)!$.*

Clearly, a 1-coset is a 1-intersecting family of size $(n-1)!$. Deza and Frankl [71] conjectured that the 1-cosets are the only 1-intersecting families of permutations with size $(n-1)!$. This conjecture was first confirmed by Cameron and Ku [41] and independently by Larose and Malvenuto [129]. As for t-intersecting families of permutations when $t \geq 2$, in the same paper, Deza and Frankl also conjectured that the t-cosets are the only largest t-intersecting families in S_n provided n is large enough. Using eigenvalue techniques together with the representation theory of S_n, Ellis, Friedgut and Pilpel [58] proved this conjecture.

Following the path led by Erdős, Ko and Rado, most of these extensions and variations concerned problems of a same type of flavour: Given a family (or families) of subsets, subspaces or permutations with some certain kind of intersecting property, people try to figure out how large this family can be. Once the maximum size of the family with given intersecting property is determined, people then turn to an immediate inverse problem — characterizing the structure of the extremal family. This gives rise to further studies of the stability and supersaturation for extremal families. For a simple example, as is shown in [64], the *full* 1-*star*, defined as the family consisting of all k-sets in $\binom{[n]}{k}$ containing a fixed element, is proved to be the only structure for intersecting families that achieve the equality in (2.1) when $n > 2k$. In recent years, there have been a lot of works concerning this kind of inverse problems, for examples, see [18, 19, 47, 48, 53, 73, 89, 121, 167].

In this chapter, with the same spirit, we consider a new type of inverse problems for intersecting families from another point of view.

To state the problem formally, first, we introduce the notion *total intersection number* of a family. Let X be the underlying set with finite members, X can be $\binom{[n]}{k}$, or $\begin{bmatrix} V \\ k \end{bmatrix}$ for an n-dimensional space V over $\mathbb{F}_q$, or S_n. Consider a family $\mathcal{F} \subseteq X$, the *total intersection number* of $\mathcal{F}$ is defined by

$$\mathcal{I}(\mathcal{F}) = \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} int(A, B), \tag{2.3}$$

where $int(A, B)$ has different meanings for different Xs. When $X = \binom{[n]}{k}$, $int(A, B) = |A \cap B|$; when $X = \begin{bmatrix} V \\ k \end{bmatrix}$, $int(A, B) = \dim(A \cap B)$; when $X = S_n$, $int(A, B) = |\{i \in [n] : A(i) = B(i)\}|$. Moreover, we denote

$$\mathcal{MI}(X, \mathcal{F}) = \max_{\mathcal{G} \subseteq X, |\mathcal{G}| = |\mathcal{F}|} \mathcal{I}(\mathcal{G}) \tag{2.4}$$

as the maximum total intersection number among all families in X with the same size of $\mathcal{F}$ and we denote it as $\mathcal{MI}(\mathcal{F})$ for short if X is clear. Similarly, for two families $\mathcal{F}_1$ and $\mathcal{F}_2$ in X, the total intersection number between $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined as

$$\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) = \sum_{A \in \mathcal{F}_1} \sum_{B \in \mathcal{F}_2} int(A, B). \tag{2.5}$$

Clearly, we have $\mathcal{I}(\mathcal{F},\mathcal{F})=\mathcal{I}(\mathcal{F})$.

Certainly, the value of $\mathcal{I}(\mathcal{F})$ reveals the level of intersections among the members of $\mathcal{F}$: the larger $\mathcal{I}(\mathcal{F})$ is, the more intersections there will be in $\mathcal{F}$. For an integer $t\geq 1$, note that being t-intersecting also indicates that $\mathcal{F}$ possesses a large amount of intersections, therefore, it is natural to ask the relationship between being t-intersecting and having large $\mathcal{I}(\mathcal{F})$:

Question 2.1.1. *For $t\geq 1$ and n large enough, denote $M(X,t)$ as the maximum size of the t-intersecting family in X. Let $\mathcal{F}\subseteq X$ with size $M(X,t)$, if $\mathcal{I}(\mathcal{F})=\mathcal{MI}(\mathcal{F})$, is $\mathcal{F}$ a t-intersecting family? Or, if $\mathcal{F}$ is a maximal t-intersecting family in X, do we have $\mathcal{I}(\mathcal{F})=\mathcal{MI}(\mathcal{F})$?*

Since the maximum total intersection number is an intrinsic parameter for any family, therefore, we can ask the following more general question:

Question 2.1.2. *For a family $\mathcal{F}\subseteq X$, if $\mathcal{I}(\mathcal{F})=\mathcal{MI}(\mathcal{F})$, what can we say about its structure?*

Aiming to solve these questions, we provide structural characterizations of the optimal family satisfying $\mathcal{I}(\mathcal{F})=\mathcal{MI}(\mathcal{F})$ for several ranges of size of $\mathcal{F}$ when $X=\binom{[n]}{k}$, $\left[\begin{smallmatrix}\mathbb{F}_q^n\\ k\end{smallmatrix}\right]$ and S_n. Moreover, we also obtain some upper bounds on $\mathcal{MI}(\mathcal{F})$ for several ranges of $|\mathcal{F}|$ for all three cases.

2.1.1 Structural characterizations

When, $X=\binom{[n]}{k}$, for $F_1,F_2\in\binom{n}{k}$, denote $F_1\Delta F_2=(F_1\setminus F_2)\cup(F_2\setminus F_1)$ as the symmetric difference of F_1 and F_2. We say F_1 succeeds F_2 under the lexicographic ordering if the minimal element of $F_1\Delta F_2$ is in F_1, and we write $F_1\leq_{lex}F_2$. Given a positive integer M, denote $\mathcal{L}_{n,k}(M)$ as the first M k-subsets of $[n]$ under the lexicographic ordering. Particularly, for $t\geq 1$, denote $\mathcal{L}_{n,k,t}^{(r)}$ as the first $\binom{n-t+1}{k-t+1}-\binom{n-(t+r-1)}{k-t+1}$ k-subsets of $[n]$ under the lexicographic ordering. Given $k\geq 2$, $r\geq 0$ and $t\geq 2$, for $1\leq s\leq t$, let $C_s=2^{2^{s-1}-1}\cdot 10^{2^{s+2}-2}\cdot(k^2t^4(r+1)^7)^{2^{s-1}}$ be a constant irrelevant to n, we have the following theorems:

Theorem 2.4. *Let $C_0 \geq 3\times10^3$ be an absolute constant and $k \geq 2$, $r \geq 0$ be two fixed integers. For any $n \geq C_0(r+1)^3(k+r)k^2$ and $\delta \in [\frac{150k^3(r+1)^2}{n}, 1-\frac{150k^3(r+1)^3}{n}] \cup \{1\}$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = \sum_{i=1}^{r}\binom{n-i}{k-1} + \delta\binom{n-(r+1)}{k-1}$ and satisfies $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$, then*

$$\mathcal{L}_{n,k,1}^{(r)} \subseteq \mathcal{F} \subseteq \mathcal{L}_{n,k,1}^{(r+1)},$$

up to isomorphism.

Theorem 2.5. *Let $k \geq 2$, $r \geq 0$ and $t \geq 2$ be three fixed integers. For any $n \geq C_1 \cdot (3tC_t)^{2t}$ and $\delta \in [\frac{60k^2(r+1)^6t^4}{C_1}, 1-\frac{60k^2(r+1)^6t^4}{C_1}] \cup \{1\}$, if $\mathcal{F} \subseteq \binom{[n]}{k}$ with $|\mathcal{F}| = \sum_{i=t}^{t+r-1}\binom{n-i}{k-t} + \delta\binom{n-(t+r)}{k-t}$ and satisfies $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Then*

$$\mathcal{L}_{n,k,t}^{(r)} \subseteq \mathcal{F} \subseteq \mathcal{L}_{n,k,t}^{(r+1)},$$

up to isomorphism.

Denote

$$R_1 = [\frac{150k^3(r+1)^2}{n}, 1-\frac{150k^3(r+1)^3}{n}] \cup \{1\}$$

and

$$R_t = [\frac{60k^2(r+1)^6t^4}{C_1}, 1-\frac{60k^2(r+1)^6t^4}{C_1}] \cup \{1\},$$

for $t \geq 2$. As a direct consequence of the above two theorems, families of proper sizes that maximize total intersection numbers are indeed t-intersecting.

Corollary 2.1.1. *Let k, r, $t \geq 1$ and n be non-negative integers defined in Theorem 2.5. If $|\mathcal{F}| = \delta\binom{n-t}{k-t}$ for some $\delta \in R_t$ satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Then, $\mathcal{F}$ is a t-intersecting family.*

Moreover, we have the following two corollaries of Theorem 2.4 and Theorem 2.5 that determine the unique structure of the optimal family for certain values of $|\mathcal{F}|$, respectively.

Corollary 2.1.2. *Let k, r and n be positive integers defined in Theorem 2.4. If $|\mathcal{F}| = \sum_{i=1}^{r}\binom{n-i}{k-1}$ satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Then, up to isomorphism, we have $\mathcal{F} = \mathcal{L}_{n,k,1}^{(r)}$.*

Corollary 2.1.3. *Let k, r, $t \geq 2$ and n be positive integers defined in Theorem 2.5. If $|\mathcal{F}| = \sum_{i=t}^{t+r-1} \binom{n-i}{k-t}$ satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Then, up to isomorphism, we have $\mathcal{F} = \mathcal{L}_{n,k,t}^{(r)}$.*

When $X = \begin{bmatrix} V \\ k \end{bmatrix}$, through similar combinatorial arguments, we have the following theorem which shows the main structure of the optimal family $\mathcal{F} \subseteq X$ with $|\mathcal{F}|$ not much larger than $\begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$.

Theorem 2.6. *Given positive integers $1 \leq t < k$ and $n \geq (4k+4)^2 \begin{bmatrix} k \\ t \end{bmatrix}_q^2$, let $\mathcal{F}$ be a family of k-dim subspaces of V with size $|\mathcal{F}| = \delta \begin{bmatrix} n-t \\ k-t \end{bmatrix}_q$ for some $\delta \in [\frac{(4k+4)^2 n}{q^{n-k}}, 1 + \frac{1}{96t \ln q(k+1)}]$ satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Then, when $\delta \leq 1$, $\mathcal{F}$ is contained in a full t-star and when $\delta > 1$, $\mathcal{F}$ contains a full t-star.*

When $X = S_n$, for an integer $s > \frac{1}{2}(n-1)!$, consider the subfamilies of X consisting of $\lfloor \frac{s}{(n-1)!} \rfloor$ pairwise disjoint 1-cosets and $s - \lfloor \frac{s}{(n-1)!} \rfloor (n-1)!$ permutations from another 1-coset disjoint with all the former 1-cosets. We denote $\mathcal{T}(n, s)$ as the family of this form with size s with maximum total intersection number. Using eigenvalue techniques together with the representation theory of S_n, we prove that families of permutations of size $\Theta((n-1)!)$ having large total intersection numbers are close to the union of 1-cosets.

Theorem 2.7. *For a sufficiently large integer n, there exist positive constants C_0 and c such that the following holds. For integer $0 \leq k \leq \frac{n-1}{2}$, let $\varepsilon \in (-\frac{1}{2}, \frac{1}{2}]$ and $\delta \geq 0$ such that $\max\{|\varepsilon|, \delta\} \leq ck$. If $\mathcal{F}$ is a subfamily of S_n with size $(k+\varepsilon)(n-1)!$ and $\mathcal{I}(\mathcal{F}) \geq \mathcal{I}(\mathcal{T}(n, |\mathcal{F}|)) - \delta((n-1)!)^2$, then there exists some $\mathcal{G} \subseteq S_n$ consisting of k 1-cosets such that*

$$|\mathcal{F} \Delta \mathcal{G}| \leq C_0 \left(\sqrt{2k(|\varepsilon| + \delta)} + \frac{k}{n} \right) |\mathcal{F}|.$$

Moreover, when $\varepsilon = \delta = 0$, $\mathcal{F} = \mathcal{G}_0$ for some $\mathcal{G}_0 \subseteq S_n$ consisting of k pairwise disjoint 1-cosets.

2.1.2 Upper bounds on $\mathcal{MI}(\mathcal{F})$

When $X = \binom{[n]}{k}$, as corollaries of Theorem 2.4 and Theorem 2.5, we have the following general upper bounds on $\mathcal{I}(\mathcal{F})$ for certain values of $|\mathcal{F}|$.

Corollary 2.1.4. *Let k, r, n and δ be the same as those defined in Theorem 2.4. For any family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size $\sum_{i=1}^{r} \binom{n-i}{k-1} + \delta\binom{n-(r+1)}{k-1}$, we have $\mathcal{I}(\mathcal{F}) \leq (r + \delta^2)\binom{n-1}{k-1}^2 + (n - r - \lfloor\delta\rfloor)(\sum_{i=2}^{r+1} \binom{n-i}{k-2})^2$.*

Corollary 2.1.5. *Let k, r, $t \geq 2$, n and δ be the same as those defined in Theorem 2.5. For any family $\mathcal{F} \subseteq \binom{[n]}{k}$ of size $\sum_{i=t}^{t+r-1} \binom{n-i}{k-t} + \delta\binom{n-(t+r)}{k-t}$, we have $\mathcal{I}(\mathcal{F}) \leq (t-1)|\mathcal{F}|^2 + (r + \delta^2)\binom{n-t}{k-t}^2 + (n - (t + r + \lfloor\delta\rfloor - 1))(\sum_{i=t+1}^{t+r} \binom{n-i}{k-(t+1)})^2$.*

Using linear programming method over association schemes, when $X = \begin{bmatrix} V \\ k \end{bmatrix}$ and S_n, we have the following upper bounds on $\mathcal{MI}(\mathcal{F})$.

Theorem 2.8. *Given positive integers n, k, M with $k \leq n$ and $M \leq \begin{bmatrix} n \\ k \end{bmatrix}_q$, for $\mathcal{F} \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ with $|\mathcal{F}| = M$, we have*

$$\mathcal{MI}(\mathcal{F}) \leq \left(\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{M} - \begin{bmatrix} n \\ 1 \end{bmatrix}_q \right) \frac{qM^2 \begin{bmatrix} k \\ 1 \end{bmatrix}_q \begin{bmatrix} n-k \\ 1 \end{bmatrix}_q}{\begin{bmatrix} n \\ 1 \end{bmatrix}_q \left(\begin{bmatrix} n \\ 1 \end{bmatrix}_q - 1 \right)} + kM^2, \tag{2.6}$$

especially, when $n \geq 2k$ and $M \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$, we have

$$\mathcal{MI}(\mathcal{F}) \leq \left[\frac{\begin{bmatrix} n \\ k \end{bmatrix}_q}{M} - \frac{(q^n - 1)(q^{n-1} - 1)}{(q-1)(q^k - 1)} \right] \frac{M^2(q^k - 1)(q^{k-1} - 1)(q^{n-k} - 1)}{(q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1)} + kM^2. \tag{2.7}$$

Theorem 2.9. *Given positive integers n and $M \leq n!$, for $\mathcal{F} \subseteq S_n$ with $|\mathcal{F}| = M$, we have*

$$\mathcal{MI}(\mathcal{F}) \leq \frac{M^2}{n-1} \left(\frac{n!}{M} + n - 2 \right).$$

2.1.3 Notations and outline

We use the following standard mathematical notations throughout this chapter.

- Denote $\mathbb{N}$ as the set of all non-negative integers. For any $n \in \mathbb{N} \setminus \{0\}$, let $[n] = \{1, 2, \cdots, n\}$. For any $a, b \in \mathbb{N}$ such that $a \leq b$, let $[a, b] = \{a, a+1, \cdots, b\}$.
- For given finite set $S \subseteq \mathbb{N}$ and any positive integer k, denote $\binom{S}{k}$ as the family of all k-subsets of S.

- For a given family $\mathcal{F}$ in $\binom{[n]}{k}$ and a t-subset $A \subseteq [n]$, we denote $\mathcal{F}(A) = \{F \in \mathcal{F} : A \in F\}$ as the subfamily of $\mathcal{F}$ containing A and call $\deg_{\mathcal{F}}(A) = |\mathcal{F}(A)|$ the degree of A in $\mathcal{F}$. Moreover, when $t = 1$ and $A = \{x\}$, we denote $\mathcal{F}(x) = \mathcal{F}(\{x\})$ for short.

- For a given family $\mathcal{F} \subseteq \binom{[n]}{k}$ and an integer $s > 0$, a subset $U \subseteq [n]$ is called an *s-cover* of $\mathcal{F}$ with size $|U|$, if for every $F \in \mathcal{F}$, $|F \cap U| \geq s$.

- For a given family $\mathcal{F}$ in $\binom{[n]}{k}$ and $A \in \mathcal{F}$, the *shifting operator* $\mathcal{S}_{i,j}$ is defined as follows:

$$\mathcal{S}_{i,j}(A) = \begin{cases} A \setminus \{i\} \cup \{j\}, & \text{if } i \in A, j \notin A \text{ and } A \setminus \{i\} \cup \{j\} \notin \mathcal{F}; \\ A, & \text{otherwise.} \end{cases} \tag{2.8}$$

  And we define $\mathcal{S}_{i,j}(\mathcal{F}) = \{\mathcal{S}_{i,j}(A) : A \in \mathcal{F}\}$.

- For a given prime power q and a positive integer n, we denote $\mathbb{F}_q$ as a finite field with q elements and $\mathbb{F}_q^n$ as the n-dimensional vector space over $\mathbb{F}_q$. Moreover, for a vector $\mathbf{x}$ with length n, we denote $\mathbf{x}_i$ as the i_{th} position of $\mathbf{x}$ for $1 \leq i \leq n$.

- For two subspaces $V_1, V_2 \subseteq \mathbb{F}_q^n$, we denote $V_1 + V_2$ as the sum of these two subspaces and V_1/V_2 as the quotient subspace of V_1 by V_2. If $V_1 \cap V_2 = \{\mathbf{0}\}$, we denote $V_1 \oplus V_2$ as the direct sum of V_1, V_2. Moreover, we have $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2)$ and $\dim(V_1/V_2) = \dim(V_1) - \dim(V_1 \cap V_2)$.

- For a given prime power q and positive integers n, k with $k \leq n$, the *Gaussian binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=0}^{k-1} \frac{q^{n-i} - 1}{q^{k-i} - 1}.$$

  Usually, the q is omitted when it is clear.

- For a given family $\mathcal{F}$ in $\begin{bmatrix} V \\ k \end{bmatrix}$ and a t-dim subspace $U \subseteq V$, we denote $\mathcal{F}(U) = \{F \in \mathcal{F} : U \subseteq F\}$ as the subfamily in $\mathcal{F}$ containing U and $\deg_{\mathcal{F}}(U) = |\mathcal{F}(U)|$ is called the degree of U in $\mathcal{F}$.

The remainder of the chapter is organized as follows. In Section § 2.2, we will introduce some basic notions and known results on general association schemes, representation theory of S_n and spectra of Cayley graphs on S_n. Moreover, we also include some preliminary lemmas for the proof of our main results. In Section § 2.3, we consider families of subsets and prove Theorem 2.4 and Theorem 2.5. In Section § 2.4, we consider families of vector spaces and prove Theorem 2.6. In Section § 2.5, we consider families of permutations and prove Theorem 2.7. And we prove Theorem 2.8 and Theorem 2.9 in Section § 2.6. Finally, we conclude this chapter and discuss some remaining problems in Section § 2.7.

§ 2.2 Preliminaries

In this section, we will introduce some necessary notions and related results to support proofs of our theorems. First, we will introduce some notions about general association schemes, which are crucial for the proof of the upper bounds on $\mathcal{MI}(X, \mathcal{F})$ for $X = \begin{bmatrix} V \\ k \end{bmatrix}$ and $X = S_n$. Then, we shall give a brief introduction on the representation theory of S_n. Finally, we will review some known results about spectra of Cayley graphs on S_n. Readers familiar with these parts are invited to skip corresponding subsections. Based on these results, we will provide some new estimations about eigenvalues of Cayley graphs on S_n for the proof of Theorem 2.7.

2.2.1 Association schemes

Association scheme is one of the most important topics in algebraic combinatorics, coding theory, etc. Many questions concerning distance-regular graphs are best solved in this framework, see [23],[24]. In 1973, by performing linear programming methods on specific association schemes, Delsarte [49] proved many of the sharpest bounds on the size of a code, which demonstrated the power of association schemes in coding theory. Since then, association schemes have been widely studied and related notions have also been extended to other objects, such as equiangular lines and special codes, etc. In this subsection, we only include some basic notions about association schemes. For more details about association schemes, we recommend [49] and [90] as standard references.

Let X be a finite set with v ($v \geq 2$) elements, and for any integer $s \geq 1$, let $\mathcal{R} = \{R_0, R_1, \ldots, R_s\}$ be a family of $s+1$ relations on X. The pair $(X, \mathcal{R})$ is called an association scheme with s classes if the following three conditions are satisfied:

1. The set $\mathcal{R}$ is a partition of $X \times X$ and R_0 is the diagonal relation, i.e., $R_0 = \{(x, x) |\ x \in X\}$.

2. For $i = 0, 1, \ldots, s$, the inverse $R_i^{-1} = \{(y, x) |\ (x, y) \in R_i\}$ of the relation R_i also belongs to $\mathcal{R}$.

3. For any triple of integers $i, j, k \in \{0, 1, \ldots, s\}$, there exists a number $p_{i,j}^{(k)} = p_{j,i}^{(k)}$ such that, for all $(x, y) \in R_k$:

$$|\{z \in X |\ (x, z) \in R_i,\ (z, y) \in R_j\}| = p_{i,j}^{(k)}.$$

   And $p_{i,j}^{(k)}$s are called the *intersection numbers* of $(X, \mathcal{R})$.

For relation $R_i \in \mathcal{R}$, the adjacency matrix of R_i is defined as follows:

$$A_i(x, y) = \begin{cases} 1, & (x, y) \in R_i, \\ 0, & (x, y) \notin R_i. \end{cases}$$

The space consisting of all complex linear combinations of the matrices $\{A_0, \ldots, A_s\}$ in an association scheme $(X, \mathcal{R})$ is called a *Bose-Mesner algebra*. Moreover, denote J as the $v \times v$ matrix with all entries 1, there is a set of pairwise orthogonal idempotent matrices $\{B_0 = \frac{J}{v}, \ldots, B_s\}$, which forms another basis of this Bose-Mesner algebra. The relations between $\{A_r\}_{r=0}^{s}$ and $\{B_r\}_{r=0}^{s}$ are shown as follows:

$$A_i = \sum_{j=0}^{s} P_i(j) B_j,\ i = 0, \ldots, s; \qquad B_j = \frac{1}{v} \sum_{i=0}^{s} Q_j(i) A_i,\ j = 0, \ldots, s, \tag{2.9}$$

where $P_i(0), \ldots, P_i(s)$ are the eigenvalues of A_i, which are called the *eigenvalues* of the association scheme; and $Q_j(i)$ are known as *dual eigenvalues* of the association scheme. Usually, $v_i := P_i(0)$ denotes the number of 1's in each row of A_i and

$u_j := Q_j(0) = tr(B_j)$. According to [90], for $1 \leq i, j \leq s$, $P_i(j)$s and $Q_j(i)$s have the following relation:

$$\frac{\overline{P_i(j)}}{v_i} = \frac{Q_j(i)}{u_j}. \tag{2.10}$$

Let $\mathcal{R} = \{R_0, R_1, \ldots, R_s\}$ be a set of $s+1$ relations on X of an association scheme. Given a subset $Y \subseteq X$ with $|Y| = M$, the *inner distribution* of Y with respect to $\mathcal{R}$ is an $(s+1)$-tuple $\mathbf{a} = (a_0, \ldots, a_s)$ of nonnegative rational numbers a_i $(0 \leq i \leq s)$ given by

$$a_i = \frac{|R_i \cap (Y \times Y)|}{M}. \tag{2.11}$$

Clearly, we have $a_0 = 1$ and $\sum_{i=0}^{s} a_i = |Y|$.

Moveover, let $\mathbf{u}$ be the indicator vector of Y with respect to X, i.e., $\mathbf{u}_x = 1$, if $x \in Y$ and $\mathbf{u}_x = 0$, if $x \notin Y$. Then, (2.11) can be rewritten as

$$a_i = \frac{1}{M}\mathbf{u}A_i\mathbf{u}^T. \tag{2.12}$$

Besides, for $0 \leq j \leq s$, define

$$b_j = \frac{v}{M^2}\mathbf{u}B_j\mathbf{u}^T, \tag{2.13}$$

and $\mathbf{b} = (b_0, \ldots, b_s)$ as the *dual distribution* of Y. By combining (2.9) and (2.13) together, we have the following lemma which provides a linear relationship between a_is and b_js.

Lemma 2.1. *Given an association scheme $(X, \mathcal{R})$ with s classes and $|X| = v$. Let $Y \subseteq X$ with size M, then for $\{a_0, \ldots, a_s\}$ and $\{b_0, \ldots, b_s\}$ defined in* (2.11) *and* (2.13) *respectively, we have*

$$a_i = \frac{M}{v}\sum_{j=0}^{s} b_j P_i(j), \quad i = 0, 1, \ldots, s.$$

As a consequence of Lemma 2.1, we have the following properties of $\{b_j : 0 \leq j \leq s\}$.

Lemma 2.2. *([144], Theorem 12 in Section 6, Chapter 21) Given an association scheme* $(X, \mathcal{R})$ *with* s *classes and* $|X| = v$*. Let* $Y \subseteq X$ *with size* M *and* $\{b_0, \ldots, b_s\}$ *be defined as* (2.13)*, then* $b_j \geq 0$ *for all* $0 \leq j \leq s$*.*

Lemma 2.3. *With the same conditions as those in Lemma 2.2, for* $\{b_0, \ldots, b_s\}$*, we have*

$$b_0 = 1 \text{ and } \sum_{j=0}^{s} b_j = \frac{v}{M}. \tag{2.14}$$

Proof of Lemma 2.3. Since $B_0 = J/v$, by the definition of b_j in (2.13), we can obtain

$$b_0 = \frac{1}{M^2}\mathbf{u}J\mathbf{u}^T = 1.$$

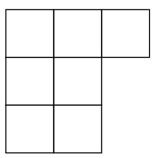
Note that $a_0 = 1$ and $P_0(j) = 1$ for $0 \leq j \leq s$, by taking $i = 0$ in Lemma 2.1, we can obtain

$$\sum_{j=0}^{s} b_j = \frac{v}{M}.$$

□

2.2.2 Background on the representation theory of S_n

A *partition* of n is a nonincreasing sequence of positive integers summing to n, i.e., a sequence $\lambda = (\lambda_1, \ldots, \lambda_l)$ with $\lambda_1 \geq \cdots \geq \lambda_l$ and $\sum_{i=1}^{l} \lambda_i = n$, and we write $\lambda \vdash n$. The *Young diagram* of λ is an array of n *cells*, having l left-justified rows, where row i contains λ_i cells. For example, the Young diagram of the partition $(3, 2^2)$ is:



If the array contains the numbers $\{1, \ldots, n\}$ in some order in place of dots, we call it λ-*tableau*, for example,

5	1	3
2	4	
6	7	

is a $(3, 2^2)$-tableau. Two λ-tableaux are said to be *row-equivalent* if they have the same numbers in each row; if a λ-tableau s has rows $R_1, \ldots, R_{l_1} \subseteq [n]$ and columns $C_1, \ldots, C_{l_2} \subseteq [n]$, then we let $R_s = S_{R_1} \times \cdots \times S_{R_{l_1}}$ be the row-stabilizer of s and $C_s = S_{C_1} \times \cdots \times S_{C_{l_2}}$ be the column-stabilizer of s.

A λ-*tabloid* is a λ-tableau with unordered row entries. Given a tableau s, denote $[s]$ as the tabloid it produces. For example, the $(3, 2^2)$-tableau above produces the following $(3, 2^2)$-tabloid:

$$\begin{matrix} \{5 & 1 & 3\} \\ \{2 & 4\} & \\ \{6 & 7\} & \end{matrix}$$

For given group G and set S, denote e as the identity in G. The left action of G on S is a function $G \times S \to S$ (denoted by $(g, x) \mapsto gx$) such that for all $x \in S$ and $g_1, g_2 \in G$:

$$ex = x \text{ and } (g_1 g_2)x = g_1(g_2 x).$$

Now, consider the left action of S_n on X^λ, the set of all λ-tabloids; let $M^\lambda = \mathbb{C}[X^\lambda]$ be the corresponding permutation module, i.e., the complex vector space with basis X^λ and the action of S_n on $\mathbb{C}[X^\lambda]$ linearly extended from the action of S_n on X^λ. Given a λ-tableau s, the corresponding λ-*polytabloid* is defined as

$$e_s := \sum_{\pi \in C_s} sgn(\pi)\pi[s].$$

We define the *Specht module* S^λ to be the submodule of M^λ spanned by the λ-polytabloids:

$$S^\lambda = \text{Span}\{e_s : s \text{ is a } \lambda\text{-tableau}\}.$$

As shown in [58], any irreducible representation ρ of S_n is isomorphic to some S^λ. This leads to a one to one correspondence between irreducible representations and partitions of n. In the following of this chapter, for convenience, we shall write $[\lambda]$ for the equivalence class of the irreducible representation S^λ, χ_λ for the character

χ_{S^λ} (The formal definition of the character of a representation will be presented in Section 2.3.1).

Let $\lambda = (\lambda_1, \ldots, \lambda_{l_1})$ be a partition of n. If its Young diagram has columns of lengths $\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{l_2} \geq 1$, then the partition $\lambda^T = (\lambda'_1, \ldots, \lambda'_{l_2})$ is called the *transpose* (or *conjugate*) of λ. Consider each cell (i, j) in the Young diagram of λ, the *hook* of (i, j) is $H_{i,j} = \{(i, j') : j' \geq j\} \cup \{(i', j) : i' \geq i\}$. The *hook length* of (i, j) is $h_{i,j} = |H_{i,j}|$. As an important parameter, the dimension $\dim[\lambda]$ of the Specht module S^λ is given by the following theorem:

Theorem 2.10. *([68]) If λ is a partition of n with hook lengths $(h_{i,j})$, then*

$$\dim[\lambda] = \frac{n!}{\prod_{i,j} h_{i,j}}. \tag{2.15}$$

As an immediate consequence of Theorem 2.10, we have $\dim[\lambda] = \dim[\lambda^T]$.

2.2.3 Spectra of Cayley graphs on S_n

2.2.3.1 Basics and known results

Given a group G and an inverse-closed subset $X \subseteq G$, the *Cayley graph* on G generated by X, denoted by $Cay(G, X)$, is the graph with vertex-set G and edge-set $\{\{u, v\} \in \binom{G}{2} : uv^{-1} \in X\}$. Cayley graphs have been studied for many years and are a class of the most important structures in algebraic graph theory. Here, we only consider a very special kind of Cayley graphs where $G = S_n$ and X is a union of conjugacy classes.

For fixed $k \geq 1$, consider the Cayley graph Γ_k on S_n with generating set

$$FPF_k = \{\sigma \in S_n : \sigma \text{ has less than } k \text{ fixed points}\}.$$

When $k = 1$, the corresponding Cayley graph Γ_1 is also called the *derangement graph* on S_n.

For $i, j \in [n]$, denote $\mathcal{C}_{i \to j}$ as the coset consisting of permutations $\sigma \in S_n$ with $\sigma(i) = j$. In [58], by taking FPF_k as a union of conjugacy classes, the authors used the representation theory of S_n and obtained the following results about the eigenvalues of Γ_k:

$$\lambda_\rho^{(k)} = \frac{1}{\dim[\rho]} \sum_{\sigma \in FPF_k} \chi_\rho(\sigma) \quad (\rho \vdash n), \tag{2.16}$$

where the character χ_ρ of ρ is the map defined by

$$\chi_\rho : S_n \to \mathbb{C},$$
$$\chi_\rho(\sigma) = Tr(\rho(\sigma)),$$

and $Tr(\rho(\sigma))$ denotes the trace of the linear map of $\rho(\sigma)$. If there is no confusion, for a partition ρ of n, we also use the notation ρ to denote the corresponding *irreducible representation* of S_n (For this correspondence, see Theorem 14 in [58].).

Let $d_n = |FPF_1(n)|$ be the number of derangements in S_n, using the inclusion-exclusion formula, we have

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!} = \left(\frac{1}{e} + o(1)\right) \cdot n!.$$

From [58], we know that for $n \geq 5$, the eigenvalues of Γ_1 satisfy:

$$\begin{aligned} \lambda_{(n)}^{(1)} &= d_n, \\ \lambda_{(n-1,1)}^{(1)} &= -\frac{d_n}{(n-1)}, \\ |\lambda_\rho^{(1)}| &< \frac{c \cdot d_n}{n^2} < \frac{d_n}{(n-1)} \text{ for all other } \rho \vdash n, \end{aligned} \tag{2.17}$$

where c is an absolute constant. And the eigenvalues of Γ_k satisfy:

$$\begin{aligned} \lambda_{(n)}^{(k)} &= \sum_{i=0}^{k-1} \left[\binom{n}{i} \cdot d_{n-i}\right], \\ \lambda_{(n-1,1)}^{(k)} &= \frac{1}{(n-1)} \cdot \sum_{i=0}^{k-1} \left[\binom{n}{i} \cdot d_{n-i} \cdot (i-1)\right], \\ |\lambda_\rho^{(k)}| &< \frac{c_k \cdot n!}{n^2} \text{ for all other } \rho \vdash n, \end{aligned} \tag{2.18}$$

where $c_k > 0$ depends on k alone. As shown in [57], for $\lambda_\rho^{(k)}$s with different ks and the same ρ, their corresponding eigenspaces are the same U_ρ with dimension $\dim[\rho]$.

For each $t \in \mathbb{N}$, define

$$U_t = \bigoplus_{\rho \vdash n : \rho_1 \geq n-t} U_\rho.$$

It was proved in [58] that U_t is the linear span of the characteristic functions of the t-cosets of S_n, i.e.,

$$U_t = Span\{\mathcal{C}_{I \to J} : I, J \text{ are ordered } t\text{-tuples of distinct elements of } [n]\},$$

where for $I = \{i_1, \ldots, i_t\}$ and $J = \{j_1, \ldots, j_t\}$, $\mathcal{C}_{I \to J} = \{\sigma \in S_n : \sigma(i_1) = j_1, \ldots, \sigma(i_t) = j_t\}$ is a t-coset of S_n. Moreover, write $V_t = \bigoplus_{\rho \vdash n : \rho_1 = n-t} U_\rho$. Clearly, V_ts are pairwise orthogonal and

$$U_t = U_{t-1} \bigoplus V_t. \tag{2.19}$$

During their study of intersecting families for permutations in [58], Ellis, Friedgut and Pilpel developed several tools to estimate the spectra of Γ_ks, we include the following three lemmas which are useful for our estimations of the eigenvalues of Γ_ks.

Lemma 2.4. *([58], Lemma 6) Let G be a finite group, let $X \subseteq G$ be inverse-closed and conjugation-invariant, and let $Cay(G, X)$ be the Cayley graph on G with generating set X. Let ρ be an irreducible representation of G with dimension d, and let λ_ρ be the corresponding eigenvalue of $Cay(G, X)$. Then*

$$|\lambda_\rho| \leq \frac{\sqrt{|G||X|}}{d}. \tag{2.20}$$

Lemma 2.5. *([58], Claim 1 in Section 3.2.1) Let $[\rho]$ be an irreducible representation whose first row or column is of length $n - t$. Then*

$$\dim [\rho] \geq \binom{n}{t} e^{-t}. \tag{2.21}$$

Theorem 2.11. *([146]) If $\alpha, \epsilon > 0$, then there exists $N(\alpha, \epsilon) \in \mathbb{N}$ such that for all $n > N(\alpha, \epsilon)$, any irreducible representation $[\lambda]$ of S_n which has all rows and columns of length at most $\frac{n}{\alpha}$ has*

$$\dim [\lambda] \geq (\alpha - \epsilon)^n. \tag{2.22}$$

The above three lemmas provide a way to control $|\lambda_\rho^{(k)}|$ based on the dimension of $[\rho]$. When the structure of the partition ρ is relatively simple, $[\rho]$'s dimension can be well bounded and therefore leads to a good control of $|\lambda_\rho^{(k)}|$. When the dimension of $[\rho]$ is relatively large, this method no longer works. Thus, we need the following results from [126] and [127].

Theorem 2.12. *([126], Theorem 3.7) Let $0 < k < n$ and $\rho \vdash n$. Let $\mu_1, \ldots, \mu_q$ be the Young diagram obtained from ρ by removing the right most box from any row of the diagram so that the resulting diagram is still a partition of $(n-1)$. Then*

$$\lambda_\rho^{(k)} = \frac{n}{k\dim[\rho]} \sum_{j=1}^{q} \dim[\mu_j]\lambda_{\mu_j}^{(k-1)}. \tag{2.23}$$

Theorem 2.13. *([127], Theorem 3.5) Let n, k be integers with $n > k \geq 0$, and $\rho = (n) \vdash n$. Then*

$$\lambda_{\rho^T}^{(k)} = \binom{n}{k}(-1)^{n-k-1}(n-k-1). \tag{2.24}$$

For positive integers n_1 and n_2, we write $n_1 = O_t(n_2)$ if $n_1 \leq C_t n_2$ for some constant C_t that depends only on t.

Theorem 2.14. *([127], Theorem 3.9) Let n, k, t be integers with $k \geq 0$, $t > 0$ and $n > k + 2t$, $\rho = (n-t, \rho_2, \ldots, \rho_r) \vdash n$ with $\sum_{i=2}^{r} \rho_i = t$, and $\beta = (\rho_2, \ldots, \rho_r) \vdash t$. Then*

$$\dim[\rho]\lambda_\rho^{(k)} = \dim[\beta]\binom{n}{k}\left(\sum_{r=0}^{t}\binom{k}{r}\frac{(-1)^{t-r}}{(t-r)!}\right)d_{n-k} + O_t(n^{2t-1+k}). \tag{2.25}$$

Let n, k, t be integers with $0 \leq k < n$ and $0 \leq 2t < n$. We define $V(n,t) = \{\rho \vdash n : \rho = (n-t, \rho_2, \ldots, \rho_l) \text{ with } \sum_{i=2}^{l} \rho_i = t\}$, and we also need the following lemma.

Lemma 2.6. *([127], Lemma 3.16) Let $t \geq 0$ and $\rho \in V(n,t)$. Then*

$$\lambda_{\rho^T}^{(k)} = O_t(n^{k+1}). \tag{2.26}$$

2.2.3.2 Some new results about $\lambda_\rho^{(k)}$s

In this part, first, we shall prove two identities of the linear combinations of $\lambda_{(n)}^{(k)}$s. Then, using aforementioned results, we will provide some new estimations about $|\lambda_\rho^{(k)}|$ for $\rho \vdash n$ and $\rho \neq (n), (n-1,1)$.

Based on the formula of d_n, we have the following simple identity.

Proposition 2.1. *For any positive integer $n \geq 5$, we have*

$$\sum_{k=2}^{n}\sum_{i=1}^{k-1} \frac{1}{(i-1)!} \cdot \left(\sum_{s=0}^{n-i} \frac{(-1)^s}{s!}\right) = n-2. \tag{2.27}$$

Proof. First, by interchanging the summation order of the LHS of (2.27), we have

$$\begin{aligned}\sum_{k=2}^{n}\sum_{i=1}^{k-1} \frac{1}{(i-1)!} \cdot \left(\sum_{s=0}^{n-i} \frac{(-1)^s}{s!}\right) &= \sum_{k=2}^{n}\sum_{i=0}^{k-2} \frac{1}{i!} \cdot \left(\sum_{s=0}^{n-i-1} \frac{(-1)^s}{s!}\right) \\ &= \sum_{i=0}^{n-2} \frac{1}{i!} \cdot \left(\sum_{k=i+2}^{n}\sum_{s=0}^{n-i-1} \frac{(-1)^s}{s!}\right) \\ &= \sum_{i=0}^{n-2} \frac{n-i-1}{i!} \cdot \left(\sum_{s=0}^{n-i-1} \frac{(-1)^s}{s!}\right).\end{aligned}$$

Now, let $a_m = \sum_{s=0}^{m} \frac{(-1)^s}{s!}$ and $A(x)$ be the generating function $\sum_{m\geq 0} a_m x^m$ of sequence $\{a_m\}_{m\geq 0}$. Then, we have $A(x) = \frac{e^{-x}}{1-x}$ and

$$\sum_{m\geq 0} m a_m x^m = \left(\frac{e^{-x}}{1-x}\right)' x = \frac{e^{-x}x^2}{(1-x)^2}.$$

Let $b_n = \sum_{i=0}^{n} \frac{n-i+1}{i!} a_{n-i+1}$ and $B(x)$ be the generating function $\sum_{n\geq 0} b_n x^n$ of sequence $\{b_n\}_{n\geq 0}$. From the above equality and the property of products of generating functions, we immediately have

$$B(x) = e^x \cdot \frac{\sum_{m\geq 0} m a_m x^m}{x} = \frac{x}{(1-x)^2} = \sum_{n\geq 1} n x^n.$$

Therefore, $b_n = n$ and

$$\sum_{k=2}^{n}\sum_{i=1}^{k-1} \frac{1}{(i-1)!} \cdot \left(\sum_{s=0}^{n-i} \frac{(-1)^s}{s!}\right) = \sum_{i=0}^{n-2} \frac{n-i-1}{i!} \cdot \left(\sum_{s=0}^{n-i-1} \frac{(-1)^s}{s!}\right) = b_{n-2} = n-2.$$

This completes the proof of (2.27). □

As an application of Proposition 2.1, we can prove the following lemma.

Lemma 2.7. *For any integer $n \geq 5$,*

$$\sum_{k=1}^{n}(\lambda_{(n)}^{(k)} + (n-1)\cdot\lambda_{(n-1,1)}^{(k)}) = n!\cdot(n-2), \tag{2.28}$$

$$\sum_{k=1}^{n}(\lambda_{(n)}^{(k)} - \lambda_{(n-1,1)}^{(k)}) = n!\cdot\left(n - \frac{n-2}{n-1}\right). \tag{2.29}$$

Proof. By (2.18), we have

$$\begin{aligned}\lambda_{(n)}^{(k)} + (n-1)\cdot\lambda_{(n-1,1)}^{(k)} &= \sum_{i=0}^{k-1}\binom{n}{i}\cdot i\cdot d_{n-i}\\ &= 0 + n!\cdot\sum_{i=1}^{k-1}\frac{1}{(i-1)!}\cdot(\sum_{s=0}^{n-i}\frac{(-1)^s}{s!}),\end{aligned}$$

where the second term $n!\cdot\sum_{i=1}^{k-1}\frac{1}{(i-1)!}\cdot(\sum_{s=0}^{n-i}\frac{(-1)^s}{s!})$ in the RHS of the above equality equals 0 when $k \leq 1$. Thus the identity (2.28) follows from Lemma 2.1.

Similarly, by (2.18), we have

$$\begin{aligned}\lambda_{(n)}^{(k)} - \lambda_{(n-1,1)}^{(k)} &= \sum_{i=0}^{k-1}\binom{n}{i}\cdot\left(1-\frac{i-1}{n-1}\right)\cdot d_{n-i}\\ &= \frac{n}{n-1}\cdot\sum_{i=0}^{k-1}\binom{n}{i}\cdot d_{n-i} - \frac{1}{n-1}\cdot\sum_{i=0}^{k-1}\binom{n}{i}\cdot i\cdot d_{n-i}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n}(\lambda_{(n)}^{(k)} - \lambda_{(n-1,1)}^{(k)}) &= \sum_{k=1}^{n}\left(\frac{n}{n-1}\cdot\sum_{i=0}^{k-1}\binom{n}{i}\cdot d_{n-i} - \frac{1}{n-1}\cdot\sum_{i=0}^{k-1}\binom{n}{i}\cdot i\cdot d_{n-i}\right)\\ &= n!\cdot\frac{n}{n-1}\cdot\sum_{k=1}^{n}\sum_{i=0}^{k-1}\frac{1}{i!}\cdot\left(\sum_{s=0}^{n-i}\frac{(-1)^s}{s!}\right) - n!\cdot\frac{n-2}{n-1}.\end{aligned}$$

From the definition of M_n in Proposition 2.1, we have $\sum_{k=1}^{n}\sum_{i=0}^{k-1}\frac{1}{i!}\cdot(\sum_{s=0}^{n-i}\frac{(-1)^s}{s!}) = M_n$. Thus we have

$$\sum_{k=1}^{n}(\lambda_{(n)}^{(k)} - \lambda_{(n-1,1)}^{(k)}) = n!\cdot\left(n-\frac{n-2}{n-1}\right).$$

This completes the proof. □

Denote $\Phi = \{\rho \vdash n : \rho \neq (n), (n-1,1)\}$. According to (2.18), for fixed k and $\rho \neq (n), (n-1,1)$, we have $|\lambda_\rho^{(k)}| \leq \frac{c_k \cdot n!}{n^2}$. However, this bound is not good enough. When the index k varies from 1 to n, the constant c_k might become relatively large. Thus, if we try to get similar identities as (2.28) and (2.29) for $\lambda_\rho^{(k)}$ with $\rho \in \Phi$, we need some more delicate evaluations about $\lambda_\rho^{(k)}$s for $\rho \in \Phi$.

According to the structure of their corresponding partitions, we can divide $\rho \in \Phi$ into the following four parts:

$$
\begin{aligned}
\Phi_1 =& \{\rho \in \Phi : \text{the first row or column of } \rho \text{ is of length at most } n-3\};\\
\Phi_2 =& \{(n)^T\};\\
\Phi_3 =& \{(n-2,1,1),(n-2,2)\};\\
\Phi_4 =& \{(n-1,1)^T,(n-2,1,1)^T,(n-2,2)^T\}.
\end{aligned}
$$

Clearly, Φ are formed by these four parts and all of them are pairwise disjoint. Based on the known results, we can prove the following bounds about $\lambda_\rho^{(k)}$s for $\rho \in \Phi$.

Lemma 2.8. *Let n, k, t be positive integers with n sufficiently large. Then,*

- *When $\rho \in \Phi_1 \sqcup \Phi_2$, we have $|\lambda_\rho^{(k)}| \leq \frac{7e^3 n!}{n^3}$ for all $1 \leq k \leq n$.*
- *When $\rho \in \Phi_3$, we have $\lambda_\rho^{(k)} \geq -\frac{c_0 n!}{n^3}$ for $3 \leq k \leq n - \frac{n}{\ln n} - 7$, where c_0 is an absolute constant; and $|\lambda_\rho^{(k)}| \leq \frac{3n!}{n^2}$ for $k = 1,2$ or $k > n - \frac{n}{\ln n} - 7$.*
- *When $\rho \in \Phi_4$, we have $|\lambda_\rho^{(k)}| \leq \frac{n!}{n^3}$ for $1 \leq k \leq n - \frac{n}{\ln n} - 7$; and $|\lambda_\rho^{(k)}| \leq \frac{3n!}{n^2}$ for $k > n - \frac{n}{\ln n} - 7$.*

Proof. Consider the eigenvalues corresponding to irreducible representations in $\Phi_1 \sqcup \Phi_2$. For each $\rho \in \Phi_1$, assume that the length of the first row or column of ρ is $n-t$. When $3 \leq t \leq \frac{n}{3}$, since $\binom{n}{t}e^{-t}$ is increasing in the range $3 \leq t \leq \frac{n-e}{e+1}$ and is decreasing in the range $\frac{n-e}{e+1} < t \leq \frac{n}{3}$, thus, we have $\binom{n}{t}e^{-t} \geq \frac{n^3}{7e^3}$. By Lemma 2.5, $\dim[\rho] \geq \binom{n}{t}e^{-t} \geq \frac{n^3}{7e^3}$. When $t \geq \frac{n}{3}$, ρ has all rows and columns of length at most $\frac{2n}{3}$. Since n is sufficiently large, by Theorem 2.11, $\dim[\rho] \geq (\frac{3}{2} - \epsilon)^n \geq \frac{n^3}{7e^3}$.

Therefore, for all $\rho \in \Phi_1$, we have $\dim[\rho] \geq \frac{n^3}{7e^3}$. Note that $|FPF_k| < n!$. By Lemma 2.4, we have

$$|\lambda_\rho^{(k)}| \leq \frac{7e^3 n!}{n^3}$$

for all $\rho \in \Phi_1$ and $1 \leq k \leq n$. According to Theorem 2.13, $\lambda_{(n)^T}^{(k)} = (-1)^{n-k-1}(n-k-1)\binom{n}{k}$. Thus, we also have $|\lambda_{(n)^T}^{(k)}| \leq \frac{7e^3 n!}{n^3}$.

Consider the eigenvalues corresponding to irreducible representations in Φ_3. Based on structures of Young diagrams of $(n-2,1,1)$ and $(n-2,2)$, one can easily get their hook lengths. Thus, by Theorem 2.10, $\dim[(n-2,1,1)] = \frac{(n-1)(n-2)}{2}$ and $\dim[(n-2,2)] = \frac{(n-1)(n-3)}{2}$. Take $t = 2$ in Theorem 2.14, for $1 \leq k \leq n-5$, we have

$$\frac{(n-1)(n-2)}{2}\lambda_{(n-2,1,1)}^{(k)} = \binom{n}{k}\frac{k^2-3k+1}{2}d_{n-k} + O_2(n^{k+3});$$
$$\frac{(n-1)(n-3)}{2}\lambda_{(n-2,2)}^{(k)} = \binom{n}{k}\frac{k^2-3k+1}{2}d_{n-k} + O_2(n^{k+3}).$$

For $k = 1, 2$ and n sufficiently large, this leads to $\lambda_{(n-2,1,1)}^{(1)} = \lambda_{(n-2,2)}^{(1)} = -\left(\frac{1}{e} + o(1)\right) \cdot \frac{n!}{n^2}$ and $\lambda_{(n-2,1,1)}^{(2)} = \lambda_{(n-2,2)}^{(2)} = -\left(\frac{1}{2e} + o(1)\right) \cdot \frac{n!}{n^2}$. For $3 \leq k \leq n - \frac{n}{\ln n} - 7$, we have $\binom{n}{k}\frac{k^2-3k+1}{2}d_{n-k} > 0$ and $n^{k+3} < \frac{n!}{n^3}$. This indicates that

$$\lambda_{(n-2,1,1)}^{(k)} \geq -c_1\frac{n!}{n^3} \text{ and } \lambda_{(n-2,2)}^{(k)} \geq -c_2\frac{n!}{n^3}$$

for all $3 \leq k \leq n - \frac{n}{\ln n} - 7$, where $c_1, c_2 \geq 0$ are absolute constants. For $k > n - \frac{n}{\ln n} - 7$, since we already have $\dim[(n-2,1,1)] = \frac{(n-1)(n-2)}{2}$ and $\dim[(n-2,2)] = \frac{(n-1)(n-3)}{2}$, by Lemma 2.4 and $|FPF_k| < n!$, we have

$$|\lambda_{(n-2,1,1)}^{(k)}|,\ |\lambda_{(n-2,2)}^{(k)}| \leq \frac{3n!}{n^2}.$$

Consider the eigenvalues corresponding to irreducible representations in Φ_4. For $1 \leq k \leq n - \frac{n}{\ln n} - 7$, by Lemma 2.6, we have $\lambda_{(n-1,1)^T}^{(k)} = O_1(n^{k+1})$ and $\lambda_{(n-2,1,1)^T}^{(k)} = \lambda_{(n-2,2)^T}^{(k)} = O_2(n^{k+1})$. Therefore, we have

$$|\lambda_{(n-1,1)^T}^{(k)}|,\ |\lambda_{(n-2,1,1)^T}^{(k)}|,\ |\lambda_{(n-2,2)^T}^{(k)}| \leq \frac{n!}{n^3},$$

for all $1 \leq k \leq n - \frac{n}{\ln n} - 7$. For $k > n - \frac{n}{\ln n} - 7$, based on the structure of $(n-1,1)^T$, we have

$$\lambda_{(n-1,1)^T}^{(k)} = \frac{n}{k\dim[\rho]} \cdot (\dim[(n-1)^T] \cdot \lambda_{(n-1)^T}^{(k-1)} + \dim[(n-2,1)^T] \cdot \lambda_{(n-2,1)^T}^{(k-1)})$$

by Theorem 2.12. Since $\dim[(n-1)^T] = \dim[(n-1)] = 1$ and $\dim[\rho] = \dim[(n-1,1)] = n-1$, we further have

$$|\lambda^{(k)}_{(n-1,1)^T}| \le \frac{n}{k(n-1)} \cdot |\lambda^{(k-1)}_{(n-1)^T}| + \frac{n}{k} \cdot |\lambda^{(k-1)}_{(n-2,1)^T}|.$$

From the first part of this proof, $|\lambda^{(k-1)}_{(n-1)^T}| \le \frac{7e^3(n-1)!}{(n-1)^3} < \frac{n!}{2n^2}$. Meanwhile, since $\dim[(n-2,1)^T] = n-2$, by Lemma 2.4, we have $|\lambda^{(k-1)}_{(n-2,1)^T}| \le \frac{(n-1)!}{n-2} < \frac{3n!}{2n^2}$. Therefore, by the choice of k, we have

$$|\lambda^{(k)}_{(n-1,1)^T}| \le \frac{3n!}{n^2}.$$

Similarly, note that $\dim[(n-2,1,1)^T] = \dim[(n-2,1,1)]$ and $\dim[(n-2,2)^T] = \dim[(n-2,2)]$, by Lemma 2.4, we also have

$$|\lambda^{(k)}_{(n-2,1,1)^T}|,\ |\lambda^{(k)}_{(n-2,2)^T}| \le \frac{3n!}{n^2}.$$

This completes the proof. □

§ 2.3 Proofs of Theorem 2.4 and Theorem 2.5

2.3.1 Proof of Theorem 2.4

In this section, we present the proof of Theorem 2.4. The main tool that we use in this proof is the *quantitative shifting method* introduced in [47]. To carry out this method, our proof is divided into the following three steps:

- First, to guarantee its optimality, we shall prove that the family $\mathcal{F}$ must contain a popular element, i.e., there is some $x \in [n]$ in many sets of $\mathcal{F}$. Based on this argument, we can prove the result when $\mathcal{F}$ contains a full 1-star by induction.

- Second, when $\mathcal{F}$ does not contain any full 1-star, we can replace the k-sets in $\mathcal{F}$ consisting of less popular elements with new k-sets containing this popular element. Through an estimation about the increment of $\mathcal{I}(\mathcal{F})$, we will show that $\mathcal{F}$ can be covered by $r+1$ elements in $[n]$.

- Finally, based on the results from former steps, we complete the proof by induction on n and r.

Lemma 2.9. *Let $k \geq 2$, r and n be non-negative integers defined in Theorem 2.4. If $\mathcal{F} \subseteq \binom{[n]}{k}$ with size*

$$|\mathcal{F}| = \sum_{i=1}^{r} \binom{n-i}{k-1} + \delta \binom{n-(r+1)}{k-1}$$

for some $\delta \in [\frac{150k^3}{n}, 1]$, and satisfies $\mathcal{I}(\mathcal{F}) \geq \frac{r+\delta^2}{(r+\delta)^2}|\mathcal{F}|^2$. Then, there exists an $x \in [n]$ with $|\mathcal{F}(x)| \geq \frac{|\mathcal{F}|}{4(r+1)}$.

Proof. First, take $X = \{x \in [n] : |\mathcal{F}(x)| \geq \frac{|\mathcal{F}|}{5k(r+1)}\}$ as the set of moderately popular elements, we show that X can not be very large.

Claim 1. $|X| < 10k(r+1)$.

Proof. Suppose not, let X_0 be a subset of X with size $10k(r+1)$, then we have

$$\begin{aligned} |\mathcal{F}| \geq |\bigcup_{x \in X_0} \mathcal{F}(x)| &\geq \sum_{x \in X_0} |\mathcal{F}(x)| - \sum_{x \neq y \in X_0} |\mathcal{F}(x,y)| \\ &\geq 2|\mathcal{F}| - \binom{|X_0|}{2}\binom{n-2}{k-2}. \end{aligned} \tag{2.30}$$

Since $|\mathcal{F}| = |\mathcal{L}_{n,k,1}^{(r)}| + \delta\binom{n-(r+1)}{k-1}$, by the choice of n and Bonferroni Inequalities, we know that

$$\begin{aligned} |\mathcal{F}| &\geq \binom{r}{1}\binom{n-1}{k-1} - \binom{r}{2}\binom{n-2}{k-2} + \delta\binom{n-(r+1)}{k-1} \\ &\geq (\frac{nr}{3k} + \delta\frac{n-(r+k)}{k-1}(1 - \frac{k(r+k)}{n-2}))\binom{n-2}{k-2}. \end{aligned} \tag{2.31}$$

Combining (2.30) and (2.31) together, we have

$$|\mathcal{F}| > (2 - \frac{150k^3(r+1)^2}{n(r+\delta)})|\mathcal{F}|,$$

which contradicts the requirement of n. Therefore, the claim holds. □

Now, we complete the proof by proving the following claim.

Claim 2. There is an $x_0 \in X$ such that $|\mathcal{F}(x_0)| \geq \frac{|\mathcal{F}|}{4(r+1)}$.

Proof. W.l.o.g., assume that $1 \in X$ is the most popular element appearing in $\mathcal{F}$. Then, we have

$$\mathcal{I}(\mathcal{F}) = \sum_{x \in X} |\mathcal{F}(x)|^2 + \sum_{x \in [n] \setminus X} |\mathcal{F}(x)|^2$$

$$\begin{aligned}
&\leq |\mathcal{F}(1)| \cdot \sum_{x\in X} |\mathcal{F}(x)| + \frac{|\mathcal{F}|}{5k(r+1)} \cdot \sum_{x\in[n]\setminus X} |\mathcal{F}(x)| \\
&\leq |\mathcal{F}(1)| \cdot (|\mathcal{F}| + \binom{|X|}{2}\binom{n-2}{k-2}) + \frac{|\mathcal{F}|}{5k(r+1)} \cdot k \cdot |\mathcal{F}| \\
&\leq |\mathcal{F}(1)| \cdot |\mathcal{F}| \cdot (1 + \frac{150k^3(r+1)^2}{n(r+\delta)}) + \frac{|\mathcal{F}|^2}{5(r+1)}. \qquad (2.32)
\end{aligned}$$

By the lower bound of $\mathcal{I}(\mathcal{F})$ and (2.32), we can obtain

$$|\mathcal{F}(1)| \geq \frac{3|\mathcal{F}|}{10(r+1)\cdot(1+\frac{150k^3(r+1)^2}{n(r+\delta)})} \geq \frac{|\mathcal{F}|}{4(r+1)}.$$

Therefore, the claim holds. □

This completes the proof. □

Given a subset $A \subseteq [n]$ and a family of subsets $\mathcal{F} \subseteq \binom{[n]}{k}$, we say A is a *cover* of $\mathcal{F}$ if for every $F \in \mathcal{F}$, $A \cap F \neq \emptyset$. Based on Lemma 2.9, we can proceed to the second step.

Lemma 2.10. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be the same family as that in Theorem 2.4. If $\mathcal{F}$ does not contain any full 1-star, then $\mathcal{F}$ has a cover $A \subseteq [n]$ of size $r+1$.*

Proof. First, we show that the set of moderately popular elements already forms a cover of $\mathcal{F}$.

Claim 3. $X = \{x \in [n] : \mathcal{F}(x) \geq \frac{|\mathcal{F}|}{5k(r+1)}\}$ is a cover of $\mathcal{F}$.

Proof. Suppose not, there exists an $F_0 \in \mathcal{F}$ such that $F_0 \cap X = \emptyset$. Thus, for every $x \in F_0$, $\mathcal{F}(x) < \frac{|\mathcal{F}|}{5k(r+1)}$. Since

$$\mathcal{I}(\mathcal{F}) = \sum_{F\in\mathcal{F}} \mathcal{I}(F,\mathcal{F}) = k + 2\mathcal{I}(F_0, \mathcal{F}\setminus\{F_0\}) + \mathcal{I}(\mathcal{F}\setminus\{F_0\}). \qquad (2.33)$$

Noted that the unpopularity of the elements in F_0 may lead to $\mathcal{I}(F_0, \mathcal{F}\setminus\{F_0\})$ being very small, thus, if it is possible, we can increase the value of $\mathcal{I}(F_0, \mathcal{F}\setminus\{F_0\})$ by replacing F_0 with another k-subset of $[n]$ containing a popular element without changing the value of $\mathcal{I}(\mathcal{F}\setminus\{F_0\})$.

In fact, this is possible. Due to the assumption that $\mathcal{F}(1) \subsetneq \mathcal{L}_{n,k,1}^{(1)}$ (i.e., $\mathcal{F}$ contains no full 1-star), we can replace F_0 with some $F_0' \in \mathcal{L}_{n,k,1}^{(1)} \setminus \mathcal{F}$. Denote the new family as $\mathcal{F}'$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}') - \mathcal{I}(\mathcal{F}) &= 2(\mathcal{I}(F_0', \mathcal{F}' \setminus \{F_0'\}) - \mathcal{I}(F_0, \mathcal{F} \setminus \{F_0\})) + \mathcal{I}(\mathcal{F}' \setminus \{F_0'\}) - \mathcal{I}(\mathcal{F} \setminus \{F_0\}) \\ &= 2(\mathcal{I}(F_0', \mathcal{F} \setminus \{F_0\}) - \mathcal{I}(F_0, \mathcal{F} \setminus \{F_0\})) \\ &\geq 2(\sum_{x \in F_0'} |\mathcal{F}(x)| - \sum_{x \in F_0} |\mathcal{F}(x)|) \\ &\geq 2(|\mathcal{F}(1)| - \frac{|\mathcal{F}|}{5(r+1)}) > 0,\end{aligned}$$

which contradicts the optimality of $\mathcal{F}$. Therefore, the claim holds. □

By Claim 3, we know that $\mathcal{F}$ has a cover X with size less than $10k(r+1)$. Let $X_0 \subseteq X$ be the minimal cover of $\mathcal{F}$ containing 1. W.l.o.g., assume that $X_0 = [m]$. For each $i \in [m]$, denote $\mathcal{F}^*(i)$ as the subfamily in $\mathcal{F}(i)$ consisting of all k-sets with i as their minimal element. Then, $\mathcal{F} = \bigsqcup_{i=1}^m \mathcal{F}^*(i)$. Thus, we have the following claim.

Claim 4. For every $i, j \in [m]$, we have $|\mathcal{F}^*(i)| \geq |\mathcal{F}^*(j)| - \frac{3mk^2}{(r+\delta)n}|\mathcal{F}|$.

Proof. First we claim that for each $i \in [m]$, there is some $F \in \mathcal{F}(i)$ such that $F \cap [m] = \{i\}$. Otherwise, suppose that for every $F \in \mathcal{F}(i)$, we have $|F \cap [m]| \geq 2$. Then,

$$|\mathcal{F}(i)| \leq (m-1)\binom{n-2}{k-2} < \frac{3mk}{(r+\delta)n}|\mathcal{F}| < \frac{|\mathcal{F}|}{5k(r+1)}.$$

This contradicts the fact that $\mathcal{F}(i) \geq \frac{|\mathcal{F}|}{5k(r+1)}$.

Now, assume there exist $i_0 \neq j_0 \in [m]$ satisfying $|\mathcal{F}^*(j_0)| > |\mathcal{F}^*(i_0)| + \frac{3mk^2}{(r+\delta)n}|\mathcal{F}|$. Thus, since 1 is the most popular element in $[n]$ and $\mathcal{F}^*(1) = \mathcal{F}(1)$, we know that $i_0 \neq 1$ and

$$|\mathcal{F}^*(1)| \geq |\mathcal{F}^*(j_0)| > |\mathcal{F}^*(i_0)| + \frac{3mk^2}{(r+\delta)n}|\mathcal{F}|.$$

Noted that $\mathcal{F}^*(1) \subsetneq \mathcal{L}_{n,k,1}^{(1)}$, therefore, we can replace the k-subset $F \in \mathcal{F}^*(i_0)$ satisfying $F \cap [m] = \{i_0\}$ with some $F' \in \mathcal{L}_{n,k,1}^{(1)} \setminus \mathcal{F}^*(1)$. Let $\mathcal{F}'$ be the resulting new family, by (2.33), we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}')-\mathcal{I}(\mathcal{F}) &= 2(\mathcal{I}(F',\mathcal{F}'\setminus\{F'\})-\mathcal{I}(F,\mathcal{F}\setminus\{F\}))\\ &\geq 2(\mathcal{I}(F',\mathcal{F}^*(1))-\sum_{x\in F}\sum_{i\in[m]}|\{A\in\mathcal{F}^*(i):x\in A\}|)\\ &\geq 2(|\mathcal{F}^*(1)|-|\mathcal{F}^*(i_0)|-k(m-1)\binom{n-2}{k-2})>0.\end{aligned}$$

This contradicts the optimality of $\mathcal{F}$. Therefore, the claim holds. □

Actually, Claim 4 shows that as the extremal family, the sizes of sub-families $\mathcal{F}^*(i)$ ($i\in[m]$) of $\mathcal{F}$ are relatively close. Since $|\mathcal{F}^*(1)|=|\mathcal{F}(1)|\geq\frac{|\mathcal{F}|}{4(r+1)}$, thus for each $i\neq 1\in[m]$,

$$|\mathcal{F}^*(i)|\geq|\mathcal{F}^*(1)|-\frac{3mk^2}{(r+\delta)n}|\mathcal{F}|\geq\frac{|\mathcal{F}|}{20(r+1)}.$$

Noticed that $\{\mathcal{F}^*(i)\}_{i=1}^m$ forms a partition of $\mathcal{F}$, this leads to a rough bound on m as: $m\leq 20(r+1)$.

Based on this rough bound, we complete the proof by proving the next claim.

Claim 5. $m=r+1$.

Proof. We only prove the case when $r>0$, for $r=0$ the proof is the same.

Given two k-uniform families $\mathcal{F}_1$ and $\mathcal{F}_2$, we define $\mathcal{I}(\mathcal{F}_1,\mathcal{F}_2)=\sum_{A\in\mathcal{F}_1,B\in\mathcal{F}_2}|A\cap B|$. Clearly, we have $\mathcal{I}(\mathcal{F})=\mathcal{I}(\mathcal{F},\mathcal{F})$. By the size of $\mathcal{F}$, $m\geq r+1$. Assume that $m\geq r+2$. First, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}) &= \sum_{i,j\in[m]}\mathcal{I}(\mathcal{F}^*(i),\mathcal{F}^*(j))=\sum_{i\in[m]}\mathcal{I}(\mathcal{F}^*(i))+\sum_{i\neq j\in[m]}\mathcal{I}(\mathcal{F}^*(i),\mathcal{F}^*(j))\\ &=\sum_{i\in[m]}\sum_{F\in\mathcal{F}^*(i)}\sum_{x\in F}|\{A\in\mathcal{F}^*(i):x\in A\}|+\sum_{i\neq j\in[m]}\sum_{F\in\mathcal{F}^*(i)}\sum_{x\in F}|\{A\in\mathcal{F}^*(j):x\in A\}|\\ &\leq\sum_{i\in[m]}(|\mathcal{F}^*(i)|+(k-1)\binom{n-2}{k-2})|\mathcal{F}^*(i)|+\sum_{i\neq j\in[m]}k\binom{n-2}{k-2}(|\mathcal{F}^*(i)|+|\mathcal{F}^*(j)|)\\ &\leq\sum_{i\in[m]}|\mathcal{F}^*(i)|^2+2km\binom{n-2}{k-2}|\mathcal{F}|\leq(|\mathcal{F}^*(1)|+2km\binom{n-2}{k-2})|\mathcal{F}|.\end{aligned}\tag{2.34}$$

By a simple averaging argument, there exists some $i_0 \in [m]$ such that $|\mathcal{F}^*(i_0)| \leq \frac{|\mathcal{F}|}{m}$. Thus, by Claim 4, we have

$$|\mathcal{F}^*(1)| \leq (\frac{1}{m} + \frac{3mk^2}{(r+\delta)n})|\mathcal{F}|.$$

This leads to

$$\mathcal{I}(\mathcal{F}) \leq (\frac{1}{m} + \frac{9mk^2}{(r+\delta)n})|\mathcal{F}|^2 \leq (\frac{1}{r+2} + \frac{360k^2}{n})|\mathcal{F}|^2. \tag{2.35}$$

Since $\mathcal{F}$ is the extremal family, we know that

$$\begin{aligned}\mathcal{I}(\mathcal{F}) \geq \mathcal{I}(\mathcal{L}_{n,k}(|\mathcal{F}|)) &= \sum_{x\in[n]} |F \in \mathcal{L}_{n,k}(|\mathcal{F}|) : x \in F|^2 \\ &\geq (r+\delta^2)\binom{n-1}{k-1}^2 \geq \frac{r+\delta^2}{(r+\delta)^2}|\mathcal{F}|^2.\end{aligned}$$

Combining with (2.35), we can obtain

$$\frac{r+\delta^2}{(r+\delta)^2}|\mathcal{F}|^2 \leq \mathcal{I}(\mathcal{F}) \leq (\frac{1}{r+2} + \frac{360k^2}{n})|\mathcal{F}|^2.$$

Since $n \geq C_0(r+1)^3(k+r)k^2$, we have $\frac{360k^2}{n} < \frac{r+\delta^2}{(r+\delta)^2} - \frac{1}{r+2}$, a contradiction.

Therefore, $m = r+1$. □

This completes the proof. □

Proof of Theorem 2.4. We prove the theorem by induction on n and r.

Consider the base case: $r = 0$. By Lemma 2.10, we know that $\mathcal{F}$ has a cover of size 1. Noted that we have already assumed that $1 \in [n]$ is the most popular element of $\mathcal{F}$, thus $\mathcal{F} = \mathcal{F}(1)$. This indicates that $\mathcal{L}^{(0)}_{n,k,1} \subseteq \mathcal{F} \subseteq \mathcal{L}^{(1)}_{n,k,1}$.

Now, suppose that $\mathcal{F}$ contains a full 1-star. W.l.o.g., assume this full 1-star consists of all k-sets containing 1. Then, we have $r \geq 1$ and

$$\begin{aligned}\mathcal{I}(\mathcal{F}) &= \mathcal{I}(\mathcal{F}(1)) + \mathcal{I}(\mathcal{F}\setminus\mathcal{F}(1), \mathcal{F}) \\ &= (\binom{n-1}{k-1}^2 + (n-1)\binom{n-2}{k-2}^2) + \sum_{A\in\mathcal{F}\setminus\mathcal{F}(1)} \mathcal{I}(A, \mathcal{F}).\end{aligned} \tag{2.36}$$

And for any $A \in \mathcal{F} \setminus \mathcal{F}(1)$,

$$\begin{aligned}\mathcal{I}(A,\mathcal{F}) &= \mathcal{I}(A,\mathcal{F}(1)) + \mathcal{I}(A,\mathcal{F}\setminus\mathcal{F}(1)) \\ &= k\binom{n-2}{k-2} + \mathcal{I}(A,\mathcal{F}\setminus\mathcal{F}(1)).\end{aligned} \tag{2.37}$$

Therefore,

$$\mathcal{I}(\mathcal{F}) = C_0(n,k) + \mathcal{I}(\mathcal{F}\setminus\mathcal{F}(1)),$$

where $C_0(n,k) = (\binom{n-1}{k-1}^2 + (n-1)\binom{n-2}{k-2}^2) + k\binom{n-2}{k-2}(|\mathcal{F}| - \binom{n-1}{k-1})$. Denote $\mathcal{F}' = \mathcal{F}\setminus\mathcal{F}(1)$, then $\mathcal{F}'$ can be viewed as a family of k-sets in $\binom{[n]\setminus\{1\}}{k}$. Due to the optimality of $\mathcal{F}$, we have

$$\mathcal{I}(\mathcal{F}') = \max_{\mathcal{G}\subseteq\binom{[n]\setminus\{1\}}{k}, |\mathcal{G}|=|\mathcal{F}'|} = \mathcal{I}(\mathcal{G}) = \mathcal{MI}_{[n]\setminus\{1\}}(\mathcal{F}').$$

Thus, by induction hypothesis, $\mathcal{F}' \subseteq \binom{[n]\setminus\{1\}}{k}$ satisfies that $\mathcal{L}^{(r-1)}_{n-1,k,1} \subseteq \mathcal{F}' \subseteq \mathcal{L}^{(r)}_{n-1,k,1}$. Joined with the full 1-star $\mathcal{F}(1)$, we have $\mathcal{L}^{(r)}_{n,k,1} \subseteq \mathcal{F} \subseteq \mathcal{L}^{(r+1)}_{n,k,1}$ as claimed.

When $\mathcal{F}$ does not contain any full 1-star, by Lemma 2.10, we know that $\mathcal{F}$ can be covered by an $(r+1)$-subset of $[n]$. W.l.o.g, assume that this $(r+1)$-subset is $[r+1]$.

Let $\mathcal{A} = \{A \in \binom{[n]}{k} : A \cap [r+1] \neq \emptyset\}$ be the family of all k-subsets that intersect $[r+1]$. When $\delta = 1$, we have $\mathcal{F} = \mathcal{A} = \mathcal{L}^{(r+1)}_{n,k,1}$. When $\delta \neq 1$, $\mathcal{F} \subsetneq \mathcal{A}$. Let $\mathcal{G} = \mathcal{A}\setminus\mathcal{F}$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}) &= \mathcal{I}(\mathcal{A}) - 2\mathcal{I}(\mathcal{G},\mathcal{A}) + \mathcal{I}(\mathcal{G}) \\ &= \mathcal{I}(\mathcal{A}) - 2\sum_{G\in\mathcal{G}}\mathcal{I}(G,\mathcal{A}) + \mathcal{I}(\mathcal{G}).\end{aligned} \tag{2.38}$$

Note that once $r+1$ is given, $\mathcal{A}$ can be viewed as the union of $r+1$ full 1-stars with cores $1, 2, \ldots, r+1$. Based on this structure, for each $x \in [r+1]$, we have $\mathcal{A}(x) = \binom{n-1}{k-1}$ and for each $x \in [n]\setminus[r+1]$, we have $\mathcal{A}(x) = \sum_{i=1}^{r+1}\binom{n-i-1}{k-2}$. Since $\mathcal{I}(\mathcal{A}) = \sum_{x\in[n]}|\mathcal{A}(x)|^2$ and for each $G \in \mathcal{A}$, $\mathcal{I}(G,\mathcal{A}) = \sum_{x\in G}|\mathcal{A}(x)|$, thus $\mathcal{I}(\mathcal{A})$ and $\mathcal{I}(G,\mathcal{A})$ are both fixed constants.

By (2.38), the optimality of $\mathcal{F}$ is actually guaranteed by $\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}}\mathcal{I}(G,\mathcal{A})$, i.e., $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$ if and only if $\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}}\mathcal{I}(G,\mathcal{A})$ reaches the maximum. Based on this observation, we have the following claim.

Claim 6. For $\mathcal{G} \subseteq \mathcal{A}$ with size $|\mathcal{G}| = |\mathcal{A}| - |\mathcal{F}|$, $\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A})$ reaches its maximum only if there exists some $i_0 \in [r+1]$ such that $G \cap [r+1] = \{i_0\}$ for all $G \in \mathcal{G}$.

Proof. First, we show that for all $G \in \mathcal{G}$, $|G \cap [r+1]| = 1$. Otherwise, assume that there exists $G_0 \in \mathcal{G}$ satisfying $|G_0 \cap [r+1]| \geq 2$. W.l.o.g., assume that 1 is the most popular element in $\mathcal{G}$ among $[r+1]$. Since $\mathcal{G}$ contains no full 1-star, we can replace G_0 with some $G_1 \in \mathcal{A}(1) \setminus \mathcal{G}(1)$. Denote the resulting new family as $\mathcal{G}'$, we have

$$\begin{aligned}
&[\mathcal{I}(\mathcal{G}') - 2\sum_{G\in\mathcal{G}'} \mathcal{I}(G, \mathcal{A})] - [\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A})] \\
&= (\mathcal{I}(\mathcal{G}') - \mathcal{I}(\mathcal{G})) + 2(\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A}) - \sum_{G\in\mathcal{G}'} \mathcal{I}(G, \mathcal{A})).
\end{aligned}$$

From (2.33), we know that

$$\begin{aligned}
\mathcal{I}(\mathcal{G}') - \mathcal{I}(\mathcal{G}) &= 2(\mathcal{I}(G_1, \mathcal{G} \setminus \{G_0\}) - \mathcal{I}(G_0, \mathcal{G} \setminus \{G_0\})) \\
&\geq 2(\sum_{x\in G_1} |\mathcal{G}(x)| - \sum_{x\in G_0} |\mathcal{G}(x)|) \\
&\geq -2|\mathcal{G}| - r(r+1)\binom{n-2}{k-2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A}) - \sum_{G\in\mathcal{G}'} \mathcal{I}(G, \mathcal{A}) &= \mathcal{I}(G_0, \mathcal{A}) - \mathcal{I}(G_1, \mathcal{A}) \\
&\geq \binom{n-1}{k-1} - (k-1)(r+1)\binom{n-2}{k-2}.
\end{aligned}$$

By combining these two estimations together, we have

$$\begin{aligned}
&[\mathcal{I}(\mathcal{G}') - 2\sum_{G\in\mathcal{G}'} \mathcal{I}(G, \mathcal{A})] - [\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A})] \\
&\geq 2\delta\binom{n-1}{k-1} - (2k+r)(r+1)\binom{n-2}{k-2} > 0,
\end{aligned}$$

where the first inequality follows from $|\mathcal{G}| = (1-\delta)\binom{n-(r+1)}{k-1}$ and the second inequality follows from the choice of δ. This contradicts the maximality of $\mathcal{I}(\mathcal{G}) - 2\sum_{G\in\mathcal{G}} \mathcal{I}(G, \mathcal{A})$.

Noticed that $|G\cap[r+1]|=1$ for all $G\in\mathcal{G}$ indicates that $\mathcal{I}(G,\mathcal{A})=\binom{n-1}{k-1}+(k-1)\sum_{i=1}^{r+1}\binom{n-i-1}{k-2}$, which is a constant irrelevant to the structure of $\mathcal{G}$. Therefore, $\mathcal{I}(\mathcal{G})-2\sum_{G\in\mathcal{G}}\mathcal{I}(G,\mathcal{A})$ attains its maximum only if

$$\mathcal{I}(\mathcal{G})=\max_{\substack{\mathcal{G}_0\subseteq\mathcal{A},|\mathcal{G}_0|=|\mathcal{G}|\\ |G\cap[r+1]|=1\text{ for all }G\in\mathcal{G}_0}}\mathcal{I}(\mathcal{G}_0).$$

Noted that for $\mathcal{G}_0\subseteq\mathcal{A}(1)$ with $|\mathcal{G}_0|=|\mathcal{G}|$, $\mathcal{I}(\mathcal{G}_0)\geq|\mathcal{G}|^2$, thus we have $\mathcal{I}(\mathcal{G})\geq|\mathcal{G}|^2$. Since $\mathcal{G}=\bigsqcup_{i=1}^{r+1}\mathcal{G}(i)$, by the upper bound of $\mathcal{I}(\mathcal{G})$ in (2.34), we have $|\mathcal{G}(1)|\geq|\mathcal{G}|-2k(r+1)\binom{n-2}{k-2}$. Therefore, through a similar shifting argument as Claim 4, the maximality of $\mathcal{I}(\mathcal{G})$ guarantees that $|\mathcal{G}(i)|\geq|\mathcal{G}(1)|-\frac{6(r+1)k^2}{(1-\delta)n}|\mathcal{G}|$ for every $i\in[r+1]$ with $|\mathcal{G}(i)|>0$. If there exists some $2\leq i'\leq r+1$ such that $|\mathcal{G}(i')|>0$, we shall have $\sum_{i\in[r+1]}|\mathcal{G}(i)|\geq(2-\frac{6(r+1)k^2}{(1-\delta)n})|\mathcal{G}|-2k(r+1)\binom{n-2}{k-2}$, which contradicts the fact that $|\mathcal{G}|\geq\sum_{i\in[r+1]}|\mathcal{G}(i)|-\binom{r+1}{2}\binom{n-2}{k-2}$. Therefore, we have $\mathcal{G}=\mathcal{G}(1)$ and the claim holds. □

By Claim 6, we know that $\mathcal{G}$ is contained in a full 1-star of $\mathcal{A}$. W.l.o.g, assume that $\mathcal{G}\subseteq\{A\in\binom{[n]}{k}:r+1\in A\}$ and this leads to $\mathcal{L}_{n,k,1}^{(r)}\subseteq\mathcal{F}\subseteq\mathcal{L}_{n,k,1}^{(r+1)}$. This completes the proof of Theorem 2.4. □

Remark 2.1. *According to the proof, one may wonder if the range that*

$$\delta\in[\frac{150k^3(r+1)^2}{n},1-\frac{150k^3(r+1)^3}{n}]\cup\{1\}$$

can be extended. Actually, the range might be improved to be a little bit larger, but anyway, δ can not be too close to 0 or too close to 1 when $\delta<1$. For example, fix $k\geq 2$ and n sufficiently large. Consider a family $\mathcal{F}_0\subseteq\binom{[n]}{k}$ with size $k+1$, one can easily verify that $\mathcal{I}(\mathcal{G})$ achieves the maximality when $\mathcal{F}_0=\binom{[k+1]}{k}$. Clearly, for this case, $\mathcal{F}_0\nsubseteq\mathcal{L}_{n,k,1}^{(1)}$.

2.3.2 Proof of Theorem 2.5

Recall the proof of Theorem 2.4. First, we showed that $\mathcal{F}$ must contain a popular element to guarantee its optimality. Then, we showed that if $\mathcal{F}$ doesn't have a small cover, $\mathcal{I}(\mathcal{F})$ can be increased through shifting arguments. This indicates that $\mathcal{F}$ must have a certain clustering property and can be covered by a few popular elements. Finally, noted that $|\mathcal{F}|$ is fixed, this small cover ensures $\mathcal{F}$ to have the desired structure.

For Theorem 2.5, since the family we shall deal with is much sparser when $t \geq 2$, it requires more delicate analysis of the family $\mathcal{F}$ to proceed the above arguments. In order to prove Theorem 2.5, we shall require a few preliminary results.

First, we need the following lemma from [47] which shows that among all unions of r full t-stars, the lexicographic ordering contains the fewest sets.

Lemma 2.11. *[47] Suppose $k \geq 2$, $t \geq 1$, r and n are given non-negative integers defined in Theorem 2.5. Let $\mathcal{F}$ be the union of r full t-stars in $\binom{[n]}{k}$. Then $|\mathcal{F}| \geq \sum_{i=t}^{t+r-1} \binom{n-i}{k-t}$, with equality to hold if and only if $\mathcal{F}$ is isomorphic to $\mathcal{L}^{(r)}_{n,k,t}$.*

With the help of Lemma 2.11, when $\mathcal{F}$ contains p full t-stars and the total intersection number of the remaining k-sets is well bounded, we have the following lemma which determines the structure of these p full t-stars and shows that the remaining family is almost cross $(t-1)$-intersecting with each of these p full t-stars.

Lemma 2.12. *Let $k \geq 2$, $t \geq 1$, $r \geq 1$ and n be given non-negative integers defined in Theorem 2.5. Let $\mathcal{F} \subseteq \binom{[n]}{k}$ with size $|\mathcal{F}| = \sum_{i=t}^{t+r-1} \binom{n-i}{k-t} + \delta_0 \binom{n-(r+t)}{k-t}$ for some $\delta_0 \in [\frac{6k(r+1)t}{C_1}, 1]$, and satisfy $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Suppose $\mathcal{F}$ contains p full t-stars $\mathcal{Y}_1, \ldots, \mathcal{Y}_p$ for some integer $1 \leq p \leq r$ and*

$$\mathcal{I}(\mathcal{F}_0) \leq (t-1)|\mathcal{F}_0|^2 + (r - p + \delta_0^2 + \frac{1}{C_t})\binom{n-t}{k-t}^2,$$

for any $\mathcal{F}_0 \subseteq \mathcal{F}$ with size $\sum_{i=t}^{t+r-(p+1)} \binom{n-i}{k-t} + \frac{r\delta_0}{r+1}\binom{n-(r+t-p)}{k-t} < |\mathcal{F}_0| \leq \sum_{i=t}^{t+r-(p+1)} \binom{n-i}{k-t} + \delta_0 \binom{n-(r+t-p)}{k-t}$. Let $\mathcal{F}_1 = \cup_{i=1}^{p} \mathcal{Y}_i$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. For each $1 \leq i \leq p$, denote $Y_i \in \binom{[n]}{t}$ as the core of $\mathcal{Y}_i$, then

- *for all $i \neq j \in [p]$, $|Y_i \cap Y_j| = t-1$;*
- *for at least $(1 - \frac{2r^2kt}{C_t})|\mathcal{F}_2|$ k-sets $F \in \mathcal{F}_2$, $|F \cap Y_i| = t-1$ for each $1 \leq i \leq p$;*
- *$\mathcal{I}(\mathcal{F}_2) \geq (t-1)|\mathcal{F}_2|^2 + (r - p + \delta_0^2 - \frac{4kr^2p^2}{C_t})\binom{n-t}{k-t}^2$.*

Let $\mathcal{F}_1 = \cup_{i=t}^{t+r}\mathcal{G}_j$ and $\mathcal{F}_2 = \cup_{j=1}^{t+1}\mathcal{H}_j$ with the same size $\sum_{i=t}^{t+r-1}\binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t}$ for some $\delta \in [\frac{6krt}{C_1}, 1-\frac{6krt}{C_1}] \cup \{1\}$, where $\mathcal{G}_i$ is a t-star with core $\{1,\ldots,t-1,i\}$ and $\mathcal{H}_j$ is a t-star with core $[t+1]\setminus\{j\}$. The following lemma shows that when the size of each star is not too small, family with the structure of $\mathcal{F}_1$ has larger total intersection number.

Lemma 2.13. *Let $k \geq 2$, $t \geq 1$, $1 \leq r \leq t$ and n be given non-negative integers defined in Theorem 2.5. Let $\mathcal{F} = \cup_{j=1}^{t+1}\mathcal{H}_j$ with $|\mathcal{F}| = \sum_{i=t}^{t+r-1}\binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t}$ for some $\delta \in [\frac{6krt}{C_1}, 1-\frac{6krt}{C_1}] \cup \{1\}$, where $\mathcal{H}_j$ is a t-star with core $[t+1]\setminus\{j\}$. Assume that $|\mathcal{H}_j| \geq \frac{\delta}{2C_1}\binom{n-t}{k-t}$ for each $j \in [t+1]$. Then, there exists a family $\mathcal{F}_0$ with size $|\mathcal{F}|$ such that $\mathcal{L}_{n,k,t}^{(r)} \subseteq \mathcal{F}_0 \subseteq \mathcal{L}_{n,k,t}^{(r+1)}$ and $\mathcal{I}(\mathcal{F}_0) > \mathcal{I}(\mathcal{F})$.*

Our proof of Theorem 2.5 will proceed according to the following steps:

- First, we show that if $\mathcal{I}(\mathcal{F})$ is large enough, $\mathcal{F}$ must have a popular t-set.

Lemma 2.14. *Let $k \geq 2$, $t \geq 1$, r and n be given non-negative integers defined in Theorem 2.5. If $\mathcal{F} \subseteq \binom{[n]}{k}$ with size*

$$|\mathcal{F}| = \sum_{i=t}^{t+r-1}\binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t}$$

for some $\delta \in [\frac{6k(r+1)t}{C_1}, 1]$, and satisfies $\mathcal{I}(\mathcal{F}) \geq (t-1)|\mathcal{F}|^2 + (r+\delta^2-\epsilon_0)\binom{n-t}{k-t}^2$ for some constant $\epsilon_0 \leq \frac{\delta^2}{10(r+1)}$. Then, there exists some $A \in \binom{[n]}{t}$ with $|\mathcal{F}(A)| \geq \frac{r+\delta^2}{2t(r+\delta)^2}|\mathcal{F}|$. Moreover, when $r = 0$, we have $|\mathcal{F}(A_0)| \geq |\mathcal{F}|(1-\frac{2k^t}{C_t})$, where $A_0 \in \binom{[n]}{t}$ is the most popular t-set in $\mathcal{F}$.

- Second, we show that if $\mathcal{F}$ contains at most one full t-star, then $\mathcal{F}$ has a small t-cover. Moreover, if $\mathcal{F}$ contains no full t-star, all $F \in \mathcal{F}$ share a common element.

Lemma 2.15. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be the same family defined in Theorem 2.5. For $t \geq 2$, if $\mathcal{F}$ contains at most one full t-star, then there exists a subset $U_t \subseteq [n]$ with $|U_t| \leq t(4r+5)$ being a t-cover of $\mathcal{F}$.*

Lemma 2.16. *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be the same family defined in Theorem 2.5. If $\mathcal{F}$ contains no full t-star, then there exists an $x_0 \in [n]$ such that $x_0 \in F$, for all $F \in \mathcal{F}$.*

- Third, with the help of Lemmas 2.12 and 2.13, by induction on r, we show that for the extremal family $\mathcal{F}$, all $F \in \mathcal{F}$ share a common element. This enables us to proceed the induction on n, k and t and therefore, Theorem 2.5 shall follow from Theorem 2.4.

Since the estimation of $\mathcal{MI}(\mathcal{F})$ requires using the property of a certain convex function, we have the following theorem which is crucial during the proof of Theorem 2.5 and related lemmas.

Theorem 2.15. *Given integer $r \geq 0$, let $f : \mathbb{R}^{r+1} \to \mathbb{R}$ be the function defined as $f(x_1, \ldots, x_{r+1}) = \sum_{i=1}^{r+1} x_i^2$. Let $C = \{(x_1, \ldots, x_{r+1}) \in \mathbb{R}^{r+1} : \sum_{i=1}^{r+1} x_i = M$ and $0 \leq a \leq x_i \leq b\}$ for some fixed a, b and $ra + b \leq M \leq (r+1)b$. Then, we have*

$$f(x_1, \ldots, x_2) \leq r_0 b^2 + (r - r_0)a^2 + (M - r_0 b - (r - r_0)a)^2,$$

where r_0 is the largest integer satisfying $M - r_0 b \geq (r + 1 - r_0)a$. Moreover, the equality holds if and only $x_1 = \ldots = x_{r_0} = b$, $x_{r_0+1} = M - r_0 b - (r - r_0)a$ and $x_{r_0+2} = \ldots = x_{r+1} = a$, up to isomorphism.

Proof. Noted that $f(x_1, \ldots, x_{r+1}) = \sum_{i=1}^{r+1}(x_i - a)^2 + 2Ma - (r+1)a^2$ over C, therefore, we only need to prove the case when $a = 0$.

When $a = 0$, C is actually the polyhedral convex set in $(r+1)$-dimensional cube $[0, b]^{r+1}$ cut by the hyperplane $\sum_{i=1}^{r+1} x_i = M$. Clearly, f is a convex function. Therefore, by Corollary 32.3.4 in [165], the supremum of f relative to C is attained at one of the extreme points of C. Denote $r_1 = \lfloor \frac{M}{b} \rfloor$, since coordinates of the extreme points of C all have the form: $x_{i_1} = \ldots = x_{i_{r_1}} = b$, $x_{i_{r_1+1}} = M - r_1 b$ and the rest x_is all equal to zero. Therefore, we have $f(x_1, \ldots, x_{r+1}) \leq r_1 b^2 + (M - r_1 b)^2$ and the equality holds if and only if $(x_1, \ldots, x_{r+1})$ is an aforementioned extreme point of C. □

Armed with all these lemmas whose proofs we defer until later in this section, we now show how to deduce Theorem 2.5.

Proof of Theorem 2.5. We prove the theorem by induction on r.

Consider the base case: $r = 0$. In this case, $\mathcal{F}$ contains at most one full t-star. By Lemma 2.15, we know that $\mathcal{F}$ has a t-cover U_t with size $|U_t| \leq t(4r+5)$. W.l.o.g., assume that $U_t = [m]$. From Lemma 2.14 and Claim 9 in the proof of Lemma 2.15, we know that as one of the most popular t-sets appearing in $\mathcal{F}$, $[t]$ has the degree $|\mathcal{F}([t])| \geq (1 - \frac{2k^t}{C_t})|\mathcal{F}|$. If $\mathcal{F} \neq \mathcal{F}([t])$, we must have $[t] \subsetneq [m]$. For $i \in [m] \setminus [t]$, by Claim 11 in the proof of Lemma 2.15, we have $|\mathcal{F}(\{1, \ldots, t-1, i\})| \geq |\mathcal{F}([t])| - \frac{2k}{C_1}|\mathcal{F}|$. Thus,

$$\begin{aligned}
|\mathcal{F}| &\geq |\mathcal{F}([t])| + |\mathcal{F}\{1, \ldots, t-1, i\})| - |\mathcal{F}(\{1, \ldots, t, i\})| \\
&\geq (2 - \frac{4k^t}{C_t} - \frac{2k}{C_1})|\mathcal{F}| - \binom{n-(t+1)}{k-(t+1)} > |\mathcal{F}|,
\end{aligned}$$

a contradiction. Therefore, $\mathcal{F} = \mathcal{F}([t]) \subseteq \mathcal{L}^{(1)}_{n,k,t}$.

Now, let r_0 be a non-negative integer. Assume that for every $r \leq r_0$, $\mathcal{F} \subseteq \binom{[n]}{k}$ with size $\sum_{i=t}^{t+r-1} \binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t}$ satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$ is isomorphic to some $\mathcal{F}_0$ with the structure $\mathcal{L}^{(r)}_{n,k,t} \subseteq \mathcal{F}_0 \subseteq \mathcal{L}^{(r+1)}_{n,k,t}$. We shall prove that this also holds when $r = r_0 + 1$ by induction on n, k and t.

Assume that $\mathcal{F}$ can be covered by a single element $x_0 \in [n]$, i.e., there exists an $x_0 \in [n]$ such that $\mathcal{F} = \mathcal{F}(x_0)$. Then, by identity (2.3), the optimality of $\mathcal{F}$ is guaranteed by the new family

$$\partial_{k-1}(\mathcal{F}(\overline{x_0})) = \{F \setminus \{x_0\} : F \in \mathcal{F}\} \subseteq \binom{[n] \setminus \{x_0\}}{k-1}$$

with the same size as $\mathcal{F}$. Noted that

$$\begin{aligned}
&\sum_{i=t}^{t+r-1} \binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t} = \\
&\sum_{i=t-1}^{(t-1)+r-1} \binom{(n-1)-i}{(k-1)-(t-1)} + \delta\binom{(n-1)-(r+t-1)}{(k-1)-(t-1)}.
\end{aligned}$$

Thus, the result follows from the induction hypothesis for the case $n-1$, $k-1$, $r = r_0 + 1$ and $t-1$. In view of this, to complete the proof, we only need to show that all $F \in \mathcal{F}$ share one common element. If $\mathcal{F}$ contains no full t-star, this result follows from Lemma 2.16. Therefore, the case left is when $\mathcal{F}$ contains at least one full t-star. For the induction process, we can assume that $\mathcal{F} \subseteq \binom{[n]}{k}$ with size $\sum_{i=t}^{t+r-1}\binom{n-i}{k-t} + \delta\binom{n-(r+t)}{k-t} > \binom{n-t}{k-t}$.

Suppose $\mathcal{F}$ contains p full t-star $\mathcal{Y}_1, \ldots, \mathcal{Y}_p$ for some integer $1 \le p \le r$. Let $\mathcal{F}_1 = \cup_{i=1}^{p}\mathcal{Y}_i$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Denote $\mathcal{G}_0$ as the optimal subfamily in $\binom{[n]}{k}$ of size $\sum_{i=t}^{t+r-p-1}\binom{n-i}{k-t} + \delta'\binom{n-(t+r-p)}{k-t}$ with respect to $\mathcal{I}(\mathcal{G}_0)$, where $\frac{r\delta}{r+1} \le \delta' \le \delta$. Since $p \ge 1$, by induction on r, we know that $\mathcal{L}_{n,k,t}^{(r-p)} \subseteq \mathcal{G}_0 \subseteq \mathcal{L}_{n,k,t}^{(r-p+1)}$. Thus,

$$\mathcal{I}(\mathcal{F}_0) \le \mathcal{I}(\mathcal{G}_0) \le (t-1)|\mathcal{G}_0|^2 + (r - p + \delta^2 + \frac{1}{C_t})\binom{n-t}{k-t}^2,$$

for any $\mathcal{F}_0 \subseteq \mathcal{F}$ with size $|\mathcal{F}_0| = |\mathcal{G}_0|$. Therefore, by Lemma 2.12,

- for all $i \neq j \in [p]$, $|Y_i \cap Y_j| = t-1$;
- for at least $(1 - \frac{2r^2kt}{C_t})|\mathcal{F}_2|$ k-sets $F \in \mathcal{F}_2$, $|F \cap Y_i| = t-1$ for each $1 \le i \le p$;
- $\mathcal{I}(\mathcal{F}_2) \ge (t-1)|\mathcal{F}_2|^2 + (r - p + \delta^2 - \frac{4kr^2p^2}{C_t})\binom{n-t}{k-t}^2$.

Furthermore, by Lemma 2.14, the most popular t-set A appearing in $\mathcal{F}_2$ has degree $|\mathcal{F}_2(A)| \ge \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2|$. Denote $\mathcal{F}_2' = \{F \in \mathcal{F}_2 : |F \cap Y_i| = t-1 \text{ for each } 1 \le i \le p\}$ and $\mathcal{F}_3 = \mathcal{F}_2 \setminus \mathcal{F}_2'$, we have $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2' \sqcup \mathcal{F}_3$. In the following, we shall determine all the possible structures of $\mathcal{F}_1$ and $\mathcal{F}_2$ through discussions of the value of p and as a consequence, we will have $\mathcal{F}_3 = \emptyset$.

To guarantee that $|Y_i \cap Y_j| = t-1$ for all $i \neq j \in [p]$, there are only two possible cases:

- The first case: $p \le t+1$, up to isomorphism, $Y_i \in \binom{[t+1]}{t}$ for all $i \in [p]$.

When $p \le 2$, structures of Y_is are the same as the second case, which will be discussed later.

When $p \geq 3$, since $|F \cap Y_i| = t-1$ for all $F \in \mathcal{F}_2'$ and $i \in [p]$, we know that for each $F \in \mathcal{F}_2'$, $|F \cap [t+1]| = t$. Therefore, assume that $Y_i = [t+1] \setminus \{i\}$ and $\mathcal{F}_2' = \cup_{i=p+1}^{t+1} \mathcal{H}_i$, where $\mathcal{H}_i$ is the t-star in $\mathcal{F}_2$ with core $[t+1] \setminus \{i\}$. Since $|\mathcal{F}_3| \leq \frac{2r^2kt}{C_t}|\mathcal{F}_2|$, w.l.o.g., we can assume $A = [t+1] \setminus \{p+1\}$ as the most popular t-set in $\mathcal{F}_2$. Then, $|\mathcal{F}_2(A)| = |\mathcal{H}_{p+1}| \geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2|$. Clearly, $A \in \binom{[t+1]}{t}$. Denote $\mathcal{Z}_0 = \{j \in [p+1, t+1] : |\mathcal{H}_j| \geq \frac{|\mathcal{F}_2|}{C_1}\}$, we claim that $\mathcal{F}_2' = \cup_{j \in \mathcal{Z}_0} \mathcal{H}_j$. Otherwise, assume that there exists a $G_0 \in \mathcal{H}_{j_0}$ for some $j_0 \in [p+1, t+1] \setminus \mathcal{Z}_0$. Since $\mathcal{F}_2$ contains no full t-star, by replacing G_0 with some F containing A, we have

$$\begin{aligned}\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(G_0, \mathcal{F}) =& \mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(G_0, \mathcal{F}_1) + \mathcal{I}(F, \tilde{\mathcal{F}}_2') - \mathcal{I}(G_0, \mathcal{F}_2') \\ &+ \mathcal{I}(F, \mathcal{F}_3) - \mathcal{I}(G_0, \mathcal{F}_3),\end{aligned}$$

where $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \mathcal{F}_3$ and $\tilde{\mathcal{F}}_2' = \mathcal{F}_2' \setminus \{G_0\} \sqcup \{F\}$ is the new "$\mathcal{F}_2'$" after shifting. Noted that for every $x \in [p]$, $\mathcal{F}_1(x) = |\mathcal{F}_1| - |\mathcal{Y}_x| + \binom{n-(t+1)}{k-(t+1)}$; for every $x \in [p+1, t+1]$, $|\mathcal{F}_1(x)| = |\mathcal{F}_1|$; and for every $x \in [t+2, n]$, $|\mathcal{F}_1(x)| = \binom{n-(t+2)}{k-(t+2)} + p\binom{n-(t+2)}{k-(t+1)}$. Therefore, $\mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(G_0, \mathcal{F}_1) = 0$ and

$$\begin{aligned}\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(G_0, \mathcal{F}) = \mathcal{I}(F, \tilde{\mathcal{F}}_2) - \mathcal{I}(G_0, \mathcal{F}_2) &\geq \sum_{x \in F} |\tilde{\mathcal{F}}_2'(x)| - \sum_{x \in G_0} (|\mathcal{F}_2'(x)| + |\mathcal{F}_3(x)|) \\ &\geq (|\mathcal{F}_2'| - |\mathcal{H}(j_0)|) - (|\mathcal{F}_2'| - |\mathcal{H}(p+1)|) - \\ &\quad \sum_{x \in G_0 \setminus [t+1]} |\mathcal{F}_2'(x)| - k|\mathcal{F}_3(x)| \\ &\geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - \frac{k-t}{C_1}|\mathcal{F}_2| - \frac{2r^2k^2t}{C_t}|\mathcal{F}_2| > 0,\end{aligned}$$

a contradiction, where the third inequality follows from $|\mathcal{H}_{j_0}| \leq \frac{|\mathcal{F}_2|}{C_1}$, $|\mathcal{H}_{p+1}| \geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2|$, $|\mathcal{F}_2'(x)| \leq (t+1)\binom{n-(t+1)}{k-(t+1)} \leq \frac{|\mathcal{F}_2|}{C_1}$ for each $x \in [n] \setminus [t+1]$ and $|\mathcal{F}_3(x)| \leq \frac{2r^2kt}{C_t}|\mathcal{F}_2|$.

Moreover, if $\mathcal{F}_3 \neq \emptyset$, let $G_1 \in \mathcal{F}_3$. Again, we can replace G_1 with some F containing A. Denote $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \tilde{\mathcal{F}}_3$ as the new family. Since $|G_1 \cap Y_{i_0}| \leq t-2$ for some $i_0 \in [p]$, thus $|G_1 \cap [t+1]| \leq t-1$. When $p+1 \notin G_1$, we have $(G_1 \cap [t+1]) \subsetneq A$. Assume $x_0 \in A \setminus G_1$, since $\mathcal{F}_1 \cup \mathcal{F}_2' = \cup_{i=1}^{t+1} \mathcal{H}_i$ ($\mathcal{H}_i = \mathcal{Y}_i$ for $i \in [p]$), we have

$$\sum_{x \in F} |\tilde{\mathcal{F}}(x)| - \sum_{x \in G_1} |\mathcal{F}(x)| \geq |\mathcal{F}_2(x_0)| - \sum_{x \in G_1 \setminus [t+1]} |\mathcal{F}(x)|$$

$$\geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - k(t+1)\binom{n-(t+1)}{k-(t+1)} - k|\mathcal{F}_3|$$
$$\geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - \frac{3r^2k^2t}{C_t}|\mathcal{F}_2| > 0.$$

When $p+1 \in G_1$, we have $|G_1 \cap A| \leq t-2$. Assume $x_1, x_2 \in A \setminus G_1$, we have

$$\sum_{x\in F}|\tilde{\mathcal{F}}(x)| - \sum_{x\in G_1}|\mathcal{F}(x)| \geq |\mathcal{F}(x_1)| + |\mathcal{F}(x_2)| - |\mathcal{F}(p+1)| - \sum_{x\in G_1\setminus[t+1]}|\mathcal{F}(x)|$$
$$\geq (|\mathcal{F}| - |\mathcal{H}_{x_1}| - |\mathcal{F}_3|) + (|\mathcal{F}| - |\mathcal{H}_{x_2}| - |\mathcal{F}_3|) -$$
$$(|\mathcal{F}| - |\mathcal{H}_{p+1}|) - \frac{3r^2k^2t}{C_t}|\mathcal{F}_2|$$
$$\geq (|\mathcal{F}| - |\mathcal{H}_{x_1}| - |\mathcal{H}_{x_2}|) + |\mathcal{H}_{p+1}| - \frac{5r^2k^2t}{C_t}|\mathcal{F}_2|.$$

Since $|\mathcal{F}| - |\mathcal{H}_{x_1}| - |\mathcal{H}_{x_2}| \geq -|\mathcal{H}_{x_1} \cap \mathcal{H}_{x_2}| \geq -\binom{n-(t+1)}{k-(t+1)}$ and $|\mathcal{H}_{p+1}| = |\mathcal{F}_2(A)|$, thus the above inequality is lower bounded by $\frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - \frac{6r^2k^2t}{C_t}|\mathcal{F}_2| > 0$. Both cases contradict the optimality of $\mathcal{F}$. Therefore, $\mathcal{F}_3 = \emptyset$ and $\mathcal{F} = \cup_{i\in\mathcal{Z}_0\cup[p]}\mathcal{H}_i$, where for $i \in [p]$, $\mathcal{H}_i = \mathcal{Y}_i$ is the full t-star with core $[t+1] \setminus \{i\}$.

- The second case: all Y_is share $t-1$ elements in common.

The second case is much more complicated. W.l.o.g., assume $Y_i = \{1, 2, \ldots, t-1, t-1+i\}$ for $1 \leq i \leq p$. To guarantee that $|F \cap Y_i| = t-1$ for all $F \in \mathcal{F}_2'$ and $i \in [p]$, we have the following claim:

Claim 14. Either $p \leq 2$ and $\mathcal{F}_2' = \cup_{j=1}^{t+1-p}\mathcal{H}_j$, or $\mathcal{F}_2' = \cup_{i=t+p}^{l}\mathcal{G}_i$ for some $l \in [t+p, n]$, where $\mathcal{H}_j$ is a t-star with core $[t+1] \setminus \{j\}$ and $\mathcal{G}_i$ is a t-star with core $\{1, \ldots, t-1, i\}$.

Proof. • Case I: $p \geq 3$.

Assume that there exists an $F_0 \in \mathcal{F}_2'$ such that $|F_0 \cap [t-1]| \leq t-2$. Since $|F_0 \cap Y_i| = t-1$, we have $[t, t+p-1] \subseteq F_0$ and $|F_0 \cap [t-1]| = t-2$. Thus, such F_0 contains at least $t+1$ fixed elements. By the choice of δ, this indicates that

$$|\{F_0 \in \mathcal{F}_2' : |F_0 \cap [t-1]| \leq t-2\}| \leq p(t-1)\binom{n-(t+1)}{k-(t+1)} < \frac{|\mathcal{F}_2'|}{C_t}.$$

Noted that $|\mathcal{F}_3| \leq \frac{2r^2kt}{C_t}$, therefore, at least $|\mathcal{F}_2|(1-\frac{3r^2kt}{C_t})$ k-sets in $\mathcal{F}_2$ contain $[t-1]$.

W.l.o.g., assume $\mathcal{F}_2([t-1]) = \cup_{j=t+p}^{l} \mathcal{G}_j$. Since the most popular t-set A in $\mathcal{F}_2$ satisfies $|\mathcal{F}_2(A)| \geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2|$, thus $[t-1] \subseteq A$. Assume that $A = \{1, 2, \ldots, t-1, t+p\}$ and denote $\mathcal{Z}_1 = \{j \in [t+p, l] : |\mathcal{G}_j| \geq \frac{|\mathcal{F}_2|}{C_1}\}$. Since

$$|\mathcal{F}_2([t-1])| \geq \sum_{j \in \mathcal{Z}_1} |\mathcal{G}_j| - \sum_{j_1 \neq j_2 \in \mathcal{Z}_1} |\mathcal{G}_{j_1} \cap \mathcal{G}_{j_2}|,$$

we have $|\mathcal{Z}_1| \leq 2C_1$. By the optimality of $\mathcal{F}$, we claim that $\mathcal{F}_2([t-1]) = \cup_{j \in \mathcal{Z}_1} \mathcal{G}_j$. Otherwise, assume that there exists a $G_0 \in \mathcal{G}_{j_0}$ for some $j_0 \in [t+p, l] \setminus \mathcal{Z}_1$. Since $\mathcal{F}_2$ contains no full t-star, by replacing G_0 with some F containing A, we have

$$\begin{aligned}\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(G_0, \mathcal{F}) = \mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(G_0, \mathcal{F}_1) + \mathcal{I}(F, \tilde{\mathcal{F}}_2') - \mathcal{I}(G_0, \mathcal{F}_2') + \\ \mathcal{I}(F, \mathcal{F}_3) - \mathcal{I}(G_0, \mathcal{F}_3),\end{aligned}$$

where $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \mathcal{F}_3$ and $\tilde{\mathcal{F}}_2' = \mathcal{F}_2' \setminus \{G_0\} \sqcup \{F\}$. The structure of $\mathcal{F}_1$ indicates that for every $x \in [t-1]$, $|\mathcal{F}_1(x)| = |\mathcal{F}_1|$; for every $x \in [t, t+p-1]$, $|\mathcal{F}_1(x)| = \binom{n-t}{k-t}$; and for every $x \in [t+p, n]$, $|\mathcal{F}_1(x)| = \sum_{i=t+1}^{t+p} \binom{n-i}{k-(t+1)}$. Therefore, $\mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(G_0, \mathcal{F}_1) = 0$ and

$$\begin{aligned}\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(G_0, \mathcal{F}) &\geq \sum_{x \in F} |\tilde{\mathcal{F}}_2'(x)| - \sum_{x \in G_0} (|\mathcal{F}_2'(x)| + |\mathcal{F}_3(x)|) \\ &\geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - \sum_{x \in G_0 \setminus [t-1]} |\mathcal{F}_2'(x)| - k|\mathcal{F}_3| \\ &\geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2| - \frac{k-t+1}{C_1}|\mathcal{F}_2'| - \frac{2r^2k^2t}{C_t}|\mathcal{F}_2| > 0,\end{aligned}$$

a contradiction.

Recall that $|F_0 \cap [t-1]| = t-2$, again, we can replace F_0 with some F containing A and denote the new family as $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \mathcal{F}_3$. The above argument actually proved that $\mathcal{F}_2([t-1])$ has a small t-cover, since $|\mathcal{G}_i \cap \mathcal{G}_j| \leq \binom{n-(t+1)}{k-(t+1)}$ for $i \neq j \in \mathcal{Z}_1$, this enables us to control the value of $\mathcal{I}(F_0, \mathcal{F}_2)$. Thus, we have

$$\begin{aligned}\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(F_0, \mathcal{F}) = \; &\mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(F_0, \mathcal{F}_1) + \mathcal{I}(F, \tilde{\mathcal{F}}_2') - \mathcal{I}(F_0, \mathcal{F}_2') + \\ &\mathcal{I}(F, \mathcal{F}_3) - \mathcal{I}(F_0, \mathcal{F}_3) \\ \geq \; &|\mathcal{F}_1| - p\binom{n-t}{k-t} + (1 + \frac{r-p+\delta^2}{2t(r-p+\delta)^2} - \frac{3r^2kt+p}{C_t})|\mathcal{F}_2| -\end{aligned}$$

$$\begin{aligned}
&\sum_{x\in F_0\setminus[t+p-1]}|\mathcal{F}_2'(x)|-k|\mathcal{F}_3|\\
&\geq (1+\frac{r-p+\delta^2}{2t(r-p+\delta)^2}-\frac{3r^2kt+p+p^2}{C_t})|\mathcal{F}_2|-\sum_{x\in F_0\cap\mathcal{Z}_1}|\mathcal{F}_2'(x)|-\\
&\frac{k}{C_1}|\mathcal{F}_2|-\frac{2r^2k^2t}{C_t}|\mathcal{F}_2|\\
&\geq (\frac{r-p+\delta^2}{2t(r-p+\delta)^2}-\frac{7r^2k^2t}{C_t}-\frac{2k}{C_1})|\mathcal{F}_2|>0,
\end{aligned}$$

where the third inequality follows from $|\mathcal{F}_2'|\geq\sum_{x\in\mathcal{Z}_1}|\mathcal{F}_2'(x)|-\sum_{x\neq y\in\mathcal{Z}_1}|\mathcal{F}_2'(x)\cap\mathcal{F}_2'(y)|$ and $|\mathcal{F}_2'(x)\cap\mathcal{F}_2'(y)|\leq|\mathcal{G}_x\cap\mathcal{G}_y|+\frac{3r^2kt}{C_t}|\mathcal{F}_2|$. This contradicts the optimality of $\mathcal{F}$ and thus disproves the existence of F_0. Therefore, $\mathcal{F}_2'=\mathcal{F}_2([t-1])=\cup_{j=t+p}^{l}\mathcal{G}_j$.

- Case II: $p=1$.

Assume $Y_1=[t]$. By Lemma 2.15, $\mathcal{F}$ has a t-cover U_t of size $|U_t|\leq t(4r+5)$. According to the proof of Lemma 2.15, $U_t=\bigcup_{A\in\mathcal{X}_t}A$, where $\mathcal{X}_t=\{A\in\binom{[n]}{t}:|\mathcal{F}(A)|\geq\frac{|\mathcal{F}|}{C_t}\}$, we have $[t]\subseteq U_t$. Therefore, denote $\mathcal{A}=\{A\in\binom{U_t}{t},|A\cap[t]|=t-1\}$, we have $|\mathcal{A}|\leq t^2(4r+5)$ and $\mathcal{F}_2'=\cup_{A\in\mathcal{A}}\mathcal{F}(A)$. First, for each $A\in\mathcal{A}$,

$$\mathcal{I}(\mathcal{F}(A))=\sum_{x\in[n]}|\mathcal{F}(A\cup\{x\})|^2\leq t|\mathcal{F}(A)|^2+(n-t)\binom{n-(t+1)}{k-(t+1)}^2;$$

and for $A_1\neq A_2\in\mathcal{A}$,

$$\begin{aligned}
\mathcal{I}(\mathcal{F}(A_1),\mathcal{F}(A_2))&=\sum_{F_1\in\mathcal{F}(A_1)}\sum_{F_2\in\mathcal{F}(A_2)}|F_1\cap F_2|=\sum_{F_1\in\mathcal{F}(A_1)}\sum_{x\in F_1}|\mathcal{F}(A_2\cup\{x\})|\\
&\leq|A_1\cap A_2||\mathcal{F}(A_1)||\mathcal{F}(A_2)|+t(|\mathcal{F}(A_1)|+|\mathcal{F}(A_2)|)\binom{n-(t+1)}{k-(t+1)}\\
&+2k\binom{n-(t+1)}{k-(t+1)}^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\mathcal{I}(\mathcal{F}_2')&\leq\sum_{A\in\mathcal{A}}\mathcal{I}(\mathcal{F}(A))+2\sum_{A_1\neq A_2\in\mathcal{A}}\mathcal{I}(\mathcal{F}(A_1),\mathcal{F}(A_2))\\
&\leq t\sum_{A\in\mathcal{A}}|\mathcal{F}(A)|^2+2(\sum_{A_1\neq A_2\in\mathcal{A}}|A_1\cap A_2||\mathcal{F}(A_1)||\mathcal{F}(A_2)|)+\frac{1}{C_t}\binom{n-t}{k-t}^2.
\end{aligned}\tag{2.39}$$

Moreover, since $\sum_{A\in\mathcal{A}}|\mathcal{F}(A)| < |\mathcal{F}_2| + \frac{\binom{n-t}{k-t}}{3C_t} < (r-p+\delta+\frac{1}{3C_t})\binom{n-t}{k-t}$ and $|\mathcal{F}(A)| \leq \binom{n-t}{k-t}$ for each $A \in \mathcal{A}$, by Theorem 2.15, we have

$$\sum_{A\in\mathcal{A}}|\mathcal{F}(A)|^2 < (r-p+\delta^2+\frac{1}{C_t})\binom{n-t}{k-t}^2. \tag{2.40}$$

On the other hand, $|\mathcal{F}_3| \leq \frac{2r^2kt}{C_t}|\mathcal{F}_2|$ leads to

$$\mathcal{I}(\mathcal{F}_2') \geq \mathcal{I}(\mathcal{F}_2) - \frac{5r^4k^2t}{C_t}\binom{n-t}{k-t}^2 \geq (t-1)|\mathcal{F}_2|^2 + (r-p+\delta^2-\frac{9r^4k^2t}{C_t})\binom{n-t}{k-t}^2. \tag{2.41}$$

Thus, combining (2.39), (2.40) and (2.41) together, we have

$$(t-1)|\mathcal{F}_2|^2 - \frac{10r^4k^2t}{C_t}\binom{n-t}{k-t}^2 \leq (t-1)(\sum_{A\in\mathcal{A}}|\mathcal{F}(A)|)^2 + 2\sum_{A_1\neq A_2\in\mathcal{A}}(|A_1\cap A_2|-(t-1))|\mathcal{F}(A_1)||\mathcal{F}(A_2)|.$$

Noted that $(t-2) \leq |A_1 \cap A_2| \leq (t-1)$ and $|\mathcal{F}(A_1)\cap\mathcal{F}(A_2)| \leq \binom{n-(t+1)}{k-(t+1)}$ for $A_1 \neq A_2 \in \mathcal{A}$, the above inequality actually shows

$$\sum_{A_1\in\mathcal{A}}\sum_{\substack{A_2\in\mathcal{A},\\ |A_1\cap A_2|=t-2}}|\mathcal{F}(A_1)||\mathcal{F}(A_2)| \leq \frac{5r^4k^2t}{C_t}\binom{n-t}{k-t}^2. \tag{2.42}$$

Denote $\mathcal{A}_1 = \{A\in\mathcal{A} : |\mathcal{F}(A)| \geq \frac{\binom{n-t}{k-t}}{C_1}\}$, since $|\mathcal{A}| \leq t^2(4r+5) \leq 9rt^2$, we have $|\bigcup_{A\in\mathcal{A}_1}\mathcal{F}(A)| \geq |\mathcal{F}_2'| - \frac{t^2(4r+5)}{C_1}\binom{n-t}{k-t}$. For each $A \in \mathcal{A}_1$, (2.42) shows that

$$|\bigcup_{\substack{B\in\mathcal{A},\\ |B\cap A|<t-1}}\mathcal{F}(A)| \leq \frac{5r^4k^2tC_1}{C_t}\binom{n-t}{k-t}.$$

Therefore, we can remove at most $\frac{45r^5k^2t^3C_1}{C_t}\binom{n-t}{k-t}$ k-sets from $\bigcup_{A\in\mathcal{A}_1}\mathcal{F}(A)$ and obtain a subfamily $\mathcal{A}' \subseteq \mathcal{A}_1$ such that $|A_1 \cap A_2| = t-1$ for all $A_1 \neq A_2 \in \mathcal{A}'$. By the choice of C_1 and C_t, $|\cup_{A\in\mathcal{A}'}\mathcal{F}(A)| \geq (1-\frac{10r^5k^2t^3}{C_1(r-1+\delta)})|\mathcal{F}_2|$. Therefore, similar to the structures of Y_is, either $\mathcal{A}' \subseteq \binom{[t+1]}{t}$ or all $A \in \mathcal{A}'$ and $[t]$ share $t-1$ common elements. Thus, either $\cup_{A\in\mathcal{A}'}\mathcal{F}(A) = \cup_{j=1}^{t}\mathcal{H}_j$ or $\cup_{A\in\mathcal{A}'}\mathcal{F}(A) = \cup_{i=t+1}^{l}\mathcal{G}_i$. For $B \in \mathcal{A}\setminus\mathcal{A}'$, if $|B\cap A| = t-1$ for all $A \in \mathcal{A}'$, then either $B \in \binom{[t+1]}{t}$ or $[t-1] \subseteq B$. Therefore,

$\cup_{A\in\mathcal{A}'\cup\{B\}}\mathcal{F}(A)$ has the same structure as $\cup_{A\in\mathcal{A}'}\mathcal{F}(A)$. W.l.o.g., we can assume that for each $B\in\mathcal{A}\setminus\mathcal{A}'$ there exists some $A\in\mathcal{A}'$ such that $|A\cap B|<t-1$.

When $\cup_{A\in\mathcal{A}'}\mathcal{F}(A)=\cup_{j=1}^{t}\mathcal{H}_j$, if $\mathcal{A}'\neq\mathcal{A}$, let $G_0\in\mathcal{F}(A_0)$ for some $A_0\in\mathcal{A}\setminus\mathcal{A}'$, we have $|A_0\cap[t]|=|A_0\cap[t+1]|=t-1$. W.l.o.g., assume the most popular t-set in $\mathcal{F}_2$ is $[t+1]\setminus\{t\}$ and $A_0\cap[t+1]=[t]\setminus\{i_0\}$. Since $\mathcal{F}_2$ contains no full t-star, we can replace G_0 with some F containing $[t+1]\setminus\{t\}$. Denote the new family as $\tilde{\mathcal{F}}$, we have

$$\begin{aligned}\sum_{x\in F}|\tilde{\mathcal{F}}(x)|-\sum_{x\in G_0}|\mathcal{F}(x)| &\geq |\mathcal{F}(t+1)|+|\mathcal{F}(i_0)|-|\mathcal{F}(t)|-\sum_{x\in G_0\setminus[t+1]}|\mathcal{F}(x)|\\ &\geq |\mathcal{F}_2(t+1)|-\sum_{x\in G_0\setminus[t+1]}|\mathcal{F}_2(x)|\\ &\geq \frac{r-1+\delta^2}{2t(r-1+\delta)^2}|\mathcal{F}_2|-\frac{k(t+1)}{C_t}|\mathcal{F}_2|-\frac{10r^5k^3t^3}{C_1(r-1+\delta)}|\mathcal{F}_2|\\ &\quad -k|\mathcal{F}_3|\\ &\geq \frac{r-1+\delta^2}{2t(r-1+\delta)^2}|\mathcal{F}_2|-\frac{15r^5k^3t^3}{C_1(r-1+\delta)}|\mathcal{F}_2|>0,\end{aligned}$$

a contradiction.

When $\cup_{A\in\mathcal{A}'}\mathcal{F}(A)=\cup_{i=t+1}^{l}\mathcal{G}_i$, if $\mathcal{A}'\neq\mathcal{A}$, let $G_0\in\mathcal{F}(A_0)$ for some $A_0\in\mathcal{A}\setminus\mathcal{A}'$, we have $|A_0\cap[t]|=t-1$ and $A_0\cap[t]\neq[t-1]$. With a similar shifting argument as above, we can also reach a contradiction.

Therefore, we have $\mathcal{A}=\mathcal{A}'$ and this indicates that either $\mathcal{F}_2'=\cup_{j=1}^{t}\mathcal{H}_j$, or $\mathcal{F}_2'=\cup_{i=t+1}^{l}\mathcal{G}_i$.

- Case III: $p=2$.

Assume that $Y_1=[t+1]\setminus\{t+1\}$ and $Y_2=[t+1]\setminus\{t\}$, since $|F\cap Y_i|=t-1$ for all $F\in\mathcal{F}_2'$ and $i\in[2]$, $\mathcal{F}_2'$ must have the following hybrid structure:

$$\mathcal{F}_2'=\mathcal{F}_{21}\sqcup\mathcal{F}_{22},$$

where $\mathcal{F}_{21}=\cup_{i=t+2}^{l}\mathcal{G}_i$ denotes the part of k-sets containing $[t-1]$, $\mathcal{F}_{22}=\cup_{j=1}^{t-1}\mathcal{H}_j$ denotes the part that contains a t-set from $[t+1]$. To guarantee the optimality of

$\mathcal{F}$, we claim that either $\mathcal{F}_{21} = \emptyset$, or $\mathcal{F}_{22} = \emptyset$. Our discussion is divided into the following three parts.

- When $|\mathcal{F}_{22}| \leq \frac{|\mathcal{F}_2|}{C_1}$, we have $|\mathcal{F}_{21}| \geq (1 - \frac{1}{C_1})|\mathcal{F}_2|$. Similar to the case when $p \geq 3$, using shifting arguments, one can prove that $\mathcal{F}_2([t-1]) = \mathcal{F}_{21}$ has a small t-cover and then derive $\mathcal{F}_2' = \mathcal{F}_{21}$ by contradiction.

- When $|\mathcal{F}_{22}| \geq (1 - \frac{1}{C_1})|\mathcal{F}_2|$, since the most popular t-set A in $\mathcal{F}_2$ satisfies $|\mathcal{F}_2(A)| \geq \frac{r-p+\delta^2}{3t(r-p+\delta)^2}|\mathcal{F}_2| > \frac{|\mathcal{F}_2|}{C_1}$, w.l.o.g., assume that $A = [t+1] \setminus \{1\}$. If there exists a $G_0 \in \mathcal{F}_{21}$, since $\mathcal{F}_2$ contains no full t-star, we can replace G_0 with some F containing A. Denote the resulting new family as $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \mathcal{F}_3$, then

$$\begin{aligned}
\mathcal{I}(F, \tilde{\mathcal{F}}) - \mathcal{I}(G_0, \mathcal{F}) &= \mathcal{I}(F, \mathcal{F}_1) - \mathcal{I}(G_0, \mathcal{F}_1) + \mathcal{I}(F, \tilde{\mathcal{F}_2}') - \mathcal{I}(G_0, \mathcal{F}_2) + \mathcal{I}(F, \mathcal{F}_3) \\
&\quad - \mathcal{I}(G_0, \mathcal{F}_3) \\
&\geq 2\binom{n-t}{k-t} - |\mathcal{F}_1| + \sum_{i=2}^{t-1}(|\mathcal{F}_2| - |\mathcal{H}_j|) + 2|\mathcal{F}_{22}| \\
&\quad - \sum_{i=1}^{t-1}(|\mathcal{F}_2| - |\mathcal{H}_j|) - \frac{3r^2kt}{C_t}|\mathcal{F}_2| \\
&\geq |\mathcal{H}_1| + (1 - \frac{2+kt}{C_1})|\mathcal{F}_2| > 0,
\end{aligned}$$

a contradiction. Therefore, $\mathcal{F}_2' = \mathcal{F}_{22}$.

- When $\frac{|\mathcal{F}_2|}{C_1} < |\mathcal{F}_{22}| < (1 - \frac{1}{C_1})|\mathcal{F}_2|$, assume that $|\mathcal{F}_{22}| = (l_2 + \delta_2)\binom{n-t}{k-t}$, where l_2 is a non-negative integer and $\delta_2 \in [0, 1)$. By the structure of $\mathcal{F}_{22}$, we have

$$\begin{aligned}
\mathcal{I}(\mathcal{F}_{22}) &\leq \sum_{i=1}^{t-1}(|\mathcal{F}_{22}| - |\mathcal{H}_i|)^2 + 2|\mathcal{F}_{22}|^2 + (n-t-1)(t-1)^2\binom{n-(t+1)}{k-(t+1)}^2 \\
&\leq (t-1)|\mathcal{F}_{22}|^2 + \sum_{i=1}^{t-1}|\mathcal{H}_i|^2 + \frac{t^2}{2C_t}\binom{n-t}{k-t}^2 \\
&\leq (t-1)|\mathcal{F}_{22}|^2 + (l_2 + \delta_2^2 + \frac{t^2}{C_t})\binom{n-t}{k-t}^2,
\end{aligned}$$

where the last inequality follows from Theorem 2.15. Combining with the lower bound of $\mathcal{I}(\mathcal{F}_2')$ from (2.41), we have

$$\mathcal{I}(\mathcal{F}_{21})+2\mathcal{I}(\mathcal{F}_{21},\mathcal{F}_{22})=\mathcal{I}(\mathcal{F}_2')-\mathcal{I}(\mathcal{F}_{22})\geq$$
$$(t-1)|\mathcal{F}_{21}|^2+2(t-1)|\mathcal{F}_{21}||\mathcal{F}_{22}|+(r-2+\delta^2-l_2-\delta_2^2-\frac{10r^4k^2t^2}{C_t})\binom{n-t}{k-t}^2.$$

On the other hand, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_{21},\mathcal{F}_{22})&=\sum_{F_1\in\mathcal{F}_{21},F_2\in\mathcal{F}_{22}}|F_1\cap F_2|\\&\leq(t-2)|\mathcal{F}_{21}||\mathcal{F}_{22}|+\sum_{F_1\in\mathcal{F}_{21},F_2\in\mathcal{F}_{22}}|(F_1\cap F_2)\setminus[t+1]|\\&\leq(t-2)|\mathcal{F}_{21}||\mathcal{F}_{22}|+(k-t+1)(t-1)\binom{n-(t+1)}{k-(t+1)}|\mathcal{F}_{21}|\\&\leq(t-2+\frac{kt}{C_t(l_2+\delta_2)})|\mathcal{F}_{21}||\mathcal{F}_{22}|.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{I}(\mathcal{F}_{21})\geq(t-1)|\mathcal{F}_{21}|^2+(2-\frac{2kt}{C_t(l_2+\delta_2)})|\mathcal{F}_{21}||\mathcal{F}_{22}|\\+(r-2+\delta^2-l_2-\delta_2^2-\frac{10r^4k^2t^2}{C_t})\binom{n-t}{k-t}^2.\end{aligned}\tag{2.43}$$

Based on this lower bound, by Lemma 2.14, the most popular t-set A' in $\mathcal{F}_{21}$ has degree $|\mathcal{F}_{21}(A')|\geq\frac{|\mathcal{F}_{21}|}{3t(r+1)}\geq\frac{|\mathcal{F}_2|}{3t(r+1)C_1}$. W.l.o.g., assume that $A'=\{1,\dots,t-1,t+2\}$ and denote $\mathcal{Z}_2=\{i\in[t+2,l]:|\mathcal{G}_i|\geq\frac{|\mathcal{F}_2|}{C_t}\}$. Then, $|\mathcal{Z}_2|\leq 2C_t$. Similar to the case $p\geq3$, using shifting arguments, we can prove that $\mathcal{F}_{21}=\cup_{i\in\mathcal{Z}_2}\mathcal{G}_i$.

Based on this structure of $\mathcal{F}_{21}$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_{21})&\leq(t-1)|\mathcal{F}_{21}|^2+\sum_{i\in\mathcal{Z}_2}|\mathcal{G}_i|^2+n|\mathcal{Z}_2|\binom{n-(t+1)}{k-(t+1)}^2\\&\leq(t-1)|\mathcal{F}_{21}|^2+[(r-2+\delta^2-l_2-\delta_2^2)+\\&\quad(2\delta_2^2-2\delta_2\delta+\min\{0,2(\delta-\delta_2)\})+\frac{2}{C_t}]\binom{n-t}{k-t}^2,\end{aligned}$$

where the second inequality follows from Theorem 2.15 and the choice of n. Since $2\delta_2^2-2\delta_2\delta+\min\{0,2(\delta-\delta_2)\}\leq0$, by the choice of C_1 and C_t, the above upper bound is always strictly less than the lower bound given by (2.43), a contradiction. Therefore, when $\frac{|\mathcal{F}_2|}{C_1} < |\mathcal{F}_{22}| < (1-\frac{1}{C_1})|\mathcal{F}_2|$, $\mathcal{I}(\mathcal{F})$ can not be optimal.

Therefore, for all three cases, either $\mathcal{F}_2' = \cup_{j=1}^{t-1}\mathcal{H}_j$ or $\mathcal{F}_2' = \cup_{i=t+2}^{l}\mathcal{G}_i$. This completes the proof of the claim. $\square$

With the same proof as that for the first case, when $\mathcal{F}_2' = \cup_{i=t+p}^{l}\mathcal{G}_i$ in the second case, we can also prove that $\mathcal{F}_2'$ is consisted of large t-stars. Denote $\mathcal{Z} = \{i \in [t+p, n] : |\mathcal{G}_i| \geq \frac{|\mathcal{F}_2|}{C_1}\}$, we claim that $\mathcal{F}_2' = \cup_{i\in\mathcal{Z}}\mathcal{G}_i$. Otherwise, assume that there exists a $G_0 \in \mathcal{G}_{i_0}$ for some $i_0 \notin \mathcal{Z}$. Since the most popular t-set A in $\mathcal{F}_2$ has degree $|\mathcal{F}_2(A)| \geq \frac{r-p+\delta^2}{2t(r-p+\delta)^2}|\mathcal{F}_2|$ and $\mathcal{F}_2$ contains no full t-star, we can replace G_0 with some $F \notin \mathcal{F}$ containing A. With a similar counting argument as the proof of Claim 14 for $p \geq 3$, this process strictly increases $\mathcal{I}(\mathcal{F})$, a contradiction. Thus, $\mathcal{F}_2' = \cup_{i\in\mathcal{Z}}\mathcal{G}_i$. Moreover, noted that $|\mathcal{F}_2| \geq \sum_{i\in\mathcal{Z}}|\mathcal{G}_i| - \sum_{i\neq j\in\mathcal{Z}}|\mathcal{G}_i \cap \mathcal{G}_j|$, we have $|\mathcal{Z}| \leq 2C_1$ and $l \leq 2C_1 + t + p$.

Now, we show that $\mathcal{F}_3 = \emptyset$ for the second case.

When $p \leq 2$ and $\mathcal{F}_2' = \cup_{j=1}^{t+1-p}\mathcal{H}_j$, we have $\mathcal{F}_1 \cup \mathcal{F}_2' = \cup_{j=1}^{t+1}\mathcal{H}_j$ which is same as the structure of $\mathcal{F}_1 \cup \mathcal{F}_2'$ in the first case when $p \geq 3$. Since the proof of $\mathcal{F}_3 = \emptyset$ in the first case only depends on the structure of $\mathcal{F}_1 \cup \mathcal{F}_2'$ and is unrelated with the value of p, therefore, with the same argument we have $\mathcal{F}_3 = \emptyset$.

When $\mathcal{F}_2' = \cup_{i=t+p}^{l}\mathcal{G}_i$, we have $\mathcal{F}_1 \cup \mathcal{F}_2' = \cup_{i=t}^{l}\mathcal{G}_i$, where $\mathcal{G}_i = \mathcal{Y}_{i-t+1}$ for $t \leq i \leq t+p-1$. If $\mathcal{F}_3 \neq \emptyset$, let $G_1 \in \mathcal{F}_3$. Replace G_1 with some F containing A and denote $\tilde{\mathcal{F}} = \mathcal{F}_1 \sqcup \tilde{\mathcal{F}}_2' \sqcup \tilde{\mathcal{F}}_3$ as the new family. Noted $|G_1 \cap Y_{i_0}| \leq t-2$ for some $i_0 \in [p]$, thus either $|G_1 \cap [t-1]| \leq t-3$ and $t-1+i_0 \in G_1$, or $|G_1 \cap [t-1]| \leq t-2$ and $t-1+i_0 \notin G_1$. When $t-1+i_0 \in G_1$, let $x_1, x_2 \in [t-1] \setminus G_1$. Since $|\mathcal{F}| - |\mathcal{F}_3| \geq \sum_{i=t}^{l}|\mathcal{G}_i| - l^2\binom{n-(t+1)}{k-(t+1)}$,we have

$$\begin{aligned}\sum_{x\in F}|\tilde{\mathcal{F}}(x)| - \sum_{x\in G_1}|\mathcal{F}(x)| &\geq |\mathcal{F}(x_1)| + |\mathcal{F}(x_2)| - \sum_{x\in G_1\setminus[t-1]}|\mathcal{F}(x)| \\ &\geq 2|\mathcal{F}| - 2|\mathcal{F}_3| - \sum_{i\in G_1\cap[t,l]}|\mathcal{G}_i| - kl\binom{n-(t+1)}{k-(t+1)} - k|\mathcal{F}_3|\end{aligned}$$

$$\geq |\mathcal{F}| - (l^2 + kl)\binom{n-(t+1)}{k-(t+1)} - (k+2)|\mathcal{F}_3| > 0.$$

When $t-1+i_0 \notin G_1$, let $x_1 \in [t-1] \setminus G_1$, since $|\mathcal{G}_{t-1+i_0}| = |\mathcal{Y}_{i_0}| = \binom{n-t}{k-t}$, we have

$$\begin{aligned}\sum_{x\in F} |\tilde{\mathcal{F}}(x)| - \sum_{x\in G_1} |\mathcal{F}(x)| &\geq |\mathcal{F}(x_1)| - \sum_{x\in G_1\setminus[t-1]} |\mathcal{F}(x)| \\ &\geq |\mathcal{F}| - |\mathcal{F}_3| - \sum_{i\in G_1\cap[t,l]} |\mathcal{G}_i| - kl\binom{n-(t+1)}{k-(t+1)} - k|\mathcal{F}_3| \\ &\geq |\mathcal{G}_{t-1+i_0}| - (l^2+kl)\binom{n-(t+1)}{k-(t+1)} - (k+1)|\mathcal{F}_3| > 0.\end{aligned}$$

Both cases contradict the optimality of $\mathcal{F}$. Therefore, $\mathcal{F}_3 = \emptyset$ and $\mathcal{F} = \cup_{i=t}^{l} \mathcal{G}_i$.

Finally, we derive the basic outlines of $\mathcal{F}$: $\mathcal{F} = \cup_{i=t}^{l} \mathcal{G}_i$ or $\mathcal{F} = \cup_{j=1}^{t+1} \mathcal{H}_j$.

When $\mathcal{F} = \cup_{i=t}^{l} \mathcal{G}_i$, all $F \in \mathcal{F}$ share $t-1$ common elements. When $\mathcal{F} = \cup_{j=1}^{t+1} \mathcal{H}_j$, if there exists some $j_0 \in [t+1]$ such that $|\mathcal{H}_{j_0}| = 0$, then all $F \in \mathcal{F}$ contain j_0. If $\mathcal{H}_j \neq \emptyset$ for all $j \in [t+1]$, from the proof of $\mathcal{F}_3 = \emptyset$ in the first case, $|\mathcal{H}_j| \geq \frac{|\mathcal{F}_2|}{C_1} > \frac{\delta}{2C_1}\binom{n-t}{k-t}$. By Lemma 2.13, there exists a family $\mathcal{F}_0$ of size $|\mathcal{F}|$ such that $\mathcal{L}_{n,k,t}^{(r)} \subseteq \mathcal{F}_0 \subseteq \mathcal{L}_{n,k,t}^{(r+1)}$ and $\mathcal{I}(\mathcal{F}_0) > \mathcal{I}(\mathcal{F})$, a contradiction. Therefore, all $F \in \mathcal{F}$ always share one common element and the result follows from the induction.

This completes the proof. □

It remains to prove the lemmas. First, with the same strategy as that of Lemma 2.9, we give a proof of Lemma 2.14.

Proof of Lemma 2.14. Fix $t \geq 1$, let $C_t = 2^{2^{t-1}-1} \cdot 10^{2^{t+2}-2} \cdot (k^2 t^4 (r+1)^7)^{2^{t-1}}$ and take $\mathcal{X}_t = \{A \in \binom{[n]}{t} : |\mathcal{F}(A)| \geq \frac{|\mathcal{F}|}{C_t}\}$ as the family of moderately popular t-sets appearing in $\mathcal{F}$. First, we show that $\mathcal{X}_t$ can not be very large.

Claim 7. $|\mathcal{X}_t| < 2C_t$.

Proof. Suppose not, let $\mathcal{X}_0$ be a subfamily of $\mathcal{X}_t$ with size $2C_t$, then we have

$$\begin{aligned}|\mathcal{F}| \geq |\bigcup_{A\in\mathcal{X}_0} \mathcal{F}(A)| &\geq \sum_{A\in\mathcal{X}_0} |\mathcal{F}(A)| - \sum_{A\neq B\in\mathcal{X}_0} |\mathcal{F}(A\cup B)| \\ &\geq 2|\mathcal{F}| - \binom{|\mathcal{X}_0|}{2}\binom{n-(t+1)}{k-(t+1)}.\end{aligned} \tag{2.44}$$

Since $|\mathcal{F}| = |\mathcal{L}^{(r)}_{n,k,t}| + \delta\binom{n-(r+t)}{k-t}$, we know that

$$|\mathcal{F}| \geq \binom{r}{1}\binom{n-t}{k-t} - \binom{r}{2}\binom{n-(t+1)}{k-(t+1)} + \delta\binom{n-(r+t)}{k-t} \tag{2.45}$$
$$\geq (\frac{nr}{3k} + \delta\frac{n-(r+k)}{k-t}(1-\frac{k(r+k)}{n-t}))\binom{n-(t+1)}{k-(t+1)}.$$

Combining (2.44) and (2.45) together, we have $|\mathcal{F}| \geq (2 - \frac{6{C_t}^2k}{n(r+\delta)})|\mathcal{F}|$, which contradicts the requirement of n. Thus, the claim holds. □

Now, we complete the proof by proving the following claim.

Claim 8. There is an $A_0 \in \mathcal{X}_t$ such that $|\mathcal{F}(A_0)| \geq \frac{r+\delta^2}{2t(r+\delta)^2}|\mathcal{F}|$.

Proof. Since

$$\mathcal{I}(\mathcal{F}) \geq (t-1)|\mathcal{F}|^2 + (r+\delta^2-\epsilon_0)\binom{n-t}{k-t}^2 \tag{2.46}$$
$$\geq (t-1+\frac{9(r+\delta^2)}{10(r+\delta)^2})|\mathcal{F}|^2,$$

where the second inequality follows from that $\mathcal{L}_{n,k} = \cup_{i=t}^{t+r}\mathcal{G}_i$, where $\mathcal{G}_i$ is the full t-star with core $[t-1]\cup\{i\}$ for $t \leq i \leq t+r-1$ and $\mathcal{G}_{t+r}$ is contained in the full t-star with core $[t-1]\cup\{t+r\}$.

W.l.o.g, assume that $[t] \in \mathcal{X}_t$ is the most popular t-subset appearing in $\mathcal{F}$. Noticed that

$$\sum_{A\in\binom{[n]}{t}} |\mathcal{F}(A)|^2 = \sum_{F_1\in\mathcal{F}}\sum_{F_2\in\mathcal{F}}\binom{|F_1\cap F_2|}{t} \tag{2.47}$$

and function $\binom{x}{t} = \frac{x(x-1)\cdots(x-t+1)}{t!}$ is convex when $x \geq t-1$. According to (2.46), we know that $\frac{\mathcal{I}(\mathcal{F})}{|\mathcal{F}|^2} > t-1$. Therefore, by Jensen's inequality, we have

$$\binom{\frac{\mathcal{I}(\mathcal{F})}{|\mathcal{F}|^2}}{t}\cdot|\mathcal{F}|^2 = \binom{\frac{\sum_{F_1,F_2\in\mathcal{F}}|F_1\cap F_2|}{|\mathcal{F}|^2}}{t}\cdot|\mathcal{F}|^2$$
$$\leq \sum_{F_1,F_2\in\mathcal{F}}\binom{|F_1\cap F_2|}{t} = \sum_{A\in\binom{[n]}{t}}|\mathcal{F}(A)|^2. \tag{2.48}$$

Since $\binom{x}{t}$ is increasing in x when $x \geq t-1$, we also have

$$\binom{\frac{\mathcal{I}(\mathcal{F})}{|\mathcal{F}|^2}}{t}\cdot|\mathcal{F}|^2 \geq \binom{t-1+\frac{9(r+\delta^2)}{10(r+\delta)^2}}{t}\cdot|\mathcal{F}|^2 \geq \frac{9(r+\delta^2)}{10t(r+\delta)^2}\cdot|\mathcal{F}|^2. \tag{2.49}$$

Therefore, by combining the above inequalities together, we can obtain

$$\begin{aligned}\frac{9(r+\delta^2)}{10t(r+\delta)^2}\cdot|\mathcal{F}|^2 &\leq \sum_{A\in\binom{[n]}{t}}|\mathcal{F}(A)|^2=\sum_{A\in\mathcal{X}_t}|\mathcal{F}(A)|^2+\sum_{A\in\binom{[n]}{t}\setminus\mathcal{X}_t}|\mathcal{F}(A)|^2\\ &\leq |\mathcal{F}([t])|\cdot\sum_{A\in\mathcal{X}_t}|\mathcal{F}(A)|+\frac{|\mathcal{F}|}{C_t}\cdot\sum_{A\in\binom{[n]}{t}\setminus\mathcal{X}_t}|\mathcal{F}(A)|\\ &\leq |\mathcal{F}([t])|\cdot(|\mathcal{F}|+\binom{|\mathcal{X}_t|}{2}\binom{n-(t+1)}{k-(t+1)})+\frac{|\mathcal{F}|}{C_t}\cdot\binom{k}{t}\cdot|\mathcal{F}|\\ &\leq |\mathcal{F}([t])|\cdot|\mathcal{F}|\cdot(1+\frac{6{C_t}^2k}{n(r+\delta)})+\frac{\binom{k}{t}}{C_t}|\mathcal{F}|^2. \end{aligned}\tag{2.50}$$

This leads to $|\mathcal{F}([t])|\geq\frac{r+\delta^2}{2t(r+\delta)^2}|\mathcal{F}|$. Therefore, the claim holds. $\square$

Moreover, when $r=0$, we have $\mathcal{I}(\mathcal{F})\geq t|\mathcal{F}|^2$, which changes the RHS of (2.49) to $|\mathcal{F}|^2$. This leads to $|\mathcal{F}([t])|\geq(1-\frac{2k^t}{C_t})|\mathcal{F}|$. $\square$

Based on Lemma 2.14, we turn to the proof of Lemma 2.15. Different from the proof of Lemma 2.10, according to the definition of $\mathcal{I}(\mathcal{F})$, it seems that the optimality of $\mathcal{F}$ can only guarantee the control of $|\mathcal{F}(x)|$. This is far from enough, since what we want is the control of $|\mathcal{F}(A)|$ for every $A\notin\mathcal{X}_t$. Therefore, besides the moderately popular t-sets $A\in\mathcal{X}_t$, we also consider the t-sets consisting of elements from every moderately popular s-sets ($1\leq s\leq t-1$).

Proof of Lemma 2.15. For each $1\leq s\leq t-1$, we define

$$\mathcal{X}_s=\{A\in\binom{[n]}{s}:|\mathcal{F}(A)|\geq\frac{|\mathcal{F}|}{C_s}\text{ and }A\nsubseteq B,\text{ for any }B\in\bigcup_{i=s+1}^{t}\mathcal{X}_i\}$$

as the family of moderately popular s-sets appearing in $\mathcal{F}$ except those already contained in some moderately popular $(s+1)$-sets, where $C_s=2^{2^{s-1}-1}\cdot10^{2^{s+2}-2}\cdot(k^2t^4(r+1)^7)^{2^{s-1}}$. Since $\frac{2{C_s}^2}{C_{s+1}}<1$, we claim that $|\mathcal{X}_s|\leq2C_s$. Otherwise, let $\mathcal{X}_0$ be a subfamily of $\mathcal{X}_s$ with size $2C_s$, we have

$$|\mathcal{F}|\geq|\bigcup_{A\in\mathcal{X}_0}\mathcal{F}(A)|\geq\sum_{A\in\mathcal{X}_0}|\mathcal{F}(A)|-\sum_{A\neq B\in\mathcal{X}_0}|\mathcal{F}(A\cup B)|.$$

A little different from (2.44), since $A, B \in \mathcal{X}_s$ are not contained in any member of $\mathcal{X}_{s+1}$, for $A \neq B \in \mathcal{X}_s$, we have $|\mathcal{F}(A \cup B)| \leq \frac{|\mathcal{F}|}{C_{s+1}}$. Then, through a similar argument as that of Claim 7, we can reach a contradiction.

Let $U_i = \bigcup_{A \in \mathcal{X}_i} A$ and $U = \bigcup_{1 \leq s \leq t} U_s$, for the convenience of our following proof, w.l.o.g., we assume that $U = [m]$ and $|\mathcal{F}(1)| \geq |\mathcal{F}(2)| \geq \ldots \geq |\mathcal{F}(m)|$. Based on this ordering, we have the following claims.

Claim 9. $[t]$ is one of the most popular t-sets appearing in $\mathcal{F}$. Moreover, if $\mathcal{F}$ contains a full t-star, then the core of this t-star is $[t]$ and $[t+1] \setminus \{t\}$ is one of the most popular t-sets appearing in $\mathcal{F} \setminus \mathcal{F}([t])$.

Proof. Let $A_0 \neq [t]$ be one of the most popular t-sets appearing in $\mathcal{F}$, by Lemma 2.14, $A_0 \subseteq [m]$. Assume that $1 \notin A_0$, we consider the new family $\mathcal{S}_{a_0,1}(\mathcal{F}(A_0))$, where $a_0 \in A_0 \setminus [t]$. If there exists some $F \in \mathcal{S}_{a_0,1}(\mathcal{F}(A_0)) \setminus \mathcal{F}$, we can replace its preimage $\mathcal{S}_{1,a_0}(F)$ in $\mathcal{F}$ with F. Denote the new family as $\mathcal{F}'$, then $\mathcal{I}(\mathcal{F}') - \mathcal{I}(\mathcal{F})$ is

$$\sum_{x \in F} |\mathcal{F}'(x)| - \sum_{(F \setminus \{1\}) \cup \{a_0\}} |\mathcal{F}(x)| = |\mathcal{F}(1)| + 1 - |\mathcal{F}(a_0)| > 0.$$

Therefore, by the optimality of $\mathcal{F}$, $\mathcal{S}_{a_0,1}(\mathcal{F}(A_0)) \subseteq \mathcal{F}(1)$.

Let $A_1 = A_0 \setminus \{a_0\} \cup \{1\}$, we know that $|\mathcal{F}(A_1)| = |\mathcal{S}_{a_0,1}(\mathcal{F}(A_0))| = |\mathcal{F}(A_0)|$. Now, assume that $2 \notin A_1$, let $A_2 = A_1 \setminus \{a_1\} \cup \{2\}$ for some $a_1 \in A_2 \setminus [t]$. With a similar argument, we have $|\mathcal{F}(A_2)| = |\mathcal{F}(A_0)|$. By repeating this process, finally, we can obtain $|\mathcal{F}([t])| = |\mathcal{F}(A_0)|$.

If $\mathcal{F}$ contains one full t-star $\mathcal{Y}_1$, we have $r \geq 1$. From the analysis above, we have $\mathcal{Y}_1 = \mathcal{F}([t])$. Denote $\mathcal{F}_1 = \mathcal{Y}_1$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$. Since $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F}) \geq (t-1)|\mathcal{F}|^2 + (r + \delta^2)\binom{n-t}{k-t}^2$ and $\mathcal{I}(\mathcal{F}_1) \leq (t + \frac{1}{C_t})\binom{n-t}{k-t}^2$, we have

$$\mathcal{I}(\mathcal{F}_2) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) \geq (t-1)|\mathcal{F}_2|^2 + 2(t-1)|\mathcal{F}_1||\mathcal{F}_2| + (r - 1 + \delta^2 - \frac{1}{C_t})\binom{n-t}{k-t}^2.$$

Moreover, noted that for each $F_2 \in \mathcal{F}_2$, $|F_2 \cap [t]| \leq t-1$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) &= \sum_{F_1 \in \mathcal{F}_1} \sum_{F_2 \in \mathcal{F}_2} |F_1 \cap F_2| \leq \sum_{F_1 \in \mathcal{F}_1} \sum_{F_2 \in \mathcal{F}_2} (|[t] \cap F_2| + |F_1 \cap (F_2 \setminus [t])|) \\ &\leq (t-1)|\mathcal{F}_1||\mathcal{F}_2| + k|\mathcal{F}_2|\binom{n-(t+1)}{k-(t+1)} \leq (t-1)|\mathcal{F}_1||\mathcal{F}_2| + \frac{kr}{C_t}\binom{n-t}{k-t}^2.\end{aligned}$$

Combining the above two inequalities, we have

$$\mathcal{I}(\mathcal{F}_2) \geq (t-1)|\mathcal{F}_2|^2 + (r - 1 + \delta^2 - \frac{3kr}{C_t})\binom{n-t}{k-t}^2. \tag{2.51}$$

By Lemma 2.14, we know that the most popular t-set A satisfies

$$|\mathcal{F}_2(A)| \geq \frac{r-1+\delta^2}{2t(r-1+\delta)^2}|\mathcal{F}_2| \geq \frac{\delta}{3t(r+\delta)}|\mathcal{F}|.$$

Let $B_0 \neq [t+1] \setminus \{t\}$ be one of the most popular t-sets appearing in $\mathcal{F} \setminus \mathcal{F}([t])$. Then, $|\mathcal{F}(B_0)| \geq |\mathcal{F}_2(A)|$, which indicates that $B_0 \subseteq [m]$. Similarly, assume that $1 \notin B_0$ and consider the new family $\mathcal{S}_{b_0,1}(\mathcal{F}(B_0))$, where $b_0 \in B_0 \setminus [t]$. Using a same shifting argument, we have $\mathcal{S}_{b_0,1}(\mathcal{F}(B_0)) \subseteq \mathcal{F}(1)$. Then, repeating this process for $2, 3, \ldots, t-1$ and $t+1$ successively, we can obtain $|\mathcal{F}([t+1] \setminus \{t\})| = |\mathcal{F}(B_0)|$. $\square$

For $A, B \in \binom{[m]}{t}$, let $A \setminus B = \{a_1, a_2, \ldots, a_l\}$ and $B \setminus A = \{b_1, b_2, \ldots, b_l\}$, where $a_1 \leq \ldots \leq a_l$ and $b_1 \leq \ldots \leq b_l$ for some $0 \leq l \leq t$. From the given ordering that $|\mathcal{F}(1)| \geq \ldots \geq |\mathcal{F}(m)|$, the shifting argument in Claim 9 actually shows that if $a_i \geq b_i$ for all $1 \leq i \leq l$, then $|\mathcal{F}(A)| \geq |\mathcal{F}(B)|$.

Claim 10. $[m]$ is a t-cover of $\mathcal{F}$.

Proof. Suppose not, there exists an $F_0 \in \mathcal{F}$ such that $|F_0 \cap [m]| \leq t-1$. Actually, by the definition of $[m]$, we know that for each $x \in F_0 \setminus [m]$ and every $1 \leq s \leq t$, there is no moderately popular s-set containing x. Thus, for each $x \in F_0 \setminus [m]$, we have $|\mathcal{F}(x)| \leq \frac{|\mathcal{F}|}{C_1}$.

Since $\mathcal{F}$ contains at most one full t-star, by Claim 9, we can assume $[t]$ as the core of this only full t-star in $\mathcal{F}$ (if exists). Thus, we can replace F_0 with some $F \in \binom{[n]}{k}$ containing $[t+1] \setminus \{t\}$. Denote the new family as $\mathcal{F}'$. Noted that $|\mathcal{F}(t+1)| \geq |\mathcal{F}([t+1] \setminus \{t\})| \geq \frac{\delta}{3t(r+\delta)}|\mathcal{F}|$. Thus, we have

$$\begin{aligned}\sum_{x \in F} |\mathcal{F}'(x)| - \sum_{x \in F_0} |\mathcal{F}(x)| &\geq \sum_{i=1}^{t-1} |\mathcal{F}(i)| + |\mathcal{F}(t+1)| - \sum_{x \in F_0 \cap [m]} |\mathcal{F}(x)| - \frac{k-t+1}{C_1}|\mathcal{F}| \\ &\geq \frac{\delta}{3t(r+\delta)}|\mathcal{F}| - \frac{k-t+1}{C_1}|\mathcal{F}| > 0,\end{aligned}$$

which contradicts the optimality of $\mathcal{F}$. Thus, $[m]$ is a t-cover of $\mathcal{F}$. $\square$

Now, we only need to show that for each $i \in [m]$, i is contained in some $A \in \mathcal{X}_t$. For $i \in [t]$, this easily follows from the fact that $[t] \in \mathcal{X}_t$. For $t+1 \leq i \leq m$, we have the following claim.

Claim 11. When $\mathcal{F}$ contains no full t-star, for $t+1 \leq i \leq m$, $|\mathcal{F}(i)| \geq |\mathcal{F}(t)| - \frac{k}{C_1}|\mathcal{F}|$ and $|\mathcal{F}(\{1,2,\ldots,t-1,i\})| \geq |\mathcal{F}([t])| - \frac{2k}{C_1}|\mathcal{F}|$. When $\mathcal{F}$ contains one full t-star, for $t+1 \leq i \leq m$, $|\mathcal{F}(i)| \geq |\mathcal{F}(t+1)| - \frac{k}{C_1}|\mathcal{F}|$ and $|\mathcal{F}(\{1,2,\ldots,t-1,i\})| \geq |\mathcal{F}([t+1] \setminus \{t\})| - \frac{2k}{C_1}|\mathcal{F}|$.

Proof. First, similar to Claim 9, by a shifting argument, we can prove that $\{1,2,\ldots,t-1,i\}$ has the largest degree in $\mathcal{F}$ among all t-sets in $\binom{[m]}{t}$ containing i. This indicates that

$$|\mathcal{F}(\{1,\ldots,t-1,i\})| \geq \frac{|\mathcal{F}(i)|}{\binom{m}{t-1}}.$$

By the definition of U, we know that $|\mathcal{F}(i)| \geq \frac{|\mathcal{F}|}{C_t}$ and $m = |U| \leq \sum_{i=1}^{t} 2iC_i$. Therefore,

$$|\mathcal{F}(\{1,\ldots,t-1,i\})| \geq \frac{|\mathcal{F}|}{(3tC_t)^t}.$$

If for every $F \in \mathcal{F}(\{1,\ldots,t-1,i\})$, $|F \cap [m]| \geq t+1$, then the size of $\mathcal{F}(\{1,\ldots,t-1,i\})$ shall be upper bounded by

$$|\mathcal{F}(\{1,\ldots,t-1,i\})| \leq m\binom{n-(t+1)}{k-(t+1)},$$

which contradicts the above lower bound since n is very large. Thus, there is an $F_0 \in \mathcal{F}(\{1,\ldots,t-1,i\})$ such that $|F_0 \cap [m]| = t$.

When $\mathcal{F}$ contains no full t-star, we can replace this F_0 with an $F \notin \mathcal{F}$ containing $[t]$. Denote the new family as $\mathcal{F}'$, then, we have

$$\begin{aligned}\sum_{x\in F}|\mathcal{F}'(x)| - \sum_{x\in F_0}|\mathcal{F}(x)| &\geq \sum_{i=1}^{t}|\mathcal{F}(i)| - \sum_{x\in F_0\cap[m]}|\mathcal{F}(x)| - \frac{k-t+1}{C_1}|\mathcal{F}| \\ &= |\mathcal{F}(t)| - |\mathcal{F}(i)| - \frac{k-t+1}{C_1}|\mathcal{F}|.\end{aligned}$$

Therefore, $|\mathcal{F}(i)| \geq |\mathcal{F}(t)| - \frac{k-t+1}{C_1}|\mathcal{F}|$ follows from the optimality of $\mathcal{F}$.

When $\mathcal{F}$ contains one full t-star $\mathcal{Y}_1$, according to Claim 9, the core of $\mathcal{Y}_1$ is $[t]$ and $[t+1]\setminus\{t\}$ is the most popular t-set with degree less than $\binom{n-t}{k-t}$. Thus, we can replace F_0 with some $F'\notin\mathcal{F}$ containing $[t+1]\setminus\{t\}$ and a same argument leads to $|\mathcal{F}(i)|\geq|\mathcal{F}(t+1)|-\frac{k-t+1}{C_1}|\mathcal{F}|$.

Fix $j\in[m]\setminus\{i\}$. For any $B\in\binom{[m]\setminus\{i\}}{t}$ containing j, we know that $\mathcal{F}(B)\subseteq\mathcal{F}(j)$ and $\mathcal{F}(B\setminus\{j\}\cup\{i\})\subseteq\mathcal{F}(i)$. Moreover, from our previous analysis, $|\mathcal{F}(B)|\geq|\mathcal{F}(B\setminus\{j\}\cup\{i\})|$ for all $j+1\leq i\leq m$. Since for each $j\in[m]$,

$$\sum_{B\in\binom{[m]}{t},j\in B}|\mathcal{F}(B)|-\sum_{B_1\neq B_2\in\binom{[m]}{t},j\in B_1,B_2}|\mathcal{F}(B_1\cup B_2)|\leq|\mathcal{F}(j)|\leq\sum_{B\in\binom{[m]}{t},j\in B}|\mathcal{F}(B)|. \tag{2.52}$$

When $\mathcal{F}$ contains no full t-star, take $j=t$. For $t+1\leq i\leq m$, combining with $|\mathcal{F}(i)|\geq|\mathcal{F}(t)|-\frac{k-t+1}{C_1}|\mathcal{F}|$, we have

$$\begin{aligned}\sum_{B\in\binom{[m]}{t},i\in B}|\mathcal{F}(B)|\geq&\sum_{B\in\binom{[m]}{t},t\in B}|\mathcal{F}(B)|-\sum_{B_1\neq B_2\in\binom{[m]}{t},t\in B_1,B_2}|\mathcal{F}(B_1\cup B_2)|\\&-\frac{k-t+1}{C_1}|\mathcal{F}|.\end{aligned}$$

Noted for each $B\in\binom{[m]}{t}$ containing $\{t\}$, $|\mathcal{F}(B)|\geq|\mathcal{F}(B\setminus\{t\}\cup\{i\})|$. Thus, we have

$$\begin{aligned}&\sum_{B_1\neq B_2\in\binom{[m]}{t},t\in B_1,B_2}|\mathcal{F}(B_1\cup B_2)|+\frac{k-t+1}{C_1}|\mathcal{F}|\\&\geq\sum_{B\in\binom{[m]}{t},t\in B}|\mathcal{F}(B)|-\sum_{B'\in\binom{[m]}{t},i\in B'}|\mathcal{F}(B')|\\&\geq|\mathcal{F}([t])|-|\mathcal{F}(\{1,2,\ldots,t-1,i\})|\end{aligned}$$

where the last inequality follows from the one to one correspondence between $B\in\binom{[m]}{t}$ containing $\{t\}$ and $B'\in\binom{[m]}{t}$ containing $\{i\}$. Since $|\mathcal{F}(B_1\cup B_2)|\leq\binom{n-(t+1)}{k-(t+1)}$, by the choice of n, we have

$$\begin{aligned}|\mathcal{F}(\{1,2,\ldots,t-1,i\})|\geq&|\mathcal{F}([t])|-\frac{k-t+1}{C_1}|\mathcal{F}|-\binom{m}{t-1}^2\binom{n-(t+1)}{k-(t+1)}\\\geq&|\mathcal{F}([t])|-\frac{2k}{C_1}|\mathcal{F}|.\end{aligned}$$

When $\mathcal{F}$ contains one full t-star $\mathcal{Y}_1 = \mathcal{F}([t])$, through a similar estimation, we have $|\mathcal{F}(\{1,2,\ldots,t-1,i\})| \geq |\mathcal{F}([t+1]\setminus\{t\})| - \frac{2k}{C_1}|\mathcal{F}|$. □

When $\mathcal{F}$ contains no full t-star, noted that $|\mathcal{F}([t])| \geq \frac{r+\delta^2}{2t(r+\delta)^2}|\mathcal{F}|$, by Claim 11, $|\mathcal{F}(\{1,2,\ldots,t-1,i\})| \geq \frac{r+\delta^2}{3t(r+\delta)^2}|\mathcal{F}|$ for $t+1 \leq i \leq m$. Therefore, $\{1,2,\ldots,t-1,i\} \in \mathcal{X}_t$ and since

$$\begin{aligned}|\mathcal{F}| &\geq \sum_{i=t}^{m} |\mathcal{F}(\{1,2,\ldots,t-1,i\})| - \sum_{i\neq j\in[t,m]} |\mathcal{F}(\{1,2,\ldots,t-1,i,j\})| \\ &\geq \sum_{i=t}^{m} |\mathcal{F}(\{1,2,\ldots,t-1,i\})| - \frac{1}{C_t}|\mathcal{F}|,\end{aligned}$$

we have $m \leq \frac{4t(r+\delta)^2}{r+\delta^2} + t \leq t(4r+5)$.

When $\mathcal{F}$ contains one full t-star which is assumed as $\mathcal{F}([t])$. From Claim 9, $|\mathcal{F}([t+1]\setminus\{t\})| \geq \frac{r-1+\delta^2}{2t(r-1+\delta)^2}|\mathcal{F}_2|$, where $\mathcal{F}_2 = \mathcal{F}\setminus\mathcal{F}([t])$. By Claim 11, $|\mathcal{F}(\{1,2,\ldots,t-1,i\})| \geq \frac{r-1+\delta^2}{3t(r-1+\delta)^2}|\mathcal{F}_2|$ for $t+1 \leq i \leq m$. Thus, we also have $\{1,2,\ldots,t-1,i\} \in \mathcal{X}_t$. Moreover, since

$$\begin{aligned}|\mathcal{F}_2| &\geq \sum_{i=t+1}^{m} |\mathcal{F}(\{1,2,\ldots,t-1,i\})| - \sum_{i\neq j\in[t+1,m]} |\mathcal{F}(\{1,2,\ldots,t-1,i,j\})| \\ &\geq \sum_{i=t+1}^{m} |\mathcal{F}(\{1,2,\ldots,t-1,i\})| - \frac{1}{C_t}|\mathcal{F}_2|,\end{aligned}$$

we have $m \leq \frac{4t(r-1+\delta)^2}{r-1+\delta^2} + t \leq t(4r+5)$.

This completes the proof of Lemma 2.15. □

We now proceed the proof of Lemma 2.16. This shows that when $\mathcal{F}$ contains no full t-star, all $F \in \mathcal{F}$ share a common element.

Proof of Lemma 2.16. For the convenience of the proof, we inherit the assumptions that $[m] = U = U_t$ and $|\mathcal{F}(1)| \geq |\mathcal{F}(2)| \geq \ldots \geq |\mathcal{F}(m)|$ in the proof of Lemma 2.15.

Noted that $[m]$ is a t-cover of $\mathcal{F}$, first, we have the following claim which says that $|\mathcal{F}(1)|$ is already fairly large.

Claim 12. $|\mathcal{F}(1)| \geq (\frac{t-1}{t} + \frac{r+\delta^2}{t(r+\delta)^2} - \frac{1}{C_t})|\mathcal{F}|$.

Proof. By the definition of $\mathcal{I}(\mathcal{F})$ and inequality (2.52), for each $A = \{a_1, a_2, \dots, a_t\} \in \binom{[m]}{t}$, we have

$$\begin{aligned}
\mathcal{I}(\mathcal{F}(A), \mathcal{F}) &= \sum_{F\in\mathcal{F}(A)} \mathcal{I}(F, \mathcal{F}) \leq \sum_{B\in\binom{[m]}{t}} \sum_{F\in\mathcal{F}(A)} \mathcal{I}(F, \mathcal{F}(B)) \\
&= \sum_{B\in\binom{[m]}{t}} \sum_{F\in\mathcal{F}(A)} (\sum_{x\in A\cap B} |\mathcal{F}(B)| + \sum_{x\in F\cap(B\setminus A)} |\mathcal{F}(B)| + \sum_{x\in F\setminus B} |\mathcal{F}(B\cup\{x\})|) \\
&\leq \sum_{B\in\binom{[m]}{t}} \big[|A\cap B||\mathcal{F}(A)||\mathcal{F}(B)| + \sum_{x\in B\setminus A} \sum_{F\in\mathcal{F}(A), x\in F} |\mathcal{F}(B)| \\
&\quad + k|\mathcal{F}(A)|\binom{n-(t+1)}{k-(t+1)}\big] \\
&\leq \sum_{B\in\binom{[m]}{t}} \big[|A\cap B||\mathcal{F}(A)||\mathcal{F}(B)| + (t|\mathcal{F}(B)| + k|\mathcal{F}(A)|)\binom{n-(t+1)}{k-(t+1)}\big] \\
&\leq |\mathcal{F}(A)| \cdot [\sum_{i=1}^{t} \sum_{B\in\binom{[m]}{t}, a_i\in B} |\mathcal{F}(B)| + k\binom{m}{t}\binom{n-(t+1)}{k-(t+1)}] \\
&\quad + t\binom{n-(t+1)}{k-(t+1)} \sum_{B\in\binom{[m]}{t}} |\mathcal{F}(B)| \\
&\leq |\mathcal{F}(A)| \cdot [\sum_{i=1}^{t} |\mathcal{F}(a_i)| \cdot (1 + \frac{1}{4C_t}) + k\binom{m}{t}\binom{n-(t+1)}{k-(t+1)}] \\
&\quad + \frac{2t(k-t)}{n-t}|\mathcal{F}|^2 \\
&\leq |\mathcal{F}(A)| \cdot \sum_{i=1}^{t} |\mathcal{F}(a_i)| \cdot (1 + \frac{1}{4C_t}) + \frac{2k^2 m^t}{n-t}|\mathcal{F}|^2, \qquad (2.53)
\end{aligned}$$

where the last inequality follows from $|\mathcal{F}(A)| \leq |\mathcal{F}(a_i)| \leq |\mathcal{F}|$ and $t \leq m \leq t(4r+5)$. This leads to

$$\begin{aligned}
\mathcal{I}(\mathcal{F}) &\leq \sum_{A\in\binom{[m]}{t}} \mathcal{I}(\mathcal{F}(A), \mathcal{F}) \leq \sum_{A\in\binom{[m]}{t}} |\mathcal{F}(A)| \cdot \sum_{i\in[t]} |\mathcal{F}(i)| \cdot (1 + \frac{1}{4C_t}) + \frac{2k^2 m^{2t}}{n-t}|\mathcal{F}|^2 \\
&\leq \sum_{i\in[t]} |\mathcal{F}(i)| \cdot |\mathcal{F}| \cdot (1 + \frac{1}{2C_t}) + \frac{t|\mathcal{F}|^2}{2C_t}.
\end{aligned}$$

Since $\mathcal{I}(\mathcal{F}) \geq \mathcal{I}(\mathcal{L}_{n,k,t}(|\mathcal{F}|)) \geq (t - 1 + \frac{r+\delta^2}{(r+\delta)^2})|\mathcal{F}|^2$, thus we have

$$\sum_{i\in[t]} |\mathcal{F}(i)| \geq \frac{1}{|\mathcal{F}|} \cdot (\mathcal{I}(\mathcal{F}) - \frac{t|\mathcal{F}|^2}{C_t}) \geq (t - 1 + \frac{r+\delta^2}{(r+\delta)^2} - \frac{t}{C_t})|\mathcal{F}|. \qquad (2.54)$$

This indicates that $|\mathcal{F}(1)| \geq (\frac{t-1}{t} + \frac{r+\delta^2}{t(r+\delta)^2} - \frac{1}{C_t})|\mathcal{F}|$.

□

Now, according to the size of $|\mathcal{F}(1)|$, we divide our arguments into the following two cases.

- When $|\mathcal{F}(1)| \geq (1 - \frac{1}{C_t})|\mathcal{F}|$.

Assume that there exists an $F_0 \in \mathcal{F}$ such that $1 \notin F_0$. Since $\mathcal{F}$ contains no full t-star, we can replace F_0 with some F containing $[t]$. Denote the new family as $\mathcal{F}'$, the gain of this shifting procedure is

$$\sum_{x\in F} |\mathcal{F}'(x)| - \sum_{x\in F_0} |\mathcal{F}(x)| \geq |\mathcal{F}(1)| - \sum_{x\in F_0\setminus[t]} |\mathcal{F}(x)|.$$

Since $[m]$ is a t-cover of $\mathcal{F}$ and $|\{F \in \mathcal{F} : |F \cap [m]| \geq t+1\}| \leq \binom{m}{t+1}\binom{n-(t+1)}{k-(t+1)}$, we have

$$t|\mathcal{F}| \leq \sum_{i\in[m]} |\mathcal{F}(i)| \leq (t + \frac{t}{C_t})|\mathcal{F}|. \tag{2.55}$$

Combined with (2.54), this indicates that

$$\sum_{i=t+1}^{m} |\mathcal{F}(i)| \leq (1 + \frac{2t}{C_t} - \frac{r+\delta^2}{(r+\delta)^2})|\mathcal{F}|.$$

Therefore, we have

$$\begin{aligned}\sum_{x\in F} |\mathcal{F}'(x)| - \sum_{x\in F_0} |\mathcal{F}(x)| &\geq |\mathcal{F}(1)| - \sum_{x=t+1}^{m} |\mathcal{F}(x)| - \frac{k}{C_1}|\mathcal{F}| \\ &\geq (\frac{r+\delta^2}{(r+\delta)^2} - \frac{3t}{C_t} - \frac{k}{C_1})|\mathcal{F}| > 0,\end{aligned}$$

a contradiction. Therefore, if $|\mathcal{F}(1)| \geq (1 - \frac{1}{C_t})|\mathcal{F}|$, then the optimality of $\mathcal{F}$ guarantees that $|\mathcal{F}(1)| = |\mathcal{F}|$, i.e., for all $F \in \mathcal{F}$, $1 \in F$.

- When $|\mathcal{F}(1)| < (1 - \frac{1}{C_t})|\mathcal{F}|$.

By the shifting argument in Claim 9, we know that $[t+1]\setminus\{1\}$ has the largest degree in $\mathcal{F}$ among all t-sets not containing 1. Thus, we have $|\mathcal{F}([t+1]\setminus\{1\})| \geq \frac{|\mathcal{F}|}{C_t\cdot\binom{m-1}{t}}$. Similar to the proof of Claim 11, we can find an $F_0 \in \mathcal{F}([t+1]\setminus\{1\})$ such that $|F_0 \cap [m]| = t$. Again, replace F_0 with some F containing $[t]$ and denote the new family as $\mathcal{F}'$. The gain of this procedure is

$$\sum_{x\in F} |\mathcal{F}'(x)| - \sum_{x\in F_0} |\mathcal{F}(x)| \geq |\mathcal{F}(1)| - |\mathcal{F}(t+1)| - \frac{k}{C_1}|\mathcal{F}|.$$

Thus, by the optimality of $\mathcal{F}$, we have $|\mathcal{F}(t+1)| \geq |\mathcal{F}(1)| - \frac{k}{C_1}|\mathcal{F}|$ and this leads to $|\mathcal{F}(2)| \geq \ldots \geq |\mathcal{F}(t)| \geq |\mathcal{F}(1)| - \frac{k}{C_1}|\mathcal{F}|$. By Claim 11 and Claim 12, for all $i \in [m]$, $|\mathcal{F}(i)| \geq |\mathcal{F}(1)| - \frac{2k}{C_1}|\mathcal{F}| \geq (\frac{t-1}{t} + \frac{r+\delta^2}{t(r+\delta)^2} - \frac{3k}{C_1})|\mathcal{F}|$. This leads to

$$\begin{aligned}\sum_{i\in[m]} |\mathcal{F}(i)| &\geq m(\frac{t-1}{t} + \frac{r+\delta^2}{t(r+\delta)^2} - \frac{3k}{C_1})|\mathcal{F}| \\ &\geq (m - \frac{m}{t} + \frac{m}{t(r+1)} - \frac{3km}{C_1})|\mathcal{F}|.\end{aligned}$$

When $m \geq t+2$, this gives a lower bound no less than $(t+1-\frac{2}{t}+\frac{t+2}{t(r+1)} - \frac{3k(t+2)}{C_1}) > (t+\frac{t}{C_t})$, which contradicts the upper bound in (2.55). Therefore, $m \leq t+1$. When $m = t$, $[t]$ is a t-cover of $\mathcal{F}$. This means $[t] \in F$ for all $F \in \mathcal{F}$.

When $m = t+1$, the above lower bound is $(t + \frac{t-r}{t(r+1)} - \frac{3k(t+1)}{C_1})$. If $r < t$, it's strictly larger than $t + \frac{t}{C_t}$. Therefore, we have $r \geq t$. Since $[m] = [t+1]$ is a t-cover of $\mathcal{F}$, we can assume that $\mathcal{F} = \cup_{i=1}^{t+1}\mathcal{H}_i$, where $\mathcal{H}_i$ is the t-star in $\mathcal{F}$ with core $[t+1]\setminus\{i\}$. Therefore, $|\mathcal{F}| \leq (t+1)\binom{n-(t+1)}{k-t} + \binom{n-(t+1)}{k-(t+1)}$. By the choice of δ, this indicates $r < t+1$.

Since $|\mathcal{F}(t+1)| \geq |\mathcal{F}(1)| - \frac{k}{C_1}|\mathcal{F}|$, by (2.55), we have $|\mathcal{F}(i)| \leq |\mathcal{F}|\cdot(\frac{t}{t+1} + \frac{k+1}{C_1})$ for every $i \in [t+1]$. Denote $\tilde{\mathcal{H}}_j = \mathcal{H}_j \setminus \mathcal{F}([t+1])$, we have $|\tilde{\mathcal{H}}_j| = |\mathcal{F}| - |\mathcal{F}(j)| \geq \frac{|\mathcal{F}|}{2(t+1)}$. By Lemma 2.13, there exists a family $\mathcal{F}_0$ of size $|\mathcal{F}|$ such that $\mathcal{I}(\mathcal{F}) < \mathcal{I}(\mathcal{F}_0)$. This contradicts the optimality of $\mathcal{F}$, therefore, $m \neq t+1$.

Finally, for both cases, we have $\mathcal{F}(1) = \mathcal{F}$, this completes the proof of Lemma 2.16. □

Now, we turn to the proof of Lemma 2.12, which determines the cross-intersecting structures of all full t-stars in $\mathcal{F}$ and their relationships with the remaining k-sets of the family.

Proof of Lemma 2.12. Let $\mathcal{F}_1 = \bigcup_{i=1}^{p} \mathcal{Y}_i$ and $\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1$, then we have

$$\mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F}_1) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) + \mathcal{I}(\mathcal{F}_2).$$

By Lemma 2.11, we know that $\sum_{i=t}^{t+p-1} \binom{n-i}{k-t} \leq |\mathcal{F}_1| < p\binom{n-t}{k-t}$. Thus, by the choice of δ_0,

$$\begin{aligned}\sum_{i=t}^{t+r-p-1} \binom{n-i}{k-t} + \frac{r\delta_0}{r+1}\binom{n-(t+r-p)}{k-t} &< |\mathcal{F}_2| \\ &< \sum_{i=t}^{t+r-p-1} \binom{n-i}{k-t} + \delta_0\binom{n-(t+r-p)}{k-t}.\end{aligned}$$

According to the requirements of $\mathcal{F}$, this leads to $\mathcal{I}(\mathcal{F}_2) \leq (t-1)|\mathcal{F}_2|^2 + (r-p+\delta_0^2 + \frac{1}{C_t})\binom{n-t}{k-t}^2$.

Since $\mathcal{I}(\mathcal{F}) \geq \mathcal{I}(\mathcal{L}_{n,k,t}(|\mathcal{F}|))$, combining with this upper bound of $\mathcal{I}(\mathcal{F}_2)$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_1) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) &\geq \mathcal{I}(\mathcal{L}_{n,k,t}(|\mathcal{F}|)) - \mathcal{I}(\mathcal{F}_2) \\ &\geq (t-1)(|\mathcal{F}|^2 - |\mathcal{F}_2|^2) + (p - \frac{1}{C_t})\binom{n-t}{k-t}^2 \\ &\geq p\big[(t-1)(2r + 2\delta_0 - p) + 1 - \frac{2}{C_t}\big]\binom{n-t}{k-t}^2. \qquad (2.56)\end{aligned}$$

On the other hand, due to the structure of $\mathcal{F}_1$, we also have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_1) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2) &\leq \sum_{i,j=1}^{p} \mathcal{I}(\mathcal{Y}_i, \mathcal{Y}_j) + 2\sum_{i=1}^{p} \mathcal{I}(\mathcal{Y}_i, \mathcal{F}_2) \\ &\leq (\sum_{i,j=1}^{p} |Y_i \cap Y_j|) \cdot \binom{n-t}{k-t}^2 \cdot (1 + \frac{1}{C_t}) + \\ &\quad 2(\sum_{i=1}^{p} \sum_{F \in \mathcal{F}_2} |F \cap Y_i|) \cdot \binom{n-t}{k-t} \cdot (1 + \frac{1}{C_t}). \qquad (2.57)\end{aligned}$$

Since $\sum_{i,j=1}^{p} |Y_i \cap Y_j| = tp + \sum_{i \neq j \in [p]} |Y_i \cap Y_j|$ and $\sum_{i=1}^{p} \sum_{F \in \mathcal{F}_2} |F \cap Y_i|) \leq p(t-1)|\mathcal{F}_2|$, we have

$$\sum_{i \neq j \in [p]} |Y_i \cap Y_j| \geq p(p-1)(t-1)(1 - \frac{2rkp}{C_t}).$$

According to the choice of C_t, $p(p-1)(t-1)(1-\frac{2rkp}{C_t}) > p(p-1)(t-1)-1$. Thus, we have $\sum_{i\neq j\in[p]} |Y_i \cap Y_j| = p(p-1)(t-1)$. Therefore, $|Y_i \cap Y_j| = t-1$ for all $i \neq j \in [p]$. Moreover, by substituting $\sum_{i\neq j\in[p]} |Y_i \cap Y_j| = p(p-1)(t-1)$ into the above two inequalities about $\mathcal{I}(\mathcal{F}_1) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2)$, we have

$$\sum_{i=1}^{p} \sum_{F\in\mathcal{F}_2} |F \cap Y_i| \geq p(t-1)(1-\frac{2rk}{C_t})|\mathcal{F}_2|. \tag{2.58}$$

Therefore, there exist at least $(1-\frac{2r^2kt}{C_t})|\mathcal{F}_2|$ k-sets in $\mathcal{F}_2$ satisfying $|F \cap Y_i| = t-1$ for all $i \in [p]$.

Moreover, (2.57) also leads to

$$\begin{aligned}\mathcal{I}(\mathcal{F}_2) &\geq \mathcal{I}(\mathcal{L}_{n,k,t}(|\mathcal{F}|)) - (\mathcal{I}(\mathcal{F}_1) + 2\mathcal{I}(\mathcal{F}_1, \mathcal{F}_2)) \\ &\geq (t-1)|\mathcal{F}_2|^2 + (r-p+\delta_0^2 - \frac{4r^2kp^2}{C_t})\binom{n-t}{k-t}^2. \end{aligned} \tag{2.59}$$

This completes the proof. □

Finally, we prove Lemma 2.13.

Proof of Lemma 2.13. We prove this lemma by evaluating $\mathcal{I}(\mathcal{F})$ and $\mathcal{I}(\mathcal{F}_0)$ directly.

Since $\mathcal{L}^{(r)}_{n,k,t} \subseteq \mathcal{F}_0 \subseteq \mathcal{L}^{(r+1)}_{n,k,t}$, we can assume $\mathcal{F}_0 = \cup_{i=t}^{t+r}\mathcal{G}_i$, where $\mathcal{G}_i$ is a t-star with core $\{1,\ldots,t-1,i\}$ and $\mathcal{G}_i$ is a full t-star for each $t \leq i \leq t+r-1$. According to this structure, we have

$$\mathcal{I}(\mathcal{F}_0) = \sum_{x\in[n]} |\mathcal{F}_0(x)|^2 = (t-1)|\mathcal{F}|^2 + r\binom{n-t}{k-t}^2 + |\mathcal{F}_0(t+r)|^2 + \sum_{x\in[t+r+1,n]} |\mathcal{F}_0(x)|^2. \tag{2.60}$$

Since $|\mathcal{F}_0(t+r)| = \binom{n-t}{k-t} - (1-\delta)\binom{n-(t+r)}{k-t}$ and $|\mathcal{F}_0(x)| > \sum_{i=t+1}^{t+r}\binom{n-i}{k-(t+1)}$ for $x \in [t+r+1,n]$, this leads to

$$\begin{aligned}\mathcal{I}(\mathcal{F}_0) \geq &(t-1)|\mathcal{F}|^2 + r\binom{n-t}{k-t}^2 + (\delta\binom{n-(t+r)}{k-t} + \sum_{i=1}^{r}\binom{n-(t+i)}{k-(t+1)})^2 + \\ &(n-t-r)(\sum_{i=1}^{r}\binom{n-(t+i)}{k-(t+1)})^2.\end{aligned}$$

On the other hand, denote $\tilde{\mathcal{H}}_j = \mathcal{H}_j \setminus \mathcal{F}([t+1])$, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}) = \sum_{x\in[n]} |\mathcal{F}(x)|^2 &= \sum_{x\in[t+1]} (|\mathcal{F}| - |\tilde{\mathcal{H}}_x|)^2 + \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2 \\ &= (t+1)|\mathcal{F}|^2 - 2|\mathcal{F}|(\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|) + \sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 + \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2. \qquad (2.61)\end{aligned}$$

Since $|\mathcal{F}([t+1])| \le \binom{n-(t+1)}{k-(t+1)}$, $|\mathcal{F}(x)| \le (t+1)\binom{n-(t+1)}{k-(t+1)}$ for $x \in [t+2, n]$ and $\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x| = |\mathcal{F}| - |\mathcal{F}([t+1])|$, thus

$$\mathcal{I}(\mathcal{F}) \le (t-1)|\mathcal{F}|^2 + \sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 + 2|\mathcal{F}||\mathcal{F}([t+1])| + (n-t-1)(t+1)^2 \binom{n-(t+1)}{k-(t+1)}^2.$$

Noted that $|\tilde{\mathcal{H}}_x| \le \binom{n-(t+1)}{k-t}$, by Theorem 2.15, we have

$$\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 \le r\binom{n-(t+1)}{k-t}^2 + (|\mathcal{F}| - |\mathcal{F}([t+1])| - r\binom{n-(t+1)}{k-t})^2 = m_0 \qquad (2.62)$$

and the inequality holds if and only if $|\tilde{\mathcal{H}}_x| = \binom{n-(t+1)}{k-t}$ for exactly r distinct xs in $[t+1]$ and the rest k-sets are all contained in another $\tilde{\mathcal{H}}_x$.

When $r < t$, since $|\mathcal{H}_j| \ge \frac{\delta}{C_1}\binom{n-t}{k-t}$ for each $j \in [t+1]$, we have $|\tilde{\mathcal{H}}_j| = |\mathcal{H}_j| - |\mathcal{F}([t+1])| \ge \frac{\delta}{2C_1}\binom{n-t}{k-t}$. By Theorem 2.15, $\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 \le m_0 - \frac{\delta^2}{4C_1}\binom{n-t}{k-t}^2$. Thus, according to the former evaluation, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_0) - \mathcal{I}(\mathcal{F}) \ge& r\binom{n-t}{k-t}^2 + (\delta\binom{n-(t+r)}{k-t} + \sum_{i=1}^{r}\binom{n-(t+i)}{k-(t+1)})^2 - \\ & \sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 - 2|\mathcal{F}|\binom{n-(t+1)}{k-(t+1)} - (n-t-1)(t+1)^2\binom{n-(t+1)}{k-(t+1)}^2 \\ \ge& \frac{\delta^2}{4C_1}\binom{n-t}{k-t}^2 - 4kt^2\binom{n-t}{k-t}\binom{n-(t+1)}{k-(t+1)}.\end{aligned}$$

Since $\binom{n-(t+1)}{k-(t+1)} \le \frac{\binom{n-t}{k-t}}{4kt^2C_t^3}$, we have $\mathcal{I}(\mathcal{F}_0) > \mathcal{I}(\mathcal{F})$.

When $r = t$, first, we have the following claim.

Claim 13. $\mathcal{F} = \cup_{j=1}^{t+1}\mathcal{H}_j$ maximizes $\mathcal{I}(\mathcal{F})$ only if $\mathcal{F}$ contains t full t-stars.

Proof. First, we show that $\mathcal{F}$ must contain t almost full t-stars. Assume that $\mathcal{F}_1 = \cup_{j=1}^{t+1}\mathcal{H}'_j$ contains t full t-stars with size $|\mathcal{F}|$. Then, from (2.61) we have

$$\mathcal{I}(\mathcal{F}_1) \geq (t-1)|\mathcal{F}|^2 + m_1 + 2|\mathcal{F}|\binom{n-(t+1)}{k-(t+1)},$$

where $m_1 = t\binom{n-(t+1)}{k-t}^2 + (|\mathcal{F}| - \binom{n-(t+1)}{k-(t+1)} - t\binom{n-(t+1)}{k-t})^2$.

W.l.o.g., assume that $|\tilde{\mathcal{H}}_1| \geq \ldots \geq |\tilde{\mathcal{H}}_{t+1}|$ and $(1-\frac{1}{C_t})\binom{n-(t+1)}{k-t} \geq |\tilde{\mathcal{H}}_t| \geq |\tilde{\mathcal{H}}_{t+1}|$. According to the size of $\mathcal{F}$, this indicates that $\delta \neq 1$. Since $\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x| = |\mathcal{F}| - |\mathcal{F}([t+1])|$, by Theorem 2.15, we have

$$\begin{aligned}\sum_{x\in[t+1]} |\tilde{\mathcal{H}}_x|^2 \leq &(t-1)\binom{n-(t+1)}{k-t}^2 + (1-\frac{1}{C_t})^2\binom{n-(t+1)}{k-t}^2 + \\ &(|\mathcal{F}| - |\mathcal{F}([t+1])| - (t-\frac{1}{C_t})\binom{n-(t+1)}{k-t})^2 \leq m_0 - \frac{(1-\delta)}{C_t}\binom{n-t}{k-t}^2.\end{aligned}$$

Since $m_1 - m_0 \geq -2\delta\binom{n-t}{k-t}(\binom{n-(t+1)}{k-(t+1)} - |\mathcal{F}([t+1])|)$, by the choices of n and δ, this leads to

$$\begin{aligned}\mathcal{I}(\mathcal{F}_1) - \mathcal{I}(\mathcal{F}) \geq &\frac{1-\delta}{C_t}\binom{n-t}{k-t}^2 + 2(\binom{n-(t+1)}{k-(t+1)} - |\mathcal{F}([t+1])|)(|\mathcal{F}| - 2\delta\binom{n-t}{k-t}) \\ &- (n-t-1)(t+1)^2\binom{n-(t+1)}{k-(t+1)}^2 > 0,\end{aligned}$$

a contradiction. Therefore, to maximize $\mathcal{I}(\mathcal{F})$, $|\tilde{\mathcal{H}}_j| > (1-\frac{1}{C_t})\binom{n-(t+1)}{k-t}$ for $j \in [t]$.

Now, we prove that $\mathcal{F}$ must contain t full t-stars. W.l.o.g., assume that $|\tilde{\mathcal{H}}_1| \geq \ldots \geq |\tilde{\mathcal{H}}_{t+1}|$ and $|\mathcal{H}_t| < \binom{n-t}{k-t}$. Then, there is an $F \notin \mathcal{H}_t$ containing $[t+1] \setminus \{t\}$. Pick some $G_0 \in \tilde{\mathcal{H}}_{t+1}$ and replace G_0 with F. Denote the resulting new family as $\mathcal{F}'$, we have

$$\begin{aligned}\sum_{x\in F} |\mathcal{F}'(x)| - \sum_{x\in G_0} |\mathcal{F}(x)| &\geq (|\mathcal{F}| - |\tilde{\mathcal{H}}_{t+1}|) - (|\mathcal{F}| - |\tilde{\mathcal{H}}_t|) - k(t+1)\binom{n-(t+1)}{k-(t+1)} \\ &\geq |\tilde{\mathcal{H}}_t| - |\tilde{\mathcal{H}}_{t+1}| - \frac{1}{C_t^2}\binom{n-t}{k-t} \\ &\geq (\frac{1}{C_1} - \frac{2}{C_t})\binom{n-(t+1)}{k-t} > 0.\end{aligned}$$

This shows that as long as $\mathcal{F}$ contains less than t full t-stars, we can get a new family $\mathcal{F}'$ with $\mathcal{I}(\mathcal{F}') > \mathcal{I}(\mathcal{F})$. Therefore, $\mathcal{F}$ must contain t full t-stars. □

Now, assume that $\mathcal{F}$ contains t full t-stars. Our aim is to find a proper $\mathcal{F}_0$ with the required structure satisfying $\mathcal{I}(\mathcal{F}_0) > \mathcal{I}(\mathcal{F})$.

According to the structure of $\mathcal{F}$, we have $m_0 = m_1$ and

$$\mathcal{I}(\mathcal{F}) = (t-1)|\mathcal{F}|^2 + 2|\mathcal{F}|\binom{n-(t+1)}{k-(t+1)} + m_0 + \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2. \tag{2.63}$$

Meanwhile,

$$\mathcal{I}(\mathcal{F}_0) = (t-1)|\mathcal{F}|^2 + t\binom{n-t}{k-t}^2 + (\binom{n-t}{k-t} - (1-\delta)\binom{n-2t}{k-t})^2 + \sum_{x\in[2t+1,n]} |\mathcal{F}_0(x)|^2. \tag{2.64}$$

Thus,

$$\begin{aligned}\mathcal{I}(\mathcal{F}_0) - \mathcal{I}(\mathcal{F}) =& t\binom{n-t}{k-t}^2 + (\binom{n-t}{k-t} - (1-\delta)\binom{n-2t}{k-t})^2 - m_0 - 2|\mathcal{F}|\binom{n-(t+1)}{k-(t+1)} \\ &+ \sum_{x\in[2t+1,n]} |\mathcal{F}_0(x)|^2 - \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2.\end{aligned}$$

Denote $\Delta_1 = \binom{n-t}{k-t} - \binom{n-2t}{k-t} = \sum_{i=t+1}^{2t}\binom{n-i}{k-(t+1)}$ and $\Delta_2 = \sum_{i=1}^{t-2} i\binom{n-(2t-i)}{k-(t+1)}$. Noted that

$$\begin{aligned}|\mathcal{F}| - \binom{n-(t+1)}{k-(t+1)} - t\binom{n-(t+1)}{k-t} &= \delta\binom{n-2t}{k-t} + \sum_{i=t+2}^{2t-1}(\binom{n-i}{k-t} - \binom{n-(t+1)}{k-t}) \\ &= \delta\binom{n-2t}{k-t} - \sum_{i=1}^{t-2} i\binom{n-(2t-i)}{k-(t+1)} \\ &= \delta\binom{n-2t}{k-t} - \Delta_2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}_0) - \mathcal{I}(\mathcal{F}) =& t(\binom{n-t}{k-t} + \binom{n-(t+1)}{k-t})\binom{n-(t+1)}{k-(t+1)} - 2|\mathcal{F}|\binom{n-(t+1)}{k-(t+1)} + \\ & (\delta\binom{n-2t}{k-t} + \Delta_1)^2 - (\delta\binom{n-2t}{k-t} - \Delta_2)^2 + \sum_{x\in[2t+1,n]} |\mathcal{F}_0(x)|^2 - \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2 \\ =& (2\Delta_2 + (t-2)\binom{n-(t+1)}{k-(t+1)})\binom{n-(t+1)}{k-(t+1)} + \Delta_1^2 - \Delta_2^2 + \\ & 2\delta\binom{n-2t}{k-t}(\Delta_1 + \Delta_2 - \binom{n-(t+1)}{k-(t+1)}) + \sum_{x\in[2t+1,n]} |\mathcal{F}_0(x)|^2 - \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2.\end{aligned} \tag{2.65}$$

Since for each $x \in [2t+1,n]$, $|\mathcal{F}_0(x)| = \sum_{i=t+1}^{2t} \binom{n-i}{k-(t+1)} + |\mathcal{G}_{2t}(x)|$ and for each $x \in [t+2,n]$, $|\mathcal{F}(x)| = \binom{n-(t+2)}{k-(t+2)} + t\binom{n-(t+2)}{k-(t+1)} + |\tilde{\mathcal{H}}_{t+1}(x)|$. Thus,

$$|\mathcal{F}_0(x)| - |\mathcal{F}(x)| \geq |\mathcal{G}_{2t}(x)| - |\mathcal{H}_{t+1}(x)| - t^2\binom{n-(t+2)}{k-(t+2)}$$

for each $x \in [2t+1,n]$. Since $\sum_{x\in[t+2,2t]} |\mathcal{F}(x)|^2 \leq t(t+1)^2\binom{n-(t+1)}{k-(t+1)}^2$, therefore, we only need to find a proper $\mathcal{G}_{2t}$ to control $|\mathcal{G}_{2t}(x)| - |\mathcal{H}_{t+1}(x)|$ for all $x \in [2t+1,n]$. Let $\mathcal{H}_{t+1} = \mathcal{H}_{t+1}^1 \sqcup \mathcal{H}_{t+1}^2$, where $\mathcal{H}_{t+1}^1 = \{F \in \mathcal{H}_{t+1} : F \cap [t+2,2t] = \emptyset\}$ and $\mathcal{H}_{t+1}^2 = \mathcal{H}_{t+1} \setminus \mathcal{H}_{t+1}^1$. Noted that $|\mathcal{G}_{2t}| \geq |\mathcal{H}_{t+1}|$, thus for each $F \in \mathcal{H}_{t+1}^1$, we can arrange a $G(F) = \{1,\ldots,t-1,2t\} \cup (F \setminus [t])$ in $\mathcal{G}_{2t}$. Denote $\mathcal{G}_{2t}^1 = \{G(F) : F \in \mathcal{H}_{t+1}^1\}$ and $\mathcal{G}_{2t}^2 = \mathcal{G}_{2t} \setminus \mathcal{G}_{2t}^1$. Clearly, for $x \in [2t+1,n]$, $|\mathcal{G}_{2t}^1(x)| = |\mathcal{H}_{t+1}^1(x)|$. Since for each $F' \in \mathcal{H}_{t+1}^2$, $([t] \cup \{i_0\}) \subseteq F'$ for some $i_0 \in [t+2,2t]$. Therefore, for $x \in [2t+1,n]$ we have

$$|\mathcal{G}_{2t}(x)| - |\mathcal{H}_{t+1}(x)| = |\mathcal{G}_{2t}^2(x)| - |\mathcal{H}_{t+1}^2(x)| \geq -(t-1)\binom{n-(t+2)}{k-(t+2)}.$$

This indicates that

$$\begin{aligned}
&\sum_{x\in[2t+1,n]} |\mathcal{F}_0(x)|^2 - \sum_{x\in[t+2,n]} |\mathcal{F}(x)|^2 \qquad (2.66)\\
=&\sum_{x\in[2t+1,n]} (|\mathcal{F}_0(x)| - |\mathcal{F}(x)|)(|\mathcal{F}_0(x)| + |\mathcal{F}(x)|) - \sum_{x\in[t+2,2t]} |\mathcal{F}(x)|^2\\
\geq &-2t(t+1)^2 n\binom{n-(t+2)}{k-(t+2)}\binom{n-(t+1)}{k-(t+1)} - t(t+1)^2\binom{n-(t+1)}{k-(t+1)}^2\\
\geq &-4t(t+1)^2 k\binom{n-(t+1)}{k-(t+1)}^2. \qquad (2.67)
\end{aligned}$$

Since $t \geq 2$, based on the choice of δ, n and (2.66), we have

$$\mathcal{I}(\mathcal{F}_0) - \mathcal{I}(\mathcal{F}) \geq 2\delta\binom{n-2t}{k-t}\binom{n-(t+2)}{k-(t+1)} - 5t(t+1)^2 k\binom{n-(t+1)}{k-(t+1)}^2 > 0.$$

This completes the proof. □

§ 2.4 Proof of Theorem 2.6

Let n be a positive integer and V be an n-dimensional vector space over $\mathbb{F}_q$. In the following, if there is no confusion, we shall omit the field size q in the *Gaussian* *binomial coefficient* and use "dim" in short for "dimensional".

Lemma 2.17. *[90] Let α be a k-dim subspace of V. Then, for integers j, l satisfying $0 \leq j \leq l$, the number of l-dim subspaces of V whose intersection with α has dimension j is*

$$q^{(k-j)(l-j)} \begin{bmatrix} n-k \\ l-j \end{bmatrix} \begin{bmatrix} k \\ j \end{bmatrix}.$$

Proposition 2.2. *For integer $1 \leq t \leq n$, denote U_0 as a t-dim subspace of V. Let $\mathcal{F}$ be the family of all k-dim subspaces of V containing U_0. Then, we have $|\mathcal{F}| = \begin{bmatrix} n-t \\ k-t \end{bmatrix}$ and*

$$\mathcal{I}(\mathcal{F}) = \left(\sum_{j=0}^{k-t} (j+t) q^{(k-t-j)^2} \begin{bmatrix} n-k \\ k-t-j \end{bmatrix} \begin{bmatrix} k-t \\ j \end{bmatrix} \right) \begin{bmatrix} n-t \\ k-t \end{bmatrix}. \tag{2.68}$$

Proof. The first statement is an immediate consequence of Lemma 2.17.

Denote $V = U_0 \oplus V_1$ and take $\mathcal{G}_0$ as the family of all $(k-t)$-dim subspaces of V_1. Therefore, $\mathcal{F} = U_0 \oplus \mathcal{G}_0 = \{U_0 \oplus G : G \in \mathcal{G}_0\}$. For $F_1 \in \mathcal{F}$, let $F_1 = U_0 \oplus G_1$. Then, we have $\mathcal{I}(F_1, \mathcal{F}) = \sum_{F \in \mathcal{F}} |\dim(F_1 \cap F)| = \sum_{G \in \mathcal{G}_0} (|\dim(G_1 \cap G)| + t)$. Combined with Lemma 2.17, this leads to

$$\mathcal{I}(F_1, \mathcal{F}) = \sum_{j=0}^{k-t} (j+t) q^{(k-t-j)^2} \begin{bmatrix} (n-t)-(k-t) \\ k-t-j \end{bmatrix} \begin{bmatrix} k-t \\ j \end{bmatrix}.$$

Therefore, (2.68) follows from $\mathcal{I}(\mathcal{F}) = \sum_{F \in \mathcal{F}} \mathcal{I}(F, \mathcal{F})$. □

Now, we present the proof of Theorem 2.6.

Proof of Theorem 2.6. First, we shall show that the number of popular t-dim subspaces is not large.

Claim .1. *Let $\mathcal{X} = \{U \in \begin{bmatrix} V \\ t \end{bmatrix} : |\mathcal{F}(U)| \geq \frac{|\mathcal{F}|}{(2k+2)\begin{bmatrix} k \\ t \end{bmatrix}}\}$, then $|\mathcal{X}| < (4k+4)\begin{bmatrix} k \\ t \end{bmatrix}$.*

Proof. Otherwise, assume that there is an $\mathcal{X}_0 \subseteq \mathcal{X}$ such that $|\mathcal{X}_0| = (4k+4)\begin{bmatrix} k \\ t \end{bmatrix}$. We have

$$|\mathcal{F}| \geq |\bigcup_{U \in \mathcal{X}} \mathcal{F}(U)| \geq \sum_{U \in \mathcal{X}_0} |\mathcal{F}(U)| - \sum_{U_1 \neq U_2 \in \mathcal{X}_0} |\mathcal{F}(U_1 + U_2)|$$

$$\geq 2|\mathcal{F}| - \binom{|\mathcal{X}_0|}{2}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}.$$

Since $|\mathcal{F}| = \delta\begin{bmatrix} n-t \\ k-t \end{bmatrix}$ and $\delta \geq \frac{(4k+4)^2 n}{q^{n-k}}$, based on the choice of n and δ, we have

$$\begin{aligned} |\mathcal{F}| &\geq \frac{(4k+4)^2 n}{q^{n-k}} \cdot \begin{bmatrix} n-t \\ k-t \end{bmatrix} = \frac{(4k+4)^2 n}{q^{n-k}} \cdot \frac{q^{n-t}-1}{q^{k-t}-1} \cdot \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\ &= (4k+4)^2 n \cdot \frac{q^{n-t}-1}{q^{n-t}-q^{n-k}} \cdot \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\ &> 8(k+1)^2 \cdot \begin{bmatrix} k \\ t \end{bmatrix}^2 \cdot \begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \geq \binom{|\mathcal{X}_0|}{2}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}. \end{aligned} \tag{2.69}$$

This leads to $2|\mathcal{F}| - \binom{|\mathcal{X}_0|}{2}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} > |\mathcal{F}|$, a contradiction. □

Claim .1 enables us to proceed further estimation on $\mathcal{I}(\mathcal{F})$. Next, we shall prove that the most popular t-dim subspace is contained in the majority members of $\mathcal{F}$.

Claim .2. *There exists a t-dim subspace $U_0 \in \mathcal{X}$ such that $|\mathcal{F}(U_0)| \geq (1 - \frac{2}{3k+3})|\mathcal{F}|$.*

Proof. Denote U_0 as the most popular t-dim subspace appearing in the members of $\mathcal{F}$.

- When $\delta \leq 1$.

Consider the new family $\mathcal{F}_0 \subseteq \begin{bmatrix} V \\ k \end{bmatrix}$ of size $|\mathcal{F}|$ and $U_0 \subseteq F$ for all $F \in \mathcal{F}_0$. According to (2.3), we have $\mathcal{I}(\mathcal{F}_0) \geq t|\mathcal{F}|^2$. Therefore, by the optimality of $\mathcal{F}$, $\mathcal{I}(\mathcal{F}) \geq t|\mathcal{F}|^2$.

Given a positive integer q, for variable $x \in \mathbb{R}^+$, define the function $\begin{bmatrix} x \\ k \end{bmatrix} = \prod_{i=0}^{k-1} \frac{q^{x-i}-1}{q^{k-i}-1}$. One can easily verify that $\begin{bmatrix} x \\ k \end{bmatrix}$ is a convex increasing function when $x \geq k-1$. Thus by Jensen Inequality, we have

$$\sum_{U \in \begin{bmatrix} V \\ t \end{bmatrix}} |\mathcal{F}(U)|^2 = \sum_{A,B \in \mathcal{F}} \begin{bmatrix} \dim(A \cap B) \\ t \end{bmatrix} \geq \begin{bmatrix} \frac{\sum_{A,B \in \mathcal{F}} \dim(A \cap B)}{|\mathcal{F}|^2} \\ t \end{bmatrix} \cdot |\mathcal{F}|^2. \tag{2.70}$$

Note that $\mathcal{I}(\mathcal{F}) = \sum_{A,B \in \mathcal{F}} \dim(A \cap B)$, (2.70) leads to $\sum_{U \in \begin{bmatrix} V \\ t \end{bmatrix}} |\mathcal{F}(U)|^2 \geq \begin{bmatrix} \frac{\mathcal{I}(\mathcal{F})}{|\mathcal{F}|^2} \\ t \end{bmatrix} \cdot |\mathcal{F}|^2 \geq |\mathcal{F}|^2$. Moreover, we also have

$$\sum_{U \in \begin{bmatrix} V \\ t \end{bmatrix}} |\mathcal{F}(U)|^2 = \sum_{U \in \mathcal{X}} |\mathcal{F}(U)|^2 + \sum_{U \notin \mathcal{X}} |\mathcal{F}(U)|^2$$

$$\leq |\mathcal{F}(U_0)| \cdot \sum_{U\in\mathcal{X}} |\mathcal{F}(U)| + \frac{|\mathcal{F}|}{(2k+2)\begin{bmatrix}k\\t\end{bmatrix}} \cdot \sum_{U\notin\mathcal{X}} |\mathcal{F}(U)|$$
$$\leq |\mathcal{F}(U_0)| \cdot \left(|\mathcal{F}| + \sum_{U_1\neq U_2\in\mathcal{X}} |\mathcal{F}(U_1+U_2)|\right) + \frac{|\mathcal{F}|}{(2k+2)\begin{bmatrix}k\\t\end{bmatrix}} \cdot \sum_{U\in\begin{bmatrix}V\\t\end{bmatrix}} |\mathcal{F}(U)|.$$

Note that $\dim(U_1+U_2) \geq t+1$ for $U_1 \neq U_2 \in \mathcal{X}$ and $\sum_{U\in\begin{bmatrix}V\\t\end{bmatrix}} |\mathcal{F}(U)| = \sum_{F\in\mathcal{F}} |\{U \subseteq F : U \in \begin{bmatrix}V\\t\end{bmatrix}\}| = |\mathcal{F}|\begin{bmatrix}k\\t\end{bmatrix}$, we can further obtain

$$\sum_{U\in\begin{bmatrix}V\\t\end{bmatrix}} |\mathcal{F}(U)|^2 \leq |\mathcal{F}(U_0)| \cdot \left(|\mathcal{F}| + \binom{|\mathcal{X}|}{2}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}\right) + \frac{|\mathcal{F}|}{(2k+2)\begin{bmatrix}k\\t\end{bmatrix}} \cdot \sum_{F\in\mathcal{F}} |\{U \subseteq F : U \in \begin{bmatrix}V\\t\end{bmatrix}\}|$$
$$\leq |\mathcal{F}(U_0)| \cdot \left(|\mathcal{F}| + \binom{|\mathcal{X}|}{2}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}\right) + \frac{|\mathcal{F}|^2}{2k+2}. \tag{2.71}$$

According to the calculation of (2.69), the choice of δ leads to $\binom{|\mathcal{X}|}{2}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix} \leq \frac{\begin{bmatrix}k\\t\end{bmatrix}^2|\mathcal{F}|}{n}$. Note that $n \geq (4k+4)^2\begin{bmatrix}k\\t\end{bmatrix}^2$, this indicates that $\binom{|\mathcal{X}|}{2}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix} \leq \frac{|\mathcal{F}|}{(4k+4)^2}$. Therefore, by (2.71), we have $|\mathcal{F}(U_0)| \geq (1 - \frac{2}{3k+3})|\mathcal{F}|$.

- When $\delta > 1$.

Write $U_0 = U_1 \oplus \langle u_0\rangle$, where U_1 is a $(t-1)$-dim subspace of U_0 and $\langle u_0\rangle$ is the 1-dim subspace spanned by some $u_0 \in U_0$. Let $U' = U_1 \oplus \langle u_1\rangle$ be another t-dim subspace of V, where $u_1 \in V \setminus U_0$. Consider the new family $\mathcal{G}_0 = \mathcal{G}_1 \sqcup \mathcal{G}_2$ with size $|\mathcal{F}|$, where $\mathcal{G}_1$ consists of all k-dim subspaces containing U_0 and $\mathcal{G}_2$ consists of $(\delta-1)\begin{bmatrix}n-t\\k-t\end{bmatrix}$ k-dim subspaces containing U'. Based on the structure of $\mathcal{G}_0$, according to (2.3), we have

$$\mathcal{I}(\mathcal{G}_0) \geq (t-1)|\mathcal{F}|^2 + |\mathcal{G}_1|^2 + |\mathcal{G}_2|^2$$
$$= t|\mathcal{F}|^2 - 2|\mathcal{G}_1||\mathcal{G}_2| = (t - \frac{2(\delta-1)}{\delta^2})|\mathcal{F}|^2.$$

Again, by the optimality of $\mathcal{F}$, we have $\mathcal{I}(\mathcal{F}) \geq \mathcal{I}(\mathcal{G}_0) \geq (t - \frac{2(\delta-1)}{\delta^2})|\mathcal{F}|^2$. Therefore, (2.70) leads to $\sum_{U\in\begin{bmatrix}V\\t\end{bmatrix}} |\mathcal{F}(U)|^2 \geq \begin{bmatrix}t-\frac{2(\delta-1)}{\delta^2}\\t\end{bmatrix} \cdot |\mathcal{F}|^2$. Now, consider the function $\begin{bmatrix}t-x\\t\end{bmatrix} = \prod_{i=0}^{t-1} \frac{q^{t-x-i}-1}{q^{t-i}-1}$ for $x \in \mathbb{R}$ satisfying $0 < x < 1$. Clearly, $\begin{bmatrix}t-x\\t\end{bmatrix}$ is a decreasing function and when x is fixed, the term $\frac{q^{t-x-i}-1}{q^{t-i}-1}$ is decreasing as i increases. Therefore, we have

$$\begin{bmatrix} t-x \\ t \end{bmatrix} = \prod_{i=0}^{t-1} \frac{q^{t-x-i}-1}{q^{t-i}-1} \geq (\frac{q^{1-x}-1}{q-1})^t.$$

Since $\delta \leq 1 + \frac{1}{96t(k+1)\ln q}$, we have $\frac{2(\delta-1)}{\delta^2} \leq \frac{1}{48t(k+1)\ln q}$. Denote $\varepsilon = \frac{1}{48t(k+1)\ln q}$. Then, we have $\begin{bmatrix} t-\frac{2(\delta-1)}{\delta^2} \\ t \end{bmatrix} \geq (\frac{q^{1-\varepsilon}-1}{q-1})^t = (1 - \frac{1-q^{-\varepsilon}}{1-q^{-1}})^t$. Note that for $q \geq 2$, $1 - q^{-\varepsilon} \leq \varepsilon \ln q$ and $1 - \frac{1}{q} \geq \frac{1}{2}$. Thus, we have

$$\begin{bmatrix} t-\frac{2(\delta-1)}{\delta^2} \\ t \end{bmatrix} \geq (1 - 2\varepsilon \ln q)^t \geq 1 - 2t\varepsilon \ln q = 1 - \frac{1}{24(k+1)}.$$

This leads to

$$\sum_{U \in \begin{bmatrix} V \\ t \end{bmatrix}} |\mathcal{F}(U)|^2 \geq \begin{bmatrix} t-\frac{2(\delta-1)}{\delta^2} \\ t \end{bmatrix} \cdot |\mathcal{F}|^2 \geq (1 - \frac{1}{24(k+1)})|\mathcal{F}|^2.$$

Combined with the upper bound given by (2.71), by the choice of n and δ, we also have $|\mathcal{F}(U_0)| \geq (1 - \frac{2}{3k+3})|\mathcal{F}|$. □

Finally, we show that when $\delta \leq 1$, U_0 is contained in all members of $\mathcal{F}$; when $\delta > 1$, all k-dim subspaces of V that contains U_0 are in $\mathcal{F}$.

- When $\delta \leq 1$.

Assume that there exists an $F_0 \in \mathcal{F}$ such that $U_0 \nsubseteq F_0$. Since for each $F \in \mathcal{F}$,

$$I(F, \mathcal{F}) = \sum_{A \in \mathcal{F}} \dim(F \cap A) = \sum_{U_0 \subseteq A, A \in \mathcal{F}} \dim(F \cap A) + \sum_{U_0 \nsubseteq A, A \in \mathcal{F}} \dim(F \cap A). \quad (2.72)$$

Take $F = F_0$ in the above equality and consider the first term $\sum_{U_0 \subseteq A, A \in \mathcal{F}} \dim(F_0 \cap A)$ in the RHS. Assume that $A = A_0 \oplus U_0$ and $F_0 = F_1 \oplus (U_0 \cap F_0)$. When $\dim(F_0 \cap A) \geq t$, knowing that $U_0 \nsubseteq F_0$, we have $|\dim(A_0 \cap F_1)| \geq 1$. Therefore, we can write $A_0 = A_1 \oplus U_1$ for some 1-dim subspace in F_1. Note that there are at most $\begin{bmatrix} k \\ 1 \end{bmatrix}$ different choices of such $U_1 \subseteq F_1$. And for each fixed U_1, there are at most $\begin{bmatrix} n-(t+1) \\ k-(t+1) \end{bmatrix}$ different choices of A satisfying $U_0 \oplus U_1 \subseteq A$. Therefore, the number of such As is at most $\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}$. When $\dim(F_0 \cap A) \le t-1$, since $A \in \mathcal{F}$, the number of such As is upper bounded by $|\mathcal{F}(U_0)|$. Therefore, we have

$$\sum_{U_0 \subseteq A, A \in \mathcal{F}} \dim(F_0 \cap A) \le (k-1)\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} + (t-1)|\mathcal{F}(U_0)|.$$

As for the second term, we have that $\sum_{U_0 \not\subseteq A, A \in \mathcal{F}} \dim(F_0 \cap A) \le k(|\mathcal{F}| - |\mathcal{F}(U_0)|)$. Therefore, combined with Claim .2, this leads to

$$\begin{aligned} I(F_0, \mathcal{F}) &\le k\left(|\mathcal{F}| + \begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}\right) - (k-t+1)|\mathcal{F}(U_0)| \\ &\le (t - \frac{2t+k+1}{3k+3})|\mathcal{F}| + k\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix}. \end{aligned} \tag{2.73}$$

From the assumption, we know that $\mathcal{F}$ is not contained in any full t-star. Therefore, we can replace F_0 with some $F' \notin \mathcal{F}$ containing U_0. Denote the resulting new family as $\mathcal{F}'$, we have

$$\begin{aligned} \mathcal{I}(\mathcal{F}') - \mathcal{I}(\mathcal{F}) &= \sum_{F \in \mathcal{F}'} \mathcal{I}(F, \mathcal{F}') - \sum_{F \in \mathcal{F}} \mathcal{I}(F, \mathcal{F}) \\ &= 2(\mathcal{I}(F', \mathcal{F} \setminus \{F_0\}) - \mathcal{I}(F_0, \mathcal{F} \setminus \{F_0\})), \end{aligned}$$

where the second equality follows from $\mathcal{F}' \setminus \{F'\} = \mathcal{F} \setminus \{F_0\}$. By (2.72) and Claim .2, we have $I(F', \mathcal{F} \setminus \{F_0\}) \ge t|\mathcal{F}(U_0)| \ge (t - \frac{2t}{3k+3})|\mathcal{F}|$. Therefore, based on (2.73) and the calculations in (2.69), we have

$$\begin{aligned} &\mathcal{I}(F', \mathcal{F} \setminus \{F_0\}) - \mathcal{I}(F_0, \mathcal{F} \setminus \{F_0\}) \ge \mathcal{I}(F', \mathcal{F} \setminus \{F_0\}) - \mathcal{I}(F_0, \mathcal{F}) \\ \ge &(t - \frac{2t}{3k+3})|\mathcal{F}| - (t - \frac{2t+k+1}{3k+3})|\mathcal{F}| - k\begin{bmatrix} k \\ 1 \end{bmatrix}\begin{bmatrix} n-t-1 \\ k-t-1 \end{bmatrix} \\ \ge &\frac{|\mathcal{F}|}{3} - \frac{k\begin{bmatrix} k \\ 1 \end{bmatrix}}{8(k+1)^2\begin{bmatrix} k \\ t \end{bmatrix}^2}|\mathcal{F}| \ge (\frac{1}{3} - \frac{1}{8(k+1)\begin{bmatrix} k \\ t \end{bmatrix}})|\mathcal{F}| > 0. \end{aligned}$$

This contradicts the fact that $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$. Thus, all $F \in \mathcal{F}$ must contain U_0.

- When $\delta > 1$.

Assume that there exists a $G' \in \begin{bmatrix} V \\ k \end{bmatrix} \setminus \mathcal{F}$ with $U_0 \subseteq G'$. Since $|\mathcal{F}| = \delta\begin{bmatrix} n-t \\ k-t \end{bmatrix}$ and $\delta > 1$, clearly, there exists some $G_0 \in \mathcal{F}$ such that $U_0 \not\subseteq G_0$. Take $F = G_0$

in (2.72), since the estimation in (2.73) is irrelevant to the choice of δ. Thus, with similar procedures, we can also obtain $\mathcal{I}(G_0, \mathcal{F}) \leq (t - \frac{2t+k+1}{3k+3})|\mathcal{F}| + k\begin{bmatrix}k\\1\end{bmatrix}\begin{bmatrix}n-t-1\\k-t-1\end{bmatrix}$. On the other hand, by (2.72), we also have $\mathcal{I}(G', \mathcal{F} \setminus \{G_0\}) \geq t|\mathcal{F}(U_0)| \geq (t - \frac{2t}{3k+3})|\mathcal{F}|$. Again, we can replace G_0 with G' and denote the resulting new family as $\mathcal{F}'$. With similar arguments as those for the case $\delta \leq 1$, this procedure increases the value of $\mathcal{I}(\mathcal{F})$ strictly, a contradiction. Therefore, all k-dim subspaces of V containing U_0 are in $\mathcal{F}$.

This completes the proof of Theorem 2.6. □

§ 2.5 Proof of Theorem 2.7

For any integer $s \geq \frac{1}{2}(n-1)!$, there exist unique $k \in \mathbb{N}$ and $\varepsilon \in (-\frac{1}{2}, \frac{1}{2}]$ such that $s = (k+\varepsilon)(n-1)!$. Denote $\mathcal{T}_0(n, s)$ as the subfamily of $\mathcal{T}(n, s)$ consisting of $\lfloor k+\varepsilon \rfloor = a_0$ pairwise disjoint 1-cosets and $\lfloor (k+\varepsilon-a_0)(n-1) \rfloor = a_1$ pairwise disjoint 2-cosets from another 1-coset disjoint from the former a_0 1-cosets.

Assume that

$$\mathcal{T}_0(n, s) = (\bigsqcup_{i=2}^{a_1+1} \mathcal{C}_{1\to 1, 2\to i}) \sqcup (\bigsqcup_{j=2}^{a_0+1} \mathcal{C}_{1\to j}), \tag{2.74}$$

where $\mathcal{C}_{1\to 1,2\to i} = \{\sigma \in S_n : \sigma(1) = 1 \text{ and } \sigma(2) = i\}$ and $\mathcal{C}_{1\to j} = \{\sigma \in S_n : \sigma(1) = j\}$. Note that for every $\mathcal{T} \subseteq S_n$,

$$\mathcal{I}(\mathcal{T}) = \sum_{i,j\in[n]} |\mathcal{T}_{i\to j}|^2,$$

where $\mathcal{T}_{i\to j} = \{\sigma \in \mathcal{T} : \sigma(i) = j\}$. Hence, when $0 \leq a_0 \leq a_1 \leq n-1$, we have

$$\begin{aligned}
&\mathcal{I}(\mathcal{T}_0(n, s)) = \sum_{i,j\in[n]} |\mathcal{T}_0(n,s)_{i\to j}|^2 \\
=&\sum_{j\in[n]} |\mathcal{T}_0(n,s)_{1\to j}|^2 + \sum_{j\in[n]} |\mathcal{T}_0(n,s)_{2\to j}|^2 + \sum_{i\in[3,n]}\sum_{j\in[n]} |\mathcal{T}_0(n,s)_{i\to j}|^2 \\
=&\big[(a_1(n-2)!)^2 + a_0((n-1)!)^2\big] + ((n-2)!)^2(a_0^2 n + 2a_0a_1 - 2a_0^2 + a_1 - a_0) + \\
&(n-2)\big[(a_0(n-2)!)^2 + a_0((a_0-1)(n-2)! + (a_1-1)(n-3)!)^2 +
\end{aligned}$$

$$(a_1 - a_0)(a_0(n-2)! + (a_1-1)(n-3)!)^2 +$$
$$(n - a_1 - 1)(a_0(n-2)! + a_1(n-3)!)^2\big]. \tag{2.75}$$

When $0 \le a_1 \le a_0 \le n-1$, similarly, we have

$$\begin{aligned}\mathcal{I}(\mathcal{T}_0(n,s)) = \big[(a_1(n-2)!)^2 + a_0((n-1)!)^2\big] + ((n-2)!)^2(a_0^2 n + 2a_0a_1 - 2a_0^2 + a_0 - a_1) \\ + (n-2)\big[(a_0(n-2)!)^2 + a_1((a_0-1)(n-2)! + (a_1-1)(n-3)!)^2 + \\ (a_0 - a_1)((a_0-1)(n-2)! + a_1(n-3)!)^2 + \\ (n - a_0 - 1)(a_0(n-2)! + a_1(n-3)!)^2\big].\end{aligned} \tag{2.76}$$

For both cases, if we denote $\eta_1 = \frac{a_1}{n-1}$, then we have

$$\begin{aligned}\mathcal{I}(\mathcal{T}_0(n,s)) \ge ((n-1)!)^2\big\{(a_0 + \eta_1^2) + \frac{(a_0^2 n + 2a_0a_1 - 2a_0^2 + a_1 - a_0)}{(n-1)^2} + \\ \frac{n-2-o(1)}{(n-1)^2}\big[a_0^2 + a_0(a_0 - 1 + \eta_1)^2 + (n - a_0 - 1)(a_0 + \eta_1)^2\big]\big\}.\end{aligned} \tag{2.77}$$

To proceed the proof of Theorem 2.7, we need some additional notations and a stability result by Ellis, Filmus and Friedgut [56] (see Theorem 1 in [56]). Assume each permutation in S_n is distributed uniformly. Then, for a function $f : S_n \to \mathbb{R}$, the expected value of f is defined by $\mathbb{E}[f] = \frac{1}{n!}\sum_{\sigma \in S_n} f(\sigma)$. The inner product of two functions $f, g : S_n \to \mathbb{R}$ is defined as $\langle f, g\rangle := \mathbb{E}[f \cdot g]$, this induces the norm $\|f\| := \sqrt{\langle f, f\rangle}$. Given $c > 0$, denote $round(c)$ as the nearest integer to c.

Theorem 2.16. *[56] There exist positive constants C_0 and ε_0 such that the following holds. Let $\mathcal{F}$ be a subfamily of S_n with $|\mathcal{F}| = \alpha(n-1)!$ for some $\alpha \le \frac{n}{2}$. Let $f = \mathbb{1}_{\mathcal{F}}$ be the characteristic function of $\mathcal{F}$ and let f_{U_1} be the orthogonal projection of f onto U_1. If $\mathbb{E}[(f - f_{U_1})^2] = \varepsilon\mathbb{E}[f]$ for some $\varepsilon \le \varepsilon_0$, then*

$$\mathbb{E}[(f-g)^2] \le C_0\alpha^2\left(\frac{1}{n^2} + \frac{\varepsilon^{\frac{1}{2}}}{n}\right),$$

where g is the characteristic function of a union of $round(\alpha)$ cosets of S_n.

Proof of Theorem 2.7. For the convenience of our proof, for $\sigma, \pi \in S_n$, we denote $\sigma \cap \pi = \{i \in [n] : \sigma(i) = \pi(i)\}$. Set $c = \min\{\frac{\varepsilon_0}{12}, \frac{1}{2}\}$ and $C = 3C_0$, where ε_0 and C_0

are the positive constants from Theorem 2.16. Let f be the characteristic vector of $\mathcal{F}$. Write $f = f_0 + f_1 + f_2$, where f_0 is the projection of f onto the eigenspace $U_{(n)}$ and f_1 is the projection of f onto the eigenspace $U_{(n-1,1)}$. By the orthogonality of the eigenspaces, we have

$$\|f\|^2 = \|f_0\|^2 + \|f_1\|^2 + \|f_2\|^2. \tag{2.78}$$

Moreover, since f is Boolean and $U_{(n)} = span\{\vec{\mathbf{1}}\}$, we also have

$$\begin{cases} \|f\|^2 = \mathbb{E}[f^2] = \mathbb{E}[f] = \frac{|\mathcal{F}|}{n!} = \frac{k+\varepsilon}{n}, \\ \|f_0\|^2 = \langle f, \vec{\mathbf{1}}\rangle^2 = \mathbb{E}[f]^2 = \frac{(k+\varepsilon)^2}{n^2}. \end{cases} \tag{2.79}$$

By the definition of $\mathcal{I}(\mathcal{F})$, we have

$$\mathcal{I}(\mathcal{F}) = \sum_{\sigma\in\mathcal{F}}\sum_{\pi\in\mathcal{F}} |\sigma \cap \pi| = f^t B f, \tag{2.80}$$

where $B = (b_{i,j})_{n!\times n!}$ is a matrix with entry $b_{i,j} = |\sigma_i \cap \sigma_j|$ under a certain ordering of all permutations in $S_n = \{\sigma_1, \dots, \sigma_{n!}\}$. According to the definition of B, we can write $B = \sum_{s=1}^{n} B_s$, where $B_s = (b^s_{i,j})_{n!\times n!}$ is the matrix with entries

$$b^s_{i,j} = \begin{cases} 1, \text{ if } |\sigma_i \cap \sigma_j| \geq s; \\ 0, \text{ otherwise.} \end{cases}$$

From a simple observation, we know that $B_s = J - A_s$, where J is the $n! \times n!$ matrix with all entries 1 and A_s is the adjacency matrix of Γ_s, i.e., the adjacency matrix of the Cayley graph on S_n with generating set FPF_s. Therefore, by (2.80), we have

$$\begin{aligned} \mathcal{I}(\mathcal{F}) &= f^t \sum_{s=1}^{n} (J - A_s) f = n f^t J f - \sum_{s=1}^{n} f^t A_s f \\ &= n|\mathcal{F}|^2 - \sum_{s=1}^{n} (f_0^t A_s f_0 + f_1^t A_s f_1 + f_2^t A_s f_2). \end{aligned} \tag{2.81}$$

Since $U_{(n)}$ and $U_{(n-1,1)}$ are eigenspaces for all A_s, $1 \leq s \leq n$, therefore,

$$\mathcal{I}(\mathcal{F}) = n|\mathcal{F}|^2 - n! \sum_{s=1}^{n} (\lambda^{(s)}_{(n)} \|f_0\|^2 + \lambda^{(s)}_{(n-1,1)} \|f_1\|^2) - \sum_{s=1}^{n} f_2^t A_s f_2$$

$$= n|\mathcal{F}|^2 - n!\sum_{s=1}^{n}\big[(\lambda_{(n)}^{(s)} - \lambda_{(n-1,1)}^{(s)})\|f_0\|^2 + \lambda_{(n-1,1)}^{(s)}\|f\|^2$$
$$- \lambda_{(n-1,1)}^{(s)}\|f_2\|^2\big] - \sum_{s=1}^{n} f_2^t A_s f_2.$$

According to Lemma 2.7, $\sum_{s=1}^{n}(\lambda_{(n)}^{(s)} - \lambda_{(n-1,1)}^{(s)}) = n!(n - \frac{n-2}{n-1})$ and $\sum_{s=1}^{n}\lambda_{(n-1,1)}^{(s)} = (n-1)!(\frac{n-2}{n-1} - 2)$. Therefore, we have

$$\begin{aligned}\mathcal{I}(\mathcal{F}) &= n|\mathcal{F}|^2 - ((n-1)!)^2\big[(k+\varepsilon)^2(n - \frac{n-2}{n-1}) + (k+\varepsilon)(\frac{n-2}{n-1} - 2)\big] - \\ &\quad ((n-1)!)^2(2n - \frac{n^2-2n}{n-1})\|f_2\|^2 - \sum_{s=1}^{n} f_2^t A_s f_2 \\ &= ((n-1)!)^2\big[(k+\varepsilon)^2\frac{n-2}{n-1} + (k+\varepsilon)\frac{n}{n-1}\big] - \frac{((n)!)^2}{n-1}\|f_2\|^2 - \sum_{s=1}^{n} f_2^t A_s f_2.\end{aligned} \tag{2.82}$$

On the other hand, write $k+\varepsilon = a_0 + \eta_1 + \frac{c}{n-1}$ for some $0 \le c \le 1$. By (2.77) and (2.82), we have

$$\mathcal{I}(\mathcal{F}) - \mathcal{I}(\mathcal{T}_0(n,s)) \le (\eta_1 - \eta_1^2 + \frac{c'}{n-1})((n-1)!)^2 - \frac{((n)!)^2}{n-1}\|f_2\|^2 - \sum_{s=1}^{n} f_2^t A_s f_2,$$

where $c' = (1+2c)(a_0+\eta_1+1)$. Note that $\mathcal{I}(\mathcal{F}) \ge \mathcal{I}(\mathcal{T}_0(n,s)) - \delta((n-1)!)^2$, which indicates that

$$\frac{(n!)^2}{n-1}\|f_2\|^2 + \sum_{s=1}^{n} f_2^t A_s f_2 \le (\eta_1 - \eta_1^2 + \delta + \frac{c'}{n-1})((n-1)!)^2. \tag{2.83}$$

To obtain an upper bound on $\|f_2\|^2$, we need the following claim.

Claim 1. $\sum_{s=1}^{n} f_2^t A_s f_2 \ge -(c_3\frac{(n!)^2}{n^2} + \frac{6(n!)^2}{n\ln n})\|f_2\|^2$ for some absolute constant c_3.

Proof. Denote $\Phi = \{\rho \vdash n : \rho \ne (n), (n-1,1)\}$. First, note that f_2 lies in $U_1^{\perp}$ and the eigenvalues corresponding to $U_1^{\perp}$ are $\{\lambda_\rho^{(s)} : \rho \in \Phi, 1 \le s \le n\}$. Thus, we have

$$f_2^t A_s f_2 = n!\sum_{\rho\in\Phi}\lambda_\rho^{(s)}\|f_\rho\|^2, \tag{2.84}$$

where f_ρ is the orthogonal projection of f_2 (or f) onto U_ρ.

Based on estimations about $\lambda_\rho^{(s)}$s for $\rho \in \Phi$ from Lemma 2.8, we have

$$\begin{cases} f_2^t A_s f_2 \geq -c_3 \frac{(n!)^2}{n^3} \|f_2\|^2, \text{ for } 3 \leq s \leq n - \frac{n}{\ln n} - 7; \\ f_2^t A_s f_2 \geq -\frac{3(n!)^2}{n^2} \|f_2\|^2, \text{ for } s = 1, 2 \text{ and } n - \frac{n}{\ln n} - 7 \leq s \leq n, \end{cases} \tag{2.85}$$

where $c_3 > 0$ is an absolute constant. This leads to

$$\sum_{s=1}^{n} f_2^t A_s f_2 \geq -(c_3 \frac{(n!)^2}{n^2} + \frac{6(n!)^2}{n \ln n}) \|f_2\|^2.$$

□

Now, with the help of Claim 1 and (2.83), we have

$$\begin{aligned} (\eta_1 - \eta_1^2 + \delta + \frac{c'}{n-1})((n-1)!)^2 &\geq \frac{(n!)^2}{n-1} \|f_2\|^2 + \sum_{s=1}^{n} f_2^t A_s f_2 \\ &\geq (n!)^2 (\frac{1}{n-1} - \frac{7}{n \ln n}) \|f_2\|^2 \\ &\geq \frac{n!(n-1)!}{1 + o(1)} \|f_2\|^2. \end{aligned}$$

From the definition, $\min\{\eta_1, 1 - \eta_1\} \leq |\varepsilon|$ and $c' \leq 3(k + \varepsilon + 1)$. Thus, we have

$$\|f_2\|^2 \leq \frac{\eta_1 - \eta_1^2 + \delta + \frac{c'}{n-1}}{n}(1 + o(1)) \leq \frac{|\varepsilon| + \delta}{k + \varepsilon}(1 + o(1)) \|f\|^2.$$

Since $\max\{|\varepsilon|, \delta\} \leq ck$, we have

$$\mathbb{E}[(f - f_{U_1})^2] = \|f_2\|^2 \leq \varepsilon_0 \|f\|^2 = \varepsilon_0 \mathbb{E}[f].$$

By Theorem 2.16, there exists $\mathcal{G}$, a union of k 1-cosets of S_n such that

$$\mathbb{E}[(f - \mathbb{1}_{\mathcal{G}})^2] \leq C_0 (k + \varepsilon)^2 \left(\sqrt{\frac{|\varepsilon| + \delta}{(k + \varepsilon) n^2}} (1 + o(1)) + \frac{1}{n^2} \right).$$

This leads to $|\mathcal{F} \Delta \mathcal{G}| = \mathbb{E}[(f - \mathbb{1}_{\mathcal{G}})^2] \cdot n! \leq C_0(\sqrt{2k(|\varepsilon| + \delta)} + \frac{k}{n}) |\mathcal{F}|$.

When $\varepsilon = \delta = 0$, we have $k + \varepsilon = k = a_0$ and $\eta_1 = 0$. Since $0 = a_1 < a_0$ for this case, we need another estimation of $\mathcal{I}(\mathcal{T}_0(n, a_0(n-1)!))$. Similar to (2.75), we have

$$\mathcal{I}(\mathcal{T}_0(n, s)) = \sum_{i,j \in [n]} |\mathcal{T}_0(n, s)_{i \to j}|^2 = \sum_{j \in [n]} |\mathcal{T}_0(n, s)_{1 \to j}|^2 + \sum_{i \in [2,n]} \sum_{j \in [n]} |\mathcal{T}_0(n, s)_{i \to j}|^2$$

$$= a_0((n-1)!)^2 + (n-1)\left[a_0((a_0-1)(n-2)!)^2 + (n-a_0)(a_0(n-2)!)^2\right]. \tag{2.86}$$

Therefore, combined with (2.82), (2.86) leads to

$$\frac{(n!)^2}{n-1}\|f_2\|^2 + \sum_{s=1}^{n} f_2^t A_s f_2 \leq 0. \tag{2.87}$$

By Claim 1, we have $\|f_2\|^2 \leq 0$. Thus, $f = \mathbb{1}_{\mathcal{F}} = f_0 + f_1 \in U_1$. As shown by Ellis et. al [58] (see Theorem 7 and Theorem 8 in [58]), this indicates that $\mathcal{F}$ is the union of k 1-cosets of S_n. Since $|\mathcal{F}| = k(n-1)!$, these k 1-cosets must be pairwise disjoint.

This completes the proof. □

Remark 2.2. *As an immediate consequence of Theorem 2.7, when* $|\varepsilon|$, $\delta = o(\frac{1}{n})$, *the optimal family* $\mathcal{F}$ *with maximum total intersection number is "almost" the union of* k *disjoint* 1*-cosets. However, due to the restrictions of parameters in Theorem 2.16, the structural characterization given by Theorem 2.7 becomes weaker as each value of* $|\varepsilon|$ *and* δ *grows.*

§ 2.6 Upper bounds on maximum total intersection numbers for families from different schemes

In this section, we will show several upper bounds on maximum total intersection numbers for families of vector spaces and permutations using linear programming method for corresponding association schemes.

2.6.1 Grassmann scheme

In this subsection, we take $(X, \mathcal{R})$ as the Grassmann scheme, which can be regarded as a q-analogue of the Johnson scheme (for explicit definition of Johnson scheme, see [90]).

For $1 \leq k \leq n$, denote $G_q(n,k)$ as the set of all subspaces in $\mathbb{F}_q^n$ with constant dimension k and $\mathcal{R} = \{R_0, \ldots, R_k\}$ as the corresponding distance relation set, where $R_i = \{(U_1, U_2) \in G_q(n,k) \times G_q(n,k) : \dim(U_1 \cap U_2) = k - i\}$. $(G_q(n,k), \mathcal{R})$ is called *the Grassmann scheme*.

Theorem 2.17. *[49] Given $0 \leq i, j \leq k$, the eigenvalues and the dual eigenvalues of the Grassmann scheme $G_q(n,k)$ are given by*

$$P_i(j) = E_i^{(q)}(j); \tag{2.88}$$

$$Q_j(i) = D_j^{(q)}(i), \tag{2.89}$$

where the generalized Eberlein polynomial $E_i^{(q)}(x)$ and the dual Hahn polynomial $D_j^{(q)}(x)$ are defined as follows:

$$E_i^{(q)}(x) = \sum_{r=0}^{i} (-1)^{i-r} q^{\binom{i-r}{2}} \begin{bmatrix} k-r \\ k-i \end{bmatrix} \begin{bmatrix} k-x \\ r \end{bmatrix} \begin{bmatrix} n+r-k-x \\ r \end{bmatrix} q^{rx}, \tag{2.90}$$

$$D_j^{(q)}(x) = \left(\begin{bmatrix} n \\ j \end{bmatrix} - \begin{bmatrix} n \\ j-1 \end{bmatrix} \right) \sum_{r=0}^{j} \left\{ (-1)^r q^{\binom{r}{2}} \begin{bmatrix} j \\ r \end{bmatrix} \begin{bmatrix} n+1-r \\ r \end{bmatrix} \begin{bmatrix} k \\ r \end{bmatrix}^{-1} \begin{bmatrix} n-k \\ r \end{bmatrix}^{-1} \right\} \begin{bmatrix} x \\ r \end{bmatrix} q^{-rx}. \tag{2.91}$$

Now, consider a family $\mathcal{F} \subseteq G_q(n,k)$ with size M. According to the definition of a_i in (2.11), we have

$$a_i = \frac{1}{M} |\{(U_1, U_2) : \ U_1, U_2 \in \mathcal{F}, \ \dim(U_1 \cap U_2) = k - i\}|.$$

This leads to

$$a_0 = 1, \ \sum_{i=0}^{k} a_i = M. \tag{2.92}$$

Then, recall the definition of $\mathcal{I}(\mathcal{F})$ from (2.3), we have

$$\begin{aligned} \mathcal{I}(\mathcal{F}) = M \sum_{i=0}^{k} (k-i) a_i &= kM \sum_{i=0}^{k} a_i - M \sum_{i=0}^{k} i a_i \\ &= kM^2 - M \sum_{i=0}^{k} i a_i. \end{aligned} \tag{2.93}$$

Based on the relationship between inner distribution a_is and dual distribution b_is, we have the following theorem.

Theorem 2.18. *Given a prime power q and positive integers n, k, M with $k \leq n$, $M \leq \begin{bmatrix} n \\ k \end{bmatrix}$. Let $\mathcal{F} \subseteq G_q(n,k)$ with $|\mathcal{F}| = M$ and $\{b_0, \ldots, b_k\}$ be the dual distribution of $\mathcal{F}$. Then, we have*

$$\mathcal{I}(\mathcal{F}) \leq \left(b_1 + 1 - \begin{bmatrix} n \\ 1 \end{bmatrix} \right) \frac{qM^2 \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ 1 \end{bmatrix} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - 1 \right)} + kM^2, \tag{2.94}$$

$$\mathcal{I}(\mathcal{F}) \leq \left(\frac{{n \brack k}}{M} - \sum_{r=2}^{k} b_r - {n \brack 1} \right) \frac{qM^2 {k \brack 1} {n-k \brack 1}}{{n \brack 1} \left({n \brack 1} - 1 \right)} + kM^2. \tag{2.95}$$

Proof. From (2.9) and (2.13), we know that $b_1 = \frac{1}{M} \sum\limits_{i=0}^{k} Q_1(i) a_i$. By (2.89) and (2.90) from Theorem 2.17, we can further obtain

$$\begin{aligned}
b_1 &= \frac{1}{M} \sum_{i=0}^{k} \left({n \brack 1} - 1 \right) \left(1 - \frac{{n \brack 1} {i \brack 1}}{{k \brack 1} {n-k \brack 1} q^i} \right) a_i \\
&\geq \frac{\left({n \brack 1} - 1 \right)}{M} \sum_{i=0}^{k} a_i - \frac{{n \brack 1} \left({n \brack 1} - 1 \right)}{qM {k \brack 1} {n-k \brack 1}} \sum_{i=0}^{k} i a_i \\
&= \left({n \brack 1} - 1 \right) - \frac{{n \brack 1} \left({n \brack 1} - 1 \right)}{qM {k \brack 1} {n-k \brack 1}} \left(kM - \frac{\mathcal{I}(\mathcal{F})}{M} \right),
\end{aligned}$$

where the last equality follows from (2.92) and (2.93). This leads to (2.94).

On the other hand, from Lemma 2.3, we know that $b_1 = \frac{{n \brack k}}{M} - 1 - \sum_{r=2}^{k} b_r$. Thus, combined with (2.94), this implies that

$$\begin{aligned}
\mathcal{I}(\mathcal{F}) &\leq \left(b_1 + 1 - {n \brack 1} \right) \frac{qM^2 {k \brack 1} {n-k \brack 1}}{{n \brack 1} \left({n \brack 1} - 1 \right)} + kM^2 \\
&= \left(\frac{{n \brack k}}{M} - \sum_{r=2}^{k} b_r - {n \brack 1} \right) \frac{qM^2 {k \brack 1} {n-k \brack 1}}{{n \brack 1} \left({n \brack 1} - 1 \right)} + kM^2,
\end{aligned}$$

which gives (2.95). □

With the help of Theorem 2.18, we now proceed the proof of Theorem 2.8.

Proof of Theorem 2.8. From Lemma 2.2, we know that $b_j \geq 0$ for $0 \leq j \leq k$. This leads to $\sum_{r=2}^{k} b_r \geq 0$. Thus, combined with (2.95), we have

$$\begin{aligned}
\mathcal{MI}(\mathcal{F}) &\leq \left(\frac{{n \brack k}}{M} - \sum_{r=2}^{k} b_r - {n \brack 1} \right) \frac{qM^2 {k \brack 1} {n-k \brack 1}}{{n \brack 1} \left({n \brack 1} - 1 \right)} + kM^2 \\
&\leq \left(\frac{{n \brack k}}{M} - {n \brack 1} \right) \frac{qM^2 {k \brack 1} {n-k \brack 1}}{{n \brack 1} \left({n \brack 1} - 1 \right)} + kM^2.
\end{aligned}$$

This proves inequality (2.6).

Next, we shall use a linear programming method to give a lower bound of $\sum_{r=2}^{k} b_s$. From Lemma 2.1, we know that for $1 \leq i \leq k$,

$$\sum_{r=0}^{k} b_r P_i(r) \geq 0. \tag{2.96}$$

Meanwhile, by Lemma 2.3, we also have $b_0 = 1$ and $b_1 = \frac{\left[{n \atop k}\right]}{M} - 1 - \sum_{r=2}^{k} b_r$. Thus, this leads to

$$\begin{aligned}\sum_{r=0}^{k} b_r P_i(r) &= b_0 P_i(0) + b_1 P_i(1) + \sum_{r=2}^{k} b_r P_i(r)\\ &= P_i(0) + \left(\frac{\left[{n \atop k}\right]}{M} - 1 - \sum_{r=2}^{k} b_r\right) P_i(1) + \sum_{r=2}^{k} b_r P_i(r)\\ &= P_i(0) + \frac{\left[{n \atop k}\right]}{M} P_i(1) - P_i(1) + \sum_{r=2}^{k} [P_i(r) - P_i(1)] b_r. \qquad (2.97)\end{aligned}$$

Combining (2.96) with (2.97), we further have

$$\sum_{r=2}^{k} b_r [P_i(1) - P_i(r)] \le P_i(0) + \frac{\left[{n \atop k}\right]}{M} P_i(1) - P_i(1),$$

for $1 \le i \le k$. To obtain a lower bound on $\sum_{r=2}^{k} b_r$ under the restrictions of the above inequality together with $b_r \ge 0$ $(2 \le r \le k)$ from Lemma 2.2, we now consider the following LP problem:

(I) Choose real variables $y_2, \ldots, y_k$ so as to

$$\Lambda(n, k, q, M) = \text{minimize } \sum_{r=2}^{k} y_r,$$

subject to

$$\begin{cases} y_r \ge 0, \text{ for } r = 2, 3, \ldots, k;\\ \sum_{r=2}^{k} y_r [P_i(1) - P_i(r)] \le P_i(0) + \frac{\left[{n \atop k}\right]}{M} P_i(1) - P_i(1), \text{ for } i = 1, 2, \ldots, k. \end{cases}$$

Note that when $M \ge \left[{n-1 \atop k-1}\right]$, by (2.88) and (2.90), we have

$$\begin{aligned} 1 + \frac{\left[{n \atop k}\right]}{M} \frac{P_i(1)}{P_i(0)} - \frac{P_i(1)}{P_i(0)} &= 1 + \left(\frac{\left[{n \atop k}\right]}{M} - 1\right) \frac{P_i(1)}{P_i(0)}\\ &= 1 + \left(\frac{\left[{n \atop k}\right]}{M} - 1\right)\left(1 - \frac{\left[{n \atop 1}\right]\left[{i \atop 1}\right]}{\left[{k \atop 1}\right]\left[{n-k \atop 1}\right] q^i}\right)\\ &\ge 1 + \left(\frac{\left[{n \atop k}\right]}{\left[{n-1 \atop k-1}\right]} - 1\right)\left(1 - \frac{\left[{n \atop 1}\right]}{\left[{n-k \atop 1}\right] q^k}\right) = 0. \end{aligned}$$

Moveover, since $P_i(0) = \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix} q^{i^2} \geq 0$, this also leads to $P_i(0) + \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} P_i(1) - P_i(1) \geq 0$, for $1 \leq i \leq k$. Therefore, by taking $y_2 = y_3 = \cdots = y_k = 0$, we can obtain the optimal solution $\Lambda(n, k, q, M) = 0$.

When $M \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, by (2.95), for $\mathcal{F} \subseteq G_q(n, k)$ with $|\mathcal{F}| = M \leq \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, we have:

$$\begin{aligned} \mathcal{MI}(\mathcal{F}) &\leq \left(\frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} - \sum_{r=2}^{k} b_r - \begin{bmatrix} n \\ 1 \end{bmatrix} \right) \frac{qM^2 \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ 1 \end{bmatrix} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - 1 \right)} + kM^2 \\ &\leq \left(\frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} - \Lambda(n, k, q, M) - \begin{bmatrix} n \\ 1 \end{bmatrix} \right) \frac{qM^2 \begin{bmatrix} k \\ 1 \end{bmatrix} \begin{bmatrix} n-k \\ 1 \end{bmatrix}}{\begin{bmatrix} n \\ 1 \end{bmatrix} \left(\begin{bmatrix} n \\ 1 \end{bmatrix} - 1 \right)} + kM^2. \end{aligned} \tag{2.98}$$

Consider the dual problem of (I), which is given as follows (see [144, Section 4 of Chapter 17]).

(II) Choose real variables $x_1, x_2, \ldots, x_k$ so as to

$$\bar{\Lambda}(n, k, q, M) = \text{maximize } \sum_{i=1}^{k} \left[P_i(1) - \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} P_i(1) - P_i(0) \right] x_i,$$

subject to

$$\begin{cases} x_i \geq 0, \text{ for } i = 1, 2, \ldots, k; \\ \sum_{i=1}^{k} x_i [P_i(1) - P_i(r)] \geq -1, \text{ for } r = 2, 3, \ldots, k. \end{cases}$$

We claim that $x_1 = \cdots = x_{k-1} = 0$, $x_k = [P_k(2) - P_k(1)]^{-1}$ is a feasible solution to the above LP problem (II). To show this, we only need to prove that

$$\frac{P_k(1) - P_k(r)}{P_k(2) - P_k(1)} \geq -1, \tag{2.99}$$

for $2 \leq r \leq k$. From (2.90) and (2.88), we know that $P_k(r) = (-1)^r q^{\binom{r}{2} + k(k-r)} \begin{bmatrix} n-k-r \\ k-r \end{bmatrix}$. Thereofore, $P_k(2) - P_k(1) > 0$ and (2.99) follows from the fact that $q^{\binom{r}{2} + k(k-r)} \begin{bmatrix} n-k-r \\ k-r \end{bmatrix}$ is decreasing on r when $k \leq n/2$. With this feasible solution, we have

$$\begin{aligned} \bar{\Lambda}(n, k, q, M) &\geq \frac{P_k(1) - \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} P_k(1) - P_k(0)}{P_k(2) - P_k(1)} \\ &= \frac{-q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} + \frac{\begin{bmatrix} n \\ k \end{bmatrix}}{M} q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix} - q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix}}{q^{1+k(k-2)} \begin{bmatrix} n-k-2 \\ k-2 \end{bmatrix} + q^{k(k-1)} \begin{bmatrix} n-k-1 \\ k-1 \end{bmatrix}} \end{aligned}$$

$$= \left(\frac{\left[{n \atop k}\right]}{M} - \frac{q^n - 1}{q^k - 1} \right) \frac{q^{k-1}(q^{n-k-1} - 1)}{q^{n-2} - 1}.$$

Therefore, it follows from (2.98) that

$$\begin{aligned} \mathcal{MI}(\mathcal{F}) &\leq \left[\frac{\left[{n \atop k}\right]}{M} - \left(\frac{\left[{n \atop k}\right]}{M} - \frac{q^n - 1}{q^k - 1} \right) \frac{q^{k-1}(q^{n-k-1} - 1)}{q^{n-2} - 1} - \begin{bmatrix} n \\ 1 \end{bmatrix} \right] \frac{qM^2 \left[{k \atop 1}\right] \left[{n-k \atop 1}\right]}{\left[{n \atop 1}\right] \left(\left[{n \atop 1}\right] - 1 \right)} + kM^2 \\ &= \left[\frac{\left[{n \atop k}\right](q^{k-1} - 1)}{M(q^{n-2} - 1)} - \frac{(q^n - 1)(q^{n-1} - 1)(q^{k-1} - 1)}{(q - 1)(q^k - 1)(q^{n-2} - 1)} \right] \frac{M^2(q^k - 1)(q^{n-k} - 1)}{(q^n - 1)(q^{n-1} - 1)} + kM^2 \\ &= \left[\frac{\left[{n \atop k}\right]}{M} - \frac{(q^n - 1)(q^{n-1} - 1)}{(q - 1)(q^k - 1)} \right] \frac{M^2(q^k - 1)(q^{k-1} - 1)(q^{n-k} - 1)}{(q^n - 1)(q^{n-1} - 1)(q^{n-2} - 1)} + kM^2. \end{aligned}$$

This completes the proof of (2.7). □

As an immediate consequence of Theorem 2.8 and Proposition 2.2, we have the following corollaries showing that bounds in Theorem 2.8 are tight for some special cases.

Corollary 2.6.1. *Given a prime power q, a positive integer n with $n \geq 2$, for $\mathcal{F} \subseteq G_q(n, 2)$ with $|\mathcal{F}| = \left[{n-1 \atop 1}\right]$, we have*

$$\mathcal{MI}(\mathcal{F}) = \frac{(q^{n-1} + q - 2)(q^{n-1} - 1)}{(q - 1)^2}. \tag{2.100}$$

Proof. By inequality (2.6), we already have

$$\mathcal{MI}(\mathcal{F}) \leq \frac{(q^{n-1} + q - 2)(q^{n-1} - 1)}{(q - 1)^2}.$$

To show that this upper bound is tight, we take $\mathcal{Y}_1 \subseteq G_q(n, 2)$ as the family of all 2-dim subspaces containing some fixed 1-dim subspace. Clearly, $|\mathcal{Y}_1| = \left[{n-1 \atop 1}\right]$. By Proposition 2.2, we have

$$\begin{aligned} \mathcal{I}(\mathcal{Y}_1) &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \sum_{j=0}^{1} (j + 1) q^{(1-j)^2} \begin{bmatrix} n-2 \\ 1-j \end{bmatrix} \begin{bmatrix} 1 \\ j \end{bmatrix} \\ &= \begin{bmatrix} n-1 \\ 1 \end{bmatrix} \frac{q^{n-1} + q - 2}{q - 1}. \end{aligned}$$

Hence, (2.100) follows from $\mathcal{MI}(\mathcal{F}) \geq \mathcal{I}(\mathcal{Y}_1)$. □

Corollary 2.6.2. *Given a prime power q, a positive integer n with $n \geq 6$, for $\mathcal{F} \subseteq G_q(n,3)$ with $|\mathcal{F}| = {n-2 \brack 1}$, we have*

$$\mathcal{MI}(\mathcal{F}) = \frac{(2q^{n-2}+q-3)(q^{n-2}-1)}{(q-1)^2}. \tag{2.101}$$

Proof. Similarly, by (2.7), we have

$$\mathcal{MI}(\mathcal{F}) \leq \frac{(2q^{n-2}+q-3)(q^{n-2}-1)}{(q-1)^2}.$$

Now, take $\mathcal{Y}_2 \subseteq G_q(n,3)$ as the family of all 3-dim subspaces containing some fixed 2-dim subspace. Clearly, $|\mathcal{Y}_2| = {n-2 \brack 1}$. By Proposition 2.2, we have

$$\begin{aligned}\mathcal{I}(\mathcal{Y}_2) &= {n-2 \brack 1} \sum_{j=0}^{1} (j+2) q^{(1-j)^2} {n-3 \brack 1-j} {1 \brack j} \\ &= \frac{(2q^{n-2}+q-3)(q^{n-2}-1)}{(q-1)^2}.\end{aligned}$$

Hence, (2.101) follows from $\mathcal{MI}(\mathcal{F}) \geq \mathcal{I}(\mathcal{Y}_2)$. □

2.6.2 The conjugacy scheme of symmetric group

Given a positive integer n, we take X as the symmetric group S_n. Denote $C_0, C_1, \ldots, C_s$ as the conjugacy classes of S_n and the relations $\mathcal{R} = \{R_0, \ldots, R_s\}$ are defined as follows:

$$R_i = \{(g,h) \in S_n \times S_n | \; gh^{-1} \in C_i\}.$$

$(S_n, \mathcal{R})$ is called *the conjugacy scheme* of S_n.

For each element σ of S_n, one can write

$$\sigma = (a_1 \ldots a_{k_1})(b_1 \ldots b_{k_2}) \ldots (c_1 \ldots c_{k_m}),$$

as a product of disjoint cycles with $k_1 \geq k_2 \geq \ldots \geq k_m \geq 1$. This m-tuple $(k_1, \ldots, k_m)$ is called the cycle-shape of σ. Then, the conjugacy classes of S_n are precisely

$$\{\sigma \in S_n : \text{ cycle-shape}(\sigma) = \lambda\}_{\lambda \vdash n}.$$

Clearly, each conjugacy class $\{C_i : 0 \le i \le s\}$ corresponds to a cycle-shape σ_i of S_n respectively. In particular, C_0 corresponds to the cycle-shape $(1, 1, \dots, 1)$. According to [90, Chapter 11.12], eigenvalues and dual eigenvalues of the conjugacy scheme of S_n are given by

$$P_i(j) = \frac{|C_i|\overline{\psi_j(c_i)}}{\psi_j(e_0)}, \quad Q_j(i) = \psi_j(c_i)\psi_j(e_0), \tag{2.102}$$

where $c_i \in C_i$ for $0 \le i \le s$, e_0 is the identity element in S_n and ψ_j $(0 \le j \le s)$ denote irreducible characters of S_n. Especially, ψ_0 denotes the trivial character, which maps all the elements of G into 1.

Given $\mathcal{F} \subseteq S_n$ with size M, consider the inner distribution of $\mathcal{F}$ with respect to $\mathcal{R}$. According to the definition of a_i, (2.11) can be rewritten as

$$a_i = \frac{1}{M}|\{(x, y) : \ x, y \in \mathcal{F}, \ xy^{-1} \in C_i\}|.$$

Thus, one can easily obtain

$$a_0 = 1 \text{ and } \sum_{i=0}^{s} a_i = M. \tag{2.103}$$

Given a cycle-shape $\sigma = (k_1, \dots, k_m)$, define $U_\sigma = |\{i \in [m] : k_i = 1\}|$. From the new expression of a_i above, we have

$$\mathcal{I}(\mathcal{F}) = M \sum_{i=1}^{s} U_\sigma a_i. \tag{2.104}$$

Now, according to the relationship between inner distribution a_is and dual distribution b_is, we have the following theorem.

Theorem 2.19. *Given positive integers n and M with $M \le n!$. Let $\mathcal{F} \subseteq S_n$ with $|\mathcal{F}| = M$ and $\{b_0, \dots, b_s\}$ be the dual distribution of $\mathcal{F}$. Then, we have*

$$\mathcal{I}(\mathcal{F}) = M^2\left(\frac{b_1}{n-1} + 1\right), \tag{2.105}$$

$$\mathcal{I}(\mathcal{F}) = \frac{M^2}{n-1}\left(\frac{n!}{M} + n - 2 - \sum_{r=2}^{s} b_r\right). \tag{2.106}$$

Proof. According to [111, Lemma 6.9], there exists an irreducible character ψ of S_n which is defined as: $\psi(c) = U_{\sigma(c)} - 1$ for $c \in S_n$, where $\sigma(c)$ is the cycle-shape of c. W.l.o.g, we can assume that $\psi_1 = \psi$. By (2.9) and (2.13), we know that $b_1 = \frac{1}{M}\sum\limits_{i=0}^{s} Q_1(i)a_i$. Then, by (2.102), we further have

$$\begin{aligned} b_1 &= \frac{1}{M}\sum_{i=0}^{s} \psi_1(c_i)\psi_1(e_0)a_i \\ &= \frac{1}{M}\sum_{i=0}^{s} (U_{\sigma(c_i)} - 1)(n-1)a_i. \end{aligned}$$

Combined with (2.103) and (2.104), this leads to

$$\begin{aligned} b_1 &= \frac{1}{M}\left(\sum_{i=0}^{s}(n-1)a_iU_{\sigma(c_i)} - \sum_{i=0}^{s}(n-1)a_i\right) \\ &= \frac{1}{M^2}(n-1)\mathcal{I}(\mathcal{F}) - (n-1), \end{aligned}$$

Therefore, we have (2.105).

On the other hand, by Lemma 2.3, we have $b_1 = \frac{n!}{M} - 1 - \sum_{r=2}^{s} b_r$. Thus, combined with (2.105), this implies that

$$\mathcal{I}(\mathcal{F}) = M^2\left(\frac{b_1}{n-1} + 1\right) = \frac{M^2}{n-1}\left(\frac{n!}{M} + n - 2 - \sum_{r=2}^{s} b_r\right).$$

□

Proof of Theorem 2.9. From Lemma 2.2, $b_j \geq 0$ for $0 \leq j \leq s$. This leads to $\sum_{r=2}^{s} b_r \geq 0$. Thus, combined with (2.106), we have

$$\begin{aligned} \mathcal{MI}(\mathcal{F}) &= \frac{M^2}{n-1}\left(\frac{n!}{M} + n - 2 - \sum_{r=2}^{s} b_r\right) \\ &\leq \frac{M^2}{n-1}\left(\frac{n!}{M} + n - 2\right). \end{aligned}$$

□

Remark 2.3. *Actually, similar to the proof of Theorem 2.8, we can also use the linear programming approach to bound $\sum_{r=2}^{s} b_r$. For interested readers, the corresponding LP problem is formulated as follows:*

(I) Choose real variables $y_2, \dots, y_k$ so as to

$$\Lambda(n, M) = \textit{minimize} \sum_{r=2}^{s} y_r,$$

subject to

$$\begin{cases} y_r \geq 0, \; \textit{for } r = 2, 3, \dots, s; \\ \sum_{r=2}^{k} y_r [P_i(1) - P_i(r)] \leq P_i(0) + \frac{n!}{M} P_i(1) - P_i(1), \; \textit{for } i = 1, 2, \dots, s. \end{cases}$$

Note that when $M \geq (n-1)!$, the optimal solution $\Lambda(n, M) = 0$ is given by taking $y_2 = y_3 = \cdots = y_s = 0$. When $M \leq (n-1)!$, we turn to the following the dual problem of (I).

(II) Choose real variables $x_1, x_2, \dots, x_s$ so as to

$$\overline{\Lambda}(n, M) = \textit{maximize} \sum_{i=1}^{s} \left[P_i(1) - \frac{n!}{M} - P_i(0) \right] x_i,$$

subject to

$$\begin{cases} x_i \geq 0, \; \textit{for } i = 1, 2, \dots, s; \\ \sum_{i=1}^{s} x_i [P_i(1) - P_i(r)] \geq -1, \; \textit{for } r = 2, 3, \dots, s. \end{cases}$$

Unfortunately, the feasible solution we find is $x_1 = \ldots = x_s = 0$, which leads to the same lower bound $\sum_{r=2}^{s} b_r \geq 0$ as Lemma 2.2. Possibly, one can find other more proper feasible solutions to improve Theorem 2.9.

As an immediate consequence of Theorem 2.9, we have the following corollary.

Corollary 2.6.3. *Given a positive integer $n \geq 2$, let $\mathcal{F} \subseteq S_n$ with $|\mathcal{F}| = (n-1)!$, we have*

$$\mathcal{MI}(\mathcal{F}) = 2\left((n-1)!\right)^2. \tag{2.107}$$

Proof. By Theorem 2.9, we have

$$\mathcal{MI}(\mathcal{F}) \leq 2\left((n-1)!\right)^2.$$

On the other hand, by taking $\mathcal{Y} = \{y \in S_n : y(1) = 1\} \subseteq S_n$, we have $\mathcal{I}(\mathcal{Y}) = 2\left((n-1)!\right)^2$. Therefore, (2.107) follows from $\mathcal{MI}(\mathcal{F}) \geq \mathcal{I}(\mathcal{Y})$. □

§ 2.7 Concluding remarks and open problems

In this chapter, we introduce a new type of intersection problems which can be viewed as inverse problems of Erdős-Ko-Rado type theorems. These problems concern the extremal structure of the family that maximizes the total intersection number among all families with the same size. For families of subsets, vector spaces and permutations, using the quantitative shifting method and spectral method, we provide structural characterizations of the optimal families satisfying $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$ for several ranges of $|\mathcal{F}|$. To some extent, these results determine the unique structure of the optimal family and characterize the relation between maximizing $\mathcal{I}(F)$ and being intersecting. Moreover, using linear programming methods, we also obtain several upper bounds on $\mathcal{MI}(\mathcal{F})$. These bounds may provide a reference for the study of structures of optimal families. However, there are several limits of our results that may require some further research.

- One can remove the uniformity requirement of the family in Question 2.1.2 and consider a more general question:

  Question 2.7.1. *For a family of subsets $\mathcal{F} \subseteq 2^{[n]}$, if $\mathcal{I}(\mathcal{F}) = \mathcal{MI}(\mathcal{F})$, what can we say about its structure?*

  It should be noted that this question is highly related to Ahlswede-Katona's [4] average distance problem in Hamming space: For every $1 \leq M \leq 2^n$, determine the minimum average Hamming distance $D_n(M)$ of binary codes with length n and size M. Based on the correspondence between binary vectors with length n and subsets of $[n]$, for $|\mathcal{F}| = M$, there is a qualitative relation between $D(M)$ and $\mathcal{MI}(\mathcal{F})$. And this qualitative relation becomes an equivalence when we consider both problems for k-uniform families (i.e., codes with constant weight k). Over the years, there are a number of papers dealing with this topic. Althöfer and Sillke [13], Fu, together with other authors (see [82, 84, 206, 207]), Mounits [149], as well as Yu and Tan [209], proved various of bounds on $D_n(M)$.

In view of $D_n(M)$ for codes with constant weight k, Corollary 2.1.4 and Corollary 2.1.5 actually provide better lower bounds on $D_n(M)$ for the required ranges of M compared to the results in [207].

- The method we use for the proof of Theorem 2.4 is the quantitative shifting arguments introduced by Das, Gan and Sudakov in [47]. While for the proof of Theorem 2.5, we do a lot of modifications about this method that involve analysis of degrees of s-sets ($1 \leq s \leq t$) in $\mathcal{F}$ from different levels. This requires n to be larger than a certain polynomial of r. As a consequence, our results cannot cover the whole range of M from 1 to $\binom{n}{k}$.

  Maybe due to the nature of the problem itself, the implementation of this method requires a great deal of countings and evaluations, which might cover the idea and intuition for dealing with this kind of problems. Therefore, as one direction for the further study, one can try to use other methods to obtain stronger results and reduce n's polynomially dependent of r.

- Given a hypergraph $\mathcal{H}$ with vertex set V, for every $v \in V$, denote $\deg_{\mathcal{H}}(v)$ as the degree of v in $\mathcal{H}$. Since families of subsets are often viewed as hypergraphs, therefore in view of hypergraphs, Question 2.1.2 actually asks the structure of the extremal hypergraph which maximizes the value of $\sum_{v\in V} \deg_{\mathcal{H}}(v)^2$ with $|\mathcal{H}|$ fixed. If we treat $|\mathcal{H}|$ as some kind of perimeter and $\sum_{v\in V} \deg_{\mathcal{H}}(v)^2$ as the area, Question 2.1.2 can be viewed as an isoperimetric problem for k-uniform hypergraphs. There are already some works concerning isoperimetric inequalities about n-dimensional Boolean cube *, see [59], [60] and references therein. In view of this, is there any connections between Question 2.1.2 and the isoperimetric inequality?

- Take $\varepsilon_0 = \frac{1}{96t \ln q(k+1)}$, Theorem 2.6 shows that for n large enough and $\delta \leq 1 + \varepsilon_0$ not too close to 0, the optimal family with maximum total intersection

*This inequality was originally proved by Harper [102], Lindsey [134], Bernstein [22] and Hart [103], see Theorem 1.1 in [60].

number is either contained in a full t-star or containing a full t-star. When $|\mathcal{F}| > (1+\varepsilon_0)\begin{bmatrix} n-t \\ k-t \end{bmatrix}$, the quantitative shifting arguments in the proof of Theorem 2.6 no longer work. So, can we obtain similar structural results for families with size larger than $(1+\varepsilon_0)\begin{bmatrix} n-t \\ k-t \end{bmatrix}$? Note that the intersection problem of vector spaces often requires tools from linear algebra or exterior algebra, maybe ideas from these areas can help us to tackle this problem.

- For families of permutations, we consider the case for $|\mathcal{F}| = \Theta((n-1)!)$. Nevertheless, for $|\mathcal{F}| = \Theta((n-t)!)$ $(t \geq 2)$, nothing is known yet. It is worth noting that, in [57], the authors provide a stability result for families of permutations with size $\Theta((n-t)!)$ similar to Theorem 2.16. Thus, it is natural to wonder if we can extend the result of Theorem 2.7 to families with size $|\mathcal{F}| = \Theta((n-t)!)$ using this stability result. Sadly, this requires more information about spectra of Γ_k, which is beyond our capability.

  Moreover, when ε becomes relatively large, the result of Theorem 2.7 seems to be trivial. Thus, for this case, more specific structural characterizations for families of permutations are also worth studying.

- Due to the choice of feasible solutions of corresponding LP problems, our upper bounds on $\mathcal{MI}(\mathcal{F})$ are no longer tight for families of subspaces with size $\Theta(\begin{bmatrix} n-t \\ k-t \end{bmatrix})$ or families of permutations with size $\Theta((n-t)!)$, for $t \geq 2$. Therefore, one can try to find other more proper feasible solutions to improve these upper bounds.

Chapter 3 Multi-part cross-intersecting families

§ 3.1 Introduction

Denote $2^{[n]}$ as the *power set* of $[n]$, let $\mathcal{A}_i \subseteq 2^{[n]}$ for each $1 \leq i \leq m$, $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are said to be cross-t-intersecting, if $|A \cap B| \geq t$ for any $A \in \mathcal{A}_i$ and $B \in \mathcal{A}_j$, $i \neq j$. Especially, we say $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ are cross-intersecting if $t = 1$.

In 1967, Hilton and Milner [105] first dealt with pairs of cross-intersecting families in $\binom{[n]}{k}$ when neither of the two families is empty:

Theorem 3.1. ([105]) *Let $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$ be non-empty cross-intersecting families with $n \geq 2k$. Then $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{k} - \binom{n-k}{k} + 1$.*

This result was generalized by Frankl and Tokushige [76] to the case when $\mathcal{A}$ and $\mathcal{B}$ are not necessarily in the same k-uniform subfamily of $2^{[n]}$:

Theorem 3.2. ([76]) *Let $\mathcal{A} \subseteq \binom{[n]}{a}$ and $\mathcal{B} \subseteq \binom{[n]}{b}$ be non-empty cross-intersecting families with $n \geq a + b$, $a \leq b$. Then $|\mathcal{A}| + |\mathcal{B}| \leq \binom{n}{b} - \binom{n-a}{b} + 1$.*

Then, in [199], Wang and Zhang generalized Theorem 3.2 to cross-t-intersecting families. Recently, using shifting techniques, Frankl and Kupavskii [74] gave another proof of Wang and Zhang's result for the case when $\mathcal{A}, \mathcal{B} \subseteq \binom{[n]}{k}$.

For general cross-intersecting families, Hilton [104] investigated families in $\binom{[n]}{k}$ and proved the following inequality:

Theorem 3.3. ([104]) *Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be cross-intersecting families in $\binom{[n]}{k}$ with $n \geq 2k$. Then*

$$\sum_{i=1}^{m} |\mathcal{A}_i| \leq \begin{cases} \binom{n}{k}, & \text{if } m \leq \frac{n}{k}; \\ m \cdot \binom{n-1}{k-1}, & \text{if } m \geq \frac{n}{k}. \end{cases}$$

In the same paper, Hilton also determined the structures of $\mathcal{A}_i$'s when the equality holds. Since then, there have been many extensions about Theorem 3.3. Borg [31] gave a simple proof of Theorem 3.3, and generalized it to labeled sets [30], signed sets [33] and permutations [32]. Using the results of the independent number about vertex-transitive graphs, Wang and Zhang [198] extended this theorem to general symmetric systems, which comprise finite sets, finite vector spaces and permutations, etc.

As another direction, the multi-part extension of the Erdős–Ko–Rado problem was introduced by Frankl [70], in connection with a similar result of Sali [168]. For positive integers $p \geq 1$ and $n_1, \ldots, n_p$, let $[\sum_{i\in[p]} n_i]$ be the ground set. Then it can be viewed as the disjoint union of p parts $\bigsqcup_{i\in[p]} S_i$, where $S_1 = [n_1]$ and $S_i = [n_{i-1}+1, n_i]$ for $2 \leq i \leq p$. Denote 2^{S_i} as the *power set* of S_i, for $A_i \in 2^{S_i}$ and $\mathcal{F}_i \subseteq 2^{S_i}$, let $\bigsqcup_{i\in[p]} A_i$ be the subset of $\bigsqcup_{i\in[p]} S_i$ with A_i in the i-th part and let $\prod_{i\in[p]} \mathcal{F}_i = \{\bigsqcup_{i\in[p]} A_i : A_i \in \mathcal{F}_i\}$. Then for $1 \leq k_i \leq n_i$, $\prod_{i\in[p]} \binom{[n_i]}{k_i}$ is the family of all subsets of $\bigsqcup_{i\in[p]} S_i$ which have exactly k_i elements in the i-th part. Therefore, families of the form $\mathcal{F} \subseteq \prod_{i\in[p]} \binom{[n_i]}{k_i}$ can be viewed as the natural generalization of k-uniform families to the multi-part setting. Similarly, a multi-part family is intersecting if any two sets of this family intersect in at least one of the p parts.

In [70], Frankl proved that for any integer $p \geq 1$, any positive integers $n_1, \ldots, n_p$ and $k_1, \ldots, k_p$ satisfying $\frac{k_1}{n_1} \leq \ldots \leq \frac{k_p}{n_p} \leq \frac{1}{2}$, if $\mathcal{F} \subseteq \prod_{i\in[p]} \binom{[n_i]}{k_i}$ is a multi-part intersecting family, then

$$|\mathcal{F}| \leq \frac{k_p}{n_p} \cdot \prod_{i\in[p]} \binom{n_i}{k_i}.$$

This bound is sharp, for example, it is attained by the following family:

$$\{A \in \binom{[n_p]}{k_p} : \ i \in A, \text{ for some } i \in [n_p]\} \times \prod_{i\in[p-1]} \binom{[n_i]}{k_i}.$$

Recently, Kwan, Sudakov and Vieira [128] considered a stability version of the Erdős–Ko–Rado theorem in the multi-part setting. They determined the maximum size of the non-trivially intersecting multi-part family when all the n_i's are sufficiently large. This disproved a conjecture proposed by Alon and Katona, which was also mentioned in [120].

In this chapter, we extend Theorem 3.3 and Theorem 3.2 to the *multi-part* version. For $S \subseteq [n]$ and $\mathcal{F} \subseteq 2^{[n]}$, denote $\bar{S}$ as the complementary set of S in $[n]$ and for $\mathcal{A} \subseteq \mathcal{F}$, denote $\mathcal{A}_{\mathcal{F}} = \{B \in \mathcal{F} : A \cap B = \emptyset \text{ for some } A \in \mathcal{A}\}$. Then, we have

Theorem 3.4. *Given a positive integer p, let $n_1, \dots, n_p$ and $k_1, \dots, k_p$ be positive integers satisfying $n_i \geq 2k_i$ for all $i \in [p]$. Let $X = \prod_{i\in[p]} \binom{[n_i]}{k_i}$ and $\mathcal{A}_1, \dots, \mathcal{A}_m$ be cross-intersecting families over X with $\mathcal{A}_1 \neq \emptyset$. Then*

$$\sum_{i=1}^{m} |\mathcal{A}_i| \leq \begin{cases} |X|, & \text{if } m \leq \min\limits_{i\in[p]} \frac{n_i}{k_i}; \\ m \cdot M, & \text{if } m \geq \min\limits_{i\in[p]} \frac{n_i}{k_i}, \end{cases} \tag{3.1}$$

where $M = \max_{i\in[p]} \binom{n_i-1}{k_i-1} \prod_{j\neq i} \binom{n_j}{k_j}$. Furthermore, the bound is attained if and only if one of the following holds:

(i) $m < \min_{i\in[p]} \frac{n_i}{k_i}$ and $\mathcal{A}_1 = X$, $\mathcal{A}_2 = \cdots = \mathcal{A}_m = \emptyset$;

(ii) $m > \min_{i\in[p]} \frac{n_i}{k_i}$ and $\mathcal{A}_1 = \cdots = \mathcal{A}_m = I$, where I is a maximum intersecting family in X;

(iii) $m = \min_{i\in[p]} \frac{n_i}{k_i}$ and $\mathcal{A}_1, \dots, \mathcal{A}_m$ are as in (i) or (ii), or there exists a non-empty set $S_1 \subseteq \{s \in [p] : \frac{n_s}{k_s} = 2\}$ and $\mathcal{F} = \prod_{s\in S_1} \binom{[n_s]}{k_s}$ such that

$$\mathcal{A}_1 = (\mathcal{A} \sqcup (\mathcal{E} \cup \mathcal{E}_{\mathcal{F}})) \times \prod_{s\in[p]\setminus S_1} \binom{[n_s]}{k_s} \text{ and } \mathcal{A}_2 = (\mathcal{A} \sqcup (\mathcal{E}' \cup \mathcal{E}'_{\mathcal{F}})) \times \prod_{s\in[p]\setminus S_1} \binom{[n_s]}{k_s} \tag{3.2}$$

for some $\mathcal{A}$, $\mathcal{E}$, $\mathcal{E}' \subseteq \mathcal{F}$, where $\mathcal{A} = \{A_1, \dots, A_{w_0}\}$ satisfying $2w_0 < |\mathcal{F}|$ and $A_i \neq \bar{A}_j$ for all $i \neq j \in [w_0]$, $\mathcal{E} \sqcup \mathcal{E}' = \{E_1, \dots, E_v\}$ and $\mathcal{E}_{\mathcal{F}} \sqcup \mathcal{E}'_{\mathcal{F}} = \{\bar{E}_1, \dots, \bar{E}_v\}$ satisfying $2(v + w_0) = \prod_{s\in S_1} \binom{n_s}{k_s}$ and $\sqcup_{j=1}^{v} \{E_j, \bar{E}_j\} = \mathcal{F} \setminus (\mathcal{A} \sqcup \mathcal{A}_{\mathcal{F}})$.

Theorem 3.5. *For any $p \geq 2$, let n_i, t_i, s_i be positive integers satisfying $n_i \geq s_i + t_i + 1$, $2 \leq s_i, t_i \leq \frac{n_i}{2}$ for every $i \in [p]$ and $n_i \leq \frac{7}{4} n_j$ for all distinct $i, j \in [p]$. If $\prod_{i\in[p]} \binom{n_i}{s_i} \geq \prod_{i\in[p]} \binom{n_i}{t_i}$ and $\mathcal{A} \subseteq \prod_{i\in[p]} \binom{[n_i]}{t_i}$, $\mathcal{B} \subseteq \prod_{i\in[p]} \binom{[n_i]}{s_i}$ are non-empty cross-intersecting families, then*

$$|\mathcal{A}| + |\mathcal{B}| \leq \prod_{i\in[p]} \binom{n_i}{s_i} - \prod_{i\in[p]} \binom{n_i - t_i}{s_i} + 1, \tag{3.3}$$

and the bound is attained if and only if the following holds:

(i) $\prod_{i\in[p]}\binom{n_i}{s_i} \geq \prod_{i\in[p]}\binom{n_i}{t_i}$, $\mathcal{A}=\{A\}$ *and* $\mathcal{B}=\{B\in\prod_{i\in[p]}\binom{[n_i]}{s_i} : B\cap A\neq\emptyset\}$ *for some* $A\in\prod_{i\in[p]}\binom{[n_i]}{t_i}$;

(ii) $\prod_{i\in[p]}\binom{n_i}{s_i} = \prod_{i\in[p]}\binom{n_i}{t_i}$, $\mathcal{B}=\{B\}$ *and* $\mathcal{A}=\{A\in\prod_{i\in[p]}\binom{[n_i]}{t_i} : B\cap A\neq\emptyset\}$ *for some* $B\in\prod_{i\in[p]}\binom{[n_i]}{s_i}$.

The remainder of this chapter is organized as follows. In Section § 3.2, we will introduce some results about the independent sets of vertex-transitive graphs and their direct products. In Section § 3.3, we prove Theorem 3.4 and in Section § 3.4, we prove Theorem 3.5. In Section § 3.5, we conclude this chapter and discuss some remaining problems. For the convenience of the proof, if there is no confusion, we will denote $\prod_{i\in[p]} A_i$ as the subset $\bigsqcup_{i\in[p]} A_i \subseteq \bigsqcup_{i\in[p]} S_i$ in the rest of this chapter.

§ 3.2 Preliminary results

3.2.1 Independent sets of vertex-transitive graphs

Given a finite set X, for $A\subseteq X$, denote $\bar{A}=X\setminus A$. For a simple graph $G=G(V,E)$, denote $\alpha(G)$ as the independent number of G and $I(G)$ as the set of all maximum independent sets of G. For $v\in V(G)$, define the neighborhood $N_G(v)=\{u\in V(G):(u,v)\in E(G)\}$. For a subset $A\subseteq V(G)$, write $N_G(A)=\{b\in V(G):(a,b)\in E(G)$ for some $a\in A\}$ and $N_G[A]=A\cup N_G(A)$, if there is no confusion, we denote them as $N(A)$ and $N[A]$ for short respectively.

A graph G is said to be vertex-transitive if its automorphism group $\Gamma(G)$ acts transitively upon its vertices. As a corollary of the "No-Homomorphism" lemma for vertex-transitive graphs in [7], Cameron and Ku [41] proved the following theorem.

Theorem 3.6. ([41]) *Let* G *be a vertex-transitive graph and* B *a subset of* $V(G)$*. Then any independent set* S *in* G *satisfies that* $\frac{|S|}{|V(G)|}\leq\frac{\alpha(G[B])}{|B|}$*, equality implies that* $|S\cap B|=\alpha(G[B])$*.*

Using the above theorem, Zhang [211] proved the following result.

Lemma 3.1. ([211]) *Let G be a vertex-transitive graph, and A be an independent set of G, then $\frac{|A|}{|N_G[A]|} \leq \frac{\alpha(G)}{|G|}$. Equality implies that $|S \cap N_G[A]| = |A|$ for every $S \in I(G)$, and in particular $A \subseteq S$ for some $S \in I(G)$.*

An independent set A in G is said to be imprimitive if $|A| < \alpha(G)$ and $\frac{|A|}{|N[A]|} = \frac{\alpha(G)}{|G|}$, and G is called IS-imprimitive if G has an imprimitive independent set. Otherwise, G is called IS-primitive. Note that a disconnected vertex-transitive graph G is always IS-imprimitive. Hence IS-primitive vertex-transitive graphs are all connected.

The following inequality about the size of an independent set A and its non-neighbors $\bar{N}[A]$ is crucial for the proof of Theorem 3.4.

Lemma 3.2. *Let G be a vertex transitive graph, and let A be an independent set of G. Then $|A|+\frac{\alpha(G)}{|G|}|\bar{N}[A]| \leq \alpha(G)$. Equality holds if and only if $A = \emptyset$ or $|A| = \alpha(G)$ or A is an imprimitive independent set.*

For the integrity of the thesis, we include the proof here. In [198], Wang and Zhang proved the same inequality for a more generalized combinatorial structure called *symmetric system* (see [198], Corollary 2.4).

Proof. If $A = \emptyset$ or $A = \alpha(G)$, the equality trivially holds. Suppose $0 < |A| < \alpha(G)$, and let B be a maximal independent set in $\bar{N}[A]$, then $|B| = \alpha(\bar{N}(A))$. Clearly, $A \cup B$ is also an independent set of G, thus we have $|A| + |B| \leq \alpha(G)$. By Theorem 3.6, we obtain that $\frac{|B|}{|\bar{N}[A]|} \geq \frac{\alpha(G)}{|G|}$. Therefore,

$$|A| + \frac{\alpha(G)}{|G|}|\bar{N}[A]| \leq |A| + |B| \leq \alpha(G),$$

the equality holds when $\alpha(G) = |A| + \frac{\alpha(G)}{|G|}|\bar{N}[A]| = |A| + \frac{\alpha(G)}{|G|}(|G| - |N[A]|)$, which leads to $\frac{|A|}{|N[A]|} = \frac{\alpha(G)}{|G|}$, i.e., A is an imprimitive independent set. □

Let X be a finite set, and Γ a group acting transitively on X. Then Γ is said to be primitive on X if it preserves no nontrivial partition of X. A vertex-transitive graph G is called primitive if the automorphism $\mathrm{Aut}(G)$ is primitive on

$V(G)$. To show the connection between the primitivity and the IS-primitivity of a vertex-transitive graph G, Zhang (see Proposition 2.4 in [211]) proved that if G is primitive, then it must be IS-primitive. As a consequence of this result, Wang and Zhang [198] derived the IS-primitivity of the Kneser graph.

Proposition 3.1. ([198]) *The Kneser graph $KG_{n,k}$ is IS-primitive except for $n = 2k \geq 4$.*

In order to deal with the multi-part case, we also need the results about the independent sets in direct products of vertex-transitive graphs. Let G and H be two graphs, the direct product $G \times H$ of G and H is defined by

$$V(G \times H) = V(G) \times V(H),$$
$$E(G \times H) = \{[(u_1, v_1), (u_2, v_2)] : (u_1, u_2) \in E(G) \text{ and } (v_1, v_2) \in E(H)\}.$$

Clearly, $G \times H$ is a graph with $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$ as its automorphism group. And, if G, H are vertex-transitive, then $G \times H$ is also vertex-transitive under the actions of $\mathrm{Aut}(G) \times \mathrm{Aut}(H)$. We say the direct product $G \times H$ is MIS-normal (maximum-independent-set-normal) if every maximum independent set of $G \times H$ is a preimage of an independent set of one factor under projections.

In [212], Zhang obtained the exact structure of the maximal independent set of $G \times H$.

Theorem 3.7. ([212]) *Let G and H be two vertex-transitive graphs with $\frac{\alpha(G)}{|G|} \geq \frac{\alpha(H)}{|H|}$. Then $\alpha(G \times H) = \alpha(G)|H|$, and exactly one of the following holds:*

(i) $G \times H$ is MIS-normal;

(ii) $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and one of G or H is IS-imprimitive;

(iii) $\frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|}$ and H is disconnected.

In fact, if $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and A is an imprimitive independent set of G, then for every $I \in I(H)$, $S = (A \times V(H)) \cup (\bar{N}[A] \times I)$ is an independent set of $G \times H$ with size $\alpha(G)|H|$.

Zhang [211] also investigated the relationship between the graph primitivity and the structures of the maximum independent sets in direct products of vertex-transitive graphs.

Theorem 3.8. ([211]) *Suppose $G \times H$ is MIS-normal and $\frac{\alpha(H)}{|H|} \leq \frac{\alpha(G)}{|G|}$. If $G \times H$ is IS-imprimitive, then one of the following two possible cases holds:*

(i) $\frac{\alpha(G)}{|G|} = \frac{\alpha(H)}{|H|}$ and one of them is IS-imprimitive or both G and H are bipartite;

(ii) $\frac{\alpha(G)}{|G|} > \frac{\alpha(H)}{|H|}$ and G is IS-imprimitive.

As an application of Theorem 3.7 and Theorem 3.8, Geng et al. [87] showed the MIS-normality of the direct products of Kneser graphs.

Theorem 3.9. ([87]) *Given a positive integer p, let $n_1, n_2, \ldots, n_p$ and $k_1, k_2, \ldots, k_p$ be $2p$ positive integers with $n_i \geq 2k_i$ for $1 \leq i \leq p$. Then the direct product of the Kneser graph*

$$KG_{n_1,k_1} \times KG_{n_2,k_2} \times \cdots \times KG_{n_p,k_p}$$

is MIS-normal except that there exist i, j and ℓ with $n_i = 2k_i \geq 4$ and $n_j = 2k_j$, or $n_i = n_j = n_\ell = 2$.

3.2.2 Nontrivial independent sets of part-transitive bipartite graphs

For a bipartite graph $G(X, Y)$ with two parts X and Y, an independent set A is said to be non-trivial if $A \nsubseteq X$ and $A \nsubseteq Y$. $G(X, Y)$ is said to be part-transitive if there is a group Γ acting transitively upon each part and preserving its adjacency relations. Clearly, if $G(X, Y)$ is part-transitive, then every vertex of X (Y) has the same degree, written as $d(X)$ ($d(Y)$). We use $\alpha(X, Y)$ and $I(X, Y)$ to denote the size and the set of the maximum-sized nontrivial independent sets of $G(X, Y)$, respectively.

Let $G(X, Y)$ be a non-complete bipartite graph and let $A \cup B$ be a nontrivial independent set of $G(X, Y)$, where $A \subseteq X$ and $B \subseteq Y$. Then $A \subseteq X \setminus N(B)$ and $B \subseteq Y \setminus N(A)$, which implies

$$|A| + |B| \leq \max\{|A| + |Y| - |N(A)|, |B| + |X| - |N(B)|\}.$$

So we have

$$\alpha(X,Y) = \max\{|Y| - \epsilon(X), |X| - \epsilon(Y)\}, \tag{3.4}$$

where $\epsilon(X) = \min\{|N(A)|-|A| : A \subseteq X, N(A) \neq Y\}$ and $\epsilon(Y) = \min\{|N(B)|-|B| : B \subseteq Y, N(B) \neq X\}$.

We call $A \subseteq X$ a fragment of $G(X,Y)$ in X if $N(A) \neq Y$ and $|N(A)| - |A| = \epsilon(X)$, and we denote $\mathcal{F}(X)$ as the set of all fragments in X. Similarly, we can define $\mathcal{F}(Y)$. Furthermore, denoting $\mathcal{F}(X,Y) = \mathcal{F}(X) \cup \mathcal{F}(Y)$, we call an element $A \in \mathcal{F}(X,Y)$ a k-fragment if $|A| = k$. And we call a fragment $A \in \mathcal{F}(X)$ trivial if $|A| = 1$ or $A = X \setminus N(b)$ for some $b \in Y$. Since for each $A \in \mathcal{F}(X)$, $Y \setminus N(A)$ is a fragment in $\mathcal{F}(Y)$. Hence, once we know $\mathcal{F}(X)$, $\mathcal{F}(Y)$ can also be determined.

Let X be a finite set, and Γ a group acting transitively on X. If Γ is imprimitive on X, then it preserves a nontrivial partition of X, called a block system, each element of which is called a block. Clearly, if Γ is both transitive and imprimitive, there must be a subset $B \subseteq X$ such that $1 < |B| < |X|$ and $\gamma(B) \cap B = B$ or $\emptyset$ for every $\gamma \in \Gamma$. In this case, B is called an imprimitive set in X. Furthermore, a subset $B \subseteq X$ is said to be *semi-imprimitive* if $1 < |B| < |X|$ and for each $\gamma \in \Gamma$ we have $\gamma(B) \cap B = B$, $\emptyset$ or $\{b\}$ for some $b \in B$.

The following theorem (cf. [110, Theorem 1.12]) is a classical result on the primitivity of group actions.

Theorem 3.10. ([110]) *Suppose that a group Γ transitively acts on X. Then Γ is primitive on X if and only if for each $a \in X$, Γ_a is a maximal subgroup of Γ. Here $\Gamma_a = \{\gamma \in \Gamma : \gamma(a) = a\}$, the stabilizer of $a \in X$.*

Noticing the similarities about cross-t-intersecting and cross-Sperner families, Wang and Zhang [199] proved the following theorem about $\alpha(G(X,Y))$ and $I(X,Y)$ of a special kind of part-transitive bipartite graphs.

Theorem 3.11. ([199]) *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| \leq |Y|$. If $G(X,Y)$ is part-transitive and every fragment of $G(X,Y)$ is primitive under the action of a group Γ. Then $\alpha(X,Y) = |Y| - d(X) + 1$. Moreover,* *(1) If $|X| < |Y|$, then X has only 1-fragments;*

(2) If $|X| = |Y|$, then each fragment in X has size 1 or $|X| - d(X)$ unless there is a semi-imprimitive fragment in X or Y.

To deal with multi-part cross-intersecting families, we introduce the following variation of Theorem 3.11.

Theorem 3.12. *Let $G(X,Y)$ be a non-complete bipartite graph with $|X| \leq |Y|$. If $G(X,Y)$ is part-transitive under the action of a group Γ. Then*

$$\alpha(X,Y) = \max\{|Y| - d(X) + 1, |A'| + |Y| - |N(A')|, |B'| + |X| - |N(B')|\}, \quad (3.5)$$

where A' and B' are minimum imprimitive subsets of X and Y respectively. By minimum, here we mean that

$$|N(A')| - |A'| = \min\{|N(A)| - |A| : A \in X \text{ (or } Y\text{) is imprimitive}\}.$$

For the proof of Theorem 3.12, we need the following two lemmas from [199].

Lemma 3.3. ([199]) *Let $G(X,Y)$ be a non-complete bipartite graph. Then, $|Y| - \epsilon(X) = |X| - \epsilon(Y)$, and*

(i) $A \in \mathcal{F}(X)$ if and only if $(Y \setminus N(A)) \in \mathcal{F}(Y)$ and $N(Y \setminus N(A)) = X \setminus A$;

(ii) $A \cap B$ and $A \cup B$ are both in $\mathcal{F}(X)$ if A, $B \in \mathcal{F}(X)$, $A \cap B \neq \emptyset$ and $N(A \cup B) \neq Y$.

Lemma 3.4. ([199]) *Let $G(X,Y)$ be a non-complete and part-transitive bipartite graph under the action of a group Γ. Suppose that $A \in \mathcal{F}(X,Y)$ such that $\emptyset \neq \gamma(A) \cap A \neq A$ for some $\gamma \in \Gamma$. Define $\phi : \mathcal{F}(X,Y) \to \mathcal{F}(X,Y)$,*

$$\phi(A) = \begin{cases} Y \setminus N(A), & \text{if } A \in \mathcal{F}(X); \\ X \setminus N(A), & \text{if } A \in \mathcal{F}(Y). \end{cases}$$

If $|A| \leq |\phi(A)|$, then $A \cup \gamma(A)$ and $A \cap \gamma(A)$ are both in $\mathcal{F}(X,Y)$.

Remark 3.1. *As a direct consequence of Lemma 3.3, a maximum-sized nontrivial independent set in $G(X,Y)$ is of the form $A \sqcup (Y \setminus N(A))$ for some $A \in \mathcal{F}(X)$ or $B \sqcup (X \setminus N(B))$ for some $B \in \mathcal{F}(Y)$. Therefore, in order to address our problems, it suffices to determine $\mathcal{F}(X)$ (or $\mathcal{F}(Y)$).*

Meanwhile, for the mapping ϕ in Lemma 3.4, we have $\phi^{-1} = \phi$ and $|A| + |\phi(A)| = \alpha(X,Y)$. When $|A| = |\phi(A)|$, we call the fragment A balanced. Thus, all balanced fragments have size $\frac{1}{2}\alpha(X,Y)$.

Proof of Theorem 3.12. The same as the original proof of Theorem 3.11 in [199], we apply Lemma 3.4 repeatedly. For any $A_0 \in \mathcal{F}(X,Y)$ satisfying $|A_0| \le |\phi(A_0)|$, if there exists $\gamma \in \Gamma$ such that $\emptyset \ne \gamma(A_0) \cap A_0 \ne A_0$, then by Lemma 3.4 we have: (1) $A_0 \cap \gamma(A_0) \in \mathcal{F}(X,Y)$ or (2) $\gamma(A_0) \cap A_0 = \emptyset$ or $\gamma(A_0) \cap A_0 = A_0$ for any $\gamma \in \Gamma$.

For case (1), denote

$$A_1 = \begin{cases} A_0 \cap \gamma(A_0), & \text{if } |A_0 \cap \gamma(A_0)| \le |\phi(A_0 \cap \gamma(A_0))|; \\ \phi(A_0 \cap \gamma(A_0)), & \text{otherwise}; \end{cases}$$

and consider the primitivity of A_1, i.e., whether there is a $\gamma' \in \Gamma$ such that $\emptyset \ne \gamma'(A_1) \cap A_1 \ne A_1$ or not.

For case (2), if $|A_0| \ne 1$, according to the definition, A_0 is an imprimitive set of X (or Y). Otherwise, $|A_0| = 1$, which means $\mathcal{F}(X,Y)$ contains a singleton.

By doing these procedures repeatedly, after r $(0 \le r \le |A_0| - 1)$ steps, we have a fragment $A_r \in \mathcal{F}(X,Y)$ such that A_r is either a singleton or an imprimitive set. Hence, we have $\alpha(X,Y) =$

$$\max\{|Y| - d(X) + 1, |X| - d(Y) + 1, |A'| + |Y| - |N(A')|, |B'| + |X| - |N(B')|\},$$

where A' and B' are minimum imprimitive subsets of X and Y respectively. Noticing that $|Y| \ge |X|$ and $d(X)|X| = d(Y)|Y|$, we have $d(X) = d(Y)|Y|/|X| \ge d(Y)$. Therefore,

$$|Y| - |X| = d(X)|X|/d(Y) - |X| = (d(X) - d(Y))|X|/d(Y) \ge d(X) - d(Y),$$

which implies that $|X| - d(Y) + 1 \le |Y| - d(X) + 1$. Finally we have

$$\alpha(X,Y) = \max\{|Y| - d(X) + 1, |A'| + |Y| - |N(A')|, |B'| + |X| - |N(B')|\}.$$

□

§ 3.3 Proof of Theorem 3.4

Throughout this section, for any nonempty subset $S \subseteq [p]$ and $A = \prod_{i\in S} A_i \in \prod_{i\in S} \binom{[n_i]}{k_i}$, denote $\bar{A} = \prod_{i\in S} \bar{A}_i$. Before we start the proof of Theorem 3.4, we introduce the following proposition about the direct product of Kneser graphs.

Proposition 3.2. *Given a positive integer p, let $n_1, n_2, \ldots, n_p$ and $k_1, k_2, \ldots, k_p$ be positive integers with $n_i \geq 2k_i$ for $1 \leq i \leq p$. Let $G = \prod_{i\in[p]} KG_{n_i,k_i}$. Then G is IS-imprimitive if and only if there exists an $i \in [p]$ such that $n_i = 2k_i \geq 4$ or there exist distinct $i, j \in [p]$ such that $n_i = n_j = 2$ and $k_i = k_j = 1$.*

Proof. Note that if the Kneser graph $KG_{n,k}$ is disconnected, then $n = 2k \geq 4$ and $KG_{n,k}$ is bipartite. Thus by Proposition 3.1, $KG_{2k,k}$ is IS-imprimitive for all $k \geq 2$. Moreover, since $\chi(KG_{n,k}) = n - 2k + 2$ for all $n \geq 2k$ (Lovász-Kneser Theorem, see [136]), we know that if $KG_{n,k}$ is bipartite, then $n = 2k \geq 2$. Now we use induction on the number of factors p.

If $p = 2$, let $G_1 = KG_{n_1,k_1}$, $G_2 = KG_{n_2,k_2}$, and $G = G_1 \times G_2$. W.l.o.g., assume that $\frac{\alpha(G_1)}{|G_1|} \geq \frac{\alpha(G_2)}{|G_2|}$. Then, by Theorem 3.7, (i) $G_1 \times G_2$ is MIS-normal, or (ii) $\frac{\alpha(G_1)}{|G_1|} = \frac{\alpha(G_2)}{|G_2|}$ and one of G_1 and G_2 is IS-imprimitive, or (iii) $\frac{\alpha(G_1)}{|G_1|} > \frac{\alpha(G_2)}{|G_2|}$ and G_2 is disconnected. For case (i), by Theorem 3.8, at least one factor of G is IS-imprimitive or both G_1 and G_2 are bipartite. Noticed that $KG_{2,1}$ is IS-primitive, therefore, either there exists an $i \in [2]$ such that $n_i = 2k_i \geq 4$ or there exist distinct $i, j \in [2]$ such that $n_i = n_j = 2k_i = 2k_j = 2$. For cases (ii) and (iii), since G is not MIS-normal, by Theorem 3.9, at least one of G_1 and G_2 is IS-imprimitive. Thus the proposition holds when $p = 2$.

Suppose the proposition holds when the number of factors is $p - 1$. Set $G'_1 = \prod_{i=1}^{p-1} KG_{n_i,k_i}$ and $G'_2 = KG_{n_p,k_p}$, by Theorem 3.8, at least one factor of G'_1 and G'_2 is IS-imprimitive or both G'_1 and G'_2 are bipartite. If G'_1 is IS-imprimitive, by the induction hypothesis, there exists an $i' \in [p-1]$ such that $n_{i'} = 2k_{i'} \geq 4$ or there exist distinct $i', j' \in [p-1]$ such that $n_{i'} = n_{j'} = 2k_{i'} = 2k_{j'} = 2$. If G'_2 is IS-imprimitive, then $n_p = 2k_p \geq 4$. Otherwise, both G'_1 and G'_2 are IS-primitive and bipartite. Thus, for G'_2, we have $n_p = 2k_p = 2$. For G'_1, since $\chi(G'_1) \cdot \alpha(G'_1) \geq |V(G'_1)|$, we know that there exists $i' \in [p-1]$ such that $n_{i'} = 2k_{i'} = 2$ by Theorem 3.7. This completes the proof. □

The idea of the proof of Theorem 3.4 is similar to that for general connected symmetric systems in [198]. Since $\prod_{i=1}^{p} \mathrm{KG}_{n_i,k_i}$ is a vertex transitive graph, by Lemma 3.2, we can prove (3.1). Then, through a careful analysis, we can obtain the structure of all imprimitive independent sets of this graph. This leads to the unique structure of $\mathcal{A}_1$ and $\mathcal{A}_2$ in (3.2).

Proof of Theorem 3.4. Define a graph G on the vertex set $X = \prod_{s\in[p]} \binom{[n_s]}{k_s}$ with $A, B \in X$ forming an edge in G if and only if $A \cap B = \emptyset$. Therefore, G is the direct product of Kneser graphs $\mathrm{KG}_{n_1,k_1} \times \cdots \times \mathrm{KG}_{n_p,k_p}$.

Assume that $2 \leq \frac{n_1}{k_1} \leq \frac{n_2}{k_2} \leq \ldots \leq \frac{n_p}{k_p}$, then $\frac{|G|}{\alpha(G)} = \frac{n_1}{k_1}$ by Theorem 3.7. Following the notations of Borg in [31–33], write $\mathcal{A}_i^* = \{A \in \mathcal{A}_i | A \cap B \neq \emptyset \text{ for any } B \in \mathcal{A}_i\}$, $\hat{\mathcal{A}}_i = \mathcal{A}_i \setminus \mathcal{A}_i^*$, $\mathcal{A}^* = \bigcup_{i=1}^m \mathcal{A}_i^*$, $\hat{\mathcal{A}} = \bigcup_{i=1}^m \hat{\mathcal{A}}_i$. Note that $\bar{N}_G[\mathcal{A}] = \{B \in X | A \cap B \neq \emptyset, \text{ for any } A \in \mathcal{A}\}$ for $\mathcal{A} \subseteq X$, it is easy to show that $\mathcal{A}^*$ is an intersecting family and $\hat{\mathcal{A}} \subseteq \bar{N}_G[\mathcal{A}^*]$. It follows that $\mathcal{A}_i \cap \mathcal{A}_j \subseteq \mathcal{A}_i^* \cap \mathcal{A}_j^*$ from the definition, therefore $\hat{\mathcal{A}}_i \cap \hat{\mathcal{A}}_j = \emptyset$ for $i \neq j$, and $|\hat{\mathcal{A}}| = \sum_{i=1}^m |\hat{\mathcal{A}}_i|$. Thus by Lemma 3.2 we have

$$\begin{aligned}
\sum_{i=1}^{m} |\mathcal{A}_i| &= \sum_{i=1}^{m} |\hat{\mathcal{A}}_i| + \sum_{i=1}^{m} |\mathcal{A}_i^*| \leq |\hat{\mathcal{A}}| + m|\mathcal{A}^*| \leq |\bar{N}_G[\mathcal{A}^*]| + m|\mathcal{A}^*| \\
&= \frac{|G|}{\alpha(G)}\left(\frac{\alpha(G)}{|G|}|\bar{N}_G[\mathcal{A}^*]| + |\mathcal{A}^*|\right) + \left(m - \frac{|G|}{\alpha(G)}\right)|\mathcal{A}^*| \\
&\leq |G| + \left(m - \frac{|G|}{\alpha(G)}\right)|\mathcal{A}^*| = |G| + \left(m - \frac{n_1}{k_1}\right)|\mathcal{A}^*|.
\end{aligned}$$

If $m < \frac{n_1}{k_1}$, then $\sum_{i=1}^m |\mathcal{A}_i| \leq |G|$, and the equality implies $\mathcal{A}^* = \emptyset$. Thus $\mathcal{A}_i = \hat{\mathcal{A}}_i$ for every $i \in [m]$, and this yields that the graph G is a disjoint union of the induced subgraph $G[\mathcal{A}_i]'s$. And by the cross-intersecting property, each $G[\mathcal{A}_i]$ is a connected component of G. Since G is connected when $\frac{n_s}{k_s} > 2$ for all $s \in [p]$ and $m \geq 2$, we know that one of $\mathcal{A}_i$ is X and the rest are empty sets, as case (i).

If $m > \frac{n_1}{k_1}$, then $\sum_{i=1}^{m} |\mathcal{A}_i| \leq m\alpha(G)$, and the equality implies that $\mathcal{A}_1^* = \ldots = \mathcal{A}_m^* = \mathcal{A}^*$, $|\mathcal{A}^*| = \alpha(G)$, as case (ii).

If $m = \frac{n_1}{k_1}$, then $\sum_{i=1}^{m} |\mathcal{A}_i| \leq |X|$, and the equality implies that $\mathcal{A}_1^* = \ldots = \mathcal{A}_m^* = \mathcal{A}^*$ and $\frac{\alpha(G)}{|G|}|\bar{N}_G[\mathcal{A}^*]| + |\mathcal{A}^*| = \alpha(G)$. By Lemma 3.2, we know that $|\mathcal{A}^*| = 0$, or $|\mathcal{A}^*| = \alpha(G)$, or $\mathcal{A}^*$ is an imprimitive independent set of G. In the last case, $\hat{\mathcal{A}}_1, \ldots, \hat{\mathcal{A}}_m$ are cross-intersecting families and form a partition of $\bar{N}_G[\mathcal{A}^*]$. In order to determine the structures of the maximum-sized cross-intersecting families in this case, we shall characterize the imprimitive independent set of G.

Claim .3. *Let $\mathcal{F} = \prod_{s\in S} \binom{[n_s]}{k_s}$ and $X' = \prod_{s\in[p]\setminus S} \binom{[n_s]}{k_s}$, where $S = \{s \in [p] : \frac{n_s}{k_s} = 2\}$. If $\mathcal{A}^*$ is an imprimitive independent set of G, then $\mathcal{A}^* = \mathcal{A} \times X'$, where $\mathcal{A} \subseteq \mathcal{F}$ is a non-maximum intersecting family.*

According to Proposition 3.2, G is IS-imprimitive if and only if there exists an $i \in S$ such that $n_i = 2k_i \geq 4$ or there exist distinct $i, j \in S$ such that $n_i = n_j = 2$ and $k_i = k_j = 1$. Thus, with the assumptions in this claim, $S \neq \emptyset$ and $S = \{i_0\}$ if and only if $n_{i_0} = 2k_{i_0} \geq 4$ for some $i_0 \in [p]$. W.l.o.g., assume that $S = [s_0]$, where $s_0 = |S|$. Under this circumstance, $m = \frac{n_1}{k_1} = 2$.

Divide $\mathcal{A}^*$ into u disjoint parts $\{C_i \times \mathcal{D}_i\}_{i=1}^{u}$, where $C_i = C_{i,1} \times \ldots \times C_{i,s_0} \in \mathcal{F}$, $\mathcal{D}_i \subseteq X'$ for all $i \in [u]$ and $C_i \neq C_j$ for any $i \neq j \in [u]$. Since $N_G(C_i \times \mathcal{D}_i) = \bar{C}_i \times \mathcal{D}'_i$, where $\mathcal{D}'_i = \{A \in X' : A \cap D_i = \emptyset \text{ for some } D_i \in \mathcal{D}_i\}$, we know that $N_G[C_i \times \mathcal{D}_i] \cap N_G[C_j \times \mathcal{D}_j] = \emptyset$ for all $i \neq j \in [u]$. Meanwhile, $C_i \times \mathcal{D}_i \cap N_G(C_j \times \mathcal{D}_j) = \emptyset$ for all $i \neq j \in [u]$. Otherwise, assume that there exists $T_1 \times T_2 \in C_i \times \mathcal{D}_i \cap N_G(C_j \times \mathcal{D}_j)$, for some $T_1 \in \mathcal{F}$ and $T_2 \in X'$. Thus we have $T_1 \times T_2 \cap C_j \times D_j = \emptyset$, for some $D_j \in \mathcal{D}_j$, which contradicts the fact that $\mathcal{A}^*$ is an intersecting family.

By projecting G onto the last $p - s_0$ factors, we obtain a graph G' with vertex set X' such that $A, B \in X'$ form an edge in G' if and only if A, B are disjoint. Consider the cross-intersecting families $\{\mathcal{D}_i, \bar{N}_{G'}(\mathcal{D}_i)\}$ in X', since $|\{\mathcal{D}_i, \bar{N}_{G'}(\mathcal{D}_i)\}| = 2 < \frac{n_{s_0+1}}{k_{s_0+1}}$, by case (i), we know that

$$|\mathcal{D}_i| + |\bar{N}_{G'}(\mathcal{D}_i)| = |\mathcal{D}_i| + |X'| - |N_{G'}(\mathcal{D}_i)| \leq |X'|,$$

thus we have $|\mathcal{D}_i| \leq |N_{G'}(\mathcal{D}_i)|$, and $|C_i \times \mathcal{D}_i| = |\mathcal{D}_i| \leq |N_{G'}(\mathcal{D}_i)| = |N_G(C_i \times \mathcal{D}_i)|$. Therefore

$$\frac{|\mathcal{A}^*|}{|N_G[\mathcal{A}^*]|} = \frac{\sum_{i\in[u]} |C_i \times \mathcal{D}_i|}{\sum_{i\in[u]} |N_G[C_i \times \mathcal{D}_i]|} \leq \frac{1}{2} = \frac{\alpha(G)}{|G|} = \frac{k_1}{n_1},$$

and the equality holds if and only if for all $i \in [u]$, $\mathcal{D}_i = X'$ or $\bar{N}_{G'}(\mathcal{D}_i) = X'$. Since $\mathcal{D}_i \neq \emptyset$, we have $\mathcal{A}^* = \bigsqcup_{i=1}^{u} C_i \times X' = \mathcal{A} \times X'$. Recall that $\frac{n_s}{k_s} > 2$ for all $s > s_0$, hence $C_i \cap C_j \neq \emptyset$ for any $i \neq j \in [u]$. Therefore, by the imprimitivity of $\mathcal{A}^*$, $\mathcal{A}^*$ is a non-maximum independent set of G, thus $\mathcal{A} \subseteq \mathcal{F}$ is a non-maximal intersecting family and the claim holds.

For every intersecting family $\mathcal{A} \subseteq \mathcal{F}$, since $\frac{n_s}{k_s} = 2$ for all $s \in S$, then $\mathcal{A} = \{A_1, A_2, \ldots, A_w\} \times \prod_{s\in S\setminus S'} \binom{[n_s]}{k_s}$ for some nonempty subset $S' \subseteq S$, where $\{A_1, \ldots, A_w\} \subseteq \prod_{s\in S'} \binom{[n_s]}{k_s}$ satisfying $A_i \neq \bar{A}_j$ for all $i \neq j \in [w]$. In particular, if $\mathcal{A}$ is a maximum intersecting family, we can obtain that $\bigsqcup_{j=1}^{w}\{A_j, \bar{A}_j\} = \prod_{s\in S'} \binom{[n_s]}{k_s}$ and $2w = \prod_{s\in S'} \binom{n_s}{k_s}$.

Therefore,

$$\mathcal{A}^* = \{A_1, A_2, \ldots, A_{w_0}\} \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X'$$

and

$$N_G(\mathcal{A}^*) = \{\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_{w_0}\} \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X'$$

for some positive integer $w_0 < \frac{\prod_{s\in S_1} \binom{n_s}{k_s}}{2}$ and nonempty subset $S_1 \subseteq S$.

From the structure of the imprimitive independent set $\mathcal{A}^*$, we know that

$$\bar{N}_G[\mathcal{A}^*] = \{E_1, \bar{E}_1, E_2, \bar{E}_2, \ldots, E_v, \bar{E}_v\} \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X',$$

where $\emptyset \neq \{E_1, \ldots, E_v\} \subseteq \prod_{s\in S_1} \binom{[n_s]}{k_s}$, and

$$\bigsqcup_{j=1}^{w_0}\{A_j, \bar{A}_j\} \sqcup \bigsqcup_{j=1}^{v}\{E_j, \bar{E}_j\} = \prod_{s\in S_1} \binom{[n_s]}{k_s}.$$

Since $E_j \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X'$ and $\bar{E}_j \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X'$ must be contained in the same one of $\hat{\mathcal{A}}_1$, $\hat{\mathcal{A}}_2$, we have

$$\hat{\mathcal{A}}_1 = (\mathcal{E} \cup \tilde{\mathcal{E}}) \times \prod_{s\in S\setminus S_1} \binom{[n_s]}{k_s} \times X',$$

$$\hat{\mathcal{A}}_2 = (\mathcal{E}' \cup \tilde{\mathcal{E}}') \times \prod_{s \in S \setminus S_1} \binom{[n_s]}{k_s} \times X',$$

where $\mathcal{E} \sqcup \mathcal{E}' = \{E_1, \ldots, E_v\}$ and $\tilde{\mathcal{E}} \sqcup \tilde{\mathcal{E}}' = \{\bar{E}_1, \ldots, \bar{E}_v\}$. Here we denote $\tilde{\mathcal{E}} = \{\bar{E}_{i_1}, \ldots, \bar{E}_{i_l}\}$ if $\mathcal{E} = \{E_{i_1}, \ldots, E_{i_l}\} \subseteq \prod_{s \in S_1} \binom{[n_s]}{k_s}$, for some subset $\{i_1, \ldots, i_l\} \subseteq [v]$.

Finally, to sum up,

$$\mathcal{A}_1 = \mathcal{A}^* \sqcup \hat{\mathcal{A}}_1 = (\mathcal{A} \times X') \sqcup ((\mathcal{E} \cup \tilde{\mathcal{E}}) \times \prod_{s \in S \setminus S_1} \binom{[n_s]}{k_s} \times X'),$$

$$\mathcal{A}_2 = \mathcal{A}^* \sqcup \hat{\mathcal{A}}_2 = (\mathcal{A} \times X') \sqcup ((\mathcal{E}' \cup \tilde{\mathcal{E}}') \times \prod_{s \in S \setminus S_1} \binom{[n_s]}{k_s} \times X').$$

□

§ 3.4 Proof of Theorem 3.5

Throughout this section, we denote S_n as the symmetric group on $[n]$ and S_C as the symmetric group on C for $C \subseteq [n]$. For each $i \in [p]$, let X_i be a finite set, then for each family $\mathcal{A} \subseteq \prod_{i \in [p]} X_i$, we denote $\mathcal{A}|_i$ as the projection of $\mathcal{A}$ onto the i-th factor.

For the proof of Theorem 3.5, we need the following proposition obtained by Wang and Zhang in [199].

Proposition 3.3. ([199]) *Let $G(X, Y)$ be a non-complete bipartite graph with $|X| = |Y|$ and $\epsilon(X) = d(X) - 1$, and let Γ be a group part-transitively acting on $G(X, Y)$. If each fragment of $G(X, Y)$ is primitive and there are no 2-fragments in $\mathcal{F}(X, Y)$, then every nontrivial fragment $A \in \mathcal{F}(X)$ (if there exists) is balanced (see Remark 3.1), and for each $a \in A$, there is a unique nontrivial fragment B such that $A \cap B = \{a\}$.*

The proof of Theorem 3.5 is divided into two parts: Firstly, we prove the bound (3.3). Consider a non-complete bipartite graph defined by the multi-part cross-intersecting family. Through discussions about the primitivity of group $\prod_{i=1}^{p} S_{n_i}$ and careful evaluations about $|\mathcal{A}| + |\mathcal{Y}| - |N(\mathcal{A})|$, the bound (3.3) follows from Theorem 3.12. Secondly, based on a characterization of all nontrivial fragments in this bipartite graph, we determine all the structures of $\mathcal{A}$ and $\mathcal{B}$ when the bound (3.3) is attained.

Proof of Theorem 3.5. With the assumptions in the theorem, we define a bipartite graph $G(\mathcal{X},\mathcal{Y})$ with $\mathcal{X}=\prod_{i=1}^{p}\binom{[n_i]}{t_i}$ and $\mathcal{Y}=\prod_{i=1}^{p}\binom{[n_i]}{s_i}$. For $A=\prod_{i=1}^{p}A_i\in\mathcal{X}$ and $B=\prod_{i=1}^{p}B_i\in\mathcal{Y}$ ($A_i\in\binom{[n_i]}{t_i}$ and $B_i\in\binom{[n_i]}{s_i}$, for every $1\le i\le p$), (A,B) forms an edge in $G(\mathcal{X},\mathcal{Y})$ if and only if $A\cap B=\emptyset$, i.e., $A_i\cap B_i=\emptyset$ for each $1\le i\le p$.

It can be easily verified that $\prod_{i=1}^{p}S_{n_i}$ acts transitively on $\mathcal{X}$ and $\mathcal{Y}$, respectively, and preserves the property of cross-intersecting. Thus we have $d(\mathcal{X})=|N(A)|$ for each $A\in\mathcal{X}$, and $d(\mathcal{Y})=|N(B)|$ for each $B\in\mathcal{Y}$. Since, for each $A=\prod_{i=1}^{p}A_i\in\mathcal{X}$,

$$N(A)=\{B=\prod_{i=1}^{p}B_i\in\mathcal{Y}:\ A_i\cap B_i=\emptyset \text{ for each } 1\le i\le p\}=\prod_{i=1}^{p}\binom{[n_i]\setminus A_i}{s_i},$$

we have $d(\mathcal{X})=|N(A)|=\prod_{i=1}^{p}\binom{n_i-t_i}{s_i}$. Similarly, $d(\mathcal{Y})=|N(B)|=\prod_{i=1}^{p}\binom{n_i-s_i}{t_i}$.

By Theorem 3.12, we obtain that

$$\alpha(\mathcal{X},\mathcal{Y})=\max\{|\mathcal{Y}|-d(\mathcal{X})+1,|\mathcal{A}'|+|\mathcal{Y}|-|N(\mathcal{A}')|,|\mathcal{B}'|+|\mathcal{X}|-|N(\mathcal{B}')|\},$$

where $\mathcal{A}'$ and $\mathcal{B}'$ are minimum imprimitive subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Therefore, in order to estimate $\alpha(\mathcal{X},\mathcal{Y})$ accurately, more discussions about the sizes and the structures of the imprimitive subsets of $\mathcal{X}$ and $\mathcal{Y}$ are necessary.

Claim .4. *Let $\mathcal{A}$ and $\mathcal{B}$ be imprimitive subsets of $\mathcal{X}$ and $\mathcal{Y}$ respectively, then*

$$\mathcal{A}=\prod_{i\in T_1}\{A_i,\bar{A}_i\}\times\prod_{i\in T_2}\{A_i\}\times\prod_{i\in[p]\setminus(T_1\sqcup T_2)}\binom{[n_i]}{t_i},\ \textit{for some disjoint } T_1,T_2\subseteq[p],$$

$$\mathcal{B}=\prod_{i\in R_1}\{B_i,\bar{B}_i\}\times\prod_{i\in R_2}\{B_i\}\times\prod_{i\in[p]\setminus(R_1\sqcup R_2)}\binom{[n_i]}{s_i},\ \textit{for some disjoint } R_1,R_2\subseteq[p],$$

where $A_i\in\binom{[n_i]}{t_i}$, $B_i\in\binom{[n_i]}{s_i}$, $T_1\sqcup T_2\neq\emptyset$, $R_1\sqcup R_2\neq\emptyset$ and $T_2,R_2\neq[p]$. Furthermore, for each $i\in T_1$, $n_i=2t_i$ and for each $i\in R_1$, $n_i=2s_i$.

If $\Gamma=\prod_{i=1}^{p}S_{n_i}$ is imprimitive on $\mathcal{X}$, then from the definition we know that Γ preserves a nontrivial partition $\{\mathcal{X}_j\}_{j=1}^{L}$ of $\mathcal{X}$. By projecting $\mathcal{X}_j$ to the i-th factor, we can obtain that $\bigsqcup_{j=1}^{L}(\mathcal{X}_j|_i)=\mathcal{X}|_i=\binom{[n_i]}{t_i}$ and $\Gamma|_i=S_{n_i}$ preserving this partition of $\binom{[n_i]}{t_i}$.

It is well known that for each $A_i \in \binom{[n_i]}{t_i}$, the stabilizer of A_i is isomorphic to $S_{t_i} \times S_{n_i-t_i}$, which is a maximal subgroup of S_{n_i} if $2t_i \neq n_i$ (see e.g. [152]). Then by Theorem 3.10, we obtain that S_{n_i} is primitive on $\binom{[n_i]}{t_i}$ unless $2t_i = n_i$, which means for the factors with $2t_i \neq n_i$ the partition $\bigsqcup_{j=1}^{L}(\mathcal{X}_j|_i)$ of $\binom{[n_i]}{t_i}$ must be a trivial partition. Thus for each $j \in L$, $\mathcal{X}_j|_i$ is either a singleton in $\binom{[n_i]}{t_i}$, or $\mathcal{X}_j|_i = \binom{[n_i]}{t_i}$.

When $2t_i = n_i$, it can be easily verified that the only imprimitive subset of $\binom{[n_i]}{t_i}$ has the form $\{A_i, \bar{A}_i\}$. Therefore, for the factors with $2t_i = n_i$, the partition $\bigsqcup_{j=1}^{L}(\mathcal{X}_j|_i)$ of $\binom{[n_i]}{t_i}$ is either a trivial partition, or each partition block has the form $\mathcal{X}_j|_i = \{A_{i,j}, \bar{A}_{i,j}\}$ for some $A_{i,j} \in \binom{[n_i]}{t_i}$.

Since each imprimitive subset of $\mathcal{X}$ can be seen as a block of a nontrivial partition of $\mathcal{X}$, we have $\mathcal{A} = \mathcal{X}_j$ for some $j \in [L]$. From the analysis above, we know that $\mathcal{A}|_i = \{A_i\}$ or $\{A_i, \bar{A}_i\}$ for some $A_i \in \binom{[n_i]}{t_i}$, or $\mathcal{A}|_i = \binom{[n_i]}{t_i}$. Therefore, set $T_1 \subseteq [p]$ such that for all $i \in T_1$, $2t_i = n_i$ and $\mathcal{A}|_i = \{A_i, \bar{A}_i\}$ for some $A_i \in \binom{[n_i]}{t_i}$; set $T_2 \subseteq [p]$ such that for all $i \in T_1$, $\mathcal{A}|_i$ is a singleton, finally, we have

$$\mathcal{A} = \prod_{i \in T_1} \{A_i, \bar{A}_i\} \times \prod_{i \in T_2} \{A_i\} \times \prod_{i \in [p] \setminus (T_1 \sqcup T_2)} \binom{[n_i]}{t_i}.$$

The proof for the imprimitive subsets of $\mathcal{Y}$ is the same as that of $\mathcal{X}$. Thus, the claim holds.

By Claim .4, we know that for the imprimitive subsets $\mathcal{A}$ and $\mathcal{B}$ above

$$|\mathcal{A}| = 2^{|T_1|} \cdot \prod_{i \in [p] \setminus (T_1 \sqcup T_2)} \binom{n_i}{t_i} \text{ and } |\mathcal{B}| = 2^{|R_1|} \cdot \prod_{i \in [p] \setminus (R_1 \sqcup R_2)} \binom{n_i}{s_i}.$$

And since

$$\begin{aligned} N(\mathcal{A}) &= \{B \in \mathcal{Y} : \ A \cap B = \emptyset \text{ for some } A \in \mathcal{A}\} \\ &= \prod_{i \in T_1} (\binom{A_i}{s_i} \sqcup \binom{\bar{A}_i}{s_i}) \times \prod_{i \in T_2} \binom{[n_i] \setminus A_i}{s_i} \times \prod_{i \in [p] \setminus (T_1 \sqcup T_2)} \binom{[n_i]}{s_i}, \\ N(\mathcal{B}) &= \{A \in \mathcal{X} : \ A \cap B = \emptyset \text{ for some } B \in \mathcal{B}\} \\ &= \prod_{i \in R_1} (\binom{B_i}{t_i} \sqcup \binom{\bar{B}_i}{t_i}) \times \prod_{i \in R_2} \binom{[n_i] \setminus B_i}{t_i} \times \prod_{i \in [p] \setminus (R_1 \sqcup R_2)} \binom{[n_i]}{t_i}, \end{aligned}$$

we have

$$|N(\mathcal{A})| = 2^{|T_1|} \cdot \prod_{i\in T_1} \binom{\frac{n_i}{2}}{s_i} \cdot \prod_{i\in T_2} \binom{n_i - t_i}{s_i} \cdot \prod_{i\in[p]\setminus(T_1\sqcup T_2)} \binom{n_i}{s_i},$$
$$|N(\mathcal{B})| = 2^{|R_1|} \cdot \prod_{i\in R_1} \binom{\frac{n_i}{2}}{t_i} \cdot \prod_{i\in R_2} \binom{n_i - s_i}{t_i} \cdot \prod_{i\in[p]\setminus(R_1\sqcup R_2)} \binom{n_i}{t_i}.$$

Now we can estimate quantities $|\mathcal{A}'| + |\mathcal{Y}| - |N(\mathcal{A}')|$ and $|\mathcal{B}'| + |\mathcal{X}| - |N(\mathcal{B}')|$.

Claim .5. *With the assumptions in the theorem, for all imprimitive subsets $\mathcal{A} \subseteq \mathcal{X}$ and $\mathcal{B} \subseteq \mathcal{Y}$, $|\mathcal{Y}| - d(\mathcal{X}) + 1 > |\mathcal{A}| + |\mathcal{Y}| - |N(\mathcal{A})|$, and $|\mathcal{Y}| - d(\mathcal{X}) + 1 > |\mathcal{B}| + |\mathcal{X}| - |N(\mathcal{B})|$.*

We prove the claim by estimating the difference directly. Denote

$$D_1 = |N(\mathcal{A})| - |\mathcal{A}| - d(\mathcal{X}) + 1 \text{ and}$$
$$D_2 = |\mathcal{Y}| - |\mathcal{X}| + |N(\mathcal{B})| - |\mathcal{B}| - d(\mathcal{X}) + 1$$

to be the differences between $|\mathcal{Y}| - d(\mathcal{X}) + 1$ and, respectively, $|\mathcal{A}| + |\mathcal{Y}| - |N(\mathcal{A})|$ and $|\mathcal{B}|+|\mathcal{X}|-|N(\mathcal{B})|$. Set $d_1 = \frac{D_1}{|N(\mathcal{A})|}$, $d_2 = \frac{D_2}{|\mathcal{X}|}$. Then, we have $d_1 = 1-\beta_1-\beta_2+\theta$, $d_2 = \delta+\eta_0\cdot(1-\eta_1-\eta_2)+\theta'$, where $\theta = |N(\mathcal{A})|^{-1}$, $\delta = \frac{|\mathcal{Y}|-|\mathcal{X}|}{|\mathcal{X}|}$, $\eta_0 = \frac{|N(\mathcal{B})|}{|\mathcal{X}|}$, $\theta' = |\mathcal{X}|^{-1}$, $\beta_1 = \frac{|\mathcal{A}|}{|N(\mathcal{A})|}$, $\beta_2 = \frac{d(\mathcal{X})}{|N(\mathcal{A})|}$, $\eta_1 = \frac{|\mathcal{B}|}{|N(\mathcal{B})|}$, and $\eta_2 = \frac{d(\mathcal{X})}{|N(\mathcal{B})|}$.

Since $\binom{n_i}{t_i}\cdot\binom{n_i-t_i}{s_i} = \binom{n_i}{s_i}\cdot\binom{n_i-s_i}{t_i}$ for each $i \in [p]$, we have $1/\binom{n_i-t_i}{s_i} = \binom{n_i}{t_i}/(\binom{n_i}{s_i}\cdot\binom{n_i-s_i}{t_i})$ for each $i \in [p]$. This yields that

$$\beta_1 = \prod_{i\in[p]} \frac{\binom{n_i}{t_i}}{\binom{n_i}{s_i}} \cdot \prod_{i\in T_1\sqcup T_2} \frac{1}{\binom{n_i-s_i}{t_i}}, \quad \beta_2 = \frac{1}{2^{|T_1|}} \cdot \prod_{i\in[p]\setminus(T_1\sqcup T_2)} \prod_{j=0}^{s_i-1} (1 - \frac{t_i}{n_i - j}),$$
$$\eta_1 = \prod_{i\in[p]} \frac{\binom{n_i}{s_i}}{\binom{n_i}{t_i}} \cdot \prod_{i\in R_1\sqcup R_2} \frac{1}{\binom{n_i-t_i}{s_i}}, \quad \eta_2 = \prod_{i\in[p]} \frac{\binom{n_i}{s_i}}{\binom{n_i}{t_i}} \cdot \frac{1}{2^{|R_1|}} \cdot \prod_{i\in[p]\setminus(R_1\sqcup R_2)} \prod_{j=0}^{t_i-1} (1 - \frac{s_i}{n_i - j}).$$

By the assumptions, we know that $n_i \geq s_i + t_i + 1 \geq 5$, $\prod_{i\in[p]} \frac{\binom{n_i}{t_i}}{\binom{n_i}{s_i}} \leq 1$ and $\binom{n_i-s_i}{t_i} \geq \binom{\lceil\frac{n_i}{2}\rceil}{t_i} \geq \frac{n_i}{2}$. Since $T_1 \sqcup T_2 \neq \emptyset$, $R_1 \sqcup R_2 \neq \emptyset$ and $T_2, R_2 \neq [p]$, we can obtain

$$\beta_1 \leq \prod_{i\in T_1\sqcup T_2} \frac{1}{\binom{n_i-s_i}{t_i}} \leq \max_{i\in(T_1\sqcup T_2)} \{(\frac{2}{n_i+2})^{|T_1|} \cdot (\frac{2}{n_i})^{|T_2|}\},$$
$$\beta_2 \leq \frac{1}{2^{|T_1|}} \cdot \max_{i\in[p]\setminus(T_1\sqcup T_2)} \{(1 - \frac{4n_i - 6}{n_i(n_i-1)})^{p-(|T_1|+|T_2|)}\},$$

and

$$\eta_1 \le (1+\delta)\cdot \prod_{i\in R_1\sqcup R_2} \frac{1}{\binom{n_i-t_i}{s_i}} \le (1+\delta)\cdot \max_{i\in(R_1\sqcup R_2)} \{(\frac{2}{n_i+2})^{|R_1|}\cdot(\frac{2}{n_i})^{|R_2|}\},$$
$$\eta_2 \le (1+\delta)\cdot \frac{1}{2^{|R_1|}}\cdot \max_{i\in[p]\setminus(R_1\sqcup R_2)} \{(1-\frac{4n_i-6}{n_i(n_i-1)})^{p-(|R_1|+|R_2|)}\}.$$

This leads to

$$\beta_1+\beta_2 \le \begin{cases} 1-\min\limits_{i\neq j\in[p]}\{\frac{6}{n_i}-\frac{2}{n_i-1}-\frac{2}{n_j}\}, & \text{if } T_2\neq\emptyset;\\ \frac{1}{2}-\min\limits_{i\neq j\in[p]}\{\frac{3}{n_i}-\frac{1}{n_i-1}-\frac{2}{n_j+2}\}, & \text{otherwise;}\end{cases}$$

and

$$\frac{\eta_1+\eta_2}{1+\delta} \le \begin{cases} 1-\min\limits_{i\neq j\in[p]}\{\frac{6}{n_i}-\frac{2}{n_i-1}-\frac{2}{n_j}\}, & \text{if } R_2\neq\emptyset;\\ \frac{1}{2}-\min\limits_{i\neq j\in[p]}\{\frac{3}{n_i}-\frac{1}{n_i-1}-\frac{2}{n_j+2}\}, & \text{otherwise.}\end{cases}$$

Since $5 \le n_i \le \frac{7}{4}n_j$ for all distinct $i,j\in[p]$, thus we have $\beta_1+\beta_2, \frac{\eta_1+\eta_2}{1+\delta} \le 1$. Therefore,

$$d_1 = 1-\beta_1-\beta_2+\theta > 1-\beta_1-\beta_2 \ge 0,$$
$$d_2 = \delta+\eta_0\cdot(1-\eta_1-\eta_2)+\theta' = \delta\cdot(1-\eta_0\cdot\frac{\eta_1+\eta_2}{1+\delta})+\eta_0\cdot(1-\frac{\eta_1+\eta_2}{1+\delta})+\theta' > 0.$$

Thus, the claim holds.

For each pair of non-empty cross-intersecting families $(\mathcal{A},\mathcal{B})\in 2^{\mathcal{X}}\times 2^{\mathcal{Y}}$, $\mathcal{A}\cup\mathcal{B}$ forms a nontrivial independent set of $G(\mathcal{X},\mathcal{Y})$. Therefore, by Claim .5, the inequality (3.3) holds.

To complete the proof, we need to characterize all the nontrivial fragments in $\mathcal{F}(\mathcal{X})$. As a direct consequence of Claim .5, every fragment of $G(\mathcal{X},\mathcal{Y})$ is primitive. Hence, by Theorem 3.11, when $\prod_{i\in[p]}\binom{n_i}{t_i} < \prod_{i\in[p]}\binom{n_i}{s_i}$, $\mathcal{X}$ has only 1-fragments.

When $\prod_{i\in[p]}\binom{n_i}{t_i} = \prod_{i\in[p]}\binom{n_i}{s_i}$, suppose there are nontrivial fragments in $\mathcal{F}(\mathcal{X})$. W.l.o.g., assume that $\mathcal{S}$ is a minimal-sized nontrivial fragment in $\mathcal{X}$. By Theorem 3.11, $\mathcal{S}$ is semi-imprimitive. Since for any two different elements $A,B\in\mathcal{X}$, $|N(A)\cap N(B)| < \prod_{i\in[p]}\binom{n_i-t_i}{s_i}-1$. Therefore, there are no 2-fragments in $\mathcal{F}(\mathcal{X})$. By Proposition 3.3, $\mathcal{S}$ is balanced.

Now we are going to prove the non-existence of such $\mathcal{S}$ by analyzing its size and structure, which will yield that $\mathcal{X}$ also has only 1-fragments when $\prod_{i\in[p]}\binom{n_i}{t_i} = \prod_{i\in[p]}\binom{n_i}{s_i}$.

For each $A = \prod_{i\in[p]} A_i \in \mathcal{S}$, let $\Gamma_A = \prod_{i\in[p]}(S_{A_i} \times S_{\bar{A}_i})$, $\Gamma_{\mathcal{S}} = \{\sigma \in \Gamma : \sigma(\mathcal{S}) = \mathcal{S}\}$ and $\Gamma_{A,\mathcal{S}} = \{\sigma \in \Gamma_A : \sigma(\mathcal{S}) = \mathcal{S}\}$. We claim that there exists a subset $C \in \mathcal{S}$ such that $\Gamma_C \neq \Gamma_{C,\mathcal{S}}$. Otherwise, for any two different subsets $B, B' \in \mathcal{S}$, we have $\Gamma_B = \Gamma_{B,\mathcal{S}}$ and $\Gamma_{B'} = \Gamma_{B',\mathcal{S}}$. Since $\Gamma_{B,\mathcal{S}}$ and $\Gamma_{B',\mathcal{S}}$ are both subgroups of $\Gamma_{\mathcal{S}}$, we have $\langle \Gamma_B, \Gamma_{B'}\rangle$ is a subgroup of $\Gamma_{\mathcal{S}}$. Let $T \subseteq [p]$ be the factors where $B'_i = B_i$ (or $\bar{B}_i$ if $2t_i = n_i$), write

$$\Gamma_B = \prod_{i\in T}(S_{B_i} \times S_{\bar{B}_i}) \times \prod_{i\in[p]\setminus T}(S_{B_i} \times S_{\bar{B}_i}),$$

then we have,

$$\Gamma_{B'} = \prod_{i\in T}(S_{B_i} \times S_{\bar{B}_i}) \times \prod_{i\in[p]\setminus T}(S_{B'_i} \times S_{\bar{B}'_i}).$$

Since $\langle S_{B_i} \times S_{\bar{B}_i}, S_{B'_i} \times S_{\bar{B}'_i}\rangle = S_{n_i}$ for each $B'_i \neq B_i$ (and $B'_i \neq \bar{B}_i$ if $2t_i = n_i$), we have

$$\langle \Gamma_B, \Gamma_{B'}\rangle = \prod_{i\in T}(S_{B_i} \times S_{\bar{B}_i}) \times \prod_{i\in[p]\setminus T} S_{n_i}.$$

Therefore, for some fixed $B \in \mathcal{S}$, $\Gamma_{\mathcal{S}}$ contains $\prod_{i\in T'}(S_{B_i} \times S_{\bar{B}_i}) \times \prod_{i\in[p]\setminus T'} S_{n_i}$ as a subgroup, where

$$T' = \{i | i \in [p], \text{ such that } A_i = B_i \text{ (or } \bar{B}_i \text{ if } 2t_i = n_i) \text{ for all } A \in \mathcal{S}\}.$$

When $T' = \emptyset$, we have $\Gamma_{\mathcal{S}} = \prod_{i\in[p]} S_{n_i}$, thus $\mathcal{S} = \mathcal{X}$, yielding a contradiction. When $T' \neq \emptyset$, if $|T'| = 1$, w.l.o.g., taking $T' = \{1\}$, we have $(S_{B_1} \times S_{\bar{B}_1}) \times \prod_{i\in[p]\setminus\{1\}} S_{n_i} \subseteq \Gamma_{\mathcal{S}}$. Therefore, since $\mathcal{S} \neq \mathcal{X}$, from the definition of T' we have

$$\mathcal{S} = \{B_1\} \times \prod_{i\in[p]\setminus\{1\}} \binom{[n_i]}{t_i}, \text{ or } S = \{B_1, \bar{B}_1\} \times \prod_{i\in[p]\setminus\{1\}} \binom{[n_i]}{t_i} \text{ when } 2t_1 = n_1.$$

In both cases, $|\mathcal{S}| < \frac{\alpha(\mathcal{X},\mathcal{Y})}{2}$. If $|T'| \geq 2$, we have

$$\mathcal{S} \subseteq \{B_{i_0}\} \times \prod_{i\in[p]\setminus\{i_0\}} \binom{[n_i]}{t_i}, \text{ or } S \subseteq \{B_{i_0}, \bar{B}_{i_0}\} \times \prod_{i\in[p]\setminus\{i_0\}} \binom{[n_i]}{t_i} \text{ when } 2t_{i_0} = n_{i_0},$$

for some $i_0 \in T'$. Therefore, when $T' \neq \emptyset$, we always have $|\mathcal{S}| < \frac{\alpha(\mathcal{X},\mathcal{Y})}{2}$, which contradicts the fact that $\mathcal{S}$ is balanced. Hence, the existence of C is guaranteed.

By Proposition 3.3 we have that $[\Gamma_C : \Gamma_{C,\mathcal{S}}]$, the index of $\Gamma_{C,\mathcal{S}}$ in Γ_C, equals 2. Now let $\Gamma_{C,\mathcal{S}}[C_i]$ be the projection of $\Gamma_{C,\mathcal{S}}$ onto S_{C_i}, $\Gamma_{C,\mathcal{S}}[C_i]$ must be a subgroup of S_{C_i} of index no greater than 2. Thus $\Gamma_{C,\mathcal{S}}[C_i] = S_{C_i}$ or A_{C_i}. Since $\Gamma_C = \prod_{i\in[p]}(S_{C_i} \times S_{\bar{C}_i})$, we know that $\Gamma_{C,\mathcal{S}} = \prod_{i\in[p]\setminus\{j\}}(S_{C_i} \times S_{\bar{C}_i}) \times (A_{C_j} \times S_{\bar{C}_j})$ or $\prod_{i\in[p]\setminus\{j\}}(S_{C_i} \times S_{\bar{C}_i}) \times (S_{C_j} \times A_{\bar{C}_j})$, for some $j \in [p]$.

Since for all $i \in [p]$, $t_i = |B_i \cap C_i| + |B_i \cap \bar{C}_i|$ for each pair $B, C \in \mathcal{S}$. If $|B_i \cap C_i| > 1$, let $s, t \in B_i \cap C_i$, then the transposition $(s\ t)$ fixes both C_i and B_i. Taking $i = j$, the semi-imprimitivity of $\mathcal{S}$ implies that $(s\ t) \in \Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}}$. This yields $\Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}} = S_{C_j} \times A_{\bar{C}_j}$. From this process it follows that, for each $B \in \mathcal{S}$, there exists at most one of $|B_j \cap C_j|$ and $|B_j \cap \bar{C}_j|$ to be greater than 1. Note that if $B_j \in \bar{C}_j$, then S_{C_j} and S_{B_j} fix both C_j and B_j, i.e., $S_{C_j} \times S_{B_j} \subseteq \Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}}$. Since $\Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}} = A_{C_j} \times S_{\bar{C}_j}$ or $S_{C_j} \times A_{\bar{C}_j}$, and neither $A_{C_j} \times S_{\bar{C}_j}$ nor $S_{C_j} \times A_{\bar{C}_j}$ contains $S_{C_j} \times S_{B_j}$. Therefore, we obtain that $|B_j \cap C_j| = 1$ for each $B \in \mathcal{S}$, or $|B_j \cap C_j| = t_j - 1$ for each $B \in \mathcal{S}$.

We claim that for both cases, $\mathcal{S}$ can not be balanced.

Suppose $|B_j \cap C_j| = 1$ for each $B \in \mathcal{S}$. W.l.o.g., assume $B_j \cap C_j = \{1\}$ for some $B \in \mathcal{S}$. From the semi-imprimitivity of $\mathcal{S}$, we know that for all $\gamma \in \Gamma$, $\gamma(\mathcal{S}) \cap \mathcal{S} = \emptyset$, $\mathcal{S}$ or $\{A\}$ for some $A \in \mathcal{S}$. Thus $(\gamma(\mathcal{S}) \cap \mathcal{S})|_j = \emptyset$, $\mathcal{S}|_j$ or $\{A_j\}$ for some $A_j \in \binom{[n_j]}{t_j}$. If $t_j > 2$, then $|B_j \cap \bar{C}_j| \geq 2$, so $\Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}} = A_{C_j} \times S_{\bar{C}_j}$. On the other hand, we can find distinct $s, t \in C_j$ such that $(1\ s\ t)(B_j) = B_j \setminus \{1\} \cup \{s\} \in \mathcal{S}|_j$ since $(1\ s\ t) \in A_{C_j}$. Then $(1\ s)(\mathcal{S}|_j)$ has more than one element of $\mathcal{S}|_j$, therefore $(1\ s) \in \Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}}$. This contradiction proves that $t_j = 2$. Thus $\mathcal{S}|_j = \mathcal{C} = \{A_j \in \binom{[n_j]}{2} : 1 \in A_j\}$. Otherwise, w.l.o.g., assume $C_j = \{1, 2\}$ and there exists $B \in \mathcal{S}$ such that $B_j \cap C_j = \{2\}$. Since $\Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}} = A_{C_j} \times S_{\bar{C}_j}$ or $S_{C_j} \times A_{\bar{C}_j}$, we have $\mathcal{C} \subseteq \mathcal{S}|_j$ and $\mathcal{C}' = \{A_j \in \binom{[n_j]}{2} : 2 \in A_j\} \subseteq \mathcal{S}|_j$. Thus $\mathcal{S}|_j = \mathcal{C} \cup \mathcal{C}'$. This yields $\Gamma_{C,\mathcal{S}}|_{S_{C_j}\times S_{\bar{C}_j}} = S_{C_j} \times S_{\bar{C}_j}$, leading to a contradiction.

Suppose now $|B_j \cap C_j| = t_j - 1 > 1$ for each $B \in \mathcal{S}$. Similarly, we can prove that $n_j - t_j = 2$, which contradicts $n_j \geq s_j + t_j + 1$ and $2 \leq s_j,\ t_j \leq \frac{n}{2}$. Therefore, for each $B \in \mathcal{S}$, $|B_j \cap C_j| = 1$.

From the analysis above, we know that for each $B \in \mathcal{S}$, $B_j = \{1, b\}$ for some $b \in [n_j]$. Thus, for each $B \in \mathcal{S}$, we have $\Gamma_{B,\mathcal{S}}|_{S_{B_j} \times S_{\bar{B}_j}} = A_{B_j} \times S_{\bar{B}_j}$, and $\Gamma_{B,\mathcal{S}} = \prod_{i\in[p]\setminus\{j\}}(S_{B_i} \times S_{\bar{B}_i}) \times (A_{B_j} \times S_{\bar{B}_j})$ since $[\Gamma_B : \Gamma_{B,\mathcal{S}}] = 2$. Therefore $\Gamma_{\mathcal{S}}$ contains

$$\langle \Gamma_{B,\mathcal{S}},\ \text{for all } B \in \mathcal{S} \rangle = \prod_{i \in T''} (S_{C_i} \times S_{\bar{C}_i}) \times \prod_{i\in[p]\setminus(T''\cup\{j\})} S_{n_i} \times S_{[n_j]\setminus\{1\}}$$

as a subgroup, where $T'' = \{i | i \in [p],$ such that $B_i = C_i$ (or $\bar{C}_i$ if $2t_i = n_i$) for all $B \in \mathcal{S}\}$. Similarly, by arguing the structure of $\mathcal{S}$, if $T'' \neq \emptyset$, we can prove that $|\mathcal{S}| < \frac{\alpha(\mathcal{X},\mathcal{Y})}{2}$. Thus we have $T'' = \emptyset$ and $\mathcal{S} = \prod_{i\in[p]\setminus\{j\}} \binom{[n_i]}{t_i} \times \mathcal{C}$.

Since $\mathcal{S}$ is balanced, $\prod_{i\in[p]} \binom{n_i}{t_i} = \prod_{i\in[p]} \binom{n_i}{s_i}$ and $|\mathcal{S}| = \prod_{i\in[p]\setminus\{j\}} \binom{n_i}{t_i} \cdot (n_j - 1)$, we have

$$2 \prod_{i\in[p]\setminus\{j\}} \binom{n_i}{t_i} \cdot (n_j - 1) = \prod_{i\in[p]\setminus\{j\}} \binom{n_i}{t_i} \cdot \binom{n_j}{2} - \prod_{i\in[p]\setminus\{j\}} \binom{n_i - s_i}{t_i} \cdot \binom{n_j - s_j}{2} + 1, \tag{3.6}$$

which means n_j must be an integral zero of the following function

$$H(x) = (1 - a_0) \cdot x^2 - (5 - a_0 \cdot (2s_j + 1)) \cdot x + (2b_0 + 4 - a_0 \cdot (s_j^2 + s_j)),$$

where $a_0 = \prod_{i\in[p]\setminus\{j\}} \frac{\binom{n_i - s_i}{t_i}}{\binom{n_i}{t_i}}$ and $b_0 = \prod_{i\in[p]\setminus\{j\}} \binom{n_i}{t_i}^{-1}$. Since $n_j \geq 3 + s_j$ and $2 \leq s_j \leq \frac{n_j}{2}$, by Vieta's formulas for quadratic polynomials, there is no such n_j satisfying $H(n_j) = 0$ when $s_j \geq 3$. Hence $\mathcal{S} = \prod_{i\in[p]\setminus\{j\}} \binom{[n_i]}{t_i} \times \mathcal{C}$ is a nontrivial balanced fragment of $\mathcal{X}$ if and only if $t_j = s_j = 2$ and equation (3.6) holds. Using the fact that $\frac{\binom{n_i - s_i}{t_i}}{\binom{n_i}{t_i}} \leq (1 - \frac{s_i}{n_i})(1 - \frac{s_i}{n_i - 1})$ and the assumption $n_i \leq \frac{7}{4} n_j$ for distinct $i, j \in [p]$, it can be easily verified that the LHS of equation (3.6) is strictly less than the RHS when $s_j = 2$. Therefore, $\mathcal{S}$ can not be balanced.

This completes the proof. □

§ 3.5 Concluding remarks

In this chapter we investigate two multi-part generalizations of the cross- intersecting theorems. For multi-part cross-intersecting families, we determining the maximum size and the corresponding structures of the families for both trivially and nontrivially (with the non-empty restriction) cross-intersecting cases.

The method of the proofs were originally introduced by Wang and Zhang in [198], which was further generalized to the bipartite case in [199]. This method can deal with set systems, finite vector spaces and permutations uniformly. It is natural to ask whether we can extend the single-part cross-intersecting theorems for finite vector spaces and permutations to the multi-part case. It is possible for permutations when considering the case without the non-empty restriction, and we believe it is also possible for finite vector spaces. But when it comes to the case where the families are non-empty, as far as we know, there is still no result for finite vector spaces and permutations.

For single-part families $\mathcal{A}$ and $\mathcal{B}$, it is natural to define cross-t-intersecting as $|A \cap B| \geq t$ for each pair of $A \in \mathcal{A}$ and $B \in \mathcal{B}$. But for multi-part families, when defining cross-t-intersecting between two families, the simple extension of the definition for single-part case can be confusing. Therefore, a reasonable definition and related problems for multi-part cross-t-intersecting families are also worth considering.

Chapter 4 Constant weighted X-codes

§ 4.1 Introduction

In this chapter, we focus on constant weighted X-codes. As an important class of codes in coding theory, constant weighted codes have been extensively studied for decades. They have played crucial roles in a number of engineering applications, including code-division multiple-access (CDMA) systems for optical fibers [45], protocol design for the collision channel without feedback [1], automatic-repeat-request error-control systems [197], and parallel asynchronous communication [25]. For the study of constant weighted codes, we recommend [3] and the reference therein.

With the development of the large scale integrated circuits (IC), corresponding circuit testing techniques also updated rapidly. Typical digital circuit testing applies test patterns to the circuit and observes the circuit's responses. The observed response to a test pattern is compared with the expected response, and a chip in the circuit is determined to be defective if the comparison mismatches. Since voltages on signal lines in digital circuit system are usually interpreted as logic values 0 or 1, therefore, the expected responses are captured as $\{0, 1\}$ vectors by test engineers when applying test patterns through fault-free simulations of the circuit. However, due to timing constraints, uninitialized memory elements, bus contention, inaccuracies of simulation models, etc (see Table 2 in [148]), for many digital systems, some simulated responses cannot be uniquely determined as 0 or 1 state. These unknown states are modeled as "X" states. In the presence of Xs, the technique of X-compact was proposed in [147] as one of the compression-based approaches that have high reliability and error detection ability in actual digital systems.

X-compact uses X-codes as to compress test responses. An (m,n,d,x) X-code is a set of m-dimensional $\{0,1\}$-vectors of size n which can also be viewed as an $m\times n$ binary matrix with column vectors as codewords. The parameters d,x correspond to the test quality of the code. The value of $\frac{n}{m}$ is called the *compaction ratio* and X-codes with large *compaction ratios* are desirable for actual IC testing. The weight of a codeword $\mathbf{c}$ is the number of 1s in $\mathbf{c}$. It corresponds to the required fan-out of the X-compactor. For an X-compactor, larger fan-in increases power requirements, area, and delay [203]. Due to the large amount of connections between X-compactors and inputs [147], compactors with smaller fan-out inputs shall reduce fan-in values. Therefore, codewords in X-codes are expected to have small weights.

Let $M(m,d,x)$ be the maximum number n of codewords for which there exists an (m,n,d,x) X-code. To obtain X-codes with large compaction ratios, studies of the behavior of $M(m,d,x)$ are unavoidable. In [79], based on a combinatorial approach, Fujiwara and Colbourn obtained a general lower bound $2^{\frac{m}{2^{x+1}(d+x)}}$ on $M(m,d,x)$ using probabilistic method (see Theorem 4.6, [79]). And this lower bound was further improved to $e^{\frac{m-c_0}{e(x+1)(d+x-1)}}$ by Tsuboda et al. in [193].

Firstly, in [148], stochastic coding techniques are employed to design constant weighted X-compactors. For $x=1$, by viewing the matrix of an $(m,n,d,1)$ X-code as an incidence matrix of a graph, Wohl and Huisman [203] built a connection between this kind of X-codes with constant weight 2 and graphs with girth at least $d+2$. For cases with multiple Xs, given an (m,n,d,x) X-code, Fujiwara and Colbourn [79] showed that a codeword of weight less than or equal to x does not essentially contribute to the compaction ratio (see also [139]). Since then, aiming to achieve a large compaction ratio while minimizing the weight of each codeword, many works have been done about (m,n,d,x) X-codes of constant weight $x+1$. Let $M_w(m,d,x)$ be the maximum number n of codewords for which there exists an (m,n,d,x) X-code of constant weight w. Using results from combinatorial design theory and superimposed codes, Fujiwara and Colbourn [79] proved that $M_3(m,d,2)=O(m^2)$ and $M_3(m,1,2)=\Theta(m^2)$. And they studied a special class of $(m,n,1,2)$ X-codes of constant weight 3 with a property that boosts test quality when there are fewer unknowable bits than anticipated. In [192], Tsunoda and Fujiwara proved that $M_3(m,d,2) = o(m^2)$ for $d \geq 4$ and they also improved the lower bound on the maximum number of codewords for the above special class of $(m,n,1,2)$ X-codes of constant weight 3 introduced in [79].

This chapter is organised as follows: In Section § 4.2, we list some necessary notations and introduce the combinatorial requirements and the definitions of X-codes, we also include a lower bound for hypergraph independent sets preparing for proofs in Section § 4.4. In Section § 4.3, we investigate the bounds and constructions for constant weighted X-codes. We prove a general result on $M_w(m,d,x)$ and a non-trivial lower bound on $M_3(m,d,2)$. We also present some explicit constructions for constant weighted X-codes with $d = 3, 7$ and $x = 2$ based on the results from additive combinatorics and finite fields. These constructions further improve the general lower bound by providing a nearly optimal lower bound $m^{2-\varepsilon}$ for $M_3(m,3,2)$ and an optimal lower bound $c'm^2$ for $M_4(m,3,2)$, when m is large enough. In Section § 4.4, we improve the lower bound on the maximum number of codewords for a special class of $(m,n,1,2)$ X-codes of constant weight 3 and extend this result to a general case. In Section § 4.5, we conclude this chapter with some remarks.

§ 4.2 Preliminaries

4.2.1 Notation

We use the following standard notations throughout this chapter.

- Let q be the power of a prime p, $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q^n$ be the vector space of dimension n over $\mathbb{F}_q$.
- For any vector $\mathbf{v} = (v_1, \cdots, v_n) \in \mathbb{F}_q^n$, let $\mathrm{supp}(\mathbf{v}) = \{i \in [n] : v_i \neq 0\}$ and $w(\mathbf{v}) = |\mathrm{supp}(v)|$. For a set $S \subseteq [n]$, define $\mathbf{v}|_S = (v_{i_1}, \ldots, v_{i_{|S|}})$, where $i_j \in S$ for $1 \leq j \leq |S|$ and $1 \leq i_1 < \cdots < i_{|S|} \leq n$.

- For positive integer $k \geq 1$, a subset $P \subseteq \mathbb{F}_q$ of size k is called an *arithmetic progression of length* k if it has the form: $P = \{x + ia : x, a \in \mathbb{F}_q \text{ and } 0 \leq i \leq k-1\}$. For simplicity, we denote k-AP as the shortened form of *arithmetic progression of length* k.

- For functions $f = f(n)$ and $g = g(n)$, we use standard asymptotic notations $\Omega(\cdot)$, $\Theta(\cdot)$, $O(\cdot)$ and $o(\cdot)$ as $n \to \infty$:

$$\begin{cases} f = O(g), \text{ if } \exists \text{ a constant } c_1 \text{ such that } |f| \leq c_1|g|; \\ f = \Omega(g), \text{ if } \exists \text{ a constant } c_2 \text{ such that } |f| \geq c_2|g|; \\ f = \Theta(g), \text{ if } f = O(g) \text{ and } f = \Omega(g); \\ f = o(g), \text{ if } \lim\limits_{n\to\infty} \frac{f}{g} = 0. \end{cases}$$

4.2.2 X-Codes and digital system test compaction

To describe the behavior of unknown value Xs, operations including addition (XOR) and multiplication (AND) for the 3-valued logic system (0, 1 and X) are formulated as X-*algebra* by Fujiwara and Colbourn [79]: The X-algebra $\mathbb{X}_2 = (\{0,1,X\}, +, \cdot)$ over $\mathbb{F}_2$ is the set $\{0,1\} \subseteq \mathbb{F}_2$ and a third element X, equipped with two binary operations "+" (addition) and "·" (multiplication) satisfying:

$$\begin{cases} \text{1) For } a, b \in \mathbb{F}_2, \ a+b \text{ and } a \cdot b \text{ are performed in } \mathbb{F}_2; \\ \text{2) For } a \in \mathbb{F}_2, \ a + X = X + a = X; \\ \text{3) For the additive identity 0, } \ 0 \cdot X = X \cdot 0 = 0; \\ \text{4) } 1 \cdot X = X \cdot 1 = X. \end{cases} \tag{4.1}$$

Now, consider a circuit with response output $c = (c_1, \ldots, c_n) \in \{0,1,X\}^n$. Assume we have a test output $b = (b_1, \ldots, b_n) \in \{0,1\}^n$, based on the property of X-algebra, the i_{th} bit is regarded as an error bit if and only if $b_i + c_i = 1$.

For these testing and response outputs vectors, the X-compact technique is performed by right multiplying an $n \times m$ binary matrix H, where the arithmetics are carried out in $\mathbb{X}_2$. Denote $c' = (c'_1, \ldots, c'_m) = cH$ and $b' = (b'_1, \ldots, b'_m) = bH$

as the X-compacted outputs of the response vector c and testing vector b above. Similarly, the i_{th} bit is regarded as an error bit if and only if $b'_i + c'_i = 1$. Here, H is called the X-compact matrix and the value of $\frac{n}{m}$ is called the compaction ratio of H. To design X-compact matrices with large compaction ratio, X-codes were introduced in [139]. Roughly speaking, an X-code can be viewed as the set of row vectors of an X-compact matrix. To give the formal definition of X-codes, first, we introduce the following two operations on vectors.

Consider two m-dimensional vectors $\mathbf{s}_1 = (s_1^{(1)}, s_2^{(1)}, \ldots, s_m^{(1)})$ and $\mathbf{s}_2 = (s_1^{(2)}, s_2^{(2)}, \ldots, s_m^{(2)})$ where $s_i^{(j)} \in \mathbb{F}_2$. The *addition* of $\mathbf{s}_1$ and $\mathbf{s}_2$ is bit-by-bit addition, denoted by $\mathbf{s}_1 \oplus \mathbf{s}_2$; that is

$$\mathbf{s}_1 \oplus \mathbf{s}_2 = (s_1^{(1)} + s_1^{(2)}, s_2^{(1)} + s_2^{(2)}, \ldots, s_m^{(1)} + s_m^{(2)}).$$

The *superimposed sum* of $\mathbf{s}_1$ and $\mathbf{s}_2$, denoted by $\mathbf{s}_1 \vee \mathbf{s}_2$, is

$$\mathbf{s}_1 \vee \mathbf{s}_2 = (s_1^{(1)} \vee s_1^{(2)}, s_2^{(1)} \vee s_2^{(2)}, \ldots, s_m^{(1)} \vee s_m^{(2)}),$$

where $s_i^{(j)} \vee s_k^{(l)} = 0$ if $s_i^{(j)} = s_k^{(l)} = 0$, otherwise 1. And we say an m-dimensional vector $\mathbf{s}_1$ *covers* an m-dimensional vector $\mathbf{s}_2$ if $\mathbf{s}_1 \vee \mathbf{s}_2 = \mathbf{s}_1$. For a finite set $S = \{\mathbf{s}_1, \ldots, \mathbf{s}_s\}$ of m-dimensional vectors, define

$$\bigoplus S = \mathbf{s}_1 \oplus \cdots \oplus \mathbf{s}_s,$$

and

$$\bigvee S = \mathbf{s}_1 \vee \cdots \vee \mathbf{s}_s.$$

When $s = 1$, $\bigoplus S = \bigvee S = \{\mathbf{s}_1\}$, and when $S = \emptyset$, define $\bigoplus S = \bigvee S = \mathbf{0}$ (i.e. the zero vector).

Definition 4.1. *[79] Let d be a positive integer and x a nonnegative integer. An (m, n, d, x) X-code $\mathcal{X} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ is a set of m-dimensional vectors over $\mathbb{F}_2$ such that $|\mathcal{X}| = n$ and*

$$(\bigvee S_1) \vee (\bigoplus S_2) \neq \bigvee S_1 \tag{4.2}$$

for any pair of mutually disjoint subsets S_1 *and* S_2 *of* $\mathcal{X}$ *with* $|S_1| = x$ *and* $1 \leq |S_2| \leq d$. *A vector* $\mathbf{s}_i \in \mathcal{X}$ *is called a codeword. The weight of the codeword* $\mathbf{s}_i$ *is* $|supp(\mathbf{s}_i)|$. *The ratio* $\frac{n}{m}$ *is called the compaction ratio of* $\mathcal{X}$.

In view of X-compaction, the parameter m of an (m, n, d, x) X-code represents the size of the shrunk data, n represents the number of bits in the raw response to be compressed at a time, d corresponds to the discrepancy detecting ability and x characterizes the unknowable bits tolerance. Generally speaking, as phrased in [193], an (m, n, d, x) X-code hashes the n-bit outputs from the circuit's test into m bits while allowing for detecting the existence of up to d bit-wise discrepancies between the actual outputs and correct responses even if up to x bits of the correct behavior are unknowable to the tester.

From the definition above, when $x = 0$, the codewords of an $(m, n, d, 0)$ X-code actually form an $m \times n$ parity check matrix of a binary linear code of length n with minimum distance $d + 1$. Therefore, $(m, n, d, 0)$ X-codes can be viewed as a special kind of traditional error-correcting codes.

When $x \geq 1$ and $d \geq 2$, according to the definition, an (m, n, d, x) X-code is also an $(m, n, d-1, x)$ X-code and an $(m, n, d, x-1)$ X-code. For the case when $x \geq 1$ and $d = 1$, as pointed out in [139], an $(m, n, 1, x)$ X-code is equivalent to a $(1, x)$-*superimposed code* of size $m \times n$.

Definition 4.2. *[122] A* $(1, x)$-superimposed code *of size* $m \times n$ *is an* $m \times n$ *matrix* S *with entries in* $\mathbb{F}_2$ *such that no superimposed sum of any* x *columns of* S *covers any other column of* S.

Superimposed codes are also called *cover-free families* and *disjunct matrices*. These kinds of structures have been extensively studied in information theory, combinatorics and group testing. Thus, the bounds and constructions of $(1, x)$-superimposed codes can also be regarded as those for $(m, n, 1, x)$ X-codes (see, for example, [36, 52, 61, 62, 81, 109, 183]).

4.2.3 Independent sets in hypergraphs

A hypergraph is a pair $(V, \mathcal{E})$, where V is a finite set and $\mathcal{E} \subseteq 2^V$ is a family of subsets of V. The elements of V are called vertices and the subsets in $\mathcal{E}$ are called hyperedges. We call $\mathcal{H}$ a k-uniform hypergraph, if all the hyperedges have the same size k, i.e., $\mathcal{E} \subseteq \binom{V}{k}$. For any vertex $v \in V$, we define the degree of v to be the number of hyperedges containing v, denoted by $d(v)$. The maximum of the degrees of all the vertices is called the maximum degree of $\mathcal{H}$ and denoted by $\Delta(\mathcal{H})$.

An independent set of a hypergraph is a set of vertices containing no hyperedges and the independence number of a hypergraph is the size of its largest independent set. There are many results on the independence number of hypergraphs obtained through different methods (see [5], [6], [51], [125]). Recall that a hypergraph $\mathcal{H}$ is linear if every pair of distinct hyperedges from $\mathcal{E}$ intersects in at most one vertex. In Section § 4.4, we shall use the following version of the famous result of Ajtai et al. [5] due to Duke et al. [51] to derive some lower bounds on $M(m, d, x)$.

Lemma 4.1. *[51] Let $k \geq 3$ and let $\mathcal{H}$ be a k-uniform hypergraph with $\Delta(\mathcal{H}) \leq D$. If $\mathcal{H}$ is linear, then*

$$\alpha(\mathcal{H}) \geq c \cdot |V| \cdot (\frac{\log D}{D})^{\frac{1}{k-1}}, \tag{4.3}$$

for some constant c that depends only on k.

§ 4.3 Bounds and constructions of constant weighted X-codes

In this section, we consider the bounds and constructions of constant weighted X-codes. This section is divided into three subsections. Section III-A includes a general result on the number of codewords of constant weighted X-codes from superimposed codes. Then in Section III-B, we give some explicit constructions for constant weighted X-codes with $d = 3, 7$ and $x = 2$. And in Section III-C, we improve the general lower bound for X-codes of constant weight 3 with $x = 2$.

4.3.1 General bounds from superimposed codes

According to the definition, in [147], the authors showed that an (m, n, d, x) X-code is also an $(m, n, d+1, x-1)$ X-code. Note that for two binary vectors, their addition corresponds to the symmetric difference of their underlying sets and their superimposed sum corresponds to the union of their underlying sets. Therefore, by the equivalence between X-codes and superimposed codes, we have the following correspondence.

Proposition 4.1. *Let d be a positive integer and x be a nonnegative integer. A $(1, x+d-1)$-superimposed code of size $m \times n$ is an (m, n, d, x) X-code.*

Denote $M_w(m, d, x)$ as the maximum number of codewords of an (m, n, d, x) X-code of constant weight w. Since the restrictions for X-codes get more rigid with the growing of d, combined with the above proposition, we have

$$M_w(m, 1, x+d-1) \leq M_w(m, d, x) \leq M_w(m, 1, x). \tag{4.4}$$

In 1985, Erdős et al. [62] proved the following bounds on the maximum number of codewords of a $(1, x)$-superimposed code of constant weight w.

Theorem 4.1. *[62] Denote $f_x(m, w)$ as the maximum number of columns of a $(1, x)$-superimposed code of constant weight w. Let $t = \lceil \frac{w}{x} \rceil$. Then, we have*

$$\frac{\binom{m}{t}}{\binom{w}{t}^2} \leq f_x(m, w) \leq \frac{\binom{m}{t}}{\binom{w-1}{t-1}}.$$

Moreover, if we take $w = x(t-1)+1+\delta$ where $0 \leq \delta < x$, then there exists a constant $m_0 = m_0(w)$ such that for $m > m_0(w)$,

$$f_x(m, w) \geq (1 - o(1)) \frac{\binom{m-\delta}{t}}{\binom{w-\delta}{t}},$$

and $f_x(m, w) \leq \frac{\binom{m-\delta}{t}}{\binom{w-\delta}{t}}$ holds in the following cases:1) $\delta = 0, 1$; 2) $\delta < \frac{x}{2t^2}$; 3) $t = 2$ and $\delta < \lceil \frac{2x}{3} \rceil$. Moreover, equality of the latter upper bound holds if and only if there exists a Steiner t-design $S(t, w-\delta, n-\delta)$.

According to this bound, by inequality (4.4), we have the following immediate consequence:

Theorem 4.2. *Let d, x be given positive integers, $w = x(t-1)+1+\delta$ where $0 \leq \delta < x$, and $m_0 = m_0(w)$ be the constant defined in Theorem 4.1. Then, for all $m \geq 1$,*

$$\frac{\binom{m}{\lceil w/(x+d-1)\rceil}}{\binom{w}{\lceil w/(x+d-1)\rceil}^2} \leq M_w(m,d,x) \leq \frac{\binom{m}{t}}{\binom{w-1}{t-1}}. \tag{4.5}$$

And for $m > m_0(w)$,

$$M_w(m,d,x) \geq (1-o(1))\frac{\binom{m}{\lceil w/(x+d-1)\rceil}}{\binom{w}{\lceil w/(x+d-1)\rceil}} \tag{4.6}$$

and $M_w(m,d,x) \leq \frac{\binom{m-\delta}{t}}{\binom{w-\delta}{t}}$ holds in the following cases:1) $\delta = 0, 1$; 2) $\delta < \frac{x}{2t^2}$; 3) $t = 2$ and $\delta < \lceil \frac{2x}{3} \rceil$.

In particular, for the case $x = 2$, when $m > m_0(w)$, Theorem 4.2 actually gives the following upper bound

$$M_w(m,d,2) \leq \begin{cases} \frac{\binom{m-1}{w/2}}{\binom{w-1}{w/2}}, \text{when } w \text{ is even;} \\ \frac{\binom{m}{(w+1)/2}}{\binom{w}{(w+1)/2}}, \text{when } w \text{ is odd.} \end{cases} \tag{4.7}$$

According to the results from design theory, Fujiwara and Colbourn [79] proved the upper bound above is tight for the case $w = 3$ and $d = 1$, when there exists a corresponding Steiner triple system. Using the well-known *graph removal lemma*, Tsunoda and Fujiwara [192] improved this upper bound on $M_3(m,d,2)$ to $o(m^2)$ for $d \geq 4$. So far as we know, for $d \geq 2$ and $x = 2$, no upper or lower bounds better than these can be found in the literature.

4.3.2 Explicit constructions of constant weighted X-codes

4.3.2.1 Constructions of constant weighted X-codes with $d = 3$ and $x = 2$

In this part, we present two explicit constructions of constant weighted X-codes with $d = 3$ and $x = 2$, which provide asymptotically optimal lower bounds on $M_4(m,3,2)$ and nearly optimal lower bounds on $M_3(m,3,2)$, respectively.

- Construction I : Let $p > w$ be a prime and q be a power of p, take w copies $X_1, X_2, \ldots, X_w$ of $\mathbb{F}_q$. Define

$$\mathcal{P}_1 = \{(x_1, x_2, \ldots, x_w) \in \prod_{i=1}^{w} X_i : \\ x_1 + (j-1) \cdot x_2 + x_{j+1} = 0 \text{ for } 2 \le j \le w-1\},$$

as a family of w-tuples in $X_1 \times \cdots \times X_w$. Clearly, $|\mathcal{P}_1| = q^2$. For $P_1 \neq P_2 \in \mathcal{P}_1$, we denote $P_1 \cap P_2 = \{i : P_1(i) = P_2(i),\ 1 \le i \le w\}$ and define the indicator vector of P_i as the concatenation of the w indicator vectors of element x_i, i.e., $\mathbf{v}_{P_i} = (\mathbf{v}_{x_1}, \mathbf{v}_{x_2}, \ldots, \mathbf{v}_{x_w})$, where $\mathbf{v}_{x_i}$ is the indicator vector of element x_i of length q. Let $\mathcal{C}_1$ be the set of all indicator vectors corresponding to w-tuples in $\mathcal{P}_1$.

Theorem 4.3. *For any $w \ge 4$ and prime $p > w$, let q be a power of p, the code $\mathcal{C}_1$ from Construction I is a $(wq, q^2, 3, 2)$ X-code of constant weight w.*

Proof of Theorem 4.3. From the definition, one can easily check that $|P_1 \cap P_2| \le 1$ for any two distinct $P_1, P_2 \in \mathcal{P}_1$. Therefore, for integer $t \ge 1$, $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ of any two distinct $P_1, P_2 \in \mathcal{P}_1$ can cover at most $2t$ distinct "1"s in $\mathbf{v}_{P_3} \oplus \cdots \oplus \mathbf{v}_{P_{t+2}}$ for other t distinct P_is in $\mathcal{P}_1$. Since $w \ge 4$ and $w(\bigoplus_{i=3}^{t+2} \mathbf{v}_{P_i}) \ge t(w-t+1)$, this guarantees that the addition of any two or fewer vectors in $\mathcal{C}_1$ can not be covered by the superimposed sum of any other two vectors.

When $t = 3$, assume there are $\{P_i\}_{i=1}^{5}$ such that $\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}$ can be covered by $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$. Since $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) \ge 3(w-2)$, thus, we have $w = 4$ and $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 6$. Note that for $i \neq j$, $|P_i \cap P_j| \le 1$. Thus, we have $|P_i \cap P_j| = 1$ for $i \in \{1,2\}, j \in \{3,4,5\}$ and $|P_{j_1} \cap P_{j_2}| = 1$ for $j_1, j_2 \in \{3,4,5\}$. Assume that $P_{j_1} \cap P_{j_2} = \theta_{j_1,j_2}$, $j_1, j_2 \in \{3,4,5\}$. Since $w = 4$, w.l.o.g., assume that $\theta_{3,4} = 1$, $\theta_{3,5} = 2$ and $\theta_{4,5} = 3$. Therefore, we have $P_3(4), P_4(4), P_5(4) \in \{P_1(4)\} \cup \{P_2(4)\}$. By pigeonhole principle, w.l.o.g., we can assume that $P_3(4) = P_4(4) = P_1(4)$, this indicates that $|P_3 \cap P_4| \ge 2$, a contradiction. Therefore, the addition of any three vectors in $\mathcal{C}_1$ can not be covered by the superimposed sum of any other two vectors. This indicates that $\mathcal{C}_1$ is a $(wq, |\mathcal{P}|, 3, 2)$ X-code of constant weight w. □

Actually, with the same spirit, there can be many other similar constructions providing the same bound. However, when $w = 3$, this kind of constructions is no longer enough to guarantee the restrictions of being an X-code. For this case, we provide a new construction. First, we need the following lemma from [63].

Lemma 4.2. *[63] For positive integers w and m, there exists a set of positive integers $A \subseteq [m]$ of size*

$$|A| \geq \frac{m}{e^{c \log w \sqrt{\log m}}}$$

for some absolute constant c, such that A contains no three terms of any arithmetic progressions of length w.

The specific construction of the set A from Lemma 4.2 can be regarded as an extension of the 3-AP-free subset of $[m]$ given by Behrend [21] and the detailed construction can be found in Section 5 of [63].

- Construction II : Let $m_1 = \lfloor \frac{m}{w} \rfloor$, $m_2 = \lfloor \frac{m}{w^2} \rfloor$ and $A \subseteq [m_2]$ be the subset constructed from Lemma 4.2 such that A contains no three terms of any progressions of length w. Take w copies $X_1, X_2, \ldots, X_w$ of $[m_1]$. Define

  $$\mathcal{P}_2 = \{(x, x+a, \ldots, x+(w-1)a) : a \in A \text{ and } x+(i-1)a \in X_i \text{ for } 1 \leq i \leq w\},$$

  as a family of w-tuples in $X_1 \times \cdots \times X_w$. Similarly, given $P_1 \neq P_2 \in \mathcal{P}_2$, denote $P_1 \cap P_2 = \{i : P_1(i) = P_2(i),\ 1 \leq i \leq w\}$ and we define the indicator vector of P_i as the concatenation of the w indicator vectors of element x_i together with an assistant zero vector, i.e., $\mathbf{v}_{P_i} = (\mathbf{v}_{x_1}, \mathbf{v}_{x_2}, \ldots, \mathbf{v}_{x_w}, \mathbf{0})$, where $\mathbf{v}_{x_i}$ is the indicator vector of element x_i of length m_1 and $\mathbf{0}$ is a zero vector of length $m - wm_1$. Let $\mathcal{C}_2$ be the set of all indicator vectors corresponding to w-tuples in $\mathcal{P}_2$.

Theorem 4.4. *For any $\varepsilon > 0$ and $w \geq 3$, there exists a constant $M = M(w, \varepsilon) > 0$, such that for $m \geq M$, the code $\mathcal{C}_2$ from Construction II is an $(m, m^{2-\varepsilon}, 3, 2)$ X-code of constant weight w.*

Proof of Theorem 4.4. By the definition of $\mathcal{P}_2$, for $P_1 \neq P_2 \in \mathcal{P}_2$, we know that $|P_1 \cap P_2| \leq 1$. To proceed the proof, we need the following claim about the structure of $\mathcal{P}_2$.

Claim. $\mathcal{P}_2$ does not contain the following triple: $\{Q_1, Q_2, Q_3\} \subseteq \mathcal{P}_2$ satisfying that $Q_1 \cap Q_2 = \{\eta_1\}$, $Q_1 \cap Q_3 = \{\eta_2\}$, $Q_2 \cap Q_3 = \{\eta_3\}$, where $\eta_1, \eta_2, \eta_3 \in \{1, 2, \dots, w\}$ are pairwise distinct.

Proof. Otherwise, assume that there are $\{Q_1, Q_2, Q_3\} \subseteq \mathcal{P}_2$ such that $Q_1 \cap Q_2 = \{\eta_1\}$, $Q_1 \cap Q_3 = \{\eta_2\}$, $Q_2 \cap Q_3 = \{\eta_3\}$ for three distinct η_1, η_2, η_3. By the definition of $\mathcal{P}_2$, for $1 \leq i \leq 3$, we can assume that $Q_i = (x_i, x_i + a_i, \dots, x_i + (w-1)a_i)$. Thus, we have

$$\begin{cases} Q_1(\eta_1) = x_1 + (\eta_1 - 1)a_1 = Q_2(\eta_1) = x_2 + (\eta_1 - 1)a_2; \\ Q_1(\eta_2) = x_1 + (\eta_2 - 1)a_1 = Q_3(\eta_2) = x_3 + (\eta_2 - 1)a_3; \\ Q_2(\eta_3) = x_2 + (\eta_3 - 1)a_2 = Q_3(\eta_3) = x_3 + (\eta_3 - 1)a_3. \end{cases} \tag{4.8}$$

Combining these three equations in (4.8) together, we have

$$(\eta_2 - \eta_1)a_1 = (\eta_2 - \eta_3)a_3 + (\eta_3 - \eta_1)a_2.$$

This means that $(\eta_2 - \eta_3)(a_3 - a_1) = (\eta_1 - \eta_3)(a_2 - a_1)$. Since η_is are pairwise distinct, thus, both $\eta_2 - \eta_3$ and $\eta_1 - \eta_3$ are non-zero integers. Moreover, the distinctness of Q_i also leads to a_1, a_2, a_3 being pairwise distinct. Thus, we have $a_3 - a_1 = \frac{\eta_1 - \eta_3}{\eta_2 - \eta_3}(a_2 - a_1)$. W.l.o.g., assume that $\gcd(\eta_1 - \eta_3, \eta_2 - \eta_3) = 1$. Then, take $D = \frac{a_2 - a_1}{\eta_2 - \eta_3}$, we have

$$a_2 = a_1 + (\eta_2 - \eta_3)D \text{ and } a_3 = a_1 + (\eta_1 - \eta_3)D.$$

Since $\eta_1, \eta_2, \eta_3 \in \{0, 1, \dots, w-1\}$, thus, $|\eta_i - \eta_j| < w$ for any $i \neq j \in [3]$. Therefore, $\{a_1, a_2, a_3\} \subseteq A$ are three pairwise distinct terms of a w-AP with common difference D. This contradicts the construction of A. □

With the help of this claim, next, for any two distinct $P_1, P_2 \in \mathcal{P}_2$, we will verify that $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ can not cover the addition of any at most three other vectors in $\mathcal{C}_2$.

First, since $|P_1 \cap P_2| \leq 1$, $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ can cover at most $2t$ distinct "1"s in $\mathbf{v}_{P_3} \oplus \cdots \oplus \mathbf{v}_{P_{t+2}}$ for other t distinct $P_i s \in \mathcal{P}_1$. Note that $w \geq 3$, thus $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ can not cover any other one vector in $\mathcal{C}_2$.

Second, assume that there exist other two distinct $P_3, P_4 \in \mathcal{P}_2$ such that $\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4}$ is covered by $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$. When $w \geq 4$, we have $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4}) \geq 2(w-1) > 4$. This indicates that one of the four intersections $|P_1 \cap P_3|$, $|P_1 \cap P_4|$, $|P_2 \cap P_3|$, $|P_2 \cap P_4|$ must be strictly larger than one, which is impossible. When $w = 3$ and $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4}) = 4$, this leads to $|P_1 \cap P_3| = |P_1 \cap P_4| = |P_2 \cap P_3| = |P_2 \cap P_4| = |P_3 \cap P_4| = 1$ and the intersection of any three of P_1, P_2, P_3, P_4 is an empty set. Thus, we can assume that $P_3 \cap P_4 = \{\theta_0\}$, $P_1 \cap P_3 = \{\theta_1\}$, $P_1 \cap P_4 = \{\theta_2\}$, where $\theta_0, \theta_1, \theta_2 \in \{1, \ldots, w\}$ are pairwise distinct. This contradicts the claim above. Thus, $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ can not cover the addition of any other two vectors in $\mathcal{C}_2$.

Now, assume that there exist other three distinct $\{P_3, P_4, P_5\} \subseteq \mathcal{P}_2$ such that $\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}$ is covered by $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$. Since $\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}$ can cover at most 6 distinct "1"s in $\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}$, thus, by $w(\mathbf{v}_{P_1} \oplus \mathbf{v}_{P_2} \oplus \mathbf{v}_{P_3}) \geq 3(w-2)$, we can assume that $w \leq 4$.

When $w = 3$, since $w(\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}) \leq 6$ and $|P_i \cap P_j| \leq 1$ $(i \neq j \in \{3, 4, 5\})$, thus, either $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 3$ or $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 5$. For the case $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 3$, we can assume that $P_3 \cap P_4 = \{\theta_0\}$, $P_3 \cap P_5 = \{\theta_1\}$, $P_4 \cap P_5 = \{\theta_2\}$, where $\theta_0, \theta_1, \theta_2 \in \{1, \ldots, w\}$ are pairwise distinct. For the case $w(\mathbf{v}_{P_1} \oplus \mathbf{v}_{P_2} \oplus \mathbf{v}_{P_3}) = 5$, we can assume that $P_3 \cap P_4 = \{\theta_0\}$, $P_3 \cap P_5 = \{\theta_1\}$, $P_1 \cap P_3 = \{\theta_2\}$, $P_1 \cap P_4 = \{\theta_3\}$, where $\{\theta_i\}_{i=0}^{3} \subseteq \{1, \ldots, w\}$ are pairwise distinct. For both cases, we have three distinct P_is pairwise intersecting at three distinct elements θ_js, which contradicts to the former claim.

When $w = 4$, since $w(\mathbf{v}_{P_1} \vee \mathbf{v}_{P_2}) \leq 8$ and $|P_i \cap P_j| \leq 1$ $(i \neq j \in \{3, 4, 5\})$, thus, either $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 6$ or $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 8$. For the case $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 8$, we have $|P_i \cap P_j| > 1$ for some $i \in \{1, 2\}$ and $j \in \{3, 4, 5\}$, a contradiction. For the case $w(\mathbf{v}_{P_3} \oplus \mathbf{v}_{P_4} \oplus \mathbf{v}_{P_5}) = 6$, we can assume that $P_3 \cap P_4 = \{\theta_{34}\}$, $P_3 \cap P_5 = \{\theta_{35}\}$, $P_4 \cap P_5 = \{\theta_{45}\}$ and $P_i \cap P_j = \{\theta_{ij}\}$ for each $i \in \{1,2\}, j \in \{3,4,5\}$, where $\theta_{ij} \in \{1,2,\ldots,w\}$ are pairwise distinct. This also leads to three distinct P_is pairwise intersecting at three distinct elements θ_{ij}s, which contradicts the construction of $\mathcal{P}_2$.

In conclusion, the addition of any three or fewer vectors in $\mathcal{C}_2$ can not be covered by the superimposed sum of any other two vectors. Since $|A| \geq \frac{m_2}{e^{c\log w\sqrt{\log m_2}}}$ for some $c > 0$, we have $|\mathcal{P}_2| \geq m_2|A| \geq m^{2-\varepsilon}$ for every $\varepsilon > 0$ and $m \geq M$, therefore, $\mathcal{C}_2$ is the desired $(m, m^{2-\varepsilon}, 3, 2)$ X-code of constant weight w. □

Remark 4.1. *According to the upper bound given by (4.7), we have*

$$\begin{cases} M_3(m,3,2) \leq \frac{m(m-1)}{6}, \\ M_4(m,3,2) \leq \frac{(m-1)(m-2)}{6}. \end{cases}$$

Therefore, for the case $w = 3$, the lower bound $m^{2-\varepsilon}$ from Theorem 4.4 is nearly optimal; and for the case $w = 4$, the lower bound $c'm^2$ from Theorem 4.3 is optimal, regardless of a constant factor. For cases when $w \geq 9$, (4.6) in Theorem 4.2 provides better lower bounds $(1-o(1))\frac{\binom{m}{\lceil w/4\rceil}}{\binom{w}{\lceil w/4\rceil}}$, but the gaps between the upper bounds and the lower bounds are still quite large.

It is also worth noting that, the construction from Theorem 4.4 was originally proposed by Erdős et al. [63] to construct w-uniform hypergraphs on m vertices such that no $3w-3$ vertices span 3 or more hyperedges. This kind of hypergraphs is a special kind of sparse hypergraphs which will be discussed later in Section III.D.

4.3.2.2 Construction of constant weighted X-codes with $d = 7$ and $x = 2$

Before we present the construction, we shall prove a proposition which establishes a connection between constant weighted X-codes with $d = 7, x = 2$ and uniform hypergraphs of girth five.

Given a k-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ and a positive integer $l \geq 2$, a cycle of length l in $\mathcal{H}$ (l-cycle in short), denoted by $\mathbb{C}_l$, is an alternating sequence of distinct vertices and hyperedges of the form: $v_1, E_1, v_2, E_2, \ldots, v_l, E_l, v_1$, such that $\{v_i, v_{i+1}\} \subseteq E_i$ for each $i \in \{1,2,\ldots,l\}$ and $\{v_l, v_1\} \subseteq E_l$. A linear path of length l (l-path in short), denoted by $\mathbb{P}_l$, is an alternating sequence of distinct vertices and hyperedges of the form: $E_1, v_2, E_2, v_3, \ldots, v_l, E_l$, such that $E_i \cap E_{i+1} = \{v_{i+1}\}$ for each i and $E_i \cap E_j = \emptyset$ whenever $|j-i| > 1$. And the *girth* of hypergraph $\mathcal{H}$ is the minimum length of a cycle in $\mathcal{H}$.

Proposition 4.2. *Let $w \geq 3$ be a positive integer. For any w-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ of girth at least 5, the set of all the indicator vectors of hyperedges in $\mathcal{E}$ forms a $(|V|, |\mathcal{E}|, 7, 2)$ X-code of constant weight w.*

Proof of Proposition 4.2. First, note that the girth of $\mathcal{H}$ is at least 5, we know that $\mathcal{H}$ is a linear hypergraph, i.e., $|E_1 \cap E_2| \leq 1$ for any $E_1, E_2 \in \mathcal{E}$. Hence, if we denote $\mathbf{v}_{E_i}$ as the indicator vector of hyperedge E_i, then for any $\{E_1, \ldots, E_7\} \subseteq \mathcal{E}$ and any s-subset $I_s \subseteq [7]$ with $1 \leq s \leq 7$, we have

$$w(\bigoplus_{i \in I_s} \mathbf{v}_{E_i}) \geq s \cdot (w - s + 1).$$

Moreover, for every $E \in \mathcal{E}$, $\mathbf{v}_E$ can't be covered by the superimposed sum of the indicator vectors of any other two edges in $\mathcal{E}$. For each $2 \leq s \leq 7$ and an s-subset $I_s \subseteq [7]$, consider the subhypergraph spanned by $\{E_i\}_{i \in I_s}$, we denote $V_0(I_s)$ as the set of vertices with even degree in this subhypergraph and $V_1(I_s)$ as the set of vertices with odd degree in this subhypergraph.

Let $\mathcal{C}$ be the set of indicator vectors of all edges in $\mathcal{E}$, according to the restrictions of the $(|V|, |\mathcal{E}|, 7, 2)$ X-code, our proof is divided into the following three parts.

Case 1. Assume that there exist $\{E_i\}_{i=1}^9 \subseteq \mathcal{E}$ such that $\bigoplus_{i \in [7]} \mathbf{v}_{E_i}$ is covered by $\mathbf{v}_{E_8} \vee \mathbf{v}_{E_9}$.

When the length of the longest linear path in the subhypergraph formed by $\{E_i\}_{i=1}^7$ is at most 3, consider a longest linear path $\mathbb{P}^{(7)}$ formed by edges $\{E_i\}_{i \in S}$ for some subset $S \subseteq [7]$ of size at most 3. Since $\mathcal{H}$ has girth at least 5, therefore, by the maximality of $\mathbb{P}^{(7)}$, the starting edge E_{i_s} and the ending edge E_{i_e} of $\mathbb{P}^{(7)}$ are disjoint with all edges in $\{E_i\}_{i \in [7] \setminus S}$. Therefore, by $w \geq 3$, we have $|V_1(S) \cap E_{i_s}|, |V_1(S) \cap E_{i_e}| \geq 2$. Note that $V_1(S)$ is covered by $\mathbf{v}_{E_8} \vee \mathbf{v}_{E_9}$. This forces E_8 (or E_9) together with $\mathbb{P}^{(7)}$ to form a cycle of length at most 4, which contradicts the requirement of $\mathcal{H}$ having girth at least 5.

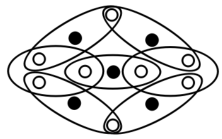


图 4.1 Subhypergraph formed by $\{E_1, \ldots, E_7\}$ with 5 odd vertices, where vertices with odd degree are denoted as "$\bullet$" and vertices with even degree are denoted as "$\circ$".

When the length of the longest linear path in the subhypergraph formed by $\{E_i\}_{i=1}^7$ is at least 4, consider a linear 3-path $\mathbb{P}_1^{(7)}$ formed by edges $\{E_i\}_{i\in S_1}$ for some 3-set $S_1 \subseteq [7]$. The vector $\bigoplus_{i\in S_1} \mathbf{v}_{E_i}$ has weight

$$w(\bigoplus_{i\in S_1} \mathbf{v}_{E_i}) = 3(w-2)+2.$$

As $\mathcal{H}$ has girth at least 5, for each $i \in [9] \setminus S_1$, $\mathbf{v}_{E_i}$ has at most one coordinate with value "1" agreeing with $\bigoplus_{i\in S_1} \mathbf{v}_{E_i}$. Then, the assumption that $\bigoplus_{i\in[7]} \mathbf{v}_{E_i}$ being covered by $\mathbf{v}_{E_8} \vee \mathbf{v}_{E_9}$ leads to $3(w-2)+2 \le 6$. Therefore, we have $w \le 3$.

Take I_s as $[7]$, then the assumption indicates that $V_1([7]) \subseteq E_8 \cup E_9$. Since $w = 3$, we have $|V_1([7])| \le 6$. One can easily check this only holds when the configuration formed by $\{E_1, \ldots, E_7\}$ is isomorphic to the subhypergraph shown in Fig. 1. Since there are 5 distinct vertices with odd degree in this configuration, the assumption that $\bigoplus_{i\in[7]} \mathbf{v}_{E_i}$ being covered by $\mathbf{v}_{E_8} \vee \mathbf{v}_{E_9}$ forces that E_8 forms a linear cycle of length at most 4 with 2 or 3 distinct hyperedges in $\{E_1, \ldots, E_7\}$, a contradiction.

Case 2. Assume that there exist $\{E_i\}_{i=1}^8 \subseteq \mathcal{E}$ such that $\bigoplus_{i\in[6]} \mathbf{v}_{E_i}$ is covered by $\mathbf{v}_{E_7} \vee \mathbf{v}_{E_8}$.

Similar to the analysis in Case 1, when the length of the longest linear path in the subhypergraph formed by $\{E_i\}_{i=1}^6$ is at most 3, the assumption that $\bigoplus_{i\in[6]} \mathbf{v}_{E_i}$ being covered by $\mathbf{v}_{E_7} \vee \mathbf{v}_{E_8}$ forces E_7 (or E_8) together with one of the longest linear path to form a cycle of length at most 4, a contradiction.

When the length of the longest linear path in the subhypergraph formed by

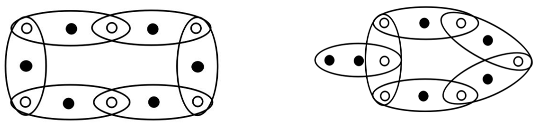


图 4.2 Subhypergraphs formed by $\{E_1, \ldots, E_6\}$ with 6 odd vertices, where vertices with odd degree are denoted as "•" and vertices with even degree are denoted as "○".

$\{E_i\}_{i=1}^6$ is at least 4, consider a 3-path $\mathbb{P}^{(6)}$ formed by $\{E_i\}_{i\in S_2}$ for some 3-subset $S_2 \subseteq [6]$, we have

$$w(\bigoplus_{i\in S_2} \mathbf{v}_{E_i}) = 3(w-2)+2.$$

As $\mathcal{H}$ has girth at least 5, for each $i \in [8] \setminus S_2$, $\mathbf{v}_{E_i}$ has at most one coordinate with value "1" agreeing with $\bigoplus_{i\in S_2} \mathbf{v}_{E_i}$. Therefore, the assumption that $\bigoplus_{i\in[6]} \mathbf{v}_{E_i}$ being covered by $\mathbf{v}_{E_7} \vee \mathbf{v}_{E_8}$ implies that $3(w-2)+2 \le 5$. Thus, we have $w \le 3$.

Take I_s as $[6]$, then we have $|V_1([6])| \le 6$. One can easily check this only holds when the configuration formed by $\{E_1, \ldots, E_6\}$ is isomorphic to one of the subhypergraphs shown in Fig. 2. Since there are 6 distinct vertices with odd degree in this configuration, the assumption that $\bigoplus_{i\in[6]} \mathbf{v}_{E_i}$ being covered by $\mathbf{v}_{E_7} \vee \mathbf{v}_{E_8}$ forces that E_8 forms a linear cycle of length at most 4 with 2 or 3 distinct hyperedges in $\{E_1, \ldots, E_6\}$, a contradiction.

Case 3. For each $4 \le l \le 7$, assume that there exist $\{E_i\}_{i=1}^l \subseteq \mathcal{E}$ such that $\bigoplus_{i\in[l-2]} \mathbf{v}_{E_i}$ is covered by $\mathbf{v}_{E_{l-1}} \vee \mathbf{v}_{E_l}$. Similar to the analysis in Case 1 and Case 2, we only have to consider the case when the length of the longest linear path in the configuration formed by $\{E_i\}_{i=1}^{l-2}$ is at least 4.

Let $\mathbb{P}^{(l-2)}$ be a 3-path in this subgraph formed by $\{E_i\}_{i\subseteq S_3}$ for some 3-subset $S_3 \subseteq [l-2]$, we have

$$w(\bigoplus_{i\in S_3} \mathbf{v}_{E_i}) = 3(w-2)+2.$$

Again, by the girth restriction of $\mathcal{H}$, for each $i \in [l] \setminus S_3$, $\mathbf{v}_{E_i}$ has at most one coordinate with value "1" agreeing with $\bigoplus_{i\in S_3} \mathbf{v}_{E_i}$. Therefore, the assumption above indicates that $3(w-2)+2 \le (l-3)$, which leads to $w \le 2$. This contradicts the fact that $w \geq 3$.

In conclusion, the addition of any seven or fewer distinct vectors in $\mathcal{C}$ can not be covered by the superimposed sum of any other two vectors in $\mathcal{C}$. Therefore, $\mathcal{C}$ is a $(|V|, |\mathcal{E}|, 7, 2)$ X-code of constant weight w. □

Based on a construction of 3-uniform hypergraphs of girth at least five in [130], by Proposition 4.2, we have the following result.

Theorem 4.5. *For any odd prime power q, there exists a $(q(q-1), \binom{q}{3}, 7, 2)$ X-code of constant weight 3.*

Proof of Theorem 4.5. For any odd prime power q, consider the finite field $\mathbb{F}_q$, let C_q denote the set of points on the curve $2x_2 = {x_1}^2$, where $(x_1, x_2) \in \mathbb{F}_q^2$.

Define a hypergraph $\mathcal{G}_q$ with vertex set $V(\mathcal{G}_q) = \mathbb{F}_q^2 \backslash C_q$. Three distinct vertices $\mathbf{a} = (a_1, a_2)$, $\mathbf{b} = (b_1, b_2)$ and $\mathbf{c} = (c_1, c_2)$ form a hyperedge $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ in $\mathcal{G}_q$ if and only if the following three equations hold:

$$\begin{cases} a_2 + b_2 = a_1 b_1; \\ b_2 + c_2 = b_1 c_1; \\ c_2 + a_2 = c_1 a_1. \end{cases}$$

As claimed in [130] (see the Remark on page 9 in [130]), $\mathcal{G}_q$ has girth at least five. Clearly, there are $\binom{q}{3}$ choices for distinct numbers a_1, b_1 and c_1, and each choice uniquely specifies a_2, b_2 and c_2 satisfying the above three equations. This indicates that any two choices of the triple $\{a_1, b_1, c_1\}$ being the same will lead to identical corresponding hyperedges. Therefore, the number of hyperedges in $\mathcal{G}_q$ is precisely $\binom{q}{3}$. By Proposition 4.2, we obtain a $(q(q-1), \binom{q}{3}, 7, 2)$ X-code of constant weight 3. □

Remark 4.2. *The construction from Theorem 4.5 actually gives a lower bound on $M_3(m, 7, 2)$ of the form*

$$M_3(m, 7, 2) = \Omega(m^{\frac{3}{2}}),$$

for sufficiently large m. This is better than the lower bound given by (4.6) in Theorem 4.2 in this case, however, compared to the upper bound $o(m^2)$ given by Tsunoda and Fujiwara [192], there is still a gap.

Unfortunately, this construction can not be extended to obtain general constant weighted X-codes. But at least, together with Proposition 4.2, it provides a way for constructing large constant weighted X-codes with $d = 7$ and $x = 2$.

4.3.3 An improved lower bound for X-codes of constant weight 3 with $x = 2$

Notice that when taking $w = x + 1$ in Theorem 4.2, the general lower bound given by (4.6) is only a linear function of m for $d \geq 2$. Through an elaborate analysis of the connection between a special kind of 3-uniform hypergraphs and X-codes of constant weight 3, we prove the following theorem, which improves this lower bound to $\Omega(m^{\frac{9}{7}})$.

Theorem 4.6. *For any positive integer $d \geq 8$ and sufficiently large m, there exists an $(m, c\cdot m^{\frac{9}{7}}, d, 2)$ X-code of constant weight 3, where $c > 0$ is an absolute constant.*

In graph theory, a k-uniform hypergraph $\mathcal{H}$ is called $\mathcal{G}_k(v, e)$-free if the union of any e distinct hyperedges contains at least $v+1$ vertices. These kinds of hypergraphs are called *sparse hypergraphs*. They are important structures in extremal graph theory and have been well-studied since 1970s (see [11, 86, 123, 184] and the reference therein). Before we present the proof of Theorem 4.6, we need the following lemma.

Lemma 4.3. *For any 3-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ that is simultaneously $\mathcal{G}_3(2s, s)$-free for each $2 \leq s \leq 4$ and $\mathcal{G}_3(\lceil \frac{3s-1}{2} \rceil + 3, s)$-free for each $8 \leq s \leq d$, the set of all the indicator vectors of hyperedges in $\mathcal{E}$ forms a $(|V|, |\mathcal{E}|, d, 2)$ X-code of constant weight 3.*

Proof of Lemma 4.3. Consider a 3-uniform hypergraph $\mathcal{H}_0 = (V_0, \mathcal{E}_0)$ that is simultaneously $\mathcal{G}_3(2s, s)$-free for each $2 \leq s \leq 4$ and $\mathcal{G}_3(\lceil \frac{3s-1}{2} \rceil + 3, s)$-free for each $8 \leq s \leq d$. Since $\mathcal{H}_0$ is $\mathcal{G}_3(2s, s)$-free for each $2 \leq s \leq 4$, we know that the girth of $\mathcal{H}_0$ is at least 5. From the result of Proposition 4.2, the set of all the indicator vectors $\mathcal{C}(\mathcal{H}_0)$ corresponding to $\mathcal{E}_0$ already forms a $(|V|, |\mathcal{E}|, 7, 2)$ X-code of constant weight 3. Therefore, we only have to show that the addition of any s $(8 \leq s \leq d)$ distinct indicator vectors in $\mathcal{C}(\mathcal{H}_0)$ can not be covered by the superimposed sum of any other two indicator vectors in $\mathcal{C}(\mathcal{H}_0)$.

For each $e \in \mathcal{E}_0$, denote $\mathbf{v}_e$ as the indicator vector of e. For each integer $8 \leq s \leq d$, consider s distinct hyperedges $\{e_1, \ldots, e_s\}$ in $\mathcal{E}$. Assume that there exist two other hyperedges f_1 and f_2, such that $\mathbf{v}_{e_1} \oplus \cdots \oplus \mathbf{v}_{e_s}$ can be covered by $\mathbf{v}_{f_1} \vee \mathbf{v}_{f_2}$. Denote V_0 as the set of vertices in $\bigcup_{i=1}^{s} e_i$ that are contained in even number of hyperedges in $\{e_1, \ldots, e_s\}$ and V_1 as the set of vertices in $\bigcup_{i=1}^{s} e_i$ that are contained in odd number of hyperedges in $\{e_1, \ldots, e_s\}$. Then the assumption indicates that $V_1 \subseteq f_1 \cup f_2$. Since $\mathcal{H}$ is a 3-uniform hypergraph, we have

$$|V_1| \leq 6 \text{ and } 2|V_0| + |V_1| \leq 3s. \tag{4.9}$$

Now, for a fixed integer $8 \leq s_0 \leq d$, according to inequality (4.9) and the parity of s_0, we have

$$|\bigcup_{i=1}^{s_0} e_i| = |V_0| + |V_1| \leq \lceil \frac{3s_0 - 1}{2} \rceil + 3.$$

This implies that these s_0 distinct hyperedges $\{e_1, \ldots, e_{s_0}\}$ are spanned by at most $\lceil \frac{3s_0-1}{2} \rceil + 3$ distinct vertices in $\mathcal{H}_0$, which contradicts the condition that $\mathcal{H}_0$ is $\mathcal{G}_3(\lceil \frac{3s-1}{2} \rceil + 3, s)$-free for each $8 \leq s \leq d$. Thus, for each $8 \leq s \leq d$, the addition of any distinct s indicator vectors in $\mathcal{C}(\mathcal{H}_0)$ can not be covered by the superimposed sum of other 2 indicator vectors. Therefore, combined with former analysis, the set of all the indicator vectors in $\mathcal{C}(\mathcal{H}_0)$ forms a $(|V|, |\mathcal{E}|, d, 2)$ X-code of constant weight 3. □

Now, we present the proof of Theorem 4.6.

Proof of Theorem 4.6. By Lemma 4.3, we only need to construct a 3-uniform hypergraph $\mathcal{H}_0$ that is simultaneously $\mathcal{G}_3(2s, s)$-free for each $2 \leq s \leq 4$ and $\mathcal{G}_3(\lceil \frac{3s-1}{2} \rceil + 3, s)$-free for each $8 \leq s \leq d$ with $\Omega(m^{\frac{9}{7}})$ hyperedges.

Let V be a finite set of points and $|V| = m$, take a subset $\mathcal{B}$ of triples by picking elements of $\binom{V}{3}$ uniformly and independently at random with probability p. Then we have

$$\mathbb{E}[|\mathcal{B}|] = p \cdot \binom{|V|}{3}.$$

For each $2 \le s \le 4$, denote D_s as the set of s-subsets in $\mathcal{B}$ that are spanned by at most $2s$ points in V, i.e., for each $\{B_1, \ldots, B_s\} \in D_s \subseteq \binom{\mathcal{B}}{s}$, $|\bigcup_{i=1}^{s} B_i| \le 2s$. Then we have

$$p^s \cdot \binom{|V|}{2s} \le \mathbb{E}[|D_s|] \le \binom{2s}{3}^s \cdot p^s \cdot \binom{|V|}{2s},$$

for each $2 \le s \le 4$.

For each $8 \le s \le d$, denote D_s as the set of s-subsets in $\mathcal{B}$ that are spanned by at most $\lceil \frac{3s-1}{2} \rceil + 3$ points in V, i.e., for each $\{B_1, \ldots, B_s\} \in D_s \subseteq \binom{\mathcal{B}}{s}$, $|\bigcup_{i=1}^{s} B_i| \le \lceil \frac{3s-1}{2} \rceil + 3$. Then we have

$$\begin{aligned} p^s \cdot \binom{|V|}{\lceil \frac{3s-1}{2} \rceil + 3} &\le \mathbb{E}[|D_s|] \\ &\le \binom{\lceil \frac{3s-1}{2} \rceil + 3}{3}^s \cdot p^s \cdot \binom{|V|}{\lceil \frac{3s-1}{2} \rceil + 3}, \end{aligned}$$

for each $8 \le s \le d$.

By deleting at most one triple from each s-subset in D_s, for $2 \le s \le 4$ and $8 \le s \le d$, the remaining triples form a 3-uniform hypergraph that is simultaneously $\mathcal{G}_3(2s, s)$-free for each $2 \le s \le 4$ and $\mathcal{G}_3(\lceil \frac{3s-1}{2} \rceil + 3, s)$-free for each $8 \le s \le d$. Now, take $p = \frac{1}{30} \cdot m^{-\frac{12}{7}}$ and $0 \le c \le \frac{1}{30}$. For m sufficiently large, we have $\mathbb{E}[|D_s|] = o(\mathbb{E}[|\mathcal{B}|])$ for $2 \le s \le d, s \ne 8$. Therefore, by the linearity of expectation, we have

$$\begin{aligned} \mathbb{E}[|\mathcal{B}| - \sum_{s=2}^{d} |D_s|] &\ge p \cdot \binom{|V|}{3} \cdot (1 - o(1)) - \frac{(455)^8}{14!} \cdot p^8 \cdot |V|^{15} \\ &\ge c \cdot m^{\frac{9}{7}}. \end{aligned}$$

Therefore, with positive probability, there exists a 3-uniform hypergraph $\mathcal{H}$ that is simultaneously $\mathcal{G}_3(2s, s)$-free for each $2 \le s \le 4$ and $\mathcal{G}_3(\lceil \frac{3s}{2} \rceil + 3, s)$-free for each $8 \le s \le d$ with vertex set V and $c \cdot m^{\frac{9}{7}}$ hyperedges. This completes the proof. □

§ 4.4 r-even-free triple packings and X-codes with higher error tolerance

To construct X-codes with $x = 2$ and weight 3, Fujiwara and Colbourn [79] introduced the notion of r-even-free triple packing, which was further studied in [192]. In this section, by obtaining an existence result of the corresponding 6-even-free triple packing, we prove a lower bound on the maximum number of codewords of an $(m, n, 1, 2)$ X-code of constant weight 3 which can detect up to three erroneous bits if there is only one X in the raw response data and up to six erroneous bits if there is no X, this improves the lower bound given in [192]. And we also extend this lower bound to a general case.

A *triple packing* of order v is a set system $(V, \mathcal{B})$ such that $\mathcal{B}$ is a family of triples of a finite set V and any pair of elements of V appears in $\mathcal{B}$ at most once. Given a triple packing $(V, \mathcal{B})$, we call subset $\mathcal{C}$ in $\mathcal{B}$ an i-configuration if $|\mathcal{C}| = i$. A configuration $\mathcal{C}$ is *even* if for every vertex $v \in V$ appearing in $\mathcal{C}$, the number $|\{B : v \in B \in \mathcal{C}\}|$ of triples containing v is even. And a triple packing $(V, \mathcal{B})$ is *r-even-free* if for every integer i satisfying $1 \leq i \leq r$, $\mathcal{B}$ contains no even i-configurations.

By carefully analysing the structure of r-even-free triple packing, Fujiwara and Colbourn [79] obtained the following theorem which relates the r-even-free triple packing to a special kind of X-codes.

Theorem 4.7. *[79] For $r \geq 4$, if there exists an r-even-free triple packing $(V, \mathcal{B})$, there exists a $(|V|, |\mathcal{B}|, 1, 2)$ X-code of constant weight 3 that is also a $(|V|, |\mathcal{B}|, 3, 1)$ X-code and a $(|V|, |\mathcal{B}|, r, 0)$ X-code.*

Using the existence results of *anti-Pasch* Steiner triple systems, Fujiwara and Colbourn [79] proved that for every $m \equiv 1, 3 \pmod 6$ and $m \notin \{7, 13\}$, there exists an $(m, m(m-1)/6, 1, 2)$ X-code of constant weight 3 that is an $(m, m(m-1)/6, 3, 1)$ X-code and an $(m, m(m-1)/6, 5, 0)$ X-code. And they also proved the existence of a 6-even-free triple packing $\mathcal{B}$ of order m with $|\mathcal{B}| = 6.31 \times 10^{-3} \times m^{1.8}$ using the probabilistic method, which gives a lower bound on the size of the corresponding X-code given by Theorem 4.7.

Recently, according to a complete characterization of all the forbidden even configurations in the 6-even-free triple packing, Tsunoda and Fujiwara [192] obtained the following result, which improves the lower bound $6.31 \times 10^{-3} \times m^{1.8}$ given in [79].

Theorem 4.8. *[192] For sufficiently large m, there exists an $(m, c' \cdot m^{1.8}, 1, 2)$ X-code of constant weight 3 that is also an $(m, c' \cdot m^{1.8}, 3, 1)$ X-code and an $(m, c' \cdot m^{1.8}, 6, 0)$ X-code, where $c' = \frac{5}{36}(\frac{1}{72})^{\frac{1}{5}}$.*

Inspired by the probabilistic hypergraph independent set approach introduced by Duke et al. [51], we prove the following theorem, which improves the order of magnitude of the lower bound in Theorem 4.8 by a factor of $(\log m)^{\frac{1}{5}}$.

Theorem 4.9. *For sufficiently large m, there exists an $(m, c_0 \cdot m^{\frac{9}{5}}(\log m)^{\frac{1}{5}}, 1, 2)$ X-code of constant weight 3 that is also an $(m, c_0 \cdot m^{\frac{9}{5}}(\log m)^{\frac{1}{5}}, 3, 1)$ X-code and an $(m, c_0 \cdot m^{\frac{9}{5}}(\log m)^{\frac{1}{5}}, 6, 0)$ X-code, where $c_0 > 0$ is an absolute constant.*

An even 4-configuration is called a *Pasch*, if it has the form $\{\{a, b, c\}, \{a, e, f\}$ $,\{b, d, f\}, \{c, d, e\}\}$. An even 6-configuration is called a *grid* if it has the form $\{\{a, b, c\}, \{d, e, f\}, \{g, h, i\}, \{a, d, g\}, \{b, e, h\}, \{c, f, i\}\}$, and a *double triangle* if it has the form $\{\{a, b, c\}, \{c, d, e\}, \{e, f, g\}, \{a, g, h\}, \{b, h, i\}, \{d, f, i\}\}$. Before we present the proof of Theorem 4.9, we need the following proposition.

Proposition 4.3. *[192] A triple packing contains no Pasches, grids or double triangles is 6-even-free.*

Proof of Theorem 4.9. By Theorem 4.7 and Proposition 4.3, we only need to construct a triple packing without Pasches, grids and double triangles.

Let V be a finite set of points and $|V| = m$, take a subset $\mathcal{B}$ of triples by picking elements of $\binom{V}{3}$ uniformly and independently at random with probability p.

Denote D_2 as the set of non-linear triple pairs in $\mathcal{B}$, i.e., for each $\{B_1, B_2\} \in D_2 \subseteq \binom{\mathcal{B}}{2}$, $|B_1 \cap B_2| \geq 2$. Then we have

$$\mathbb{E}[|D_2|] \leq \binom{4}{3}^2 \cdot p^2 \cdot \binom{|V|}{4}.$$

Denote D_4 as the set of Pasches, D_{61} as the set of grids and D_{62} as the set of double triangles in $\mathcal{B}$, we have

$$\mathbb{E}[|D_4|] \leq 6! \cdot p^4 \cdot \binom{|V|}{6},$$

and

$$\binom{9}{3} \cdot \binom{6}{3} \cdot p^6 \cdot \binom{|V|}{9} \leq \mathbb{E}[|D_{61}|], \mathbb{E}[|D_{62}|] \leq 9! \cdot p^6 \cdot \binom{|V|}{9}.$$

Let $Y = \{(\mathcal{C}_1, \mathcal{C}_2) : \mathcal{C}_1, \mathcal{C}_2 \in D_{61} \sqcup D_{61} \text{ and } |\mathcal{C}_1 \cap \mathcal{C}_2| \geq 2\}$, then $Y = Y_1 \sqcup Y_2 \sqcup Y_3$, where $Y_1 = Y \cap (D_{61} \times D_{61})$, $Y_2 = Y \cap (D_{62} \times D_{62})$ and $Y_3 = Y \cap (D_{61} \times D_{62} \cup D_{62} \times D_{61})$. Through a routine analysis about intersection patterns of pairs in Y_i, since $p \leq 1$, we have

$$\begin{cases} \mathbb{E}[|Y_1|] \leq c_1 \cdot (p^{10}m^{13} + p^9 m^{11} + p^8 m^{10}); \\ \mathbb{E}[|Y_2|] \leq c_2 \cdot (p^{10}m^{13} + p^9 m^{12} + p^8 m^{10}); \\ \mathbb{E}[|Y_3|] \leq c_3 \cdot (p^{10}m^{13} + p^9 m^{11} + p^8 m^{10}), \end{cases}$$

for three absolute constants c_1, c_2, c_3. This leads to

$$\mathbb{E}[|Y|] \leq C_0 \cdot (p^{10}m^{13} + p^9 m^{12} + p^8 m^{10}),$$

for some absolute constant $C_0 \geq (c_1 + c_2 + c_3)$.

Now, take $\mathcal{H}$ as a random 6-uniform hypergraph with vertex set $\mathcal{B}$ and hyperedge set

$$\mathcal{E}(\mathcal{H}) = \{\{B_1, \ldots, B_6\} : \{B_1, \ldots, B_6\} \text{ forms a grid or a double triangle in } \mathcal{B}\},$$

and set $p = m^{-(\frac{9}{8}+\varepsilon)}$ for some ε small enough such that $0 < \varepsilon < \frac{3}{40}$.

Then, for m large enough, we have

$$\mathbb{E}[|D_2|], \mathbb{E}[|D_4|], \mathbb{E}[|Y|] \ll \mathbb{E}[|\mathcal{B}|].$$

Thus, with probability at least $\frac{3}{4}$, we can delete at most one triple from each non-linear pair, Pasch and $\mathcal{C}_1 \cup \mathcal{C}_2$ for $(\mathcal{C}_1, \mathcal{C}_2) \in Y$, obtaining a linear induced 6-uniform subhypergraph $\mathcal{H}'$ of $\mathcal{H}$ with at least $\frac{3}{4} \cdot |V(\mathcal{H})|$ vertices such that the vertex set of $\mathcal{H}'$ contains no non-linear triple pairs and Pasches.

Meanwhile, since

$$\mathbb{E}[|V(\mathcal{H})|] = \mathbb{E}[|\mathcal{B}|] = (\frac{1}{6} - o(1)) \cdot m^{\frac{15}{8}-\varepsilon}$$

and

$$\frac{m^{\frac{9}{4}-6\varepsilon}}{532} \leq \mathbb{E}[|\mathcal{E}(\mathcal{H})|] = \mathbb{E}[|D_{61} \cup D_{62}|] \leq (2 - o(1)) \cdot m^{\frac{9}{4}-6\varepsilon},$$

by Chernoff bound, for m large enough, we have

$$\begin{cases} \frac{m^{\frac{15}{8}-\varepsilon}}{12} \leq |V(\mathcal{H})| \leq \frac{m^{\frac{15}{8}-\varepsilon}}{3}; \\ \frac{m^{\frac{9}{4}-6\varepsilon}}{10^3} \leq |\mathcal{E}(\mathcal{H})| \leq 3m^{\frac{9}{4}-6\varepsilon}, \end{cases}$$

with probability at least $\frac{7}{8}$. Therefore, the average degree of $\mathcal{H}$

$$\bar{d}_{\mathcal{H}} \leq 216 m^{\frac{3}{8}-5\varepsilon}$$

with probability at least $\frac{7}{8}$. Thus, by Markov's inequality, with probability at least $\frac{3}{4}$, the hypergraph $\mathcal{H}$ contains at most $\frac{1}{4} \cdot |V(\mathcal{H})|$ vertices of degree exceeding $10^4 \cdot m^{\frac{3}{8}-5\varepsilon}$. Therefore, with probability at least $\frac{1}{2}$, we can delete these vertices and obtain a linear subhypergraph $\mathcal{H}''$ of $\mathcal{H}'$ with at least $(\frac{1}{24}) \cdot m^{\frac{15}{8}-\varepsilon}$ vertices and maximum degree at most $10^4 \cdot m^{\frac{3}{8}-5\varepsilon}$.

Finally, by Lemma 4.1, we have

$$\alpha(\mathcal{H}'') \geq c_0 \cdot m^{\frac{9}{5}} (\log m)^{\frac{1}{5}},$$

for some absolute constant $c_0 > 0$. Since an independent set I in $\mathcal{H}''$ is a triple packing that contains no Pasch, grid or double triangle, thus the above inequality guarantees the existence of a 6-even-free triple packing of order $c_0 \cdot m^{\frac{9}{5}} (\log m)^{\frac{1}{5}}$. This completes the proof. $\square$

The above approach can also be applied to obtain general r-even-free triple packings.

Note that for any even i-configuration $\mathcal{C}$, $1 \leq i \leq r$, we have

$$\deg_{\mathcal{C}}(v) \equiv 0 \mod 2,$$

for every $v \in V$. Since $(V, \mathcal{C})$ is a triple system, we also have

$$\sum_{v \in V} \deg_{\mathcal{C}}(v) = 3 \cdot |\mathcal{C}| = 3i. \tag{4.10}$$

Thus, for odd i, an i-configuration $\mathcal{C}$ cannot be even, and for even i, an i-configuration $\mathcal{C}$ involves at most $\frac{3i}{2}$ points in V.

Now, take a triple packing $(V, \mathcal{B})$ as a 3-uniform linear hypergraph with vertex set V, from the perspective of sparse hypergraphs, for even i, a $\mathcal{G}_3(\frac{3i}{2}, i)$-free 3-uniform linear hypergraph is a triple packing that contains no even i-configurations. Ranging i from 1 to r, we have the following proposition.

Proposition 4.4. *If a 3-uniform linear hypergraph $\mathcal{H}$ is simultaneously $\mathcal{G}_3(\frac{3i}{2}, i)$-free for every even $1 \leq i \leq r$, then $\mathcal{H}$ is an r-even-free triple packing.*

Let $r' = \lfloor \frac{r}{2} \rfloor$ and V be a finite set of points, consider a random triple system $(V, \mathcal{B})$ by picking elements of $\binom{V}{3}$ uniformly and independently with a proper probability p. First, estimate the expectations of the number of non-linear triple pairs and the number of forbidden $\mathcal{G}_3(\frac{3i}{2}, i)$s for every even $1 \leq i \leq r$. Then, construct a $2r'$-uniform random hypergraph with the set of triples $\mathcal{B}$ as its vertex set such that any $2r'$ triples form a hyperedge if and only if they involve at most $3r'$ points in V. Using a similar probabilistic hypergraph independent set approach as that for Theorem 4.9, one can obtain the following theorem.

Theorem 4.10. *For sufficiently large m, there exists an r-even-free triple packing $\mathcal{B}$ of order m such that*

$$|\mathcal{B}| = \Omega(m^{\frac{3r'}{2r'-1}} (\log m)^{\frac{1}{2r'-1}}),$$

where $r' = \lfloor \frac{r}{2} \rfloor$.

Combining the above result with Theorem 4.7, we immediately have

Corollary 4.4.1. *For sufficiently large* m*, there exists an* $(m, \Omega(m^{\frac{3r'}{2r'-1}}(\log m)^{\frac{1}{2r'-1}}), 1, 2)$ X*-code of constant weight* 3 *that is also an* $(m, \Omega(m^{\frac{3r'}{2r'-1}}(\log m)^{\frac{1}{2r'-1}}), 3, 1)$ X*-code and an* $(m, \Omega(m^{\frac{3r'}{2r'-1}}(\log m)^{\frac{1}{2r'-1}}), r, 0)$ X*-code, where* $r' = \lfloor \frac{r}{2} \rfloor$*.*

Remark 4.3. *A little different from the case* $r = 6$*, for general* r*, we can not fully characterize the specific even configurations that shall be forbidden to obtain an* r*-even-free triple packing. Thus, a stronger restriction has been required in Proposition 4.4.*

§ 4.5 Concluding remarks and further research

In this chapter, we investigate the maximum number $M_w(m, d, x)$ of an X-code of constant weight w with testing quality parameters d and x. We obtain general lower and upper bounds for $M_w(m, d, x)$ and further improve the lower bound for the case with $w = 3$ and $x = 2$. Using tools from additive combinatorics and finite fields, we also obtain some explicit constructions for cases $d = 3, 7$ and $x = 2$, which improve the corresponding general lower bounds. Moreover, we study a special class of $(m, n, 1, 2)$ X-codes of constant weight 3 which can also detect many erroneous bits if there is at most one X.

We summarize our lower bounds for $M_w(m, d, x)$ in Table 4.1, and for convenience, we also include the best known corresponding upper bounds.

Although many works have been done about bounding $M_w(m, d, x)$, in most cases, the gaps between the upper bounds and the lower bounds are still quite large. For cases $d = 3$, $x = 2$ and $w = 3$, constructions given by Theorem 4.4 narrow the gaps between the upper bounds and the lower bounds to an ε over the exponent. We expect methods from other aspects can provide some better constructions.

表 4.1 **Upper and lower bounds for** $M_w(m,d,x)$

	Lower Bounds	Upper Bounds
$M_w(m,d,x)$	$(1-o(1))\frac{\binom{m}{\lceil w/(x+d-1)\rceil}}{\binom{w}{\lceil w/(x+d-1)\rceil}}$	$\frac{\binom{m}{\lceil \frac{w}{x}\rceil}}{\binom{w-1}{\lceil \frac{w}{x}\rceil-1}}$ (see (4.5) in Theorem 4.2)
$M_3(m,d,2)$	$\Omega(m^{\frac{9}{7}})$	$o(m^2)$ (see [192])
$M_w(m,3,2)$ $(w \geq 4)$	$\Omega(m^2)$	$O(m^{\lceil w/2\rceil})$ (see (4.7) in Section III.A)
$M_3(m,3,2)$	$\Omega(m^{2-\varepsilon})$	$O(m^2)$ (see (4.7) in Section III.A)
$M_3(m,7,2)$	$\Omega(m^{\frac{3}{2}})$	$o(m^2)$ (see [192])

Chapter 5 Two kinds of codes in distributed storage systems: locally recoverable codes and maximally recoverable codes

§ 5.1 Introduction

The ever-increasing amounts of data created and transported through the internet is urgently demanding efficient and reliable storage, which resulted in distributed storage systems relying on distinct storage nodes. Traditional large scale distributed storage systems used to store data in a redundant form to ensure reliability against node failures. However, this strategy entails large storage overhead and is costly and nonadaptive for modern systems. To ensure the reliability with better efficiency, erasure coding schemes are employed, such as in Windows Azure [107] and in Facebook's Hadoop cluster [185]. However, in such schemes, if one node fails, which is the most common failure scenario, we may recover it by accessing a large amount of the remaining nodes. This is a time consuming recovery process, especially in large-scale distributed file systems.

To maintain high repair efficiency with less bandwidth, locally repairable codes (LRCs) were introduced in [91]. A block code is called a locally repairable code with locality r if any failed code symbol can be recovered by accessing at most r survived ones. Moreover, if this code is linear, r should be much smaller than code dimension k. Therefore, LRCs can guarantee efficient recovery of single node failures with low repair bandwidth. As a result, LRCs have been implemented in many large scale systems e.g., Microsoft Azure [107] and Hadoop HDFS [171].

Along with locality, the notion of maximally recoverability was first introduced by Chen et al. [43] for multi-protection group codes, and then extended by Gopalan et al. [91] to general settings. Different from locality, maximally recoverability contains global constraints and can be considered for general coding schemes. To specify constraints of the repair coding schemes, Gopalan et. al [91] introduced the concept of *topology* of a code. Based on this, they obtained a general upper bound on the minimal size of the field over which maximally recoverable codes (MRCs) for general topology exist.

Over the past few years, the concepts of locality and maximally recoverability have been generalized and studied in many different aspects. As one major generalization of LRCs, codes with (r, δ)-locality ((r, δ)-LRCs) was introduced by Prakash et al. [160], which extends the capability of repairing one erasure within each repair set to $\delta - 1$ erasures. Like original LRCs, a Singleton-type upper bound on the minimum distance of (r, δ)-LRCs was given in [160]. Recently, finding constructions of the optimal LRCs and optimal (r, δ)-LRCs with respect to such bounds has become an interesting and challenging work, which attracted lots of researchers. For examples, see [112, 113, 132, 142, 178, 186, 188, 208] for constructions of optimal LRCs and see [39, 42, 213, 214] for constructions of optimal (r, δ)-LRCs. For the study of availabilities of LRCs, see [38, 39, 164, 177, 187, 200], and for the study of codes with hierarchical locality (H-LRCs), see [17, 141, 170, 214]. For other generalizations of LRCs, we refer to the survey [15]. As for MRCs, see [91, 93, 95, 96, 99, 162].

Due to their efficient transmission performances and fast implementations in practical applications, longer codes over smaller fields are favored. In this spirit, Guruswami et al. [98] asked how long optimal (r, δ)-LRCs can be when the size q of the underlying field and other parameters are given. They considered this question for the case $\delta = 2$ and proved an upper bound on the code length. Through a greedy algorithm, they also constructed optimal $(r, 2)$-LRCs with super-linear (in q) length, which confirmed the tightness of their upper bound for some cases. Latter in [39], Cai et al. considered this question for the general case $\delta > 2$ and derived a general upper bound on the length of optimal (r,δ)-LRCs. Furthermore, using combinatorial objects such as union-intersection-bounded families, packings and Steiner systems, they obtained optimal (r,δ)-LRCs with length $\Omega(q^{\delta})$, which meet their upper bound on the code length when the minimal distance d satisfies $2\delta+1 \le d \le 3\delta$. Very recently, Cai and Schwartz [40] extended their results in [39] to codes that not only have information (r,δ)-locality but also can recover some erasure patterns beyond the minimum distance. They also introduced a new kind of array codes called *generalized sector-disk* (GSD) codes, which can recover special erasure patterns mixed of whole disk erasures together with additional sector erasures that are beyond the minimum distance.

In this chapter, through parity-check matrix approach, we provide general constructions for both optimal (r,δ)-LRCs with all symbol locality and optimal (r,δ)-LRCs with information locality and extra global recoverability. Our constructions are built on a connection between sparse hypergraphs in extremal combinatorics and optimal (r,δ)-LRCs, which can be viewed as a generalization of a work of Xing and Yuan [208]. Based on known results and a probabilistic construction about sparse hypergraphs, we obtain optimal $(r,\delta)_a$-LRCs (codes with all symbol (r,δ)-locality) and optimal $(r,\delta)_i$-LRCs (codes with information (r,δ)-locality) with length super-linear in q. Compared to the results in [39] and [40], our results provide longer codes for $d \ge 3\delta+1$. Furthermore, as two applications of our constructions, we construct optimal H-LRCs with super-linear length, which improves the results given by [213]; and we also provide a construction of generalized sector-disk codes with unbounded length.

With the same purpose of deploying longer codes in storage, Gopalan et al. [93] proposed *grid-like topologies*, which unified a number of topologies considered both in theory and practice: Consider an $m \times n$ matrix, each entry storing a data from a finite field $\mathbb{F}$. Every row satisfies a parity constraints, every column satisfies b parity constraints and all mn entries satisfies h additional global parity constraints. This topology is denoted by $T_{m\times n}(a,b,h)$. In [93], the authors considered MRCs for general grid-like topologies and established a super-polynomial lower bound on the field size required for the existence of MRCs instantiating topologies $T_{m\times n}(a,b,h)$ with $a,b,h \geq 1$. They also obtained a full combinatorial characterization of correctable erasure patterns for topology $T_{m\times n}(1,b,0)$. Recently, by relating the problem to the independence number of the Birkhoff polytope graph, Kane et al. [117] improved the lower bound to $q \geq 2^{(\frac{n}{2}-2)}$ using the representation theory of the symmetric group. They also obtained an upper bound $q \leq 2^{3n}$ using recursive constructions. As for other related works, Gandikota et al. [85] considered the maximal recoverability for erasure patterns of bounded size. Shivakrishna et al. [176] considered the recoverability of the extended erasure patterns for topologies $T_{(m+m')\times n}(2,b,0)$. It is worth noting that, Gopi et al. [95] recently obtained a super-linear lower bound for maximally recoverable LRCs which can be viewed as the MRCs for topology $T_{\frac{n}{r}\times r}(a,0,h)$.

In this chapter, we focus on the MRCs instantiating topologies of the form $T_{m\times n}(1,b,0)$. Based on *pseudo-parity check matrix* approach, we prove a general upper bound on the size of the field required for the existence of MRCs instantiating topologies $T_{m\times n}(1,b,0)$. For special topologies $T_{4\times n}(1,2,0)$ and $T_{3\times n}(1,3,0)$, this upper bound is further improved. Moreover, we also obtain a polynomial lower bound on the size of the field required for MRCs instantiating $T_{m\times n}(1,2,0)$. As far as we know, this is the first super linear lower bound on field size of MR tensor codes under any setting.

The remainder of this chapter is organized as follows. In Section § 5.2, we fix some notations and provide preliminaries on locally repairable codes, grid-like topologies and maximal recoverability. In Section § 5.3, we present our constructions of optimal (r,δ)-LRCs with all symbol locality and optimal (r,δ)-LRCs with information locality and extra global recoverability. In Section § 5.4, we first give a brief introduction about Tuŕan-type problems for sparse hypergraphs, and then based on constructions of a special kind of sparse hypergraphs, we obtain optimal $(r,\delta)_a$-LRCs and optimal $(r,\delta)_i$-LRCs with super-linear length. In Section § 5.5, we provide two applications of our constructions for H-LRCs and GSD codes. In Section § 5.6, we present our general upper bound on the field size required for the existence of MRCs instantiating $T_{m\times n}(1,b,0)$. In Section § 5.7, we further study M-RCs for two special topologies: $T_{m\times n}(1,2,0)$ and $T_{m\times n}(1,3,0)$. Finally, we conclude this chapter with some remarks in Section § 5.8.

§ 5.2 Preliminaries

5.2.1 Notation

Firstly, we introduce some notations and terminologies that will be frequently used throughout this chapter:

- Let q be the power of a prime p, $\mathbb{F}_q$ be the finite field with q elements, $\mathbb{F}_q^n$ be the vector space of dimension n over $\mathbb{F}_q$ and $\mathbb{F}_q^{m\times n}$ be the collection of all $m\times n$ matrices with elements in $\mathbb{F}_q$.
- $\mathcal{C}$ is said to be an $[n,k,d]_q$ code (or $[n,k,d]$ code for short when q is clear) if $\mathcal{C}$ is a linear code over $\mathbb{F}_q$ with length n, dimension k and minimum distance d.
- Let $\mathcal{C}$ be an $[n,k,d]$ code and $S\subseteq[n]$, $|S|=k$. We say that S is an information set if the restriction $\mathcal{C}|_S=\mathbb{F}_q^k$.
- Let $\mathcal{C}_1$ be an $[n_1,k_1,d_1]$ code and $\mathcal{C}_2$ be an $[n_2,k_2,d_2]$ code. The tensor product $\mathcal{C}_1\otimes\mathcal{C}_2$ is an $[n_1n_2,k_1k_2,d_1d_2]$ code such that the codewords of $\mathcal{C}_1\otimes\mathcal{C}_2$ are matrices of size $n_1\times n_2$, where each column belongs to $\mathcal{C}_1$ and each row belongs to $\mathcal{C}_2$. If $U\subseteq[n_1]$ is an information set of $\mathcal{C}_1$ and $V\subseteq[n_2]$ is an information set of $\mathcal{C}_2$, then $U\times V$ is an information set of $\mathcal{C}_1\otimes\mathcal{C}_2$ (see [144]).
- Let $\mathbf{I}_n$ be the $n\times n$ identity matrix. And let $\mathbf{1}_n$ and $\mathbf{0}_n$ be the all-one and all-zero vectors, respectively.
- For $x\geq 0$, we use $\lfloor x\rfloor$ and $\lceil x\rceil$ to denote the floor function and ceiling function of x, respectively.

- We use O to denote the zero matrix with proper size according to the context.

- A vector over $\mathbb{F}_q$ is said to be Vandemonde-type with generator (or generating element) a if it has the form $b(1, a, a^2, \cdots)^T$ for some $a, b \in \mathbb{F}_q^*$. For a set of Vandemonde-type vectors $\{\mathbf{v}_i\}_{i=1}^s$, the generating set of $\{\mathbf{v}_i\}_{i=1}^s$ consists of all the generators for every $\mathbf{v}_i$, $1 \le i \le s$.

- For positive integers m and n, let E be a subset of $[n]$ with size s. Write $E = \{i_1, \ldots, i_s\}$ when $s \ge 1$ and $E = \emptyset$ when $s = 0$. Let $\mathbf{H} = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n)$ be a matrix of size $m \times n$, where $\mathbf{h}_i \in \mathbb{F}_q^m$ for $1 \le i \le n$. Then, the restriction of $\mathbf{H}$ over E is defined as $\mathbf{H}|_E = (\mathbf{h}_{i_1}, \mathbf{h}_{i_2}, \ldots, \mathbf{h}_{i_s})$ when $s \ge 1$ and $\mathbf{H}|_E = ()$, i.e., the empty matrix, when $s = 0$.

- We use the standard Bachmann-Landau notations $\Omega(\cdot)$, $\theta(\cdot)$, $O(\cdot)$ and $o(\cdot)$, whenever the constant factors are not important.

5.2.2 (r, δ)-locality

Now we state the formal definition of (r, δ)-locality.

Definition 5.1. ([160]) *Let $\mathcal{C}$ be an $[n, k, d]_q$ code. The ith code symbol c_i of $\mathcal{C}$ is called to have locality (r, δ) if there exists a subset $S_i \subset [n]$ satisfying*

- *$i \in S_i$ and $|S_i| \le r + \delta - 1$,*

- *the minimum distance of the code $\mathcal{C}|_{S_i}$ obtained by deleting code symbols c_i, $i \in [n] \backslash S_i$, is at least δ.*

An $[n, k, d]_q$ code $\mathcal{C}$ is said to have all symbol (r, δ)-locality ($(r, \delta)_a$-locality) if all symbols of $\mathcal{C}$ have locality (r, δ) and it is said to have information (r, δ)-locality ($(r, \delta)_i$-locality), if there exists a k-set $I \subseteq [n]$ with $rank(I) = k$ such that for every $i \in I$, the i_{th} symbol has (r, δ)-locality. As shown in [160], for both $[n, k, d]_q$ codes with $(r, \delta)_a$-locality and $[n, k, d]_q$ codes with $(r, \delta)_i$-locality, their minimal distance d satisfies the following Singleton-type bound:

$$d \le n - k + 1 - (\left\lceil \frac{k}{r} \right\rceil - 1)(\delta - 1). \tag{5.1}$$

When the equality in (5.1) holds, the code $\mathcal{C}$ is called optimal. For the sake of our construction, we change the form of the Singleton-type bound as follows.

Lemma 5.1. *Assume that* $(r+\delta-1)|n$. *If the Singleton-type bound* (5.1) *is achieved, then*

$$n-k=(\delta-1)\frac{n}{r+\delta-1}+d-\delta-(\delta-1)\left\lfloor\frac{d-\delta}{r+\delta-1}\right\rfloor. \tag{5.2}$$

Proof. Suppose that $d=n-k+1-(\lceil\frac{k}{r}\rceil-1)(\delta-1)$. Write $k=ar-b$ for some integers $a\geq 1$ and $0\leq b\leq r-1$. Substituting $k=ar-b$ back into (5.1), we can get $d=n-(ar-b)-(\delta-1)a+\delta=n-(r+\delta-1)a+b+\delta$. This implies that $a=\frac{n}{r+\delta-1}-\frac{d-\delta-b}{r+\delta-1}$. Since $\frac{n}{r+\delta-1}$ is an integer, thus $(r+\delta-1)|d-b-\delta$. Therefore, we further have $a=\frac{n}{r+\delta-1}-\lfloor\frac{d-\delta}{r+\delta-1}\rfloor$. Finally, the result follows from

$$d=n-k-(a-1)(\delta-1)+1=n-k-\frac{n}{r+\delta-1}(\delta-1)-\left\lfloor\frac{d-\delta}{r+\delta-1}\right\rfloor(\delta-1)+\delta. \quad \square$$

Remark 5.1. *Similar results are shown in [98] and [208] for the case* $\delta=2$.

Remark 5.2. *When* $r=d-\delta$ *and* $(r+\delta-1)|n$, *the Singleton-type bound* (5.1) *can't be met. Indeed, let* x *be the least nonnegative integer satisfying*

$$d+x=n-k+1-(\left\lceil\frac{k}{r}\right\rceil-1)(\delta-1).$$

Since we can assume $k=ar-b$ *for some integers* $a\geq 1$ *and* $0\leq b\leq r-1$, *thus we have*

$$d+x=n-a(r+\delta-1)+b+\delta.$$

As $r=d-\delta$, *it follows that*

$$r+x-b=n-a(r+\delta-1).$$

Then $r+x-b$ *must be divisible by* $r+\delta-1$. *So* $x=\delta-1+b$. *This indicates that the minimum distance* d *of* $\mathcal{C}$ *is upper bounded by*

$$d\leq n-k+1-\left\lceil\frac{k}{r}\right\rceil(\delta-1). \tag{5.3}$$

When $\mathcal{C}$ *meets the bound (5.3), we say it is optimal for this case. When* $\delta=2$, *such phenomenon has already appeared in [91].*

5.2.3 Maximal recoverability for general topologies

Let $z_1, \ldots, z_m$ be variables over the field $\mathbb{F}_q$. Consider an $(n-k) \times n$ matrix $P = \{p_{ij}\}$ where each $p_{ij} \in \mathbb{F}_p[z_1, \ldots, z_m]$ is an affine function of the z_is over $\mathbb{F}_p$:

$$p_{ij}(z_1, \ldots, z_m) = c_{ij0} + \sum_{k=1}^{m} c_{ijk} z_k, \qquad c_{ijk} \in \mathbb{F}_p. \tag{5.4}$$

We refer the matrix P as a topology. Fix an assignment $\{z_i = \alpha_i\}_{i=1}^m$, where $\alpha_i \in \mathbb{F}_q$. Viewing $P(\alpha_1, \ldots, \alpha_m)$ as a parity check matrix, then it defines a linear code which is denoted by $\mathcal{C}(\alpha_1, \ldots, \alpha_m)$. And we say code $\mathcal{C}$ instantiates P. Intuitively, the topology defined above characterizes the structure of the linear dependency among all the coordinates and represents a family of parity check matrices sharing same structure. For example, let $m = n$, take $c_{ijj} = a_j^{i-1}$ for n distinct $a_j \in \mathbb{F}_q \setminus \{0\}$ and $c_{ijk} = 0$ for all the other $(i, j, k) \in [n-k] \times [n] \times ([n] \cup \{0\})$. Then for an assignment $\{z_i = \alpha_i\}_{i=1}^m \subseteq \mathbb{F}_q \setminus \{0\}$, $p_{ij} = \alpha_j a_j^{i-1}$ and $P(\alpha_1, \ldots, \alpha_m)$ is a parity check matrix of an $[n, k]$-RS code.

A set $S \subseteq [n]$ of columns of P is called *potentially independent* if there exists an assignment $\{z_i = \alpha_i\}_{i=1}^m$ where $\alpha_i \in \mathbb{F}_q$ such that the columns of $P(\alpha_1, \ldots, \alpha_m)$ indexed by S are linearly independent.

Definition 5.2. *[91] The code $\mathcal{C}(\alpha_1, \ldots, \alpha_m)$ instantiating the topology P is called maximally recoverable if every set of columns that is potentially independent in P is linearly independent in $P(\alpha_1, \ldots, \alpha_m)$.*

Using the *Sparse Zeros Lemma* (see Theorem 6.13 in [133]), Gopalan et al. [91] proved the following upper bound on the size of field over which the maximally recoverable codes for any topologies P exist.

Theorem 5.1. *[91] Let $P \in (\mathbb{F}_p[z_1, \ldots, z_m])^{(n-k)\times n}$ be an arbitrary topology. If $q > (n-k) \cdot \binom{n}{\leq n-k}$, then there exists an MR instantiation of P over the field $\mathbb{F}_q$.*

5.2.4 Grid-like topologies

Unifying and generalizing a number of topologies considered both in coding theory and practice, Gopalan et al. [93] proposed the following family of topologies called *grid-like topologies* via dual constraints.

Definition 5.3. *[93] Let $m \le n$ be integers. Consider an $m \times n$ array of symbols $\{x_{ij}\}_{i\in[m],j\in[n]}$ over the field $\mathbb{F}_q$. Let $0 \le a \le m-1$, $0 \le b \le n-1$, and $0 \le h \le (m-a)(n-b)-1$. Let $T_{m\times n}(a,b,h)$ denote the topology where there are a parity check equations per column, b parity check equations per row, and h global parity check equations that depend on all symbols. Topologies of the form $T_{m\times n}(a,b,h)$ are called grid-like topologies.*

Furthermore, we say a collection of arrays $\mathcal{C}$ in $\mathbb{F}_q^{m\times n}$ to be a code that instantiates the topology $T_{m\times n}(a,b,h)$, if there exist $\{\alpha_i^{(k)}\}_{i\in[m],k\in[a]}$, $\{\beta_j^{(k)}\}_{j\in[n],k\in[b]}$ and $\{\gamma_{ij}^{(k)}\}_{i\in[m],j\in[n],k\in[h]}$ in $\mathbb{F}_q$ such that for each codeword $C = (c_{ij})_{i\in[m],j\in[n]} \in \mathcal{C}$:

1. *Each column $j \in [n]$ satisfies the constraints*
$$\sum_{i=1}^{m} \alpha_i^{(k)} c_{ij} = 0, \quad \forall k \in [a]. \tag{5.5}$$
2. *Each row $i \in [m]$ satisfies the constraints*
$$\sum_{j=1}^{n} \beta_j^{(k)} c_{ij} = 0, \quad \forall k \in [b]. \tag{5.6}$$
3. *All the symbols satisfy h global constraints*
$$\sum_{i=1}^{m}\sum_{j=1}^{n} \gamma_{ij}^{(k)} c_{ij} = 0, \quad \forall k \in [h]. \tag{5.7}$$

Definition 5.4. *An erasure pattern is a set $E \subseteq [m]\times[n]$ of symbols. Pattern E is correctable for the topology $T_{m\times n}(a,b,h)$ if there exists a code instantiating the topology where the variables $\{x_{ij}\}_{(i,j)\in E}$ can be recovered from the parity check equations (5.5), (5.6) and (5.7).*

Clearly, constraints in (5.5) and (5.6) guarantee the local dependencies in each column and row respectively, and constraints in (5.7) ensure some additional recoverability. Notably, constraints in (5.5) specify a code $\mathcal{C}_{col} \subseteq \mathbb{F}_q^m$ and constraints in (5.6) specify a code $\mathcal{C}_{row} \subseteq \mathbb{F}_q^n$. If $h = 0$, i.e., there are no extra global constraints for all symbols, then the code specified with the settings from Definition 5.3 is exactly the tensor product code $\mathcal{C}_{col} \otimes \mathcal{C}_{row}$.

Definition 5.5. *A code $\mathcal{C}$ that instantiates the topology $T_{m\times n}(a,b,h)$ is Maximally Recoverable (MR) if it can correct every erasure pattern that is correctable for the topology.*

The maximally recoverability requires a code that instantiates the topology $T_{m\times n}(a,b,h)$ to have many good properties, especially the MDS property.

Proposition 5.1. *[93] Let $\mathcal{C}$ be an MR instantiation of the topology $T_{m\times n}(a,b,h)$. We have*

1. The dimension of $\mathcal{C}$ is given by

$$dim\ \mathcal{C} = (m-a)(n-b) - h. \tag{5.8}$$

Moreover,

$$dim\ \mathcal{C}_{col} = m - a \ \ and \ \ dim\ \mathcal{C}_{row} = n - b. \tag{5.9}$$

2. Let $U \subseteq [m]$, $|U| = m-a$ and $V \subseteq [n]$, $|V| = n-b$ be arbitrary. Then $\mathcal{C}|_{U\times V}$ is an

$$[(m-a)(n-b),\ (m-a)(n-b)-h,\ h+1]$$

MDS code. Any subset $S \subseteq U \times V$, $|S| = (m-a)(n-b)-h$ is an information set.

3. Assume

$$h \leq (m-a)(n-b) - \max\{(m-a),(n-b)\}, \tag{5.10}$$

then the code $\mathcal{C}_{col}$ is an $[m, m-a, a+1]$ MDS code and the code $\mathcal{C}_{row}$ is an $[n, n-b, b+1]$ MDS code. Moreover, for all $j \in [n]$, $\mathcal{C}$ restricted to column j is the code $\mathcal{C}_{col}$, and for all $i \in [m]$, $\mathcal{C}$ restricted to row i is the code $\mathcal{C}_{row}$.

Considering the topology $T_{m\times n}(a,b,0)$, the MRC $\mathcal{C}$ that instantiates this topology can be viewed as the tensor product code $\mathcal{C}_{col} \otimes \mathcal{C}_{row}$. Based on the MDS properties for both $\mathcal{C}_{col}$ and $\mathcal{C}_{row}$, for a corresponding erasure pattern, we know that if some column has less than $a+1$ erasures or some row has less than $b+1$ erasures, we can decode it. Therefore, the erasure pattern that really matters shall have at least $a+1$ erasures in each column and at least $b+1$ erasures in each row.

Definition 5.6. *An erasure pattern* $E \subseteq [m] \times [n]$ *for the topology* $T_{m\times n}(a, b, 0)$ *is called irreducible, if for any* $(i, j) \in E$, $|I(j)| = |\{i' \in [m] : (i', j) \in E\}| \geq a + 1$ *and* $|J(i)| = |\{j' \in [n] : (i, j') \in E\}| \geq b + 1$.

These kinds of patterns were originally mentioned in [93] and also appeared in [176]. While Gopalan et al. [93] were trying to characterize the correctable erasure patterns for grid-like topologies, they considered the natural question: *are irreducible patterns uncorrectable?* In order to address this question, they introduced the following notion of *regularity* for erasure patterns.

Definition 5.7. *[93] Consider the topology* $T_{m\times n}(a, b, 0)$ *and an erasure pattern* E. *We say that* E *is regular if for all* $U \subseteq [m]$, $|U| = u$ *and* $V \subseteq [n]$, $|V| = v$ *we have*

$$|E \cap (U \times V)| \leq va + ub - ab. \tag{5.11}$$

By reducing the regular erasure patterns to the irreducible case, the authors proved the following equivalent condition of the correctable erasure patterns for the topology $T_{m\times n}(1, b, 0)$.

Theorem 5.2. *[93] An erasure pattern* E *is correctable for the topology* $T_{m\times n}(1, b, 0)$ *if and only if it is regular for* $T_{m\times n}(1, b, 0)$.

5.2.5 Pseudo-parity check matrix

Let $\mathcal{C}$ be an $[n, k]$ linear code with a parity check matrix $\mathbf{H} \in \mathbb{F}_q^{(n-k)\times n}$, then we have the following well-known fact about $\mathbf{H}$.

Fact .1. *[144] Assume a subset* $E \subseteq [n]$ *of the coordinates of* $\mathcal{C}$ *are erased, then they can be recovered if and only if the parity check matrix* $\mathbf{H}$ *restricted to coordinates in* E *has full rank.*

Take $\mathcal{C} = \mathcal{C}_{col} \otimes \mathcal{C}_{row}$ as the tensor product code that instantiates the topology $T_{m\times n}(a, b, 0)$, where $\mathcal{C}_{col}$ and $\mathcal{C}_{row}$ are codes specified by (5.5) and (5.6), respectively. For simplicity, for each codeword $c \in \mathcal{C}$ write

$$c = (c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{2n}, \ldots, c_{m1}, \ldots, c_{mn}),$$

where for each $j \in [n]$, $(c_{1j}, \dots, c_{mj})$ is a codeword in $\mathcal{C}_{col}$ and for each $i \in [m]$, $(c_{i1}, \dots, c_{in})$ is a codeword in $\mathcal{C}_{row}$.

Denote $\mathbf{H}_{col}$ and $\mathbf{H}_{row}$ as the parity check matrices of $\mathcal{C}_{col}$ and $\mathcal{C}_{row}$ respectively, assume

$$\mathbf{H}_{col} = \begin{pmatrix} \alpha_1^{(1)} & \alpha_2^{(1)} & \cdots & \alpha_m^{(1)} \\ \alpha_1^{(2)} & \alpha_2^{(2)} & \dots & \alpha_m^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{(a)} & \alpha_2^{(a)} & \dots & \alpha_m^{(a)} \end{pmatrix} \text{ and } \mathbf{H}_{row} = \begin{pmatrix} \beta_1^{(1)} & \beta_2^{(1)} & \cdots & \beta_n^{(1)} \\ \beta_1^{(2)} & \beta_2^{(2)} & \dots & \beta_n^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^{(b)} & \beta_2^{(b)} & \dots & \beta_n^{(b)} \end{pmatrix}.$$

Then consider the following $(an + bm) \times mn$ matrix:

$$\mathbf{H}_{(a,b,0)} = \begin{pmatrix} \mathbf{H}_1 & \mathbf{H}_2 & \cdots & \mathbf{H}_m \\ \mathbf{H}_{row} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{row} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{H}_{row} \end{pmatrix}, \tag{5.12}$$

where

$$\mathbf{H}_i = \begin{pmatrix} \vec{\alpha}_i & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \vec{\alpha}_i & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \vec{\alpha}_i \end{pmatrix}_{(a \cdot n) \times n} \text{ and } \vec{\alpha}_i = \begin{pmatrix} \alpha_i^{(1)} \\ \alpha_i^{(2)} \\ \vdots \\ \alpha_i^{(a)} \end{pmatrix}. \tag{5.13}$$

From the above construction, we can see that $\mathbf{H}_{(a,b,0)}$ includes all the parity check constraints of $\mathcal{C}$, and it can be easily verified that $\mathbf{H}_{(a,b,0)} \cdot c^T = 0$ for each codeword $c \in \mathcal{C}$. Since the size of $\mathbf{H}_{(a,b,0)}$ is $(an + bm) \times mn$, instead of the parity check matrix of $\mathcal{C}$, it can only be regarded as an approximation of the parity check matrix. Therefore, we call $\mathbf{H}_{(a,b,0)}$ a *pseudo-parity check matrix* of the code $\mathcal{C}$.

Similar to Fact .1, using basic linear algebra arguments, we have the following proposition for *pseudo-parity check matrix* of code $\mathcal{C}$.

Proposition 5.2. *Assume a subset $E \subseteq [mn]$ of the coordinates of $\mathcal{C}$ are erased, then they can be recovered if and only if the* pseudo-parity check matrix $\mathbf{H}_{(a,b,0)}$ *restricted to coordinates in E has full column rank.*

When $a = 1$, $\mathbf{H}_{col}$ has rank 1. Especially, when considering the existence of MRC for topologies $T_{m\times n}(1, b, 0)$, w.l.o.g, we can fix $\mathcal{C}_{col}$ to be the simple parity code $\mathcal{P}_m$, i.e., $\mathbf{H}_{col} = (1\ 1\ \cdots\ 1)$. Hence, the *pseudo-parity check matrix* $\mathbf{H}_{(1,b,0)}$ of $\mathcal{C} = \mathcal{P}_m \otimes \mathcal{C}_{row}$ has the form:

$$\mathbf{H}_{(1,b,0)} = \begin{pmatrix} \mathbf{I}_n & \mathbf{I}_n & \cdots & \mathbf{I}_n \\ \mathbf{H}_{row} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{row} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{H}_{row} \end{pmatrix} = \begin{pmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{pmatrix}. \tag{5.14}$$

Remark 5.3. *Let $r|n$ and $g = \frac{n}{r}$, an (n, r, h, a, q)-MR LRC can be viewed as an MRC for topology $T_{g\times r}(a, 0, h)$. Therefore, it has simpler erasure patterns compared to the tensor product cases. (Briefly speaking, an (n, r, h, a, q)-MR LRC is an $[n, n-ga-h]$ linear code with (r, a)-locality which can correct any erasure pattern E consisting of a erasures from each local group and any other h more erasures. For specific definition, please see [95].) And instead of using the* pseudo-parity check matrix*, it can be verified that the parity check matrix of any (n, r, h, a, q)-MR LRC admits the form*

$$\mathbf{H} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_g \\ \mathbf{H}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{H}_g \end{pmatrix},$$

where for each $i \in [g]$, $\mathbf{H}_i$ is a parity check matrix of an $[r, r-a, a+1]$ MDS code and $\mathbf{A}_i$ is an $h \times r$ matrix over $\mathbb{F}_q$ corresponding to the global parities.

Compared to MR LRCs, MRC for topologies $T_{m\times n}(a, b, 0)$ have another difference. For an (n, r, h, a, q)-MR LRC, the $[r, r-a, a+1]$ MDS codes within each local group can be different, this results in that the corresponding parity check matrix $\mathbf{H}$ above can admit different $\mathbf{H}_i$s. However, since an MRC for topology $T_{m\times n}(a, b, 0)$ is *actually a tensor product code* $\mathcal{C} = \mathcal{C}_{col} \otimes \mathcal{C}_{row}$*. Thus, for each* $i \in [m]$*, if we take coordinates in* $\{n(i-1)+1, \ldots, ni\}$ *as a local group, once the code* $\mathcal{C}_{row}$ *is fixed, the corresponding* $[n, n-b, b+1]$ *MDS codes within each local group are all* $\mathcal{C}_{row}$ *and the corresponding parity check matrices in* $\mathbf{H}_{(a,b,0)}$ *are all* $\mathbf{H}_{row}$*.*

5.2.6 Regular irreducible erasure patterns

Let $E \in [m] \times [n]$ be an erasure pattern of the topology $T_{m\times n}(a, b, 0)$, then it can be presented in the following form:

$$E = \begin{pmatrix} * & * & * & * & \cdots & \circ & \circ \\ * & * & \circ & * & \cdots & \circ & * \\ \circ & * & \circ & * & \cdots & * & \circ \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \circ & \circ & \circ & * & \cdots & * & * \end{pmatrix},$$

where $*$ stands for the erasure and $\circ$ stands for the non-erasure. Give two different erasure patterns E_1 and E_2, we say that E_1 and E_2 are of the same type, if E_2 can be obtained from E_1 by applying elementary row and column transformations.

For a reducible erasure pattern E, there exists some $i_0 \in [m]$ or $j_0 \in [n]$, such that the number of the erasures in $E \cap [i_0] \times [n]$ or $E \cap [m] \times [j_0]$ is less than $b + 1$ or $a + 1$. Therefore, from the MDS properties of the code $\mathcal{C}_{row}$ and $\mathcal{C}_{col}$, erasures in $E \cap [i_0] \times [n]$ or $E \cap [m] \times [j_0]$ can be simply repaired by using only the parities within $\mathbf{H}_{row}$ or $\mathbf{H}_{col}$. Hence, the very erasure patterns that affect the MR property of the code $\mathcal{C}$ are irreducible erasure patterns. In other words, if we can construct a code $\mathcal{C}$ instantiating the topology $T_{m\times n}(a, b, 0)$ that can correct all correctable irreducible erasure patterns, then this code $\mathcal{C}$ is an MR instantiation for the topology $T_{m\times n}(a, b, 0)$.

Now, we focus on the irreducible erasure patterns that are correctable. Given an irreducible erasure pattern E, denote $|E|$ as the number of $*$s in E, $U_E = \{i \in [m] : \exists j \in [n] \text{ such that } E(i, j) = *\}$ and $V_E = \{j \in [n] : \exists i \in [m] \text{ such that } E(i, j) = *\}$.

From the irreducibility of E, we have

$$|E| \geq (a+1)|V_E| \textit{ and } |E| \geq (b+1)|U_E|.$$

Meanwhile, from Theorem 5.2, we know that for topology $T_{m\times n}(1,b,0)$, an erasure pattern E is correctable if and only if E is regular. Thus we have

$$|E| = |E \cap (U_E \times V_E)| \leq a|V_E| + b|U_E| - ab = |V_E| + b|U_E| - b.$$

Combining the above three inequalities together, we have

$$|U_E| + b \leq |V_E| \leq b|U_E| - b, \tag{5.15}$$

for every correctable irreducible erasure patterns E in $T_{m\times n}(1,b,0)$. Therefore,

$$\max\{2(|U_E| + b), (b+1)|U_E|\} \leq |E| \leq 2b(|U_E| - 1), \tag{5.16}$$

which indicates that once $|U_E|$ (or $|V_E|$) is given, the magnitude of $|E|$ can not be too large.

Denote $\mathcal{E}$ as the set of all the types of regular irreducible erasure patterns for topology $T_{m\times n}(1,b,0)$, i.e., for each $E \in \mathcal{E}$, one can regard E as a representative of all the erasure patterns that have the same type as E. Since $U_E \subseteq [m]$ and $V_E \subseteq [n]$, for each regular irreducible erasure pattern E, (5.15) and (5.16) show that $|V_E| \leq b(m-1)$ and $|E| \leq 2b(m-1)$. For convenience, we can take each type of erasure patterns in $\mathcal{E}$ as a submatrix of an $m \times b(m-1)$ matrix with elements from $\{*, \circ\}$. Therefore, we can obtain the following upper bound of $|\mathcal{E}|$:

$$|\mathcal{E}| \leq \binom{m \cdot b(m-1)}{\leq 2b(m-1)}. \tag{5.17}$$

For general topology $T_{m\times n}(1,b,0)$, due to complexity for examining regularity, a fully characterization of all types of regular irreducible erasure patterns can be very difficult. However, the following proposition shows that joints of erasure patterns maintain the regularity and the irreducibility, which might be useful for finding large regular irreducible erasure patterns.

Proposition 5.3. *Let E_1 and E_2 be two regular irreducible erasure patterns for $T_{m\times n}(a,b,0)$. If $m \geq |U_{E_1}| + |U_{E_2}|$ and $n \geq |V_{E_1}| + |V_{E_2}|$, then the erasure pattern of the form*

$$E' = \begin{pmatrix} E_1 & \circ \\ \circ & E_2 \end{pmatrix},$$

is also regular irreducible for $T_{m\times n}(a,b,0)$.

Proof. The irreducibility of E' follows easily from those of E_1 and E_2, thus we only have to consider the regularity.

Now, consider the index set $U \times V \subseteq [m] \times [n]$. If $U \setminus U_{E'} \neq \emptyset$ or $V \setminus V_{E'} \neq \emptyset$, then we can delete the corresponding rows and columns outside of $U_{E'}$ and $V_{E'}$, this leads to a smaller index set with the same number of erasures. Therefore, w.l.o.g., assume that $U \times V \subseteq U_{E'} \times V_{E'} = (U_{E_1} \sqcup U_{E_2}) \times (V_{E_1} \sqcup V_{E_2})$, where $\sqcup$ means the union of two disjoint sets. Then, we have the following partition of $U \times V$:

$$\begin{aligned} U \times V = (U \cap U_{E_1}) \times (V \cap V_{E_1}) \sqcup (U \cap U_{E_1}) \times (V \cap V_{E_2}) \sqcup \\ (U \cap U_{E_2}) \times (V \cap V_{E_1}) \sqcup (U \cap U_{E_2}) \times (V \cap V_{E_2}). \end{aligned}$$

Thus

$$|E' \cap (U \times V)| = \sum_{(i,j)\in[2]\times[2]} |E_{(i,j)}|,$$

where $E_{(i,j)} = E' \cap (U \cap U_{E_i}) \times (V \cap V_{E_j})$. By the form of E', we have $|E_{(1,2)}| = |E_{(2,1)}| = 0$. Since the regularity of E_1 and E_2 implies that $|E_{(1,1)}| \leq a|V \cap V_{E_1}| + b|U \cap U_{E_1}| - ab$ and $|E_{(2,2)}| \leq a|V \cap V_{E_2}| + b|U \cap U_{E_2}| - ab$. Therefore, combining these two inequalities with the former identities, we have

$$\begin{aligned} |E' \cap (U \times V)| &\leq (a|V \cap V_{E_1}| + a|V \cap V_{E_2}|) + b(|U \cap U_{E_1}| + |U \cap U_{E_2}|) - 2ab \\ &< a|V| + b|U| - ab. \end{aligned}$$

Thus, E' is also a regular erasure pattern. $\square$

5.2.7 Independent sets in hypergraphs

There are many results on the independence number of hypergraphs (see [5], [6], [51], [125]). In the following section, we will apply the lower bound derived by Kostochka et al. [125]. Before stating their theorem, we need a few definitions and notations. Let $H(V, \mathfrak{E})$ be a hypergraph with vertex set V and hyperedge set $\mathfrak{E}$. We call H a k-uniform hypergraph, if all the hyperedges have the same size k, i.e., $\mathfrak{E} \subseteq \binom{V}{k}$. For any vertex $v \in V$, we define the degree of v to be the number of hyperedges containing v, denoted by $d(v)$. The maximum of the degrees of all the vertices is called the maximum degree of H and denoted by $\Delta(H)$. The independence number of H is denoted by $\alpha(H)$. For a set R of r vertices, define the r-degree of R to be the number of hyperedges containing R.

Theorem 5.3. *[125] Fix $r \geq 2$. There exists $c_r > 0$ such that if H is an $(r+1)$-graph on n vertices with maximum r-degree $\Delta_r < n/(\log n)^{3r^2}$, then*

$$\alpha(H) \geq c_r(\frac{n}{\Delta_r} \log \frac{n}{\Delta_r})^{\frac{1}{r}}, \tag{5.18}$$

where $c_r > 0$ and $c_r \sim r/e$ as $r \to \infty$.

§ 5.3 Constructions of optimal (r, δ)-LRCs

In this section, we consider linear codes with all symbol (r, δ)-locality and information (r, δ)-locality. We provide general constructions for optimal $[n, k, d; (r, \delta)_a]_q$-LRCs and $[n, k, d; (r, \delta)_i]$-LRCs through parity-check matrix approach. Compared to the constructions in [214], [213] and [40], the restrictions of the parity-check matrix in our construction are more relaxed and therefore, our construction will lead to longer codes.

5.3.1 Construction A

Let $d \geq \delta + 1$, $R = r + \delta - 1$ and $n = mR$. For $i \in [m]$, let $G_i = \{g_{i,1}, g_{i,2}, \cdots, g_{i,R}\}$ be an R-subset of $\mathbb{F}_q$. Then, for each $i \in [m]$, we can construct a

$(d-1)\times R$ Vandermonde matrix with generating set G_i of the form $\begin{pmatrix} \mathbf{U}_i \\ \mathbf{V}_i \end{pmatrix}$, where

$$\mathbf{U}_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ g_{i,1} & g_{i,2} & \cdots & g_{i,R} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i,1}^{\delta-2} & g_{i,2}^{\delta-2} & \cdots & g_{i,R}^{\delta-2} \end{pmatrix} \text{ and } \mathbf{V}_i = \begin{pmatrix} g_{i,1}^{\delta-1} & g_{i,2}^{\delta-1} & \cdots & g_{i,R}^{\delta-1} \\ g_{i,1}^{\delta} & g_{i,2}^{\delta} & \cdots & g_{i,R}^{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i,1}^{d-2} & g_{i,2}^{d-2} & \cdots & g_{i,R}^{d-2} \end{pmatrix}.$$

Note that $\mathbf{U}_i$ is a Vandermonde matrix of size $(\delta-1)\times R$ and $\mathbf{V}_i$ is of size $(d-\delta)\times R$. Put

$$\mathbf{H} = \begin{pmatrix} \mathbf{U}_1 & O & \cdots & O \\ O & \mathbf{U}_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{U}_m \\ \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_m \end{pmatrix}. \tag{5.19}$$

Let $\mathcal{C}$ be the linear code with parity-check matrix $\mathbf{H}$. Due to the structure of $\mathbf{H}$ and the property of Vandermonde matrix $\mathbf{U}_i$, it is immediate from Definition 5.1 that $\mathcal{C}$ has all symbol (r,δ)-locality. On the other hand, $\mathcal{C}$ has dimension

$$k(\mathcal{C}) \geq n-(\delta-1)m-(d-\delta) = rm-(d-\delta).$$

When $r > d-\delta$, we have $\lceil\frac{k(\mathcal{C})}{r}\rceil \geq m$. Thus, $\mathcal{C}$ has minimum distance

$$d(\mathcal{C}) \leq n-k(\mathcal{C})+\delta-\lceil\frac{k(\mathcal{C})}{r}\rceil(\delta-1) \leq d.$$

As for $r = d-\delta$, there is still $d(\mathcal{C}) \leq d$ with respect to (5.3). Therefore, for $r \geq d-\delta$, in order to obtain an optimal $[n,k,d;(r,\delta)_a]$-LRC from the above construction, it suffices to show that the minimum distance of $\mathcal{C}$ equals d. More precisely, our following aim is to find m R-subsets $G_1, G_2, \cdots, G_m$ in $\mathbb{F}_q$ such that any $d-1$ columns from the matrix $\mathbf{H}$ are linearly independent. For brevity, we refer to the ith block as the set of columns from $\mathbf{H}$ where $\mathbf{U}_i$ arises. So there are m column blocks and each one is made up of R columns.

We finish this subsection with the following two observations about $\mathbf{H}$:

Obs.1 : any $d-1$ columns in a single block are linearly independent;

Obs.2 : any $\delta-1$ columns from one block are linearly independent from columns belonging to other blocks.

5.3.2 Optimal LRCs with $(r,\delta)_a$-locality from Construction A

In this subsection, we put some sufficient conditions on the generating sets $G_1, G_2, \cdots, G_m$ in Construction A to guarantee the optimality of minimum distances w.r.t. bounds (5.1) and (5.3). As a warm up, we start with the construction of optimal $(r,\delta)_a$-LRCs with small minimum distance and unbounded length. It is worth noting that Zhang and Liu also proved the following result in [214], for the completeness of this thesis, we include the result here.

Theorem 5.4. *[214] Let $\delta+1 \le d \le 2\delta$, set $G_1 = G_2 = \cdots = G_m$, then any $d-1$ columns of $\mathbf{H}$ are linearly independent.*

Proof. Pick any $d-1$ columns from H. To see whether these columns are linearly independent, it suffices to consider the case where only one block contains at least δ columns because of $\delta+1 \le d \le 2\delta$ and **Obs.2**. Then combining **Obs.1** with **Obs.2**, we can conclude that these $d-1$ columns of $\mathbf{H}$ are linearly independent. □

Corollary 5.3.1. *Let $q \ge r+\delta-1$. Assume that $\delta+1 \le d \le 2\delta$, $r \ge d-\delta$ and $(r+\delta-1)|n$, then there exist optimal $[n,k,d;(r,\delta)_a]$-LRCs with $n = m(r+\delta-1)$ for any positive integer m.*

For $d \ge 2\delta+1$, we have the following theorem.

Theorem 5.5. *Let $d \ge 2\delta+1$ and $r \ge d-\delta$. Suppose that for any subset $S \subseteq [m]$ with $2 \le |S| \le \lfloor \frac{d-1}{\delta} \rfloor$, we have*

$$|\bigcup_{i\in S} G_i| \ge (r+\frac{\delta}{2}-1)|S|+\frac{\delta}{2}, \tag{5.20}$$

then any $d-1$ columns of $\mathbf{H}$ are linearly independent. As a result, the code $\mathcal{C}$ generated by Construction A in Section 5.3.1 is an optimal $[n,k,d;(r,\delta)_a]_q$-LRC.

Proof. For $1 \le i \le m$ and $1 \le j \le R$, let $\mathbf{h}_{i,j}$ be the j_{th} column from the i_{th} block of $\mathbf{H}$, i.e.,

$$\mathbf{h}_{i,j} = (\underbrace{0,0,\ldots,0}_{(i-1)(\delta-1)},1,g_{i,j},\ldots,g_{i,j}^{\delta-2},\underbrace{0,0,\ldots,0}_{(m-i)(\delta-1)},g_{i,j}^{\delta-1},g_{i,j}^{\delta},\ldots,g_{i,j}^{d-2})^T.$$

Assume that there exist $d-1$ columns $\{\mathbf{h}_{i_1,j_1},\mathbf{h}_{i_2,j_2},\ldots,\mathbf{h}_{i_{d-1},j_{d-1}}\}$ in $\mathbf{H}$ that are linearly dependent. Then, we have

$$\sum_{l=1}^{d-1}\lambda_l\mathbf{h}_{i_l,j_l} = \mathbf{0}. \tag{5.21}$$

For $1 \le i \le m$, denote $E_i = \{j_l : \lambda_l \ne 0 \text{ and } i_l = i\}$. Clearly, we have $\sum_{i\in[m]}|E_i| \le d-1$. According to the structure of $\mathbf{H}$, we know that either $|E_i| = 0$ or $|E_i| \ge \delta$. Otherwise, one shall get $|E_i|$ distinct columns linearly dependent in $\mathbf{U}_i$, which contradicts to the property of $\mathbf{U}_i$.

W.l.o.g., assume that $\{i : E_i \ne \emptyset\} = [t]$ and for each $i \in [t]$, $|E_i| = s_i$. Clearly, we have $t \le \frac{d-1}{\delta}$. For each $i \in [t]$, denote $F_i = \{g_{i,j} \in G_i : j \in E_i\}$ as the generating set of columns corresponding to E_i and $F = \bigcup_{i\in[t]} F_i$. Denote $\mathbf{H}'$ as the $(m(\delta-1)+d-\delta)\times(\sum_{i\in[t]} s_i)$ submatrix of $\mathbf{H}$ consisting of columns indexed by $\bigcup_{i\in[t]}\{(i,j) : j \in E_i\}$. Write $F_i = \{a_{i,1},\ldots,a_{i,s_i}\}$, then, $\mathbf{H}'$ has the following form:

$$\mathbf{H}' = \begin{pmatrix} \mathbf{A}_1 & O & \cdots & O \\ O & \mathbf{A}_2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_t \\ O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_t \end{pmatrix}, \tag{5.22}$$

where

$$\mathbf{A}_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{i,1} & a_{i,2} & \cdots & a_{i,s_i} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1}^{\delta-2} & a_{i,2}^{\delta-2} & \cdots & a_{i,s_i}^{\delta-2} \end{pmatrix} \text{ and } \mathbf{B}_i = \begin{pmatrix} a_{i,1}^{\delta-1} & a_{i,2}^{\delta-1} & \cdots & a_{i,s_i}^{\delta-1} \\ a_{i,1}^{\delta} & a_{i,2}^{\delta} & \cdots & a_{i,s_i}^{\delta} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1}^{d-2} & a_{i,2}^{d-2} & \cdots & a_{i,s_i}^{d-2} \end{pmatrix}.$$

Denote $F_1^1 = F_1$ and $F_i^1 = F_i \setminus \bigcup_{j=1}^{i-1} F_j$ for $2 \le i \le t$. Then, we have $F = \sqcup_{i=1}^{t} F_i^1$. By permutating the columns of $\mathbf{H}'$, we can obtain a matrix of the following form:

$$\mathbf{H}_2 = (\mathbf{H}_L || \ \mathbf{H}_R) = \left(\begin{array}{ccccc||cccc} \mathbf{A}_1 & O & O & \cdots & O & O & O & \cdots & O \\ O & \mathbf{A}_2^1 & O & \cdots & O & \mathbf{A}_2^2 & O & \cdots & O \\ O & O & \mathbf{A}_3^1 & \cdots & O & O & \mathbf{A}_3^2 & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \mathbf{A}_t^1 & O & O & \cdots & \mathbf{A}_t^2 \\ O & O & O & \cdots & O & O & O & \cdots & O \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \mathbf{B}_3^1 & \cdots & \mathbf{B}_t^1 & \mathbf{B}_2^2 & \mathbf{B}_3^2 & \cdots & \mathbf{B}_t^2 \end{array}\right),$$

where for $2 \le i \le t$, $\mathbf{A}_i^1 = \mathbf{A}_i|_{F_i^1}$, $\mathbf{A}_i^2 = \mathbf{A}_i|_{F_i \setminus F_i^1}$ and $\mathbf{B}_i^1 = \mathbf{B}_i|_{F_i^1}$, $\mathbf{B}_i^2 = \mathbf{B}_i|_{F_i \setminus F_i^1}$. * Similarly, denote $E_i^1 = \{j \in E_i : g_{i,j} \in F_i^1\}$ as the index set of columns generated by F_i^1. Then, (5.21) can be written as

$$\sum_{i=1}^{t} \sum_{j \in E_i^1} \lambda_{i,j} \mathbf{h}_{i,j} + \sum_{i=1}^{t} \sum_{j \in E_i \setminus E_i^1} \lambda_{i,j} \mathbf{h}_{i,j} = \mathbf{0}, \tag{5.23}$$

where $\lambda_{i,j} \neq 0$ is the relabeled λ_l for $(i,j) = (i_l, j_l)$.

Note that for $2 \le i \le t$ and each column in $\mathbf{B}_i^2$, its generating element in F has already appeared in $F_{i'}^1$ for some $1 \le i' < i$. Therefore, we can do the following elementary row and column operations on $\mathbf{H}_2$:

- First, for each $2 \le i \le t$ and each column $\mathbf{h}_{i,j}$ in $(O \ \cdots \ O \ (\mathbf{A}_i^2)^T \ O \ \cdots \ (\mathbf{B}_i^2)^T)^T$ of $\mathbf{H}_R$, subtract the column $\mathbf{h}_{i',j'}$ in $(O \ \cdots \ O \ (\mathbf{A}_{i'}^1)^T \ O \ \cdots \ (\mathbf{B}_{i'}^1)^T)^T$ of $\mathbf{H}_L$ from it, where (i', j') satisfies $i' < i$, $j' \in E_{i'}$ and $g_{i,j} = g_{i',j'} \in F_{i'}^1$. This leads to a

*Given a Vandermonde matrix $\mathbf{A}$ with generating set G, for simplicity, we denote $\mathbf{A}|_F$ as the matrix obtained by restricting $\mathbf{A}$ to the columns corresponding to those elements in $F \subseteq G$.

matrix equivalent to $\mathbf{H}_2$:

$$\mathbf{H}_2' = (\mathbf{H}_L || \, \mathbf{H}_R') = \left(\begin{array}{cccc||cccc} \mathbf{A}_1 & O & \cdots & O & -\mathbf{A}_{2,1}^2 & -\mathbf{A}_{3,1}^2 & \cdots & -\mathbf{A}_{t,1}^2 \\ O & \mathbf{A}_2^1 & \cdots & O & \mathbf{A}_2^2 & -\mathbf{A}_{3,2}^2 & \cdots & -\mathbf{A}_{t,2}^2 \\ O & O & \cdots & O & O & \mathbf{A}_3^2 & \cdots & -\mathbf{A}_{t,3}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_t^1 & O & O & \cdots & \mathbf{A}_t^2 \\ O & O & \cdots & O & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_t^1 & O & O & \cdots & O \end{array}\right),$$

where for $1 \leq j < i \leq t$, the l_{th} column of $\mathbf{A}_{i,j}^2$ is identical to the l_{th} column of $\mathbf{A}_i^2$ if the corresponding generating element appears in $F_j^1 \cap F_i$ and is identical to the zero vector, otherwise. Clearly, we have $\mathbf{A}_{2,1}^2 = \mathbf{A}_2^2$ and $\sum_{j=1}^{i-1} \mathbf{A}_{i,j}^2 = \mathbf{A}_i^2$. Moreover, (5.23) turns into:

$$\sum_{i=1}^{t} \sum_{j \in E_i^1} \lambda_{i,j}' \mathbf{h}_{i,j} + \sum_{i=1}^{t} \sum_{j \in E_i \setminus E_i^1} \lambda_{i,j} \mathbf{h}_{i,j}' = \mathbf{0}, \tag{5.24}$$

where for $(i, j) \in [t] \times E_i^1$,

$$\lambda_{i,j}' = \lambda_{i,j} + \sum_{i' > i} \sum_{\substack{j' \in E_{i'}: \\ g_{i',j'} = g_{i,j}}} \lambda_{i',j'};$$

and for $(i, j) \in [t] \times E_i \setminus E_i^1$,

$$\mathbf{h}_{i,j}' = \mathbf{h}_{i,j} - \mathbf{h}_{i',j'} \in \mathbf{H}_R'$$

for some (i', j') satisfying $i' < i$, $j' \in E_{i'}$ and $g_{i,j} = g_{i',j'} \in F_{i'}^1$.

- Second, for each $1 \leq i \leq \delta - 1$, add the $(i + j(\delta - 1))_{th}$ row to the i_{th} row for all

$1 \leq j \leq t-1$. Then, we have:

$$\mathbf{H}_2'' = (\mathbf{H}_L'' || \, \mathbf{H}_R'') = \left(\begin{array}{cccc||cccc} \mathbf{A}_1 & \mathbf{A}_2^1 & \cdots & \mathbf{A}_t^1 & O & O & \cdots & O \\ O & \mathbf{A}_2^1 & \cdots & O & \mathbf{A}_2^2 & -\mathbf{A}_{3,2}^2 & \cdots & -\mathbf{A}_{t,2}^2 \\ O & O & \cdots & O & O & \mathbf{A}_3^2 & \cdots & -\mathbf{A}_{t,3}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_t^1 & O & O & \cdots & \mathbf{A}_t^2 \\ O & O & \cdots & O & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_t^1 & O & O & \cdots & O \end{array} \right). \tag{5.25}$$

Since elementary row operations don't affect linear relations among columns, therefore for $\mathbf{H}_2''$, (5.24) turns into

$$\sum_{i=1}^{t} \sum_{j \in E_i^1} \lambda_{i,j}' \mathbf{h}_{i,j}'' + \sum_{i=1}^{t} \sum_{j \in E_i \setminus E_i^1} \lambda_{i,j} \mathbf{h}_{i,j}'' = \mathbf{0}, \tag{5.26}$$

where $\mathbf{h}_{i,j}''$s are the new columns in $\mathbf{H}_2''$: for $(i,j) \in [t] \times E_i^1$,

$$\mathbf{h}_{i,j}'' = \begin{cases} \mathbf{h}_{i,j}, \ \text{if } i = 1; \\ \mathbf{h}_{i,j} + (1, g_{i,j}, \ldots, g_{i,j}^{\delta-2}, 0, \ldots, 0)^T, \ \text{otherwise;} \end{cases}$$

and for $(i,j) \in [t] \times E_i \setminus E_i^1$,

$$\mathbf{h}_{i,j}'' = \begin{cases} \mathbf{h}_{i,j}' + (1, g_{i,j}, \ldots, g_{i,j}^{\delta-2}, 0, \ldots, 0)^T, \ \text{if } g_{i,j} \in F_1^1; \\ \mathbf{h}_{i,j}', \ \text{otherwise.} \end{cases}$$

Consider the following submatrix consisting of the first $\delta - 1$ rows and the last $d - \delta$ rows of $\mathbf{H}_L''$:

$$\mathbf{H}_0 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2^1 & \cdots & \mathbf{A}_t^1 \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_t^1 \end{pmatrix}.$$

Clearly, $\mathbf{H}_0$ is a $(d-1) \times |F|$ Vandermonde matrix and the construction of $\mathbf{A}_i^1$ guarantees that columns in $\mathbf{H}_0$ are pairwise distinct. Since $|F| \leq d-1$, it follows that columns in $\mathbf{H}_0$ are linearly independent. On the other hand, according to the structure of $\mathbf{h}''_{i,j}$ for $(i,j) \in [t] \times E_i^1$, (5.26) indicates that

$$\sum_{i=1}^{t} \sum_{j \in E_i^1} \lambda'_{i,j} (1, g_{i,j}, \ldots, g_{i,j}^{\delta-2}, g_{i,j}^{\delta-1}, \ldots, g_{i,j}^{d-2})^T = \mathbf{0}.$$

Therefore, we have $\lambda'_{i,j} = 0$ for every $(i,j) \in [t] \times E_i^1$. This leads to

$$\sum_{i=1}^{t} \sum_{j \in E_i \setminus E_i^1} \lambda_{i,j} \mathbf{h}''_{i,j} = \mathbf{0}, \tag{5.27}$$

where $\lambda_{i,j}$s are the original non-zero coefficients in (5.23).

Now, in the following context, based on (5.27), we shall derive a contradiction by estimating $\sum_{i=2}^{t} |F_i \setminus F_i^1|$.

For $2 \leq i \leq t$ with $E_i \neq E_i^1$ and $1 \leq l \leq |E_i \setminus E_i^1|$, let $\mathbf{h}''_{i,j_l}$ be the l_{th} column in

$$(O \ (-\mathbf{A}_{i,2}^2)^T \ \cdots \ (-\mathbf{A}_{i,i-1}^2)^T \ (-\mathbf{A}_i^2)^T \ O \ \cdots \ O)^T,$$

i.e., the $(i-1)_{th}$ block of $\mathbf{H}''_R$. For simplicity of presentation, we rewrite (5.27) in the following form:

$$\begin{pmatrix} \mathbf{A}_2^2 & -\mathbf{A}_{3,2}^2 & \cdots & -\mathbf{A}_{t,2}^2 \\ O & \mathbf{A}_3^2 & \cdots & -\mathbf{A}_{t,3}^2 \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_t^2 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{v}_2^T \\ \mathbf{v}_3^T \\ \vdots \\ \mathbf{v}_t^T \end{pmatrix} = \mathbf{0}, \tag{5.28}$$

where $\mathbf{v}_i = (\mu_{i,1}, \mu_{i,2}, \ldots, \mu_{i,|E_i \setminus E_i^1|})$ and $\mu_{i,l} = \lambda_{i,j_l} \neq 0$. Given $2 \leq i' \leq t$, for $1 \leq i \leq i'-1$, define

$$\mathbf{v}_{i',i}(l) = \begin{cases} \mathbf{v}_{i'}(l), & \text{if the } l_{th} \text{ column in } \mathbf{A}_{i',i}^2 \text{ is a non-zero vector;} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\sum_{i=1}^{i'-1} \mathbf{A}_{i',i}^2 = \mathbf{A}_{i'}^2$, thus we also have $\sum_{i=1}^{i'-1} \mathbf{v}_{i',i} = \mathbf{v}_{i'}$. With the help of this observation, (5.28) is actually the following system of equations:

$$\mathbf{A}_i^2 \cdot \mathbf{v}_i^T - \sum_{i'>i} \mathbf{A}_{i',i}^2 \cdot \mathbf{v}_{i',i}^T = \mathbf{0}, \ 2 \leq i \leq t. \tag{5.29}$$

According to the constructions of $\mathbf{A}_i^2$ and $\mathbf{A}_{i',i}^2$, columns in $\mathbf{A}_i^2$ are distinct from those columns in $\mathbf{A}_{i',i}^2$ for all $i' > i$. Note that $\mu_{i,j} \neq 0$ for every $2 \leq i \leq t$ and $1 \leq j \leq |E_i \setminus E_i^1|$. Therefore, despite the fact that there might be identical columns in different $\mathbf{A}_{i',i}^2$s, (5.29) and the property of Vandermonde matrix force that

$$\omega(\mathbf{v}_i) + \sum_{i<i'\leq t} \omega(\mathbf{v}_{i',i}) \geq \delta,$$

for every $2 \leq i \leq t$. Therefore, we further have

$$\sum_{i=2}^{t} (\omega(\mathbf{v}_i) + \sum_{i<i'\leq t} \omega(\mathbf{v}_{i',i})) \geq (t-1)\delta. \tag{5.30}$$

Denote $\mathbf{v} = (\mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_t)$. Note that $\sum_{i=2}^{t} |F_i \setminus F_i^1| = \omega(\mathbf{v})$ and the LHS of (5.30) is actually $2\omega(\mathbf{v}) - \sum_{i=2}^{t} \omega(\mathbf{v}_{i,1})$, thus we have

$$\sum_{i=2}^{t} |F_i \setminus F_i^1| \geq \frac{\delta}{2}(t-1) + \frac{\sum_{i=2}^{t} \omega(\mathbf{v}_{i,1})}{2}.$$

On the other hand, for each $g_{i,j} \in F$, let $c(g_{i,j}) = |\{i' \in [t] : g_{i,j} \in F_{i'} \setminus F_{i'}^1\}|$. Through a simple double counting argument, we have

$$\sum_{g_{i,j} \in F} c(g_{i,j}) = \sum_{i=2}^{t} |F_i \setminus F_i^1|.$$

- When $\sum_{i=2}^{t} \omega(\mathbf{v}_{i,1}) = 0$, we have $\omega(\mathbf{v}_{i,1}) = 0$ for each $2 \leq i \leq t$. This indicates that $(F_i \setminus F_i^1) \cap F_1 = \emptyset$ for each $2 \leq i \leq t$, which further leads to $F_1 \cap \bigcup_{i=2}^{t} F_i = \emptyset$. Since $F_i \subseteq G_i$ for each $i \in [t]$, thus, we have

$$|\bigcup_{i=2}^{t} G_i| \leq \sum_{i=2}^{t} |G_i| - \sum_{g_{i,j} \in \bigcup_{i=2}^{t} F_i} c(g_{i,j}) = (r+\delta-1)(t-1) - \sum_{i=2}^{t} |F_i \setminus F_i^1|.$$

  Combined with $|\bigcup_{i=2}^{t} G_l| \geq (r + \frac{\delta}{2} - 1)(t-1) + \frac{\delta}{2}$, this leads to $\sum_{i=2}^{t} |F_i \setminus F_i^1| \leq \frac{\delta}{2}(t-2)$, a contradiction.

- When $\sum_{i=2}^{t} \omega(\mathbf{v}_{i,1}) > 0$, consider $\bigcup_{i\in[t]} G_i$, we have

$$|\bigcup_{i\in[t]} G_i| \leq \sum_{i\in[t]} |G_i| - \sum_{g_{i,j}\in F} c(g_{i,j}) = (r+\delta-1)t - \sum_{i=2}^{t} |F_i \setminus F_i^1|.$$

  Combined with $|\bigcup_{i\in[t]} G_l| \geq (r+\frac{\delta}{2}-1)t+\frac{\delta}{2}$, this leads to $\sum_{i=2}^{t} |F_i \setminus F_i^1| \leq \frac{\delta}{2}(t-1)$, a contradiction.

Therefore, any $d-1$ columns are linearly independent. This completes the proof of Theorem 5.5. □

Remark 5.4. *(i) Theorem 5.5 can be viewed as a generalization of Theorem 3.1 in [208], when we take $\delta = 2$, the sufficient part of Theorem 3.1 in [208] follows from Theorem 5.5.*

(ii) In [213], the author proved a similar result under the condition that

$$|\bigcup_{i\in S} G_i| \geq (r+\delta-2)|S|+1, \tag{5.31}$$

for any $S \subseteq [m]$ with $2 \leq |S| \leq \lfloor \frac{d-1}{\delta} \rfloor$. Compared to this condition, (5.20) is more relaxed and weakens the restriction of intersections among different repair groups.

5.3.3 Construction B

Let $1 \leq v \leq r$, $R = r+\delta-1$ and $n = (l+1)R+h+v-r$ with $h \geq 0$. Let $G_{l+2} = \{g_{l+2,1}, g_{l+2,2}, \ldots, g_{l+2,h}\}$ be an h-subset of $\mathbb{F}_q$, $G_i = \{g_{i,1}, g_{i,2}, \ldots, g_{i,R}\}$ for $1 \leq i \leq l$ and $G_{l+1} = \{g_{l+1,1}, g_{l+1,2}, \ldots, g_{l+1,v+\delta-1}\}$ be other $l+1$ subsets of $\mathbb{F}_q \setminus G_{l+2}$. Define $f(x) = \prod_{i=1}^{h}(x - g_{l+2,i})$ and consider the following $((l+1)(\delta-1)+h) \times n$ matrix:

$$\mathbf{H} = \begin{pmatrix} \mathbf{U}_1 & O & \cdots & O & O \\ O & \mathbf{U}_2 & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & \mathbf{U}_{l+1} & O \\ \mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_{l+1} & \mathbf{V}_{l+2} \end{pmatrix}, \tag{5.32}$$

where for $1 \leq i \leq l+1$,

$$\mathbf{U}_i = \begin{pmatrix} f(g_{i,1}) & f(g_{i,2}) & \cdots & f(g_{i,|G_i|}) \\ g_{i,1}f(g_{i,1}) & g_{i,2}f(g_{i,2}) & \cdots & g_{i,|G_i|}f(g_{i,|G_i|}) \\ \vdots & \vdots & \ddots & \vdots \\ g_{i,1}^{\delta-2}f(g_{i,1}) & g_{i,2}^{\delta-2}f(g_{i,2}) & \cdots & g_{i,|G_i|}^{\delta-2}f(g_{i,|G_i|}) \end{pmatrix}$$

and

$$\mathbf{V}_i = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ g_{i,1} & g_{i,2} & \cdots & g_{i,|G_i|} \\ \vdots & \vdots & \ddots & \vdots \\ g_{i,1}^{h-1} & g_{i,2}^{h-1} & \cdots & g_{i,|G_i|}^{h-1} \end{pmatrix}, \mathbf{V}_{l+2} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ g_{l+2,1} & g_{l+2,2} & \cdots & g_{l+2,h} \\ \vdots & \vdots & \ddots & \vdots \\ g_{l+2,1}^{h-1} & g_{l+2,2}^{h-1} & \cdots & g_{l+2,h}^{h-1} \end{pmatrix}.$$

Let $\mathcal{C}$ be the $[n,k]$ linear code with parity-check matrix $\mathbf{H}$. Note that for each $1 \le i \le l$, $\mathbf{U}_i$ is a $(\delta-1)\times R$ matrix with rank $\delta-1$ and U_{l+1} is a $(\delta-1)\times(v+\delta-1)$ matrix with rank $\delta-1$. Therefore, U_is can be viewed as parity-check matrices of generalized Reed-Solomon codes which guarantee that for $1 \le i \le n-h$, each code symbol c_i has (r,δ)-locality. On the other hand, $\mathcal{C}$ has dimension

$$k(\mathcal{C}) \ge n-(\delta-1)(l+1)-h = lr+v.$$

Since $1 \le v \le r$, we have $\lceil \frac{k(\mathcal{C})}{r} \rceil \ge l+1$. Thus, $\mathcal{C}$ has minimum distance

$$d(\mathcal{C}) \le n-k(\mathcal{C})+\delta-\lceil\frac{k(\mathcal{C})}{r}\rceil(\delta-1) \le h+\delta. \tag{5.33}$$

Therefore, in order to obtain an optimal LRC with $(r,\delta)_i$-locality from the above construction, it suffices to show that the minimum distance of $\mathcal{C}$ equals to $h+\delta$. The same as Section III, our following aim is to find $l+2$ subsets $G_1, G_2, \cdots, G_{l+2}$ in $\mathbb{F}_q$ such that any $h+\delta-1$ columns from the matrix $\mathbf{H}$ are linearly independent.

5.3.4 Optimal LRCs with $(r,\delta)_i$-locality from Construction B

In this subsection, sufficient conditions on generating sets $G_1, G_2, \cdots, G_{l+2}$ in Construction B are discussed to guarantee the optimality of the minimum distance. Actually, as we shall see later, $\mathcal{C}$ can recover more than $h+\delta-1$ erasures under proper restrictions on G_is.

For convenience, we use the evaluation points (instead of the indices of code symbols) to denote erasure patterns. Denote $\mathcal{E} = \{E_1, \ldots, E_{l+2}\}$ as an erasure pattern, where $E_i \subseteq G_i$ corresponding to the set of erasure points in G_i, $1 \le i \le l+2$.

Theorem 5.6. *Let $\mathcal{C}$ be the linear code with parity-check matrix* **H** *from construction B. Let $\mathcal{E} = \{E_1, \ldots, E_{l+2}\}$ be an erasure pattern with $E_i \subseteq G_i$ for $1 \le i \le l+2$. Denote $S = \{i \in [l+1] : |E_i| \ge \delta\}$. If the erasure pattern $\mathcal{E}$ satisfies*

$$|\bigcup_{i \in S} E_i| + |E_{l+2}| \le h + \delta - 1 \tag{5.34}$$

and

$$|\bigcup_{i \in S} G_i| \ge \begin{cases} (r + \frac{\delta}{2} - 1)|S| + \frac{\delta}{2}, & \text{if } l+1 \notin S; \\ (r + \frac{\delta}{2} - 1)|S| + \frac{\delta}{2} + v - r, & \text{otherwise,} \end{cases} \tag{5.35}$$

then the erasure pattern $\mathcal{E}$ can be recovered.

Proof. Note that for any $\mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ and each $1 \le i \le n-h$, code symbol c_i in $\mathcal{C}$ has (r, δ)-locality. Therefore, $\mathcal{C}$ is capable of recovering all the erasures in $E_i \in \mathcal{E}$ with $|E_i| \le \delta - 1$ for every $1 \le i \le l+1$ independently. Thus, we only need to consider erasures from $E_i \in \mathcal{E}$ with $|E_i| \ge \delta$ and the erasures from E_{l+2}. Let $s = |S|$, w.l.o.g., we can assume that $S = [s]$.

Take $\mathcal{E}' = \{E_1, \ldots, E_s, E_{l+2}\}$ and define $\mathbf{H}|_{\mathcal{E}'}$ as

$$\mathbf{H}|_{\mathcal{E}'} = \begin{pmatrix} \mathbf{U}_1|_{E_1} & O & \cdots & O & O \\ O & \mathbf{U}_2|_{E_2} & \cdots & O & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & \cdots & \mathbf{U}_s|_{E_s} & O \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{V}_1|_{E_1} & \mathbf{V}_2|_{E_2} & \cdots & \mathbf{V}_s|_{E_s} & \mathbf{V}_{l+2}|_{E_{l+2}} \end{pmatrix},$$

for simplicity of presentation, here, $U_i|_{E_i}$ ($V_i|_{E_i}$) denotes the restriction of U_i (V_i) to the set of columns generated by elements in E_i. Note that an erasure pattern $\mathcal{E}$ can be recovered by the code $\mathcal{C}$ with parity-check matrix **H** if and only if the restriction of **H** to $\mathcal{E}$ has full column rank. Therefore, we only need to show that $\mathbf{H}|_{\mathcal{E}'}$ has full column rank.

Assume not, i.e., there exists a non-zero vector $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s, \mathbf{v}_{l+2})$ with

$\mathbf{v}_i \in \mathbb{F}_q^{|E_i|}$ such that

$$\mathbf{H}|_{\mathcal{E}'} \cdot \begin{pmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_s^T \\ \mathbf{v}_{l+2}^T \end{pmatrix} = \mathbf{0}. \tag{5.36}$$

Write $\mathbf{H}|_{E_i} = (\mathbf{h}_{i,1} \ \mathbf{h}_{i,2} \ \cdots \ \mathbf{h}_{i,|E_i|})$, $\mathbf{v}_i = (\lambda_{i,1}, \ldots, \lambda_{i,|E_i|})$ and assume that $\mathbf{h}_{i,j}$ is generated by $a_{i,j} \in E_i \subseteq G_i$. Denote $E_i' = \{a_{i,j} : a_{i,j} \in E_i \text{ and } \lambda_{i,j} \neq 0\}$, $\mathcal{E}'' = \{E_1', \ldots, E_s', E_{l+2}'\}$ and $\mathbf{v}_i'$ as the vector of length $\omega(\mathbf{v}_i)$ by puncturing $\mathbf{v}_i$ on its non-zero coordinates. Then, (5.36) turns into the following form

$$\mathbf{H}|_{\mathcal{E}''} \cdot \begin{pmatrix} (\mathbf{v}_1')^T \\ \vdots \\ (\mathbf{v}_s')^T \\ (\mathbf{v}_{l+2}')^T \end{pmatrix} = \mathbf{0}.$$

This reduces the problem to a sub-erasure pattern $\mathcal{E}''$ of $\mathcal{E}'$. Thus, w.l.o.g., we can assume that $\lambda_{i,j} \neq 0$ for every $i \in [s] \cup \{l+2\}$ and $1 \leq j \leq |E_i|$.

For $i \in [s]$, let

$$\mathbf{A}_i = \mathbf{U}_i|_{E_i} = \begin{pmatrix} f(a_{i,1}) & f(a_{i,2}) & \cdots & f(a_{i,|E_i|}) \\ a_{i,1}f(a_{i,1}) & a_{i,2}f(a_{i,2}) & \cdots & a_{i,|E_i|}f(a_{i,|E_i|}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1}^{\delta-2}f(a_{i,1}) & a_{i,2}^{\delta-2}f(a_{i,2}) & \cdots & a_{i,|E_i|}^{\delta-2}f(a_{i,|E_i|}) \end{pmatrix}$$

and for $i \in [s] \cup \{l+2\}$, let

$$\mathbf{B}_i = \mathbf{V}_i|_{E_i} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{i,1} & a_{i,2} & \cdots & a_{i,|E_i|} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1}^{h-1} & a_{i,2}^{h-1} & \cdots & a_{i,|E_i|}^{h-1} \end{pmatrix}.$$

Denote $E_1^1 = E_1$, $E_i^1 = E_i \setminus \bigcup_{j=1}^{i-1} E_i$ for $2 \leq i \leq s$ and $E = \sqcup_{i=1}^{s} E_i^1$. By permutating the columns of $\mathbf{H}|_{\mathcal{E}'}$, we can obtain an equivalent matrix of the following form:

$$\mathbf{H}_2 = (\mathbf{H}_L||\mathbf{H}_R) = \left(\begin{array}{ccccc||cccc} \mathbf{A}_1 & O & \cdots & O & O & O & O & \cdots & O \\ O & \mathbf{A}_2^1 & \cdots & O & O & \mathbf{A}_2^2 & O & \cdots & O \\ O & O & \cdots & O & O & O & \mathbf{A}_3^2 & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_s^1 & O & O & O & \cdots & \mathbf{A}_s^2 \\ O & O & \cdots & O & O & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_s^1 & \mathbf{B}_{l+2} & \mathbf{B}_2^2 & \mathbf{B}_3^2 & \cdots & \mathbf{B}_s^2 \end{array}\right),$$

where for $2 \leq i \leq s$, $\mathbf{A}_i^1 = \mathbf{A}_i|_{E_i^1}$, $\mathbf{A}_i^2 = \mathbf{A}_i|_{E_i \setminus E_i^1}$ and $\mathbf{B}_i^1 = \mathbf{B}_i|_{E_i^1}$, $\mathbf{B}_i^2 = \mathbf{B}_i|_{E_i \setminus E_i^1}$. For each $i \in [s]$, denote $I_i^1 = \{j \in [|E_i|] : a_{i,j} \in E_i^1\}$ and $I_i^2 = \{j \in [|E_i|] : a_{i,j} \in E_i \setminus E_i^1\}$. Then, (5.36) can be written as

$$\sum_{i=1}^{s} \sum_{j \in I_i^1} \lambda_{i,j} \mathbf{h}_{i,j} + \sum_{j \in [|E_{l+2}|]} \lambda_{l+2,j} \mathbf{h}_{l+2,j} + \sum_{i=1}^{s} \sum_{j \in I_i^2} \lambda_{i,j} \mathbf{h}_{i,j} = \mathbf{0}. \tag{5.37}$$

Similar to the proof of Theorem 5.5, we can do the following elementary row and column operations:

- First, for each $2 \leq i \leq s$ and each column $\mathbf{h}_{i,j}$ in $(O \cdots O\ (\mathbf{A}_i^2)^T\ O \cdots (\mathbf{B}_i^2)^T)^T$ of $\mathbf{H}_R$, subtract the column $\mathbf{h}_{i',j'}$ in $(O \cdots O\ (\mathbf{A}_{i'}^1)^T\ O \cdots (\mathbf{B}_{i'}^1)^T)^T$ of $\mathbf{H}_L$ from it, where (i', j') satisfies $i' < i$ and $a_{i,j} = a_{i',j'} \in E_{i'}^1$. This leads to a matrix equivalent to $\mathbf{H}_2$:

$$\mathbf{H}_2' = (\mathbf{H}_L||\mathbf{H}_R') = \left(\begin{array}{ccccc|cccc} \mathbf{A}_1 & O & \cdots & O & O & -\mathbf{A}_{2,1}^2 & -\mathbf{A}_{3,1}^2 & \cdots & -\mathbf{A}_{s,1}^2 \\ O & \mathbf{A}_2^1 & \cdots & O & O & \mathbf{A}_2^2 & -\mathbf{A}_{3,2}^2 & \cdots & -\mathbf{A}_{s,2}^2 \\ O & O & \cdots & O & O & O & \mathbf{A}_3^2 & \cdots & -\mathbf{A}_{s,3}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_s^1 & O & O & O & \cdots & \mathbf{A}_s^2 \\ O & O & \cdots & O & O & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_s^1 & \mathbf{B}_{l+2} & O & O & \cdots & O \end{array}\right),$$

where for $1 \leq j < i \leq s$, the l_{th} column of $\mathbf{A}_{i,j}^2$ is identical to the l_{th} column of $\mathbf{A}_i^2$ if the corresponding generating element appears in $E_j^1 \cap E_i$ and is identical to the zero vector, otherwise. Clearly, we have $\mathbf{A}_{2,1}^2 = \mathbf{A}_2^2$ and $\sum_{j=1}^{i-1} \mathbf{A}_{i,j}^2 = \mathbf{A}_i^2$. Moreover, (5.37) turns into

$$\sum_{i=1}^{s} \sum_{j \in I_i^1} \lambda'_{i,j} \mathbf{h}_{i,j} + \sum_{j \in [|E_{l+2}|]} \lambda_{l+2,j} \mathbf{h}_{l+2,j} + \sum_{i=1}^{s} \sum_{j \in I_i^2} \lambda_{i,j} \mathbf{h}'_{i,j} = \mathbf{0}, \tag{5.38}$$

where for $(i,j) \in [s] \times I_i^1$,

$$\lambda'_{i,j} = \lambda_{i,j} + \sum_{i < i' \leq s} \sum_{\substack{j' \in I_{i'}^2: \\ a_{i',j'} = a_{i,j}}} \lambda_{i',j'},$$

and for $(i,j) \in [s] \times I_i^2$,

$$\mathbf{h}'_{i,j} = \mathbf{h}_{i,j} - \mathbf{h}_{i',j'} \in \mathbf{H}'_R$$

for some (i',j') satisfying $i' < i$ and $j' \in I_{i'}^1$ such that $a_{i,j} = a_{i',j'}$.

- Second, for each $1 \leq i \leq \delta - 1$, add the $(i + j(\delta-1))_{th}$ row to the i_{th} row for all $1 \leq j \leq s-1$. This leads to

$$\mathbf{H}''_2 = (\mathbf{H}''_L || \mathbf{H}''_R) = \left(\begin{array}{ccccc||cccc} \mathbf{A}_1 & \mathbf{A}_2^1 & \cdots & \mathbf{A}_s^1 & O & O & O & \cdots & O \\ O & \mathbf{A}_2^1 & \cdots & O & O & \mathbf{A}_2^2 & -\mathbf{A}_{3,2}^2 & \cdots & -\mathbf{A}_{s,2}^2 \\ O & O & \cdots & O & O & O & \mathbf{A}_3^2 & \cdots & -\mathbf{A}_{s,3}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{A}_s^1 & O & O & O & \cdots & \mathbf{A}_s^2 \\ O & O & \cdots & O & O & O & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_s^1 & \mathbf{B}_{l+2} & O & O & \cdots & O \end{array} \right). \tag{5.39}$$

Since elementary row operations don't affect linear relations among columns, therefore for $\mathbf{H}''_2$, (5.38) turns into

$$\sum_{i=1}^{s} \sum_{j \in I_i^1} \lambda'_{i,j} \mathbf{h}''_{i,j} + \sum_{j \in [|E_{l+2}|]} \lambda_{l+2,j} \mathbf{h}_{l+2,j} + \sum_{i=1}^{s} \sum_{j \in I_i^2} \lambda_{i,j} \mathbf{h}''_{i,j} = \mathbf{0}, \tag{5.40}$$

where for $(i,j) \in [s] \times I_i^1$,

$$\mathbf{h}''_{i,j} = \begin{cases} \mathbf{h}_{i,j}, \text{ when } i = 1; \\ \mathbf{h}_{i,j} + (f(a_{i,j}), a_{i,j} f(a_{i,j}), \ldots, a_{i,j}^{\delta-2} f(a_{i,j}), 0, \ldots, 0)^T, \text{ when } i \geq 2, \end{cases}$$

and for $(i,j) \in [s] \times I_i^2$,

$$\mathbf{h}''_{i,j} = \begin{cases} \mathbf{h}'_{i,j} + (f(a_{i,j}), a_{i,j} f(a_{i,j}), \ldots, a_{i,j}^{\delta-2} f(a_{i,j}), 0, \ldots, 0)^T, \text{ if } a_{i,j} \in E_1^1; \\ \mathbf{h}'_{i,j}, \text{ otherwise.} \end{cases}$$

Now, consider the following submatrix consisting of the first $\delta - 1$ rows and the last h rows of $\mathbf{H}''_L$:

$$\mathbf{H}_0 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2^1 & \cdots & \mathbf{A}_s^1 & O \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_s^1 & \mathbf{B}_{l+2} \end{pmatrix}.$$

Clearly, $\mathbf{H}_0$ is of size $(h + \delta - 1) \times (|E| + |E_{l+2}|)$.

Claim .6. $\mathbf{H}_0$ *has full column rank.*

Proof. Since $f(g_{l+2,j}) = 0$ for every $g_{l+2,j} \in G_{l+2}$, thus, we can treat the zero submatrix in the top-right corner of $\mathbf{H}_0$ as:

$$\mathbf{A}_{l+2} = \begin{pmatrix} f(a_{l+2,1}) & f(a_{l+2,2}) & \cdots & f(a_{l+2,|E_{l+2}|}) \\ a_{l+2,1} f(a_{l+2,1}) & a_{l+2,2} f(a_{l+2,2}) & \cdots & a_{l+2,|E_{l+2}|} f(a_{l+2,|E_{l+2}|}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{l+2,1}^{\delta-2} f(a_{l+2,1}) & a_{l+2,2}^{\delta-2} f(a_{l+2,2}) & \cdots & a_{l+2,|E_{l+2}|}^{\delta-2} f(a_{l+2,|E_{l+2}|}) \end{pmatrix},$$

where $\{a_{l+2,1}, a_{l+2,2}, \ldots, a_{l+2,|E_{l+2}|}\} = E_{l+2}$.

For ease of notations, we let $E_{l+2}^1 = E_{l+2}$. Note that

$$|E| + |E_{l+2}| = |\bigcup_{i \in [s]} E_i| + |E_{l+2}| \leq h + \delta - 1,$$

when $|E| + |E_{l+2}| \geq h + 1$, we can consider the square submatrix of $\mathbf{H}_0$ consisting of the first $d_0 - h + 1 = |E| + |E_{l+2}| - h$ rows and the last h rows:

$$\mathbf{H}'_0 = \begin{pmatrix} \tilde{\mathbf{A}}_1 & \tilde{\mathbf{A}}_2 & \cdots & \tilde{\mathbf{A}}_s & \tilde{\mathbf{A}}_{l+2} \\ \mathbf{B}_1 & \mathbf{B}_2^1 & \cdots & \mathbf{B}_s^1 & \mathbf{B}_{l+2} \end{pmatrix},$$

where for $i \in [s] \cup \{l+2\}$,

$$\tilde{\mathbf{A}}_i = \begin{pmatrix} f(a_{i,1}) & f(a_{i,2}) & \cdots & f(a_{i,|E_i^1|}) \\ a_{i,1}f(a_{i,1}) & a_{i,2}f(a_{i,2}) & \cdots & a_{i,|E_i^1|}f(a_{i,|E_i^1|}) \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1}^{d_0-h}f(a_{i,1}) & a_{i,2}^{d_0-h}f(a_{i,2}) & \cdots & a_{i,|E_i^1|}^{d_0-h}f(a_{i,|E_i^1|}) \end{pmatrix}.$$

For any integer $d \geq 0$, denote $\mathbb{F}_q^{\leq d}[x]$ as the linear space of polynomials with degree at most d in $\mathbb{F}_q[x]$. Since $\{1, x, \ldots, x^{h-1}\}$ together with $\{f(x), xf(x), \ldots, x^{d_0-h}f(x)\}$ form a basis of $\mathbb{F}_q^{\leq d_0}[x]$, therefore, for any non-zero vector $\mathbf{u} \in \mathbb{F}_q^{d_0+1}$,

$$\mathbf{u} \cdot (f(x), xf(x), \ldots, x^{d_0-h}f(x), 1, x, \ldots, x^{h-1})^T \in \mathbb{F}_q^{\leq d_0}[x]$$

has at most $d_0 = |E| + |E_{l+2}| - 1$ different zeros in $\mathbb{F}_q$. Since $a_{i,j}$s from $\bigsqcup_{i=1}^{s} E_i^1 \sqcup E_{l+2}$ are pairwise distinct, thus, $\mathbf{u} \cdot \mathbf{H}_0' \neq \mathbf{0}$ for any non-zero vector $\mathbf{u} \in \mathbb{F}_q^{d_0+1}$. This shows that $rank(\mathbf{H}_0') = |E| + |E_{l+2}|$.

When $|E| + |E_{l+2}| \leq h$, consider the square submatrix $\mathbf{H}_0'$ consisting of the last $|E| + |E_{l+2}|$ rows of $\mathbf{H}_0$. Similarly, by the property of Vandermonde-type matrices, we can also obtain $rank(\mathbf{H}_0') = |E| + |E_{l+2}|$.

To sum up, for both cases, $\mathbf{H}_0$ contains a square submatrix of rank $|E| + |E_{l+2}|$, therefore, $\mathbf{H}_0$ has full column rank. □

According to the structure of $\mathbf{h}_{i,j}''$, (5.40) together with Claim .6 actually indicates that

$$\begin{cases} \lambda_{i,j}' = 0, \text{ for } (i,j) \in [s] \times I_i^1; \\ E_{l+2} = \emptyset. \end{cases}$$

Therefore, we have

$$\sum_{i=1}^{s} \sum_{j \in I_i^2} \lambda_{i,j} \mathbf{h}_{i,j}'' = \mathbf{0}. \tag{5.41}$$

Note that the structures of $\mathbf{H}_R''$s in (5.25) and (5.39) are the same and $\mathbf{A}_i$s here are also Vandermonde-like matrices. Therefore, through an analogous argument to the latter part of the proof of Theorem 5.5, we can also derive a contradiction by estimating $\sum_{i=2}^{t} |E_i \setminus E_i^1|$, which shows that $\mathbf{H}|_{\mathcal{E}'}$ has full column rank.

This completes the proof of Theorem 5.6. □

In the same vine, for small h, we can obtain optimal $(r,\delta)_i$-LRCs with arbitrarily long length by Theorem 5.6.

Theorem 5.7. *Let $1 \le h \le \delta$, set $G_1 = G_2 = \cdots = G_l$ and G_{l+1} as any $v+\delta-1$-subset of G_1 in Construction B, then the code $\mathcal{C}$ generated by Construction B can correct any $h+\delta-1$ erasures.*

Proof. Given any erasure pattern $\mathcal{E} = \{E_1, \ldots, E_{l+2}\}$ satisfying $\sum_{i=1}^{l+2} |E_i| = h+\delta-1$. Since $1 \le h \le \delta$, thus, there is only one block which contains at least δ columns. This indicates that $|S| = 1$ and thus, (5.35) holds naturally. Therefore, $\mathcal{E}$ can be recovered by $\mathcal{C}$. □

Corollary 5.3.2. *Let $q \ge r+\delta-1$. Assume that $1 \le h \le \delta$, then there exists an optimal $[n,k,h+\delta;(r,\delta)_i]$-LRC with length $n = (l+1)(r+\delta-1)+h+v-r$ for any positive integer l.*

As another consequence of Theorem 5.6, for general h, we have the following corollary.

Corollary 5.3.3. *If the system $\mathcal{G} = \{G_1, \ldots, G_{l+1}\}$ from Construction B satisfies*

$$|\bigcup_{i\in S} G_i| \ge \begin{cases} (r+\frac{\delta}{2}-1)|S| + \frac{\delta}{2}, & \text{if } l+1 \notin S; \\ (r+\frac{\delta}{2}-1)|S| + \frac{\delta}{2} + v - r, & \text{otherwise,} \end{cases} \tag{5.42}$$

for every subset $S \subseteq [l+1]$ of size at most $\lfloor \frac{h+\delta-1}{\delta} \rfloor$, then the code $\mathcal{C}$ generated by Construction B is an optimal $[n,k,h+\delta;(r,\delta)_i]$-LRC.

Proof. According to (5.33), we only need to show that the code $\mathcal{C}$ can recover any erasure pattern $\mathcal{E} = \{E_i : 1 \le i \le l+2\}$ with $\sum_{i\in S} |E_i| + |E_{l+2}| \le h+\delta-1$, where $E_i \subseteq G_i$. For any $\mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ and $1 \le j \le n-h$, the structure of $\mathbf{H}$ in Construction B ensures that the code symbol c_j has (r,δ)-locality. Therefore, for $i \in S$ with $|E_i| < \delta$, E_i can be recovered. Denote $S' = \{i' \in S : |E_{i'}| \ge \delta\}$. Then, we have $|S'| \le \lfloor \frac{h+\delta-1}{\delta} \rfloor$ and

$$\sum_{i'\in S'} |E_{i'}| + |E_{l+2}| \le h+\delta-1.$$

This leads to $|\bigcup_{i'\in S'} E_{i'}| + |E_{l+2}| \leq h + \delta - 1$ and the result follows from Theorem 5.6. □

Remark 5.5. *In [40], based on ideas of polynomial interpolation, Cai and Schwartz provide a construction of LRCs with $(r,\delta)_i$-locality with the same recovering capability (see Theorem 1 in [40]). From the perspective of parity-check matrix, Theorem 1 in [40] requires the generating sets G_is to satisfy*

$$|G_i \cap (\bigcup_{i\neq j\in S} G_j)| \leq \delta - 1,$$

which is a local condition for G_is. However, the minimal distance is a global parameter of the code. Therefore, due to the advantage of the intrinsic combinatorial property of G_is satisfying (5.35), Theorem 5.6 and Corollary 5.3.3 can provide longer codes.

§ 5.4 Optimal LRCs based on sparse hypergraphs

5.4.1 Tuŕan-type problems for sparse hypergraphs

Throughout this section, we will use some standard notations of sparse hypergraph from [175]. An R-uniform hypergraph (R-graph for short) on n vertices $\mathcal{H} := (V(\mathcal{H}), E(\mathcal{H}))$ is a pair of vertices and edges, where the vertex set $V(\mathcal{H})$ is a finite set (denoted as $[n]$) and edge set $E(\mathcal{H})$ is a collection of R-subsets of $V(\mathcal{H})$. For convenience, we often use $\mathcal{H}$ to denote its edge set $E(\mathcal{H})$ if there is no confusion.

For positive integers v and e, let $\mathcal{G}_R(v,e)$ be the family of all R-graphs consisting of e edges and at most v vertices, i.e.,

$$\mathcal{G}_R(v,e) = \{\mathcal{H} \subseteq \binom{[n]}{R} : |E(\mathcal{H})| = e, |V(\mathcal{H})| \leq v\}.$$

Then, an R-graph $\mathcal{H}$ is said to be $\mathcal{G}_R(v,e)$-free if it does not contain a copy of any member in $\mathcal{G}_R(v,e)$. In relevant literatures, such R-graphs are called sparse hypergraphs. Usually, we denote $f_R(n,v,e)$ as the maximum number of edges in a $\mathcal{G}_R(v,e)$-free R-graph on n vertices.

In [34], Brown, Erdős and Sós first made the following estimation about the value of $f_R(n,v,e)$.

Lemma 5.2. *[34] For $R \geq 2, e \geq 2, v \geq R+1$, there exist constants c_1, c_2 depending only on R, e, v such that*

$$c_1 n^{\frac{eR-v}{e-1}} \leq f_R(n,v,e) \leq c_2 n^{\lceil \frac{eR-v}{e-1} \rceil}.$$

When $e-1|eR-v$, this already determined the order of $f_R(n,v,e)$ up to a constant factor. However, for $e-1 \nmid eR-v$, it turns out to be extremely difficult to determine the correct exponent. With additional condition that $v = 3(R-l)+l+1$ and $e=3$, Alon and Shapira [11] proved the next result.

Lemma 5.3. *[11] For $2 \leq l < R$, we have*

$$n^{l-o(1)} < f_R(n, 3(R-l)+l+1, 3) = o(n^l).$$

Furthermore, there exists an explicit construction of R-graph which is both $\mathcal{G}_R(3(R-l)+l+1,3)$-free and $\mathcal{G}_R(2(R-l)+l,2)$-free with $n^{l-o(1)}$ edges.

Later in 2017, Ge and Shangguan [86] provided a construction for hypergraphs forbidding small rainbow cycles with order-optimal edges w.r.t. Lemma 5.2 (see Theorem 1.6 in [86]). For general lower bound on $f_R(n,v,e)$, very recently, Shangguan and Tamo [175] proved the following result.

Theorem 5.8. *[175] For $R \geq 2, e \geq 3, v \geq R+1$ satisfying $\gcd(e-1, eR-v) = 1$ and sufficiently large n, there exists an R-graph with*

$$\Omega(n^{\frac{eR-v}{e-1}} (\log n)^{\frac{1}{e-1}})$$

edges, which is also $\mathcal{G}_R(iR - \lceil \frac{(i-1)(eR-v)}{e-1} \rceil, i)$-free for every $2 \leq i \leq e$, and in particular,

$$f_R(n,v,e) = \Omega(n^{\frac{eR-v}{e-1}} (\log n)^{\frac{1}{e-1}})$$

as $n \to \infty$. Here the constants in $\Omega(\cdot)$ are independent of n.

In the same paper, Shangguan and Tamo also considered this type of problems for hypergraphs that are simultaneously $\mathcal{G}_R(v_i, e_i)$-free for a series of $\{(v_i, e_i)\}_{i=1}^s$.

Lemma 5.4. *[175] Let $s \geq 1$, $R \geq 3$ and (v_i, e_i), $1 \leq i \leq s$ be fixed integers satisfying $v_i \geq R+1$, $e_i \geq 2$. Suppose further that $e_1 \geq 3$, $\gcd(e_1-1, e_1R-v_1) = 1$ and $\frac{e_1R-v_1}{e_1-1} < \frac{e_iR-v_i}{e_i-1}$ for $2 \leq i \leq s$. Then there exists an R-graph with $\Omega(n^{\frac{e_1R-v_1}{e_1-1}}(\log n)^{\frac{1}{e_1-1}})$ edges which is $\mathcal{G}_R(v_i, e_i)$-free for each $1 \leq i \leq s$.*

According to Theorem 5.5 and Theorem 5.6, constructions of both optimal $(r,\delta)_a$-LRCs and optimal $(r,\delta)_i$-LRCs require the generating sets G_is to form a special kind of sparse hypergraph which is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2}\rfloor - 1, i)$-free for $2 \leq i \leq \mu$ (for some given integer $\mu \geq 3$). Armed with the above results, we have the following existence theorem for such hypergraphs with $|E(\mathcal{H})|$ growing super-linearly in n.

Theorem 5.9. *Let $\delta \geq 2$, $\mu \geq 3$ and $R \geq \min\{\delta, 3\}$ be fixed integers. Then, for n sufficiently large, there exists an R-uniform hypergraph $\mathcal{H}(V,E)$ that is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2}\rfloor - 1, i)$-free for every $2 \leq i \leq \mu$ with $|V| = n$ and*

$$|E| = \begin{cases} \Omega(n^{\frac{\delta}{2}+\frac{1}{\mu-1}}(\log n)^{\frac{1}{\mu-1}}), & \text{when } \delta \text{ is even;} \\ \Omega(n^{\frac{\delta}{2}+\frac{1}{2(\mu-1)}}(\log n)^{\frac{1}{2(\mu-1)}}), & \text{when } \delta \text{ is odd and } \mu \text{ is even;} \\ \Omega(n^{\frac{\delta}{2}+\frac{1}{2(\mu-2)}}(\log n)^{\frac{1}{2(\mu-2)}}), & \text{when both } \delta \text{ and } \mu > 3 \text{ are odd;} \\ \Omega(n^{\frac{\delta+1}{2}}), & \text{when } \delta \text{ is odd and } \mu = 3. \end{cases} \tag{5.43}$$

Proof. For each $2 \leq i \leq \mu$, let $v_i' = iR - \lfloor (i-1)\frac{\delta}{2}\rfloor - 1$. Consider the sequence $\{\frac{iR-v_i'}{i-1}\}_{i=2}^{\mu} = \{\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor+1}{i-1}\}_{i=2}^{\mu}$: For each $2 \leq i \leq \mu$,

$$\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor + 1}{i-1} = \begin{cases} \frac{\delta}{2} + \frac{1}{i-1}, & \text{if } (i-1)\delta \text{ is even;} \\ \frac{\delta}{2} + \frac{1}{2(i-1)}, & \text{if } (i-1)\delta \text{ is odd.} \end{cases}$$

Therefore, when δ is even, $\{\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor+1}{i-1}\}_{i=2}^{\mu} = \{\frac{\delta}{2} + \frac{1}{i-1}\}_{i=2}^{\mu}$ is a strictly decreasing sequence and $\frac{\delta}{2}+\frac{1}{\mu-1}<\frac{\delta}{2}+\frac{1}{i-1}$ for all $2\le i\le \mu-1$. When δ is odd, we have

$$\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor+1}{i-1}=\begin{cases}\frac{\delta}{2}+\frac{1}{i-1}, & \text{if } i \text{ is odd;}\\ \frac{\delta}{2}+\frac{1}{2(i-1)}, & \text{if } i \text{ is even.}\end{cases}$$

Therefore, based on the monotone decreasing property of both $\frac{\delta}{2}+\frac{1}{i-1}$ (for odd i) and $\frac{\delta}{2}+\frac{1}{2(i-1)}$ (for even i), we have

$$\begin{cases}\frac{\delta}{2}+\frac{1}{2(\mu-1)}<\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor+1}{i-1} \text{ for } 2\le i\le \mu-1, \text{ when } \mu \text{ is even;}\\ \frac{\delta}{2}+\frac{1}{2(\mu-2)}<\frac{\lfloor (i-1)\frac{\delta}{2}\rfloor+1}{i-1} \text{ for } 2\le i\le \mu-2 \text{ and } i=\mu, \text{ when } \mu>3 \text{ and } \mu \text{ is odd;}\\ \frac{\delta}{2}+\frac{1}{2(\mu-2)}=\frac{\delta}{2}+\frac{1}{\mu-1}, \text{ when } \mu=3.\end{cases}$$

When δ is even, clearly, we have $\gcd(\mu-1,(\mu-1)\frac{\delta}{2}+1)=1$. By applying Lemma 5.4 with $s=\mu-1$, $(v_1,e_1)=(\mu R-(\mu-1)\frac{\delta}{2}-1,\mu)$ and $\{(v_i,e_i)\}_{i=2}^{\mu-1}=\{(v'_j,j)\}_{j=2}^{\mu-1}$, there exists an R-graph with $\Omega(n^{\frac{\delta}{2}+\frac{1}{\mu-1}}(\log n)^{\frac{1}{\mu-1}})$ edges. This proves the first part of (5.43).

When δ is odd, μ is even. Assume that $\mu=2u$ for some $u\ge 2$. Then, we have

$$(\mu-1)\frac{\delta}{2}+\frac{1}{2}=(\mu-1)\frac{\delta-1}{2}+u.$$

Since δ is odd, thus $(\mu-1)|(\mu-1)\frac{\delta-1}{2}$. Therefore, we have $\gcd(\mu-1,(\mu-1)\frac{\delta}{2}+\frac{1}{2})=1$. By applying Lemma 5.4 with $s=\mu-1$, $(v_1,e_1)=(\mu R-((\mu-1)\frac{\delta}{2}+\frac{1}{2}),\mu)$ and $\{(v_i,e_i)\}_{i=2}^{\mu-1}=\{(v'_j,j)\}_{j=2}^{\mu-1}$, there exists an R-graph with $\Omega(n^{\frac{\delta}{2}+\frac{1}{2(\mu-1)}}(\log n)^{\frac{1}{2(\mu-1)}})$ edges. This proves the second part of (5.43).

When δ, $\mu>3$ are both odd. Assume that $\mu=2u+1$ for some $u\ge 2$. Then, we have

$$(\mu-2)\frac{\delta}{2}+\frac{1}{2}=(\mu-2)\frac{\delta-1}{2}+u.$$

Thus, we also have $\gcd(\mu-2,(\mu-2)\frac{\delta}{2}+\frac{1}{2})=1$. By applying Lemma 5.4 with $s=\mu-1$, $(v_1,e_1)=((\mu-1)R-((\mu-2)\frac{\delta}{2}+\frac{1}{2}),\mu-1)$ and $\{(v_i,e_i)\}_{i=2}^{\mu-1}=\{(v'_j,j)\}_{j=2,j\neq\mu-1}^{\mu}$, there exists an R-graph with $\Omega(n^{\frac{\delta}{2}+\frac{1}{2(\mu-2)}}(\log n)^{\frac{1}{2(\mu-2)}})$ edges. This proves the third part of (5.43).

Now, we turn to the proof of the rest part of (5.43). When δ is odd and $\mu = 3$, the conditions of Lemma 5.4 no longer hold, thus we shall use the standard probabilistic method to prove the existence of such sparse hypergraph. Actually, we are going to prove the following stronger result.

Claim .7. *When both δ and μ are odd, there exists an R-uniform hypergraph $\mathcal{H}(V, E)$ that is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for every $2 \le i \le \mu$ with $|V| = n$ and $|E| = \Omega(n^{\frac{\delta}{2} + \frac{1}{2(\mu-2)}})$.*

Proof. Set $p := p(n) = \varepsilon n^{\frac{\delta}{2} + \frac{1}{2(\mu-2)} - R}$ where $\varepsilon = \varepsilon(R, \delta, \mu) > 0$ is a small constant to be determined. Construct an R-graph $\mathcal{H}_0 \subseteq \binom{V}{R}$ randomly by choosing each member of $\binom{V}{R}$ independently with probability p. Let X denote the number of edges in $\mathcal{H}_0$. Clearly, for n sufficiently large,

$$E[X] = p\binom{n}{R} \ge \frac{\varepsilon n^{\frac{\delta}{2} + \frac{1}{2(\mu-2)}}}{2R!}.$$

For $2 \le i \le \mu$, let $\mathcal{Y}_i$ be the collection of all i distinct edges of $\mathcal{H}_0$ whose union contains at most $iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1$ vertices. Denote Y_i as the size of $\mathcal{Y}_i$. Then,

$$\begin{aligned} E[Y_i] &\le p^i \binom{n}{iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1} \binom{iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1}{R}^i \\ &\le \varepsilon^i \binom{iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1}{R}^i n^{(\frac{\delta}{2} + \frac{1}{2(\mu-2)})i - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1}. \end{aligned}$$

Take $\varepsilon = (\mu R)^{-(3R)}$, since $\frac{\delta}{2} + \frac{1}{2(\mu-2)} \le \frac{\lfloor (i-1)\frac{\delta}{2} \rfloor + 1}{i-1}$ and $\binom{iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1}{R}^i \le \binom{\mu R}{R}^i$, thus we have

$$\begin{aligned} E[Y_i] &\le \varepsilon^i \binom{iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1}{R}^i n^{(\frac{\delta}{2} + \frac{1}{2(\mu-2)})i - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1} \\ &< \frac{\varepsilon n^{\frac{\delta}{2} + \frac{1}{2(\mu-2)}}}{\mu^{3i-3}(R!)^i} \le \frac{E[X]}{\mu^2 R!} \end{aligned} \tag{5.44}$$

for every $2 \le i \le \mu$.

Applying Chernoff's inequality (see Corollary A.1.14 in [12]) for X and Markov's inequality for Y_i, it is easy to see that for each $2 \le i \le \mu$ and sufficiently large n, we have

$$Pr[X < 0.9E[X]] < \frac{1}{2\mu} \text{ and } Pr[Y_i > 2\mu E[Y_i]] < \frac{1}{2\mu}.$$

Therefore, with positive probability, there exists an R-graph $\mathcal{H}_0 \subseteq \binom{V}{R}$ such that

$$X \geq 0.9E[X] \text{ and } Y_i \leq 2\mu E[Y_i] \text{ for each } 2 \leq i \leq \mu.$$

Fix such $\mathcal{H}_0$, we construct a subgraph $\mathcal{H}_1$ from $\mathcal{H}_0$ by removing one edge from each member of $\mathcal{Y}_i$ in $\mathcal{H}_0$ for every $2 \leq i \leq \mu$. By (5.44), $\mathcal{H}_1$ satisfies $|E(\mathcal{H}_1)| = \Omega(n^{\frac{\delta}{2}+\frac{1}{2(\mu-2)}})$ and for each $2 \leq i \leq \mu$, the union of any i distinct edges in $\mathcal{H}_1$ contains at least $iR - \lfloor (i-1)\frac{\delta}{2} \rfloor$ vertices. Therefore, $\mathcal{H}_1$ is the desired R-graph and this proves the claim. □

Take $\mu = 3$ in Claim .7, we have the fourth part of (5.43). This completes the proof of Theorem 5.9. □

5.4.2 Optimal locally repairable codes with super-linear length

In this subsection, we are going to achieve our code constructions with the help of sparse hypergraphs. For LRCs with all symbol (r, δ)-locality, we have the following result.

Theorem 5.10. *For positive integers $\delta \geq 2$, $r \geq d - \delta$ and $d \geq 2\delta + 1$. Let $R = r + \delta - 1$, $\mu = \lfloor \frac{d-1}{\delta} \rfloor$ and $\mathcal{H}(V, E)$ be an R-uniform hypergraph with $V = \mathbb{F}_q$ that is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for every $2 \leq i \leq \mu$. Then, there exists an optimal $[n, k, d; (r, \delta)_a]_q$-LRC with length $n = R|E|$.*

Proof. Let $m = |E|$ and for each $e_i \in E(\mathcal{H})$, take e_i as the generating set of Vandermonde matrices $\begin{pmatrix} \mathbf{U}_i \\ \mathbf{V}_i \end{pmatrix}$ in Construction A. Note that $V(\mathcal{H}) = \mathbb{F}_q$ and $\mathcal{H}$ is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for $2 \leq i \leq \mu$, therefore, for any subset $S \subseteq [m]$ with $2 \leq |S| \leq \lfloor \frac{d-1}{\delta} \rfloor$, we have

$$|\bigcup_{i \in S} e_i| \geq R|S| - \lfloor (|S|-1)\frac{\delta}{2} \rfloor \geq (r + \frac{\delta}{2} - 1)|S| + \frac{\delta}{2}.$$

Thus, the conclusion easily follows from Theorem 5.5. □

As for LRCs with information (r, δ)-locality, we have a similar result.

Theorem 5.11. *For integers $r \geq 1$, $1 \leq v \leq r$, $\delta \geq 2$ and $h \geq 0$. Let $R = r+\delta-1$, $\mu = \lfloor \frac{h+\delta-1}{\delta} \rfloor$ and $\mathcal{H}(V,E)$ be an R-uniform hypergraph with $V = \mathbb{F}_q \setminus G_{l+2}$ for some h-subset $G_{l+2} \subseteq \mathbb{F}_q$ that is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for every $2 \leq i \leq \mu$. Then, there exists an optimal $[n,k,h+\delta;(r,\delta)_i]_q$-LRC with length $n = R|E| - r + v$.*

Proof. Similarly, let $l+1 = |E|$. Take any $e_{l+1} \in E(\mathcal{H})$, choose a $(v+\delta-1)$-subset of e_{l+1} as the generating set of matrices $\mathbf{U}_{l+1}$ and $\mathbf{V}_{l+1}$, and for the rest $e_i \in E(\mathcal{H})$, take e_i as the generating set of matrices $\mathbf{U}_i$ and $\mathbf{V}_i$ $(1 \leq i \leq l)$ in Construction B. Note that $\mathcal{H}$ is simultaneously $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for $2 \leq i \leq \mu$, therefore, for any subset $S \subseteq [m]$ with $2 \leq |S| \leq \lfloor \frac{h+\delta-1}{\delta} \rfloor$, we have

$$|\bigcup_{i \in S} e_i| \geq \begin{cases} (r+\frac{\delta}{2}-1)|S| + \frac{\delta}{2}, & \text{when } l+1 \notin S; \\ (r+\frac{\delta}{2}-1)|S| + \frac{\delta}{2} + v - r, & \text{when } l+1 \in S. \end{cases}$$

Thus, the conclusion easily follows from Corollary 5.3.3. □

Recall that in Theorem 5.5, $r \geq d-\delta$ and $R|n$. When $2\delta+1 \leq d \leq 3\delta$, one can get optimal LRCs with length $\Omega(q^{\delta})$ and minimum distance d via packings or Steiner systems as in [112] and [39]. For $3\delta+1 \leq d \leq 4\delta$, we have the following explicit construction.

Corollary 5.4.1. *For $3\delta+1 \leq d \leq 4\delta$ and $r \geq d-\delta$, there exist explicit constructions of optimal $[n,k,d;(r,\delta)_a]_q$-LRCs with length*

$$n = \begin{cases} Rq^{\frac{\delta}{2}+1-o(1)}, & \text{if } \delta \text{ is even;} \\ Rq^{\frac{\delta+1}{2}-o(1)}, & \text{if } \delta \text{ is odd.} \end{cases}$$

Proof. When δ is even, take $l = \frac{\delta}{2}+1$ in Lemma 5.3, there exists an R-graph $\mathcal{H}_0$ which is both $\mathcal{G}_R(3R-\delta-1,3)$-free and $\mathcal{G}_R(2R-\frac{\delta}{2}-1,2)$-free with

$$q^{l-o(1)} = q^{\frac{\delta}{2}+1-o(1)}$$

edges.

When δ is odd, take $l = \frac{\delta+1}{2}$ in Lemma 5.3, there exists an R-graph $\mathcal{H}_0$ which is both $\mathcal{G}_R(3R-\delta, 3)$-free and $\mathcal{G}_R(2R-\frac{\delta+1}{2}, 2)$-free with

$$q^{l-o(1)} = q^{\frac{\delta+1}{2}-o(1)}$$

edges.

Therefore, the conclusion follows from Theorem 5.10. □

Corollary 5.4.2. *For $2\delta+1 \leq h \leq 3\delta$, there exist explicit constructions of optimal $[n, k, h+\delta; (r,\delta)_i]$-LRCs with length*

$$n = \begin{cases} Rq^{\frac{\delta}{2}+1-o(1)}, & \textit{if } \delta \textit{ is even;} \\ Rq^{\frac{\delta+1}{2}-o(1)}, & \text{if } \delta \textit{ is odd.} \end{cases}$$

Proof. Based on the sparse hypergraph given by Lemma 5.3 and Construction B, the conclusion easily follows from Theorem 5.11. □

For LRCs with larger minimal distance, we have the following results from Theorem 5.9, Theorem 5.10 and Theorem 5.11.

Corollary 5.4.3. *For $\delta \geq 2$, $r \geq d-\delta$, $d \geq 3\delta+1$ and q large enough. Let $\mu = \lfloor \frac{d-1}{\delta} \rfloor$, then there exists an optimal $[n, k, d; (r,\delta)_a]_q$-LRC of length*

$$n = \begin{cases} \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{\mu-1}}), & \textit{when } \delta \textit{ is even;} \\ \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\mu-1)}}), & \textit{when } \delta \textit{ is odd and } \mu \textit{ is even;} \\ \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\mu-2)}}), & \textit{when both } \delta \textit{ and } \mu > 3 \textit{ are odd;} \\ \Omega(q^{\frac{\delta+1}{2}}), & \textit{when } \delta \textit{ is odd and } \mu = 3. \end{cases}$$

Corollary 5.4.4. *For $1 \leq v \leq r$, $\delta \geq 2$ and $h \geq 2\delta+1$ and q large enough. Let $\mu = \lfloor \frac{h+\delta-1}{\delta} \rfloor$, then there exists an optimal $[n, k, h+\delta; (r,\delta)_i]_q$-LRC of length*

$$n = \begin{cases} \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{\mu-1}}), & \textit{when } \delta \textit{ is even;} \\ \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\mu-1)}}), & \textit{when } \delta \textit{ is odd and } \mu \textit{ is even;} \\ \Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\mu-2)}}), & \textit{when both } \delta \textit{ and } \mu > 3 \textit{ are odd;} \\ \Omega(q^{\frac{\delta+1}{2}}), & \textit{when } \delta \textit{ is odd and } \mu = 3. \end{cases}$$

Remark 5.6. *For more details about sparse hypergraphs and other related applications, we recommend [35] and [175] for interested readers.*

In Table I and Table II, we have listed all the known parameters of optimal LRCs of super-linear length together with our results. As one can see, for optimal $(r,\delta)_a$-LRCs:

- when $\delta = 2$, our results from Corollary 5.4.1 and Corollary 5.4.3 agree with those in [208] for $d = 7, 8$ and $d \geq 11$; for $d = 9, 10$, Xing and Yuan [208] provided longer codes;
- when $\delta > 2$ and $2\delta + 1 \leq d \leq 3\delta$, Cai et.al [39] provided the longest known codes of length $\Omega(q^\delta)$ which meets the upper bound for the case $d = 2\delta + 1$;
- when $\delta > 2$ and $d \geq 3\delta + 1$, Corollary 5.4.1 gives the longest known codes for $d \leq 4\delta$ and δ is even; Corollary 5.4.3 gives the longest known codes for other cases.

For optimal $(r,\delta)_i$-LRCs:

- when $\delta > 2$ and $\delta + 1 \leq d \leq 2\delta$, Corollary 5.3.2 provides a code of arbitrarily long length;
- when $\delta > 2$ and $2\delta+1 \leq d \leq 3\delta$, Cai and Schwartz [40] provided codes of order optimal length $\Omega(q^\delta)$;
- when $\delta > 2$ and $d \geq 3\delta + 1$, Corollary 5.4.2 gives the longest known codes for $d \leq 4\delta$ and δ is even; Corollary 5.4.4 gives the longest known codes for other cases.

§ 5.5 Applications: Constructions of H-LRCs and generalized sector-disk codes

In this section, we present two applications of Constructions A and B, respectively. In Subsection 5.5.1, based on Construction A, we construct optimal H-LRCs

表 5.1 Optimal $(r,\delta)_a$-LRCs over $\mathbb{F}_q$ with super-linear lengths and corresponding upper bounds

Distance	Other conditions	Length	Upper Bound
$d=5,6$	$\delta=2$, $r\geq d-2$, $r+1\mid n$	$\Omega(q^2)$ ([112], [98] and [39])	$\begin{cases} O(q^2), d=5 \\ O(q^3), d=6 \end{cases}$ ([98])
$d=7,8$	$\delta=2$, $r\geq d-2$, $r+1\mid n$	$\Omega(q^{2-o(1)})$ ([208])	$\begin{cases} O(q^3), d=7 \\ O(q^4), d=8 \end{cases}$ ([98])
$d=9,10$	$\delta=2$, $r\geq d-2$, $r+1\mid n$	$\Omega(q^{\frac{3}{2}-o(1)})$ ([208])	$\begin{cases} O(q^{\frac{5}{2}}), d=9 \\ O(q^3), d=10 \end{cases}$ ([98])
$d\geq 11$	$\delta=2$, $r\geq d-2$, $r+1\mid n$	$\Omega(q(q\log q)^{\frac{1}{\lfloor (d-3)/2\rfloor}})$ ([208] and [175])	$\begin{cases} O(dq^3), 4\nmid d \\ O(dq^{3+\frac{4}{d-4}}), 4\mid d \end{cases}$ ([98])
$\delta+1\leq d\leq 2\delta$	$d\leq r+\delta-1\leq q$, $r+\delta-1\mid n$	∞ ([39] and [214])	∞
$d\geq 2\delta+1$	$r\geq d-\delta+1$, $r+\delta-1\mid n$	$\Omega(q^{\frac{\delta}{\lceil d/\delta\rceil-2}})$ ([39])	$\begin{cases} O(q^{\frac{2(d-\delta-1)}{\lfloor (d-1)/\delta\rfloor}-1}), \lfloor\frac{d-1}{\delta}\rfloor \text{ odd} \\ O(q^{\frac{2(d-\delta)}{\lfloor (d-1)/\delta\rfloor}-1}), \lfloor\frac{d-1}{\delta}\rfloor \text{ even} \end{cases}$ ([39])
$d\geq 2\delta+1$	$r\geq d-\delta+1$, $r+\delta-1\mid n$	$\Omega(q^{1+\lfloor\frac{\delta^2}{d-\delta}\rfloor})$ ([39])	——
$3\delta+1\leq d\leq 4\delta$	δ even, $r\geq d-\delta$, $r+\delta-1\mid n$	$\Omega(q^{1+\frac{\delta}{2}-o(1)})$ (Corollary 5.4.1)	——
$3\delta+1\leq d\leq 4\delta$	δ odd, $r\geq d-\delta$, $r+\delta-1\mid n$	$\Omega(q^{\frac{\delta+1}{2}})$ (Corollary 5.4.3)	——
$d\geq 3\delta+1$	δ even, $r\geq d-\delta$, $r+\delta-1\mid n$	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{\lfloor\frac{d-1}{\delta}\rfloor-1}})$ (Corollary 5.4.3)	——
$d\geq 3\delta+1$	δ odd, $\lfloor\frac{d-1}{\delta}\rfloor$ even, $r\geq d-\delta$, $r+\delta-1\mid n$	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\lfloor\frac{d-1}{\delta}\rfloor-1)}})$ (Corollary 5.4.3)	——
$d\geq 3\delta+1$	δ odd, $\lfloor\frac{d-1}{\delta}\rfloor>3$ odd, $r\geq d-\delta$, $r+\delta-1\mid n$	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\lfloor\frac{d-1}{\delta}\rfloor-2)}})$ (Corollary 5.4.3)	——

表 5.2 Optimal $(r,\delta)_i$-LRCs over $\mathbb{F}_q$ with super-linear lengths and corresponding upper bounds

Distance	Other conditions	Length	Upper Bound
$\delta+1 \le d \le 2\delta$	$q \ge r+\delta-1$	∞ (Corollary 5.3.2)	∞
$d \ge 2\delta+1$	Null	$\Omega(q^{\tau+1})$, where $\tau = \max\{x \in \mathbb{N}^* : \lceil \frac{\delta}{x} \rceil = \lceil \frac{d-\delta}{\delta} \rceil\}$ ([40])	$\begin{cases} O(q^{\frac{2(d-\delta-a-1)}{T(a)-1}-1}), T(a) \text{ odd} \\ O(q^{\frac{2(d-\delta-a)}{T(a)-1}-1}), T(a) \text{ even} \end{cases}$ ([40]) when $r \mid k$ and $T(a) \ge 2$, where $T(a) = \lfloor \frac{d-a-1}{\delta} \rfloor$, for any $0 \le a \le d-\delta$
$3\delta+1 \le d \le 4\delta$	δ even	$\Omega(q^{1+\frac{\delta}{2}-o(1)})$ (Corollary 5.4.2)	——
$3\delta+1 \le d \le 4\delta$	δ odd	$\Omega(q^{\frac{\delta+1}{2}})$ (Corollary 5.4.4)	——
$d \ge 3\delta+1$	δ even	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{\lfloor \frac{d-1}{\delta} \rfloor -1}})$ (Corollary 5.4.4)	——
$d \ge 3\delta+1$	δ odd and $\lfloor \frac{d-1}{\delta} \rfloor$ even	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\lfloor \frac{d-1}{\delta} \rfloor -1)}})$ (Corollary 5.4.4)	——
$d \ge 3\delta+1$	δ odd and $\lfloor \frac{d-1}{\delta} \rfloor > 3$ odd	$\Omega(q^{\frac{\delta}{2}}(q\log q)^{\frac{1}{2(\lfloor \frac{d-1}{\delta} \rfloor -2)}})$ (Corollary 5.4.4)	——

with super-linear length, which improves the results given by [213]. In Subsection 5.5.2, based on Construction B, we provide two constructions of generalized sector-disk codes, which provide a code of unbounded length.

5.5.1 Optimal codes with hierarchical locality

The conception of *hierarchical locality* was first introduced by Sasidharan et al. in [170]. The authors considered the intermediate situation when the code can correct a single erasure by contacting a small number of helper nodes, while at the same time maintains local recovery of multiple erasures. Such codes are called locally recoverable codes with hierarchical locality, its formal definition is given as follows.

Definition 5.8. *[170] Let $2 \leq \delta_2 < \delta_1$ and $r_2 \leq r_1$ be positive integers. An $[n, k, d]_q$ code $\mathcal{C}$ is an H-LRC with parameters $[(r_1, \delta_1), (r_2, \delta_2)]$ if for every $i \in [n]$, there is a punctured code $\mathcal{C}_i$ such that $i \in Supp(\mathcal{C}_i)$ and the following conditions hold*

1) $\dim(\mathcal{C}_i) \leq r_1$*;*

2) $d(\mathcal{C}_i) \geq \delta_1$*;*

3) $\mathcal{C}_i$ *is a code with* (r_2, δ_2)*-locality.*

For each $i \in [n]$, the punctured code $\mathcal{C}_i$ associated with c_i is referred to as a *middle code* of $\mathcal{C}$. In [170], the authors extended the Singleton-type bound (5.1) and proved the following bound for H-LRCs with parameters $[(r_1, \delta_1), (r_2, \delta_2)]$:

$$d \leq n - k + 1 - (\left\lceil \frac{k}{r_2} \right\rceil - 1)(\delta_2 - 1) - (\left\lceil \frac{k}{r_1} \right\rceil - 1)(\delta_1 - \delta_2). \qquad (5.45)$$

H-LRCs with parameters $[(r_1, \delta_1), (r_2, \delta_2)]$ that attain this bound are called optimal. Using Reed-Solomon codes, Sasidharan et al. [170] construct optimal H-LRCs of length $n \leq q - 1$. Later in [17], Ballentine et al. presented a general construction of H-LRCs via maps between algebraic curves. From the perspective of parity-check matrices, Zhang and Liu [214] obtained a family of optimal H-LRCs with parameters

$[(r_1, \delta_1), (r_2, \delta_2)]$ and minimum distance $d \leq 3\delta_2$. Recently, in [213], Zhang extended the constructions in [214] and obtained H-LRCs with new parameters.

Based on Construction A and a correspondence between optimal $(r_2, \delta_2)_a$-LRCs and optimal H-LRCs with parameters $[(r_1, \delta_1), (r_2, \delta_2)]$ from [214], we have the following result.

Theorem 5.12. *For positive integers m_2, r_2, $\delta_2 \geq 2$ and $d_2 < r_2 + \delta_2$. Let $\mathbf{H}_{middle} = \mathbf{H}_{middle}(m_2, r_2, \delta_2, d_2)$ be the parity-check matrix in (5.19) from Construction A with parameters $m = m_2$, $r = r_2$, $\delta = \delta_2$ and $d = d_2$. For positive integer $m_1 < \frac{r_2}{d_2 - \delta_2}$, define*

$$\mathbf{H}(m_1, \mathbf{H}_{middle}) = \begin{pmatrix} \mathbf{H}_{middle} & O & \cdots & O \\ O & \mathbf{H}_{middle} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{H}_{middle} \end{pmatrix}, \tag{5.46}$$

where there are m_1 $\mathbf{H}_{middle}$s on the diagonal. Let r_1 and δ_1 be positive integers satisfying

$$r_1(1 - \frac{1}{m_1}) < m_2 r_2 - d + \delta_2 \leq r_1, \text{ and } \delta_1 = d_2. \tag{5.47}$$

If for any subset $S \subseteq [m_2]$ with $2 \leq |S| \leq \lfloor \frac{d_2 - 1}{\delta_2} \rfloor$, we have $|\bigcup_{i \in S} G_i| \geq (r_2 + \frac{\delta_2}{2} - 1)|S| + \frac{\delta_2}{2}$. Then, the code $\mathcal{C}$ with parity-check matrix $\mathbf{H}(m_1, \mathbf{H}_{middle})$ is an optimal $[n, k, d]_q$ H-LRC with parameters $[(r_1, \delta_1), (r_2, \delta_2)]$, where $n = m_1 m_2 (r_2 + \delta_2 - 1)$, $k = m_1(m_2 r_2 - d_2 + \delta_2)$ and $d = d_2$.

Proof. The proof is a routine check of the conditions in Definition 5.8 and the equality in (5.45).

Clearly, $n = m_1 m_2 (r_2 + \delta_2 - 1)$ and $rank(\mathbf{H}(m_1, \mathbf{H}_{middle})) = m_1 rank(\mathbf{H}_{middle})$. By Theorem 5.5, the code $\mathcal{C}_{middle}$ with parity-check matrix $\mathbf{H}_{middle}$ is an optimal $[m_2(r_2+\delta_2-1), m_2 r_2 - d_2 + \delta_2, d_2]_q$-LRC with $(r_2, \delta_2)_a$-locality. This verifies condition 3) in Definition 5.1. And conditions 1), 2) in Definition 5.8 follow from (5.47). Moreover, we also have $rank(\mathbf{H}_{middle}) = m_2(\delta_2 - 1) + d_2 - \delta_2$, which leads to $k = m_1(m_2 r_2 - d_2 + \delta_2)$.

It remains to verify the optimality of $\mathcal{C}$ w.r.t. bound (5.45). From $d_2 < r_2 + \delta_2$, $m_1 < \frac{r_2}{d_2-\delta_2}$ and $r_1(1-\frac{1}{m_1}) < m_2r_2 - d + \delta_2$, we have $\left\lceil \frac{k}{r_2} \right\rceil = m_1m_2$ and $\left\lceil \frac{k}{r_1} \right\rceil = m_1$. Therefore,

$$\begin{aligned}
&n-k+1-(\left\lceil \frac{k}{r_2} \right\rceil - 1)(\delta_2 - 1) - (\left\lceil \frac{k}{r_1} \right\rceil - 1)(\delta_1 - \delta_2)\\
=&m_1(m_2(\delta_2-1)+d_2-\delta_2)+1-(m_1m_2-1)(\delta_2-1)-(m_1-1)(d_2-\delta_2)\\
=&m_1(d_2-\delta_2)+\delta_2-(m_1-1)(d_2-\delta_2)=d_2.
\end{aligned}$$

This completes the proof of Theorem 5.12. □

Analogous to the case for $(r,\delta)_a$-LRCs, as immediate consequences of Theorem 5.12, we have the following corollaries.

Corollary 5.5.1. *Let r_1, r_2, δ_1, δ_2, $3\delta_2+1 \le d_2 \le 4\delta_2$ be those parameters defined in Theorem 5.12. Then, there exist explicit constructions of optimal $[n,k,d_2]_q$ H-LRCs with parameters $[(r_1,\delta_1),(r_2,\delta_2)]$ of length*

$$n = \begin{cases} (r_2+\delta_2-1)q^{\frac{\delta_2}{2}+1-o(1)}, & \text{if } \delta_2 \text{ is even;}\\ (r_2+\delta_2-1)q^{\frac{\delta_2+1}{2}-o(1)}, & \text{if } \delta_2 \text{ is odd.}\end{cases}$$

Corollary 5.5.2. *Let r_1, r_2, δ_1, $d_2 \ge 3\delta_2+1$ be those parameters defined in Theorem 5.12 and $\mu = \lfloor \frac{d_2-1}{\delta_2} \rfloor$. For q sufficiently large, there exists an optimal $[n,k,d_2]_q$ H-LRC with parameters $[(r_1,\delta_1),(r_2,\delta_2)]$ of length*

$$n = \begin{cases} \Omega(q^{\frac{\delta_2}{2}}(q\log q)^{\frac{1}{\mu-1}}), & \text{when } \delta_2 \text{ is even;}\\ \Omega(q^{\frac{\delta_2}{2}}(q\log q)^{\frac{1}{2(\mu-1)}}), & \text{when } \delta_2 \text{ is odd and } \mu \text{ is even;}\\ \Omega(q^{\frac{\delta_2}{2}}(q\log q)^{\frac{1}{2(\mu-2)}}), & \text{when both } \delta_2 \text{ and } \mu>3 \text{ are odd;}\\ \Omega(q^{\frac{\delta_2+1}{2}}), & \text{when } \delta_2 \text{ is odd and } \mu=3.\end{cases} \tag{5.48}$$

Remark 5.7. *(i) As mentioned in Remark 5.4, the construction of optimal $(r,\delta)_a$-LRCs in [213] is under the condition (5.31). As a consequence, H-LRCs generated from this construction require the generating sets of the middle code satisfying (5.31). Therefore, compared to the construction in [213], Theorem 5.12 provides a way to construct H-LRCs under a more relaxed condition.*

(ii) When $\delta_2 + 1 \leq d_2 \leq 2\delta_2$, like Theorem 5.4, Zhang and Liu [214] provide a construction of optimal $[n,k,d]_q$ H-LRCs with parameters $[(r_1,\delta_1),(r_2,\delta_2)]$ with unbounded length. When $2\delta_2+1 \leq d_2 \leq 3\delta_2$, H-LRCs obtained from Theorems IV.3 and IV.4 in [214] can have length $\Omega(q^2)$ through $(q, r_2+\delta_2-1, 1)$-packings. When $d_2 \geq 3\delta_2+1$, Corollary 5.5.1 and Corollary 5.5.2 give the longest known optimal H-LRCs for these cases.

5.5.2 Generalized sector-disk codes

Aiming to construct codes that can recover erasure patterns beyond the minimum distance, Cai and Schwartz [40] relaxed the restrictions of sector-disk codes [159] and considered the following array codes.

Definition 5.9. *[40] Let $\mathcal{C}$ be an optimal $[n,k,d;(r,\delta)_i]_q$-LRC. Then the code $\mathcal{C}$ is said to be a (γ, s)-generalized sector disk code (GSD code) if the codewords can be arranged into an array*

$$C = \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,a} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,a} \\ \vdots & \vdots & \ddots & \vdots \\ c_{b,1} & c_{b,2} & \cdots & c_{b,a} \end{pmatrix}$$

such that:

(i) *all the erasure patterns consisting of any γ columns and additional s sectors can be recovered;*

(ii) $\gamma b + s > d - 1$.

In [40], based on locally repairable codes with information locality constructed from regular packings, Cai and Schwartz obtained GSD codes with super-linear length for several different (γ, s)s. As an application of Theorem 5.7, we have the following construction.

Construction C: For positive integers r, $\delta \geq 2$ and $1 \leq h \leq \delta$. Let S be an h-subset of $\mathbb{F}_q$ and G be an $(r+\delta-1)$-subset of $\mathbb{F}_q \setminus S$. For any positive integer $l \geq 1$, let $n=(l+1)(r+\delta-1)+h$ and $k=(l+1)r$. Then take $G_{l+2}=S$ and $G_i=G$ for $1\le i\le l+1$, we can obtain an $[n,k,h+\delta;(r,\delta)_i]$-LRC $\mathcal{C}_0$ by Construction B. Denote $G=\{x_1,\dots,x_{r+\delta-1}\}$ and $c=(c_{1,1},\dots,c_{1,r+\delta-1},\dots,c_{l+1,1},\dots,c_{l+1,r+\delta-1},c_{l+2,1},\dots,c_{l+2,h})$ for any $c\in\mathcal{C}_0$. Define column vectors $\mathbf{v}_{x_a}\in\mathbb{F}_q^{l+2}$ for $a\in[r+\delta-1]$ as

$$\mathbf{v}_{x_a}^T=(c_{i_{x_a,1},j_{x_a,1}},c_{i_{x_a,2},j_{x_a,2}},\dots,c_{i_{x_a,l+1},j_{x_a,l+1}},c'_{l+2,a}),$$

where the generating element corresponding to $c_{i_{x_a,b},j_{x_a,b}}$ satisfies $g_{i_{x_a,b},j_{x_a,b}}=x_a$ for $1\le b\le l+1$, $c'_{l+2,a}=c_{l+2,a}$ for $1\le a\le h$ and $c'_{l+2,a}=0$ for $h+1\le a\le r+\delta-1$.

Theorem 5.13. *Let $\mathcal{C}$ be the $(l+2)\times(r+\delta-1)$ array code generated by Construction C. Then,*

- *when $\gamma\le h$, the code $\mathcal{C}$ is a $(\gamma,h+\delta-1-2\gamma)$-GSD code;*
- *when $h<\gamma<\delta-1$, the code $\mathcal{C}$ is a $(\gamma,\delta-1-\gamma)$-GSD code.*

Proof. According to Definition 5.9 and Theorem 5.7, we only need to show that erasure patterns consisting of γ columns and any other s erasures satisfy (5.34) and (5.35), where

$$s=\begin{cases}h+\delta-1-2\gamma, & \text{when } \gamma\le h;\\ \delta-1-\gamma, & \text{when } h<\gamma<\delta-1.\end{cases}$$

Let $\mathcal{F}=\{F_1,\dots,F_{l+2}\}$ be the erasure pattern formed by given γ columns and other s erasures. Denote $\mathcal{F}'=\{F_i\in\mathcal{F}:|F_i|\ge\delta\}\cup\{F_{l+2}\}$. Clearly, we have

$$\begin{aligned}|\bigcup_{F_i\in\mathcal{F}'}F_i|+|F_{l+2}|\le|\bigcup_{F_i\in\mathcal{F}}F_i|+|F_{l+2}|&\le\begin{cases}2\gamma+s, & \text{when } \gamma\le h;\\ \gamma+h+s, & \text{when } h<\gamma<\delta-1\end{cases}\\ &\le h+\delta-1.\end{aligned}$$

Therefore, $\mathcal{F}$ satisfies (5.34). Moreover, since $G_i=G$ for all $i\in[l+1]$, thus (5.35) holds naturally. □

From Corollary 5.3.2, the code $\mathcal{C}$ generated by Construction C can be arbitrarily long and its minimal distance $d=h+\delta$ satisfies $\delta+1\le d\le 2\delta$. For general

$d \geq 2\delta + 1$, based on Theorem 5.6, we can extend Cai and Schwartz's construction as follows.

Construction D: Given positive integers r, $1 \leq v < r$ and $\delta \geq 2$. Let S be an $(r-v)$-subset of $\mathbb{F}_q$ and $\mathcal{H}(V, E)$ be a t-regular ($t \geq 2$) R-uniform hypergraph with $V = \mathbb{F}_q \setminus S$ that is $\mathcal{G}_R(iR - \lfloor (i-1)\frac{\delta}{2} \rfloor - 1, i)$-free for every $2 \leq i \leq \binom{r-v+\delta-1}{\delta}$. Let $E = \{e_i\}_{i=1}^{|E|}$, $G_i = e_i$ for $1 \leq i \leq |E| - 1$ and $G_{|E|}$ be a $(v + \delta - 1)$-subset of $e_{|E|}$. Let $n = t(q - r + v)$ and $k = (|E| - 1)r + v$. Based on S and $\mathcal{H}$, we can obtain an $[n, k, r - v + \delta; (r, \delta)_i]$-LRC $\mathcal{C}_0$ by Construction B. Denote $e_{|E|} \setminus G_{|E|} = \{x_1, \ldots, x_{r-v}\}$, $V = \{x_1, \ldots, x_{q-r+v}\}$ and $c = (c_{1,1}, \ldots, c_{1,r+\delta-1}, \ldots, c_{|E|,1}, \ldots, c_{|E|,v+\delta-1}, c_{|E|+1,1}, \ldots, c_{|E|+1,r-v})$ for any $c \in \mathcal{C}_0$. Define column vectors $\mathbf{v}_{x_a} \in \mathbb{F}_q^t$ for $a \in [q - r + v]$ as

$$\mathbf{v}_{x_a}^T = \begin{cases} (c_{i_{x_a,1},j_{x_a,1}}, c_{i_{x_a,2},j_{x_a,2}}, \ldots, c_{i_{x_a,t-1},j_{x_a,t-1}}, c_{|E|+1,a}), & \text{if } 1 \leq a \leq r - v; \\ (c_{i_{x_a,1},j_{x_a,1}}, c_{i_{x_a,2},j_{x_a,2}}, \ldots, c_{i_{x_a,t},j_{x_a,t}}), & \text{otherwise,} \end{cases}$$

where the generating element corresponding to $c_{i_{x_a,b},j_{x_a,b}}$ satisfies $g_{i_{x_a,b},j_{x_a,b}} = x_a$, $1 \leq b \leq t - 1$ for $1 \leq a \leq r - v$ and $1 \leq b \leq t$ for $r - v + 1 \leq a \leq q - r + v$.

Theorem 5.14. *Let $\mathcal{C}$ be the $t \times (q - r + v)$ array code generated by Construction D. Then,*

- *when $\gamma \leq r - v$, the code $\mathcal{C}$ is a $(\gamma, r - v + \delta - 1 - 2\gamma)$-GSD code;*
- *when $r - v < \gamma < \delta - 1$, the code $\mathcal{C}$ is a $(\gamma, \delta - 1 - \gamma)$-GSD code.*

Proof. According to Definition 5.9 and Theorem 5.6, we only need to show that erasure patterns consisting of γ columns and any other s erasures satisfy (5.34) and (5.35), where

$$s = \begin{cases} r - v + \delta - 1 - 2\gamma, & \text{when } \gamma \leq r - v; \\ \delta - 1 - \gamma, & \text{when } r - v < \gamma < \delta - 1. \end{cases}$$

Let $\mathcal{F} = \{F_1, \ldots, F_{|E|+1}\}$ be the erasure pattern formed by γ given columns and other s erasures. Denote $\mathcal{F}' = \{F_i \in \mathcal{F} : |F_i| \geq \delta\} \cup \{F_{|E|+1}\}$ and $|\bigcup_{F_i \in \mathcal{F}'} F_i| + |F_{|E|+1}| \leq r - v + \delta - 1$ follows from the choice of s. Therefore, $\mathcal{F}$ satisfies (5.34).

On the other hand, denote $I_{\mathcal{F}'} = \{i \in [|E|] : F_i \in \mathcal{F}'\}$. Note that $|\bigcup_{i \in I_{\mathcal{F}'}} F_i| \leq r - v + \delta - 1$ and $|F_i| \geq \delta$ for each $i \in I_{\mathcal{F}'}$, which indicates that

$$|I_{\mathcal{F}'}| \leq \binom{r - v + \delta - 1}{\delta}.$$

Therefore, (5.35) follows from the sparsity of $\mathcal{H}$. □

Remark 5.8. *Unfortunately, all known results about large sparse hypergraphs can not guarantee the regularity of every vertex of $\mathcal{H}$. A standard probabilistic argument like Claim .7 can only provide sparse hypergraphs with bounded degree. Thus, more advanced methods are required to construct large regular sparse hypergraphs.*

§ 5.6 An upper bound on the minimal field size required for MRCs

In this section, we take the prime $p = 2$, which is the natural setting for distributed storage. And we will establish our upper bound on the minimal field size required for MRCs that instantiate the topology $T_{m\times n}(1, b, 0)$.

Theorem 5.15. *Let $m, b \geq 1$. Then for any $q \geq (m-1) \cdot \binom{m \cdot b(m-1)}{2b(m-1)} \cdot n^{bm-b} + n^{b-1}$, there exists an MRC $\mathcal{C}$ that instantiates the topology $T_{m\times n}(1, b, 0)$ over the field $\mathbb{F}_q$.*

In order to do this, we will exhibit a column code $\mathcal{C}_{col}$ and a row code $\mathcal{C}_{row}$ over a relative small field, so that for every correctable irreducible erasure pattern E, the code $\mathcal{C}_{col} \otimes \mathcal{C}_{row}$ can correct E. Thus the tensor product code $\mathcal{C} = \mathcal{C}_{col} \otimes \mathcal{C}_{row}$ is an MRC that instantiates the topology $T_{m\times n}(1, b, 0)$. We need the following lemma known as the Combinatorial Nullstellensatz.

Lemma 5.5. *(Combinatorial Nullstellensatz) [9] Let $\mathbb{F}$ be an arbitrary field, let $P \in \mathbb{F}[t_1, \ldots, t_n]$ be a polynomial of degree d which contains a non-zero coefficient at $t_1^{d_1} \cdots t_n^{d_n}$ with $d_1 + \cdots + d_n = d$, and let $S_1, \ldots, S_n$ be subsets of $\mathbb{F}$ such that $|S_i| > d_i$ for all $1 \leq i \leq n$. Then there exist $x_1 \in S_1, \ldots, x_n \in S_n$ such that $P(x_1, \ldots, x_n) \neq 0$.*

Proof of Theorem 5.15. Since the case $m = 1$ is trivial, w.l.o.g., we assume $m \geq 2$. For simplicity, we fix $\mathcal{C}_{col}$ as the simple parity code $\mathcal{P}_m$ and focus on obtaining the code $\mathcal{C}_{row}$.

Denote $\mathcal{E}$ as the set of all the types of regular irreducible erasure patterns for topology $T_{m\times n}(1, b, 0)$. Assume the parity check matrix of the code $\mathcal{C}_{row}$ is $\mathbf{H}_{row}$, then the *pseudo-parity check matrix* $\mathbf{H}$ is of the form in (5.14). Thus, our goal is to construct a $b \times n$ matrix $\mathbf{H}_{row}$ such that:

(i) Every b distinct columns of $\mathbf{H}_{row}$ are linearly independent.

(ii) For each regular irreducible erasure pattern E, the *pseudo-parity check matrix* $\mathbf{H} \in \mathbb{F}_q^{(n+bm)\times mn}$ of $\mathcal{C}$ satisfies: $rank(\mathbf{H}|_E) = |E|$.

The requirement (i) is to guarantee that the code $\mathcal{C}_{row}$ is an $[n, n-b]$-MDS code and by Proposition 3.2, the requirement (ii) guarantees that $\mathcal{C}$ can correct all regular irreducible erasure patterns.

Given a regular irreducible erasure pattern $E \in [m] \times [n]$, w.l.o.g., assume $U_E = [u_0] \subseteq [m]$ and $V_E = [v_0] \subseteq [n]$, then E has the form

$$E = \begin{pmatrix} * & * & * & * & \cdots & \circ & \circ \\ * & * & * & \circ & \cdots & * & \circ \\ \circ & * & \circ & * & \cdots & * & \circ \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \circ & \circ & \circ & * & \cdots & * & * \end{pmatrix} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ \vdots \\ E_{u_0} \end{pmatrix},$$

where E_i represents the sub-erasure pattern of E over the i_{th} row. Thus

$$\mathbf{H}|_E = \begin{pmatrix} \mathbf{I}_n|_{E_1} & \mathbf{I}_n|_{E_2} & \cdots & \mathbf{I}_n|_{E_{u_0}} \\ \mathbf{H}_{row}|_{E_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{row}|_{E_2} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_{row}|_{E_{u_0}} \end{pmatrix} = \begin{pmatrix} \mathbf{H}_1|_E \\ \mathbf{H}_2|_E \end{pmatrix}.$$

Let $supp(E_i) = \{j \in [n] : E_i(j) = *\}$. Since $\bigcup_{i=1}^{u_0} supp(E_i) = [v_0]$, by applying elementary row and column transformations, we have

$$\mathbf{H}|_E = \begin{pmatrix} \mathbf{I}_{v_0} & \mathbf{0}_{v_0 \times (|E|-v_0)} \\ \mathbf{A}_{u_0 b \times v_0} & \mathbf{B}_{u_0 b \times (|E|-v_0)} \\ \mathbf{0}_{(n-v_0)\times(v_0)} & \mathbf{0}_{(n-v_0)\times(|E|-v_0)} \end{pmatrix}, \tag{5.49}$$

where $\mathbf{A}$ consists of all the columns in $\mathbf{H}_2|_E$ corresponding to an $\mathbf{I}_{v_0}$ in $\mathbf{H}_1|_E$ and $\mathbf{B}$ consists of all the rest columns in $\mathbf{H}_2|_E$ by substituting columns of $\mathbf{A}$ with the same parts in $\mathbf{H}_1|_E$. Thus a non-zero element of $\mathbf{A}$ equals to some h_{ij} in $\mathbf{H}_{row}$ and a non-zero element of $\mathbf{B}$ equals to h_{ij} or $-h_{ij}$ for some h_{ij} in $\mathbf{H}_{row}$. For example, take $U_E = V_E = \{1, 2, 3\}$, $\mathbf{H}_{row} = (\mathbf{h}_1, \ldots, \mathbf{h}_n)$ and

$$E = \begin{pmatrix} * & * & \circ \\ * & \circ & * \\ \circ & \circ & * \end{pmatrix},$$

then

$$\mathbf{H}|_E = \begin{pmatrix} 1 & & 1 & & \\ & 1 & & & \\ & & & 1 & 1 \\ \mathbf{h}_1 & \mathbf{h}_2 & & & \\ & & \mathbf{h}_1 & \mathbf{h}_3 & \\ & & & & \mathbf{h}_3 \\ \multicolumn{3}{c}{\mathbf{0}_{(n-3)\times 3}} & \multicolumn{2}{c}{\mathbf{0}_{(n-3)\times 2}} \end{pmatrix} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ \mathbf{h}_1 & \mathbf{h}_2 & & -\mathbf{h}_1 & \\ & & \mathbf{h}_3 & \mathbf{h}_1 & -\mathbf{h}_3 \\ & & & & \mathbf{h}_3 \\ \multicolumn{3}{c}{\mathbf{0}_{(n-3)\times 3}} & \multicolumn{2}{c}{\mathbf{0}_{(n-3)\times 2}} \end{pmatrix}.$$

From the above simplified form of $\mathbf{H}|_E$ in (5.49), we have

$$\begin{aligned} rank(\mathbf{H}|_E) &= rank(\mathbf{I}_{v_0}) + rank(\mathbf{B}_{u_0 b \times (|E|-v_0)}) \\ &= v_0 + rank(\mathbf{B}_{u_0 b \times (|E|-v_0)}). \end{aligned}$$

By Definition 5.7 and (5.16), we have $|E| \leq v_0 + u_0 b - b$. Thus $rank(\mathbf{H}|_E) = |E|$ if and only if there exists an $(|E|-v_0)\times(|E|-v_0)$ minor $\mathbf{B}'$ in $\mathbf{B}$ such that $det(\mathbf{B}') \neq 0$.

Now, take

$$\mathbf{H}_{row} = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2n} \\ x_{31} & x_{32} & x_{33} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{b1} & x_{b2} & x_{b3} & \cdots & x_{bn} \end{pmatrix},$$

where each x_{ij} is a variable over $\mathbb{F}_q$. Therefore, our goal is to find a proper valuation of these $x'_{ij}s$ over $\mathbb{F}_q$ such that the resulting matrix $\mathbf{H}_{row}$ satisfies both requirement (i) and requirement (ii).

- For requirement (i)

For any $J = \{j_1, \ldots, j_b\} \subseteq [n]$, let $\mathbf{M}_J$ be the $b \times b$ submatrix of $\mathbf{H}_{row}$ formed by the b columns indicated by J, i.e.,

$$\mathbf{M}_J = \begin{pmatrix} x_{1j_1} & x_{1j_2} & x_{1j_3} & \cdots & x_{1j_b} \\ x_{2j_1} & x_{2j_2} & x_{2j_3} & \cdots & x_{2j_b} \\ x_{3j_1} & x_{3j_2} & x_{3j_3} & \cdots & x_{3j_b} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{bj_1} & x_{bj_2} & x_{bj_3} & \cdots & x_{bj_b} \end{pmatrix}.$$

Define

$$P = \prod_{J \in \binom{[n]}{b}} det(\mathbf{M}_J).$$

Since each $det(\mathbf{M}_J)$ is a homogeneous polynomial of degree b, we know that P is a homogeneous polynomial of degree $b\binom{n}{b}$, and each variable x_{ij} has degree at most $\binom{n-1}{b-1}$. According to the definition of P, if there is a valuation $(h_{11}, \ldots, h_{bn})$ of $(x_{11}, \ldots, x_{bn})$ such that $P(h_{11}, \ldots, h_{bn}) \neq 0$, then the resulting matrix $\mathbf{H}_{row} = (h_{ij})_{i\in[b], j\in[n]}$ satisfies requirement (i).

- For requirement (ii)

For each regular irreducible erasure pattern $E \in [m] \times [n]$, set $|U_E| = u_0$ and $|V_E| = v_0$ and consider the $u_0 b \times (|E| - v_0)$ submatrix $\mathbf{B}(E)$ of $\mathbf{H}|_E$ in (5.49). For each $(|E| - v_0) \times (|E| - v_0)$ minor $\mathbf{B}'(E)$ in $\mathbf{B}(E)$, $det(\mathbf{B}'(E))$ can be viewed as a multi-variable polynomial in $\mathbb{F}_q[x_{11}, \ldots, x_{bn}]$ with degree at most $|E| - v_0$. Since each non-zero element of $\mathbf{B}$ equals to x_{ij} or $-x_{ij}$ for some x_{ij} in $\mathbf{H}_{row}$, and each variable x_{ij} appears in at most $u_0 - 1$ columns of $\mathbf{B}$, thus for each minor $\mathbf{B}'(E)$ we have

$$det(\mathbf{B}'(E)) = \sum_{\substack{\sum_{1\le i\le b, 1\le j\le n} a_{ij} = |E|-v_0, \\ 0 \le a_{ij} \le u_0 - 1}} c_{(a_{11},\ldots,a_{bn})} \cdot x_{11}^{a_{11}} x_{12}^{a_{12}} \cdots x_{bn}^{a_{bn}}, \tag{5.50}$$

where $c_{(a_{11},\ldots,a_{bn})}$ equals to $0, 1$ or -1.

Since elementary row and column transformations don't change the form of $\mathbf{H}|_E$ (In fact, these transformations only affect the indices of $\mathbf{h}_i$s, and therefore, switch $x_{i,j}$ to $x_{i,j'}$ for $j' \neq j \in [n]$), the structure of $\mathbf{B}'(E)$ is determined by the type of the erasure pattern. Therefore, once E is given, for each minor $\mathbf{B}'(E)$ in $\mathbf{B}(E)$, $det(\mathbf{B}'(E))$ can be viewed as a polynomial in $\mathbb{F}_2[x_{11}, \ldots, x_{bn}]$ with a fixed form.

Since for each regular irreducible erasure pattern E, $|E| \le v_0 + bm - b$. Thus, when $q > bm - b \ge \deg(det(\mathbf{B}'(E)))$, $det(\mathbf{B}'(E))|_{\mathbb{F}_q^{(bn)}} \equiv 0$ if and only if $det(\mathbf{B}'(E)) = \mathbf{0}$ (i.e. the zero polynomial).

According to the proof of Theorem 5.2 in [93], when the size of the field is large enough, there exists a code $\mathcal{C}_0$ such that the tensor product code $\mathcal{C} = \mathcal{P}_m \otimes \mathcal{C}_0$ can correct E. This means that there exists a valuation of the bn variables in $\mathbf{H}_{row}$ such that $det(\mathbf{B}'(E)) \neq 0$ for some $(|E| - |V_E|) \times (|E| - |V_E|)$ minor $\mathbf{B}'(E)$ in $\mathbf{B}(E)$. By this, we know that the multi-variable polynomial $det(\mathbf{B}'(E))$ corresponding to this minor $\mathbf{B}'(E)$ can not be zero polynomial. From the previous analysis, we know that the form of this polynomial $det(\mathbf{B}'(E))$ is irrelevant to the size of the field. Therefore, for any $q > bm - b$ as a power of 2, this $det(\mathbf{B}'(E))$ is a non-zero polynomial in $\mathbb{F}_q[x_{11}, \ldots, x_{bn}]$.

For each regular irreducible erasure pattern E, denote f_E as the non-zero determinant polynomial corresponding to some $(|E| - |V_E|) \times (|E| - |V_E|)$ minor $\mathbf{B}'(E)$

in $\mathbf{B}(E)$. Define

$$F = \prod_{\substack{E \in [m]\times[n], \\ E \text{ is a regular irreducible erasure pattern}}} f_E. \tag{5.51}$$

Similarly, if there is a valuation $(h_{11}, \ldots, h_{bn})$ of $(x_{11}, \ldots, x_{bn})$ such that $F(h_{11}, \ldots, h_{bn}) \neq 0$, then the resulting matrix $\mathbf{H}_{row} = (h_{ij})_{i\in[b], j\in[n]}$ satisfies requirement (ii).

In order to apply the Combinatorial Nullstellensatz, we shall estimate the degree of each variable in F. Noted that

$$F = \prod_{E^* \in \mathcal{E}} \prod_{\substack{E \text{ is a regular irreducible} \\ \text{erasure pattern of the same type with } E^*}} f_E, \tag{5.52}$$

and for each $E^* = (E_1, E_2, \ldots, E_m)^T \in \mathcal{E}$, we can find at most $\binom{n}{|V_E|} \leq \binom{n}{bm-b}$ different regular irreducible erasure patterns of type E^* in $[m] \times [n]$. By (5.50), for every regular irreducible erasure pattern E, we have the degree of each variable x_{ij} in f_E to be at most $m-1$. Therefore, the degree of each variable x_{ij} in F is at most $(m-1) \cdot |\mathcal{E}| \cdot \binom{n}{bm-b}$.

Now, consider the polynomial $P \cdot F$, by Lemma 5.5, there is a valuation $(h_{11}, \ldots, h_{bn})$ of $(x_{11}, \ldots, x_{bn})$ over a field $\mathbb{F}_q$ of size

$$q = (m-1) \cdot \binom{m \cdot b(m-1)}{2b(m-1)} \cdot n^{bm-b} + n^{b-1}$$

such that $P \cdot F(h_{11}, \ldots, h_{bn}) \neq 0$. Therefore, the corresponding matrix $\mathbf{H}_{row} = (h_{ij})_{i\in[b], j\in[n]}$ is the objective matrix satisfying both requirement (i) and requirement (ii). This completes the proof. □

Remark 5.9. *In the proof above, according to the property of regular irreducible erasure patterns, we can bound the degree of each variable in F. During this bounding process, we also showed that* $\deg(f_E) \leq b(m-1)$*, which will lead to an upper bound on* $\deg(F)$*. Based on this, one can apply Schwartz-Zippel lemma with $P \cdot F$ and obtain $q \leq (bm-b) \cdot \binom{m\cdot b(m-1)}{2b(m-1)} \cdot n^{bm-b} + n^{b-1}$, which is slightly weaker than the bound given in Theorem 5.15.*

Remark 5.10. *Considering the MRCs for topologies $T_{m\times n}(1,b,0)$, the general bound given by Gopalan et al. [91] is*

$$q > (n+bm-b)\cdot\binom{mn}{\le n+bm-b} = \Omega((n+bm-b)^2(\frac{mn}{n+bm-b})^{(n+bm-b)}), \quad (5.53)$$

which is exponentially increasing for both m and n, while the bound given by Theorem 5.15 is only a polynomial of n.

But, even so, when considering the growth rate corresponding to m,

$$q = (m-1)\cdot\binom{m\cdot b(m-1)}{2b(m-1)}\cdot n^{bm-b} + n^{b-1} = \Omega(m^{2b(m-1)+m}n^{bm-b})$$

grows exponentially.

Actually, m is often considered as the number of data centers in practice, which is very small compared to n. Therefore, the when $n \gg m$, the bound given by Theorem 5.15 is better than that in [91].

§ 5.7 MRCs for topologies $T_{m\times n}(1,2,0)$ and $T_{m\times n}(1,3,0)$

In this section, for topology $T_{m\times n}(1,2,0)$, we give a full characterization of the regular irreducible erasure patterns when $m=4$ and obtain an improved upper bound on the field size for MRCs instantiating topology $T_{4\times n}(1,2,0)$. Based on a special type of regular irreducible erasure patterns, we prove a polynomial lower bound for general m. For topology $T_{m\times n}(1,3,0)$, with the same method, we obtain an improved upper bound on the field size for MRCs instantiating topology $T_{3\times n}(1,3,0)$.

5.7.1 MRCs for topologies $T_{m\times n}(1,2,0)$

In this part, first, using the results from Section 3.2, we will give a characterization of the regular irreducible erasure patterns for topology $T_{m\times n}(1,2,0)$. For $m=4$, all types of regular irreducible erasure patterns are determined. For each $m\geq 5$, we find a particular type of regular irreducible erasure patterns, which is crucial for the proof of the lower bound.

Denote $\mathcal{E}$ as the set of all the types of regular irreducible erasure patterns for topology $T_{m\times n}(1,2,0)$. For each $E \in \mathcal{E}$, by (5.15) and (5.16), we have $|U_E| + 2 \leq |V_E| \leq 2|U_E| - 2$ and $|E| \leq |V_E| + 2|U_E| - 2$. Therefore, $|U_E| \geq 4$ and by $U_E \subseteq [m]$, we have $m \geq 4$.

From the irreducibility, each erasure pattern has at least 2 erasures in each column and at least 3 erasures in each row. Noted that the more erasures each row (column) contains, the harder the corresponding erasure pattern can meet the regularity. In this spirit, we find the following 2 types of erasure patterns for $m = 4$.

- Type I

$$E_1 = \begin{pmatrix} * & * & * & \circ & \circ & \circ \\ * & * & \circ & * & \circ & \circ \\ \circ & \circ & * & \circ & * & * \\ \circ & \circ & \circ & * & * & * \end{pmatrix}_{4\times 6},$$

- Type II

$$E_2 = \begin{pmatrix} * & * & * & \circ & \circ & \circ \\ * & \circ & \circ & * & * & \circ \\ \circ & * & \circ & * & \circ & * \\ \circ & \circ & * & \circ & * & * \end{pmatrix}_{4\times 6}.$$

Proposition 5.4. *The above* 2 *types of erasure patterns are regular and irreducible for corresponding topology* $T_{m\times n}(1,2,0)$.

Proof. Noted that the irreducibility follows from that all these erasure patterns have at least 2 erasures in each column and at least 3 erasures in each row, thus we only have to prove the regularity. We just prove the regularity of erasure patterns of Type I here, the proof for the other type is the same.

Recall that erasure pattern E is called regular if for all $U \subseteq [m]$ and $V \subseteq [n]$, we have

$$|E \cap (U \times V)| \leq |V| + 2|U| - 2.$$

Since each submatrix indexed by $U \times V \subseteq [m] \times [n]$ can be obtained by executing first $m - |U|$ row deletions and then $n - |V|$ column deletions, and the violation of the above inequality only can occur when E has the maximum density in $[m] \times [n]$. Therefore, w.l.o.g., for erasure patterns of Type I, assume that $m = 4$ and $n = 6$.

Consider the row and column deletion process in E_1. Every deletion of $4 - |U|$ rows leads to a decrease of $12 - 3|U|$ in the LHS and $8 - 2|U|$ in the RHS, thus to maintain the inequality, the following column deletions can contribute at most $4 - |U|$ more decreases in the LHS. Noted that each pair of rows shares at most 2 erasures having common coordinates. Therefore, for $|U| \leq 3$, the column deletions can contribute at most $4 - |U|$ more decreases in the LHS than that in the RHS. Therefore, the above inequality always holds and this leads to the regularity of erasure patterns of Type I. □

Remark 5.11. *When $m > 4$, since $4 \leq |U_E| \leq m$ and $|U_E| + 2 \leq |V_E| \leq 2|U_E| - 2$, the type of erasure patterns varies with the growing of $|U_E|$. In the same spirit, one can construct similar regular irreducible erasure patterns for $m = 5, 6, 7$. And with the help of Proposition 5.3, one can extend these constructions recursively for general m.*

5.7.1.1 Improved upper bound for topologies $T_{4\times n}(1, 2, 0)$

Now, we are going to prove the following existence result for MRCs instantiating the topology $T_{4\times n}(1, 2, 0)$, which improves the general upper bound from Theorem 5.15 for this special topology.

Theorem 5.16. *For any $q > \frac{n^5}{\log(n)} \cdot C_1$, there exists an MRC $\mathcal{C}$ that instantiates the topology $T_{4\times n}(1, 2, 0)$ over the field $\mathbb{F}_q$, where $C_1 \geq (\frac{10}{c_5})^5$ is an absolute constant.*

Proof. Similar to the proof of Theorem 5.15, let $\mathcal{C}_{col}$ be the simple parity code $\mathcal{P}_4$. Our goal is to construct a $2 \times n$ matrix $\mathbf{H}_{row}$ such that:

(i) Every 2 distinct columns of $\mathbf{H}_{row}$ are linearly independent.

(ii) For each regular irreducible erasure pattern E of Type I or Type II, the *pseudo-parity check matrix* $\mathbf{H} \in \mathbb{F}_q^{(n+8)\times 4n}$ of $\mathcal{P}_4 \otimes \mathcal{C}_{row}$ satisfies: $rank(\mathbf{H}|_E) = 12$.

Different from the general strategy, we are going to obtain an objective matrix based on the Vandermonde matrix.

Suppose there exists an objective matrix $\mathbf{A}_0$ of the form

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix},$$

where $\{a_i\}_{i\in[n]}$ are pairwise distinct elements in $\mathbb{F}_q$. Then the distinctness of $\{a_i\}_{i\in[n]}$ guarantees that $\mathbf{A}_0$ satisfies (i).

Now take $\mathbf{H}_{row} = \mathbf{A}_0$ and consider the *pseudo-parity check matrix* $\mathbf{H}_{\mathbf{A_0}}$. For each $s \in [2]$, we have

$$\mathbf{H}_{\mathbf{A_0}}|_{E_s} = \begin{pmatrix} \mathbf{I}_6 & \mathbf{0}_{6\times 6} \\ \mathbf{A}^{(s)}_{8\times 6} & \mathbf{B}^{(s)}_{8\times 6} \\ \mathbf{0}_{(n-6)\times 6} & \mathbf{0}_{(n-6)\times 6} \end{pmatrix},$$

where

$$\mathbf{A}^{(1)} = \begin{pmatrix} 1 & 1 & 1 & & & \\ a_1 & a_2 & a_3 & & & \\ & & & 1 & & \\ & & & a_4 & & \\ & & & & 1 & 1 \\ & & & & a_5 & a_6 \\ \multicolumn{6}{c}{\mathbf{0}_{2\times 6}} \end{pmatrix} \text{ and } \mathbf{A}^{(2)} = \begin{pmatrix} 1 & 1 & 1 & & & \\ a_1 & a_2 & a_3 & & & \\ & & & 1 & 1 & \\ & & & a_4 & a_5 & \\ & & & & & 1 \\ & & & & & a_6 \\ \multicolumn{6}{c}{\mathbf{0}_{2\times 6}} \end{pmatrix},$$

$$\mathbf{B}^{(1)} = \begin{pmatrix} -1 & -1 & -1 & & & \\ -a_1 & -a_2 & -a_3 & & & \\ 1 & 1 & & -1 & & \\ a_1 & a_2 & & -a_4 & & \\ & & 1 & & -1 & -1 \\ & & a_3 & & -a_5 & -a_6 \\ & & & 1 & 1 & 1 \\ & & & a_4 & a_5 & a_6 \end{pmatrix}$$

and

$$\mathbf{B}^{(2)} = \begin{pmatrix} -1 & -1 & -1 & & & \\ -a_1 & -a_2 & -a_3 & & & \\ 1 & & & -1 & -1 & \\ a_1 & & & -a_4 & -a_5 & \\ & 1 & & 1 & & -1 \\ & a_2 & & a_4 & & -a_6 \\ & & 1 & & 1 & 1 \\ & & a_3 & & a_5 & a_6 \end{pmatrix}.$$

Since $\mathbf{B}^{(s)}$ can be simplified as

$$\mathbf{B}^{(1)} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & a_2 - a_1 & a_4 - a_1 & \\ & & & & a_3 - a_4 & \\ & & & & a_5 - a_4 & a_6 - a_5 \\ & & & \mathbf{0}_{2\times 6} & & \end{pmatrix}$$

and

$$\mathbf{B}^{(2)} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & a_1 - a_4 & a_1 - a_5 & \\ & & & a_4 - a_2 & & a_2 - a_6 \\ & & & & a_5 - a_3 & a_6 - a_3 \\ & & & \mathbf{0}_{2\times 6} & & \end{pmatrix},$$

we have

- $rank(\mathbf{B}^{(1)}) = 6$ if and only if $(a_2 - a_1)(a_4 - a_3)(a_6 - a_5) \neq 0$.

- $rank(\mathbf{B}^{(2)}) = 6$ if and only if $(a_1 - a_4)(a_2 - a_6)(a_3 - a_5) - (a_2 - a_4)(a_1 - a_5)(a_3 - a_6) \neq 0$.

Take $f(x_1, x_2, \ldots, x_6) = (x_1 - x_4)(x_2 - x_6)(x_3 - x_5) - (x_2 - x_4)(x_1 - x_5)(x_3 - x_6)$, we have $deg(f) = 3$. From the assumption that $\{a_i\}_{i\in[n]}$ are pairwise distinct, we know that erasure patterns of Type I can be easily corrected. Then if we also want to correct all erasure patterns of Type II, $\{a_i\}_{i\in[n]}$ only need to have the property that for any $\{a_{i_1}, a_{i_2}, \ldots, a_{i_6}\} \subseteq \{a_i\}_{i\in[n]}$ and each $\pi \in S_6$, $f(a_{i_{\pi(1)}}, \ldots, a_{i_{\pi(6)}}) \neq 0$.

Different from the proof of Theorem 5.15, here we use the hypergraph independent set approach.

Let $\mathcal{H}$ be a 6-uniform hypergraph with vertex set $\mathbb{F}_q$, each set of 6 vertices $\{v_1, \ldots, v_6\}$ forms a 6-hyperedge if and only if $f(v_{\pi(1)}, \ldots, v_{\pi(6)}) = 0$ for some $\pi \in S_6$. From the construction of the hypergraph $\mathcal{H}$, if there exists an independent set I such that $|I| \geq n$, then we can construct an objective matrix $\mathbf{A}_0$ by arbitrarily choosing n different vertices from I as elements for its 2_{nd} row.

Since $deg_{x_i}(f) = 1$ for each x_i, and $f(v_1, \ldots, x_i, \ldots, v_6)$ is a non-zero polynomial for any 5-subset $\{v_j\}_{j\in[6]\setminus\{i\}} \subseteq \mathbb{F}_q$. Thus the maximal 5-degree of $\mathcal{H}$ $\Delta_5(\mathcal{H}) \leq 6!$. By Theorem 5.3,

$$\alpha(\mathcal{H}) \geq \frac{c_5}{5}(q \log q)^{\frac{1}{5}} > n.$$

Denote $I(\mathcal{H})$ as the maximum independent set in $\mathcal{H}$, therefore, there exists a subset $A = \{a_1 \ldots, a_n\} \subseteq \mathbb{F}_q$ such that the matrix $\mathbf{A}_0$ of the form

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \end{pmatrix}$$

satisfies both (i) and (ii). Thus, the resulting tensor product code $\mathcal{C} = \mathcal{P}_4 \otimes \mathcal{C}_{row}$ is an MRC instantiating topology $T_{4\times n}(1, 2, 0)$. □

5.7.1.2 Lower bound for topologies $T_{m\times n}(1, 2, 0)$

The above theorem says that for any $q > \frac{n^5}{\log(n)} \cdot C_0$, there exists an MRC $\mathcal{C}$ for topology $T_{4\times n}(1, 2, 0)$ over $\mathbb{F}_q$. This actually gives an upper bound $\frac{n^5}{\log(n)} \cdot C_0$ on the minimal field size required for the existence of an MRC. But is this polynomial trend really necessary? Recall the *MDS Conjecture*:

MDS Conjecture. *If there is a nontrivial* $[n, k]$ *MDS code over* $\mathbb{F}_q$*, then* $n \le q+1$*, except when* q *is even and* $k = 3$ *or* $k = q-1$ *in which case* $n \le q+2$*.*

Since the code $\mathcal{C}_{row}$ is always an MDS code, thus from the *MDS Conjecture* we know that a linear lower bound is necessary, but will it be sufficient? Sadly not. The next theorem gives a quadratic lower bound on the smallest field size required for the existence of an MRC for the topology $T_{m\times n}(1,2,0)$ ($m \ge 4$ and $n \ge 6$).

Theorem 5.17. *If* $q < \frac{(n-3)^2}{4} + 2$*, then for any tensor product code* $\mathcal{C} = \mathcal{C}_{col} \otimes \mathcal{C}_{row}$ *over* $\mathbb{F}_q$ *with* $\mathcal{C}_{col}$ *as an* $[m, m-1, 2]$ *MDS code* ($m \ge 4$) *and* $\mathcal{C}_{row}$ *as an* $[n, n-2, 3]$ *MDS code* ($n \ge 6$)*,* $\mathcal{C}$ *can not be an MRC that instantiates the topology* $T_{m\times n}(1,2,0)$*.*

To present the proof, we need the following two propositions:

Proposition 5.5. *Take* $\omega \in \mathbb{F}_q^*$ *as the primitive element. If there exist six distinct* $t_i \in \mathbb{Z}_{q-1}$ *such that* $t_1+t_6 = t_2+t_5 = t_3+t_4$*, then the polynomial* $f(x_1, x_2, \ldots, x_6) = (x_1-x_4)(x_2-x_6)(x_3-x_5) - (x_2-x_4)(x_1-x_5)(x_3-x_6)$ *has a zero of the form* $(\omega^{t_1}, \ldots, \omega^{t_6})$*.*

Proof. By substituting $(\omega^{t_1}, \ldots, \omega^{t_6})$ to $f(x_1, x_2, \ldots, x_6)$ directly, we have

$$\begin{aligned} &(\omega^{t_1}-\omega^{t_5})(\omega^{t_2}-\omega^{t_4})(\omega^{t_3}-\omega^{t_6}) - (\omega^{t_1}-\omega^{t_4})(\omega^{t_2}-\omega^{t_6})(\omega^{t_3}-\omega^{t_5}) \\ =&\omega^{t_1+t_2+t_3}[(1-\omega^{t_5-t_1})(1-\omega^{t_4-t_2})(1-\omega^{t_6-t_3}) - (1-\omega^{t_6-t_2})(1-\omega^{t_5-t_3})(1-\omega^{t_4-t_1})]. \end{aligned}$$

Since $t_1+t_6 = t_2+t_5 = t_3+t_4$, then we have

$$\begin{cases} t_5 - t_1 = t_6 - t_2 \\ t_4 - t_2 = t_5 - t_3 \\ t_6 - t_3 = t_4 - t_1 \end{cases}.$$

Using these three identities, we have $f(\omega^{t_1}, \ldots, \omega^{t_6}) = 0$. □

Let $N \geq 2$ be a positive integer, for any subset $A \subseteq \mathbb{Z}_N$, we say A is a *2-Sidon set* if for any 2-subset $\{a_1, b_1\} \subseteq A$ there exists at most one other $\{a_2, b_2\} \subseteq A$ different from $\{a_1, b_1\}$ such that $a_1 + b_1 = a_2 + b_2$.

Proposition 5.6. *For any $A \subseteq \mathbb{Z}_N$, if A is a* 2-Sidon set*, then we have* $|A| \leq 2\sqrt{N} + 1$.

Proof. Since $A + A \subseteq \mathbb{Z}_N$, by a simple double counting, we have

$$\binom{|A|}{2} \leq 2N.$$

Thus $|A| \leq 2\sqrt{N} + 1$. □

Proof of Theorem 5.17. Different from the proof of the upper bound, since we want to obtain a necessary condition for the existence of an MRC, we have to deal with the general case.

For any $[m, m-1, 2]$ MDS code $\mathcal{C}_1$ and $[n, n-2, 3]$ MDS code $\mathcal{C}_2$, take

$$\mathbf{H}_1 = (a_1, a_2, \ldots, a_m) \text{ and } \mathbf{H}_2 = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \end{pmatrix}$$

as their parity check matrices. Then the *pseudo-parity check matrix* of $\mathcal{C} = \mathcal{C}_1 \otimes \mathcal{C}_2$ has the following form

$$\mathbf{H} = \begin{pmatrix} a_1 \cdot \mathbf{I}_n & a_2 \cdot \mathbf{I}_n & \cdots & a_m \cdot \mathbf{I}_n \\ \mathbf{H}_2 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{H}_2 \end{pmatrix}.$$

Take $\mathbf{H}_1 = (a_1, a_2, \ldots, a_m)$ as a vector in $\mathbb{F}_q$ and consider its Hamming weight $w(\mathbf{H}_1)$. Since $\mathcal{C}_1$ is an $[m, m-1, 2]$ MDS code, we have $w(\mathbf{H}_1) = m$. Therefore, $a_i \neq 0$ for each $i \in [m]$.

Now, consider erasure patterns of type II, we have

$$\mathbf{H}|_{E_2} = \begin{pmatrix} \mathbf{A'}_{6\times 6} & \mathbf{0}_{6\times 6} \\ \mathbf{A}_{8\times 6} & \mathbf{B}_{8\times 6} \\ \mathbf{0}_{(n-6)\times 6} & \mathbf{0}_{(n-6)\times 6} \end{pmatrix},$$

where

$$\mathbf{A}' = \begin{pmatrix} a_1' & & & & & \\ & a_1' & & & & \\ & & a_1' & & & \\ & & & a_2' & & \\ & & & & a_2' & \\ & & & & & a_3' \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & & & \\ & & & \beta_4 & \beta_5 & \\ & & & & & \beta_6 \\ & & & \mathbf{0}_{2\times 6} & & \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} -\frac{a_2'}{a_1'}\beta_1 & -\frac{a_3'}{a_1'}\beta_2 & -\frac{a_4'}{a_1'}\beta_3 & & & \\ \beta_1 & & & -\frac{a_3'}{a_2'}\beta_4 & -\frac{a_4'}{a_2'}\beta_5 & \\ & \beta_2 & & \beta_4 & & -\frac{a_4'}{a_3'}\beta_6 \\ & & \beta_3 & & \beta_5 & \beta_6 \end{pmatrix},$$

for some $\{a_1', a_2', a_3', a_4'\} \subseteq \{a_1, a_2, \ldots, a_m\}$ and column vectors $\{\beta_1, \ldots, \beta_6\} \subseteq \mathbf{H}_2$ corresponding to E_2.

It can be easily verified that the first row of $\mathbf{B}$ can be linearly expressed by the other three rows. Thus $rank(\mathbf{B}) = 6$ if and only if the following system of linear equations only have zero solutions.

$$\begin{cases} x_1 \cdot \beta_1 - x_4 \cdot \frac{a_3'}{a_2'}\beta_4 - x_5 \cdot \frac{a_4'}{a_2'}\beta_5 = 0 \\ x_2 \cdot \beta_2 + x_4 \cdot \beta_4 - x_6 \cdot \frac{a_4'}{a_3'}\beta_6 = 0 \\ x_3 \cdot \beta_3 + x_5 \cdot \beta_5 + x_6 \cdot \beta_6 = 0 \end{cases} \tag{5.54}$$

For (5.54), it has non-zero solution $(d_1, \ldots, d_6)$ which does not violate the MDS property of $\mathbf{H}_2$. For example, take $\omega \in \mathbb{F}_q^*$ as the primitive element, if $\beta_i = (1, \omega^{t_i})$ for some distinct $t_i \in [q-1]$ such that $t_1+t_6 = t_2+t_5 = t_3+t_4$, then the resulting $\mathbf{B}_2$ has $rank(\mathbf{B}_2) \leq 5$ and this guarantees the existence of non-zero solution for (5.54).

W.o.l.g., assume $n \geq 8$, then from the MDS property of $\mathbf{H}_2$, we know that $\mathbf{H}_2$ contains at least $n-2$ weight-2 columns. Since any six distinct elements of $[n]$ can be chosen to form an erasure pattern E_2 of Type II, therefore, the maximal recoverability requires that $rank(\mathbf{B}) = 6$ for any six distinct columns in $\mathbf{H}_2$. Especially, we can take all these columns with weight equal to 2. Assume $\beta_i = (b_{i1}, b_{i2})^T$ with $b_{is} \neq 0$ for each $i \in [6], s \in [2]$. Then $\mathbf{B}$ can be formulated as

$$\mathbf{B} = \begin{pmatrix} -\begin{pmatrix} \frac{a_2' b_{11}}{a_1} \\ \frac{a_2' b_{11} r_1}{a_1} \end{pmatrix} & -\begin{pmatrix} \frac{a_3' b_{21}}{a_1} \\ \frac{a_3' b_{21} r_2}{a_1} \end{pmatrix} & -\begin{pmatrix} \frac{a_4' b_{31}}{a_1} \\ \frac{a_4' b_{31} r_3}{a_1} \end{pmatrix} & & & \\ \begin{pmatrix} b_{11} \\ b_{11} r_1 \end{pmatrix} & & & -\begin{pmatrix} \frac{a_3' b_{41}}{a_2} \\ \frac{a_3' b_{41} r_4}{a_2} \end{pmatrix} & -\begin{pmatrix} \frac{a_4' b_{51}}{a_2} \\ \frac{a_4' b_{51} r_5}{a_2} \end{pmatrix} & \\ & \begin{pmatrix} b_{21} \\ b_{21} r_2 \end{pmatrix} & & \begin{pmatrix} b_{41} \\ b_{41} r_4 \end{pmatrix} & & -\begin{pmatrix} \frac{a_4' b_{61}}{a_3} \\ \frac{a_4' b_{61} r_6}{a_3} \end{pmatrix} \\ & & \begin{pmatrix} b_{31} \\ b_{31} r_3 \end{pmatrix} & & \begin{pmatrix} b_{51} \\ b_{51} r_5 \end{pmatrix} & \begin{pmatrix} b_{61} \\ b_{61} r_6 \end{pmatrix} \end{pmatrix},$$

where $r_i = \frac{b_{i2}}{b_{i1}}$ for each $i \in [6]$. Since the first row of $\mathbf{B}$ can be linearly expressed by the other three rows and the scaling of each column doesn't affect the linear dependency, we have $rank(\mathbf{B}) = rank(\tilde{\mathbf{B}})$, where

$$\tilde{\mathbf{B}} = \begin{pmatrix} \begin{pmatrix} 1 \\ r_1 \end{pmatrix} & & & -\frac{a_3'}{a_2'} \cdot \begin{pmatrix} 1 \\ r_4 \end{pmatrix} & -\frac{a_4'}{a_2'} \cdot \begin{pmatrix} 1 \\ r_5 \end{pmatrix} & \\ & \begin{pmatrix} 1 \\ r_2 \end{pmatrix} & & \begin{pmatrix} 1 \\ r_4 \end{pmatrix} & & -\frac{a_4'}{a_3'} \cdot \begin{pmatrix} 1 \\ r_6 \end{pmatrix} \\ & & \begin{pmatrix} 1 \\ r_3 \end{pmatrix} & & \begin{pmatrix} 1 \\ r_5 \end{pmatrix} & \begin{pmatrix} 1 \\ r_6 \end{pmatrix} \end{pmatrix}.$$

And it can be simplified as

$$\tilde{\mathbf{B}} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \frac{a_3'}{a_2'}(r_1 - r_4) & \frac{a_4'}{a_2'}(r_1 - r_5) & \\ & & & r_4 - r_2 & & \frac{a_4'}{a_3'}(r_2 - r_6) \\ & & & & r_5 - r_3 & r_6 - r_3 \end{pmatrix}.$$

Therefore, $rank(\mathbf{B}) = rank(\tilde{\mathbf{B}}) = 6$ if and only if $(r_1 - r_4)(r_2 - r_6)(r_3 - r_5) - (r_2 -$

$r_4)(r_1 - r_5)(r_3 - r_6) \neq 0$. [†]

In order to show that the tensor product code $\mathcal{C}$ can't correct all erasure patterns of Type II, we need to prove that if q isn't large enough, there will always be six distinct columns $\{b_{i1} \cdot (1, r_i)^T\}_{i \in [6]}$ with $(r_1 - r_4)(r_2 - r_6)(r_3 - r_5) - (r_2 - r_4)(r_1 - r_5)(r_3 - r_6) = 0$, which is shown as follows.

Consider $n-2$ distinct columns of $\mathbf{H}_2$ with weight 2, $\{b_{i1} \cdot (1, r_i)^T\}_{i \in [n-2]}$, according to the MDS property, we know that $r_i \neq r_j$ for all $i \neq j \in [n-2]$. Therefore, if we take $r_i = \omega^{t_i}$ for each $i \in [n-2]$, we know that $t_i \neq t_j$ for all $i \neq j \in [n-2]$. Denote $A = \{t_i\}_{i \in [n-2]}$, then A is an $(n-2)$-subset of $\mathbb{Z}_{q-2}$. Since $q < \frac{(n-3)^2}{4} + 2$, by Proposition 5.6, we know that A can't be a *2-Sidon set*. Thus, there are at least three different 2-subsets $\{t_1, t_6\}, \{t_2, t_5\}, \{t_3, t_4\} \in A$, such that $t_1 + t_6 = t_2 + t_5 = t_3 + t_4$ and t_is are all distinct. By Proposition 5.5, the corresponding $\{r_j\}_{j \in [6]}$ such that $r_j = \omega^{t_j}$ for each $j \in [6]$, satisfies $(r_1 - r_4)(r_2 - r_6)(r_3 - r_5) - (r_2 - r_4)(r_1 - r_5)(r_3 - r_6) = 0$.

Therefore, $\mathcal{C}$ can not correct the erasure patterns of Type II formed by the corresponding six columns $\{b_{j1} \cdot (1, r_j)^T\}_{j \in [6]}$, which means $\mathcal{C}$ is not an MRC that instantiates the topology $T_{m \times n}(1, 2, 0)$. □

Remark 5.12. *Since we have only considered the restrictions brought by certain erasure patterns, the lower bounds of the field size given by Theorem 5.17 can certainly be improved if one can find other non-trivial regular irreducible erasure patterns for general m.*

5.7.2 MRCs for topologies $T_{m \times n}(1, 3, 0)$

Denote $\mathcal{E}$ as the set of all the types of regular irreducible erasure patterns for topology $T_{m \times n}(1, 3, 0)$. For each $E \in \mathcal{E}$, by (5.15), we have $|U_E| + 3 \leq |V_E| \leq 3|U_E| - 3$, which leads to $|U_E| \geq 3$. Since $U_E \subseteq [m]$, we have $m \geq 3$ and $n \geq 6$.

From the irreducibility, each erasure pattern has exactly 2 erasures in each column and 4 erasures in each row. With the same spirit as that for topologies

[†] Recall the condition of $\mathbf{B}$ having full rank and the polynomial $f(x_1, x_2, \ldots, x_6)$ we defined in the proof of Theorem 5.16, the condition we obtain here for the general case is actually the same.

$T_{m\times n}(1,2,0)$, we find the following type of erasure patterns for $m=3$:

$$E_0 = \begin{pmatrix} * & * & * & * & \circ & \circ \\ * & * & \circ & \circ & * & * \\ \circ & \circ & * & * & * & * \end{pmatrix}.$$

Using the same argument as that for E_1 and E_2, one can prove that erasure patterns of type E_0 are regular and irreducible for topologies $T_{m\times n}(1,3,0)$. Also, since $m=3$ leads to $U_E=3$, $V_E=6$ and $|E|=12$ for every $E\in\mathcal{E}$, one can easily check that E_0 is the only type of regular irreducible erasure pattern for topologies $T_{3\times n}(1,3,0)$.

Based on this characterization, we have the following improved upper bound on the field size required for the existence of MRCs instantiating $T_{3\times n}(1,3,0)$.

Theorem 5.18. *For any $q>\frac{n^5}{\log(n)}\cdot C_2$, there exists an MRC $\mathcal{C}$ that instantiates the topology $T_{3\times n}(1,3,0)$ over the field $\mathbb{F}_q$, where $C_2\geq(\frac{10}{c_5})^5$ is an absolute constant.*

Sketch of the proof. Since the idea of the proof is the same as that of Theorem 5.16, we only sketch the main steps here.

Let $\mathcal{C}_{col}$ be the simple parity code $\mathcal{P}_3$, we are going to construct a $3\times n$ matrix $\mathbf{H}_{row}$ such that:

(i) Every 3 distinct columns of $\mathbf{H}_{row}$ are linearly independent.

(ii) For each erasure pattern E of type E_0, the *pseudo-parity check matrix* $\mathbf{H}\in\mathbb{F}_q^{(n+9)\times 3n}$ of $\mathcal{P}_3\otimes\mathcal{C}_{row}$ satisfies: $rank(\mathbf{H}|_E)=12$.

Suppose there exists an objective matrix $\mathbf{A}_0$ of the form

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ a_1 & a_2 & a_3 & \cdots & a_n \\ a_1^2 & a_2^2 & a_3^2 & \cdots & a_n^2 \end{pmatrix},$$

where $\{a_i\}_{i\in[n]}$ are pairwise distinct elements in $\mathbb{F}_q$. Then we have the corresponding *pseudo-parity check matrix*

$$\mathbf{H}_{\mathbf{A_0}}|_{E_0} = \begin{pmatrix} \mathbf{I}_6 & \mathbf{0}_{6\times 6} \\ \mathbf{A}_{9\times 6} & \mathbf{B}_{9\times 6} \\ \mathbf{0}_{(n-6)\times 6} & \mathbf{0}_{(n-6)\times 6} \end{pmatrix},$$

where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & & \\ a_1 & a_2 & a_3 & a_4 & & \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 & & \\ & & & & 1 & 1 \\ & & & & a_5 & a_6 \\ & & & & a_5^2 & a_6^2 \\ & & & & & \\ & & \mathbf{0}_{3\times 6} & & & \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} -1 & -1 & -1 & -1 & & \\ -a_1 & -a_2 & -a_3 & -a_4 & & \\ -a_1^2 & -a_2^2 & -a_3^2 & -a_4^2 & & \\ 1 & 1 & & & -1 & -1 \\ a_1 & a_2 & & & -a_5 & -a_6 \\ a_1^2 & a_2^2 & & & -a_5^2 & -a_6^2 \\ & & 1 & 1 & 1 & 1 \\ & & a_3 & a_4 & a_5 & a_6 \\ & & a_3^2 & a_4^2 & a_5^2 & a_6^2 \end{pmatrix}.$$

Since $\mathbf{B}$ can be simplified as

$$\mathbf{B} = \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & a_2 - a_1 & & a_1 - a_5 & a_1 - a_6 \\ & & a_2^2 - a_1^2 & & a_1^2 - a_5^2 & a_1^2 - a_6^2 \\ & & & a_4 - a_3 & a_5 - a_3 & a_6 - a_3 \\ & & & a_4^2 - a_3^2 & a_5^2 - a_3^2 & a_6^2 - a_3^2 \\ & & & & & \\ & & & \mathbf{0}_{3\times 6} & & \end{pmatrix},$$

thus we have

- $rank(\mathbf{B}) = 6$ if and only if $f(a_1, \ldots, a_6) \neq 0$, where $f(x_1, \ldots, x_6) = (x_1 - x_2)(x_3 - x_4)[(x_1 - x_6)(x_2 - x_6)(x_3 - x_5)(x_4 - x_6) - (x_1 - x_5)(x_2 - x_5)(x_3 - x_6)(x_4 - x_6)]$.

Let $\mathcal{H}$ be a 6-uniform hypergraph with vertex set $\mathbb{F}_q$, each set of 6 vertices $\{v_1, \ldots, v_6\}$ forms a 6-hyperedge if and only if $f(v_{\pi(1)}, \ldots, v_{\pi(6)}) = 0$ for some $\pi \in S_6$. If there exists an independent set $I \subseteq \mathbb{F}_q$ such that $|I| \geq n$, then we can construct an objective matrix $\mathbf{A}_0$ by arbitrarily choosing n different vertices from I as $\{a_i\}_{i\in[n]}$.

Since $deg_{x_i}(f) \leq 2$ for each x_i, and $f(v_1, \ldots, x_i, \ldots, v_6)$ is a non-zero polynomial for any 5-subset $\{v_j\}_{j\in[6]\setminus\{i\}} \in \mathbb{F}_q$, we have $\Delta_5(\mathcal{H}) \leq 2 \cdot 6!$. Thus, by Theorem 5.3,

$$\alpha(\mathcal{H}) \geq \frac{c_5}{5}(q \log q)^{\frac{1}{5}} > n,$$

therefore, there exists a subset $A = \{a_1, \ldots, a_n\} \subseteq \mathbb{F}_q$ such that the corresponding Vandermonde matrix $\mathbf{A}_0$ is the objective parity check matrix of the row code $\mathcal{C}_{row}$.

□

Remark 5.13. *Based on the same idea as that appeared in the proof of Theorem 5.17, we can obtain a lower bound* $q \geq \frac{\sqrt{n^2-11n+34}}{2}$ *on the field size ensuring the existence of MRCs for topologies* $T_{m\times n}(1, 3, 0)$*. Unfortunately, this doesn't beat the lower bound* $q \geq n-1$ *or* $q \geq n-2$ *given by the* MDS conjecture.

§ 5.8 Conclusions and further research

In this chapter, we considered two kinds of codes for distributed storage systems–(r, δ)-LRCs and MRCs. For (r, δ)-LRCs, we provide general constructions for both optimal $(r, \delta)_a$-LRCs and optimal $(r, \delta)_i$-LRCs. Based on a connection between sparse hypergraphs and optimal (r, δ)-LRCs, we obtain optimal $(r, \delta)_a$-LRCs and optimal $(r, \delta)_i$-LRCs with super-linear (in q) length. This improves all known results when the minimal distance d satisfies $d \geq 3\delta + 1$. Moreover, as applications, we provide new constructions for H-LRCs and GSD codes. For MRCs, we obtain a new upper bound on the minimal size of fields required for the existence of MRCs instantiating the topology $T_{m\times n}(1, b, 0)$, which improves the general upper bound given by Gopalan et al. [91]. We also consider some special cases with fixed m and b, and obtain a polynomial lower bound and some new upper bounds.

Though many works have been done, there is still a wide range of questions that remain open. Here we highlight some of the questions related to our work.

- As shown in Theorem 5.6, codes generated by Construction B can recover special erasure patterns beyond the minimal distance, which enables us to further construct GSD codes. This phenomenon also appears in codes from Construction A. Note that the parity check matrix in (5.19) have similar structure as that for MR-LRC (see [95]), therefore, it's worth trying to obtain longer MR-LRCs using similar approaches.

- According to Tables I and II, there are gaps between our constructions and upper bounds on the code length given in [39] and [40]. Therefore, improvements of the upper bounds and constructions of longer codes will be interesting topics for future work. Moreover, to our knowledge, explicit constructions of large sparse hypergraphs are very rare. Results of Lemma 5.4 and therefore results of Theorem 5.10 are both from the perspective of probabilistic existence. Therefore, explicit constructions or algorithmic constructions in polynomial time (like Theorem 4.2 in [208]) for optimal (r, δ)-LRCs with super-linear length are also worth studying.

- Due to the rough estimation on the number of regular erasure patterns, the upper bound given by Theorem 5.15 still grows exponentially with m. If one can give a better characterization of the regular erasure patterns (probably using tools from extremal graph theory), we believe the general upper bound in Theorem 5.15 can also be improved.

- As for the lower bound, we only considered the cases when $b = 2$ and 3. For general case, due to the complexity of the erasure patterns, our method might not work. Therefore, a general non-trivial lower bound on the field size of codes achieving the MR property for topologies $T_{m\times n}(1, b, 0)$ remains widely open.

- Under the limitations of the methods themselves, the Combinatorial Nullstellensatz and the hypergraph independent set approach can only give existence results. Therefore, explicit constructions of MRCs for topologies $T_{m\times n}(1,b,0)$ over small fields are still interesting open problems. In particular, is it possible to give an explicit construction of MRCs for the topology $T_{4\times n}(1,2,0)$ over a field of size between $\Omega(n^2)$ and $\mathcal{O}(n^5/\log(n))$?

Chapter 6 Other research

In this chapter, we discuss some other research during the study for the candidate's doctorate. These research topics are still under investigation and are all closely related to some of the topics we've discussed above. However, owing to the limitation of the extent, in this chapter, after briefly introducing their research backgrounds, we only list our corresponding results and omit detailed proofs and discussions.

§ 6.1 A new type of Bollobás's two families theorem

Let $(A_1, B_1), \ldots, (A_m, B_m)$ be pairs of sets with $|A_i| = a$ and $|B_i| = b$ for every $1 \leq i \leq m$. In 1965, Bollobás [28] proved the following theorem about cross-intersecting set pairs, which became one of the cornerstones in extremal set theory.

Theorem 6.1 (Bollobás Theorem). *[28] Suppose that $A_i \cap B_i = \emptyset$, for $1 \leq i \leq m$, and $A_i \cap B_j \neq \emptyset$, for $i \neq j$. Then*

$$m \leq \binom{a+b}{a}. \tag{6.1}$$

Furthermore, equality holds if and only if there is some set X of cardinality $a + b$ such that A_is are all subsets of X of size a and $B_i = X \setminus A_i$ for each i.

Over the years, different proofs involving various kinds of methods together with all kinds of generalizations of this theorem have been discovered (see [8, 10, 26, 69, 80, 97, 114–116, 118, 119, 135, 158, 172, 190, 191, 194–196]). Among these proofs, using tools from *exterior algebra* (or *wedge product*), Lovász's proof [137] turns out to be strikingly elegant and provides a brand-new perspective for dealing with pairs of sets or subspaces with these types of constraints.

Using Lovász's method, in 1984, Füredi [80] proved a threshold version of Bollobás theorem for linear subspaces. Recently, following the path led by Lovász and Füredi, Scott and Wilmer [172] established a new correspondence between exterior algebra and hypergraphs. It turns out to be an effective way to tackle pairs of set systems with the Bollobás-type cross-intersecting requirements. As an application of their method, they proved a weighted Bollobás theorem for finite-dimensional real vector spaces.

Using exterior algebra method together with the new correspondence in [172], we also prove a new weighted Bollobás-type theorem for two families in real vector spaces. Comparing to Scott and Wilmer's result, we generalize the original constraints to $\dim(A_i \cap B_i) \le t$ for $1 \le i \le m$ and $\dim(A_i \cap B_j) > t$ for some integer $t \ge 0$ and $1 \le i < j \le m$. Besides, there is an extra constraint about $\{A_i\}_{i=1}^m$ being $(t+1)$-intersecting.

Theorem 6.2. *Let $\{(A_i, B_i)\}_{i=1}^m$ be a collection of pairs of subspaces of a finite dimensional real vector space $\mathbb{R}^n$, such that $\dim(A_i) = a_i$ and $\dim(B_i) = b_i$ with $a_i \le b_i$ for every $1 \le i \le m$. Suppose that for some $t \ge 0$*

- $\dim(A_i \cap A_j) > t$ *for all* $1 \le i, j \le m$,
- $\dim(A_i \cap B_i) \le t$ *for all* $1 \le i \le m$,
- $\dim(A_i \cap B_j) > t$ *for all* $1 \le i < j \le m$,
- $a_i + b_i = N$ *for all* $1 \le i \le m$ *and some positive integer* N, *with* $a_1 \le \ldots \le a_m$.

Then

$$\sum_{i=1}^{m} \binom{N-(2t+1)}{a_i-(t+1)}^{-1} \le 1. \tag{6.2}$$

When $a_i < b_i$ for every $1 \le i \le m$, equality holds only if $a_1 = a_2 = \ldots = a_m$ and $b_1 = b_2 = \ldots = b_m$.

As a direct corollary of Theorem 6.2, we have the following theorem for pairs of subsets which settles a recent conjecture proposed by Gerbner et al. [88]

Theorem 6.3. *Let* $\{(A_i, B_i)\}_{i=1}^m$ *be a collection of pairs of sets such that for every* $1 \leq i \leq m$, $|A_i| = a_i \leq |B_i| = b_i$. *Suppose that for some* $t \geq 0$,

- $|A_i \cap A_j| > t$ *for all* $1 \leq i, j \leq m$,
- $|A_i \cap B_i| \leq t$ *for all* $1 \leq i \leq m$,
- $|A_i \cap B_j| > t$ *for all* $1 \leq i < j \leq m$,
- $a_i + b_i = N$ *for all* $1 \leq i \leq m$ *and some positive integer* N, *with* $a_1 \leq a_2 \leq \ldots \leq a_m$.

Then

$$\sum_{i=1}^{m} \binom{N-(2t+1)}{a_i-(t+1)}^{-1} \leq 1. \tag{6.3}$$

As shown in Theorem 6.1, Bollobás proved that the equality in (6.1) holds if and only if the ground set X has cardinality $a+b$, $\{A_1, \ldots, A_m\} = \binom{X}{a}$ and $B_i = X \setminus A_i$. In the same spirit, we determine the only structure of $\{(A_i, B_i)\}_{i=1}^m$ such that the equality holds in Theorem 6.3 when $t = 0$ and $a < b$.

Theorem 6.4. *Let* $\{(A_i, B_i)\}_{i=1}^m$ *be a collection of pairs of sets such that for every* $1 \leq i \leq m$, $|A_i| = a < |B_i| = b$. *Suppose that*

- $|A_i \cap A_j| > 0$ *for all* $1 \leq i, j \leq m$,
- $|A_i \cap B_j| = 0$ *if and only if* $i = j$.

Then, $m = \binom{a+b-1}{a-1}$ *if and only if the ground set* $X = \bigcup_{i=1}^m (A_i \cup B_i)$ *has cardinality* $a+b$, $\{A_i\}_{i=1}^m$ *is a family of all subsets of* X *of size* a *containing a fixed element and* $B_i = X \setminus A_i$ *for each* i.

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§ 6.2 Quaternary locally repairable codes attaining the Singleton-type bound

Modern distributed storage systems have been transitioning to erasure coding based schemes with good storage efficiency in order to cope with the explosion in the amount of data stored online. Locally Repairable Codes (LRCs) have emerged as the codes of choice for many such scenarios and have been implemented in a number of large scale systems.

The concept of codes with locality was introduced by Gopalan et al. [92], Oggier and Datta [153], and Papailiopoulos et al. [155]. LRCs are capable of very efficient erasure recovery for the typical case in distributed storage systems where a single node fails, while still allowing the recovery of data from a larger number of erasures. Like traditional error-correcting codes, there is also a Singleton-type bound for locally repairable codes relating its length n, dimension k, minimum distance d and locality r, which was first shown in the highly influential work [92]:

$$d(\mathcal{C}) \leq n - k - \lceil \frac{k}{r} \rceil + 2, \tag{6.4}$$

which reduces to the classical Singleton bound when $r = k$. Later, the bound was generalized to vector codes and nonlinear codes in [67], [155]. Although it certainly holds for all LRCs, it is not tight in many cases. The tightness of bound (6.4) was studied in [182], [201].

We say an LRC is optimal if it satisfies bound (6.4) with equality for given parameters n, k, d and r. Many works have been done for the constructions of optimal LRCs, for examples, see [20, 178, 186–188]. For the convenience of computer hardware implementation, LRCs over small alphabets are of particular interest. In 2016, based on a construction of quasi-random codes, Ernvall et al. [65] constructed optimal LRCs over a small alphabet. By studying the properties of the corresponding parity-check matrices, Hao and Xia [100] gave high rate optimal LRCs with $q \geq r+2$ and minimum distances 3 and 4. Then, with the same parity-check matrix approach, Hao et al. [100, 101] determined all possible parameters of optimal binary and ternary r-LRCs.

Combining tools from finite geometry, we employ the parity-check matrix approach to study the classification for parameters of optimal (n,k,r)-LRCs over the quaternary field (finite field of order 4), and we obtain the following main result.

Theorem 6.5. *Let $r \geq 1$, $k > r$ and $d \geq 2$. There are 26 classes of optimal quaternary (n,k,r) LRCs with minimum distance d meeting the Singleton-type bound, whose parameters are listed as follows respectively*

- $(n,k,r) = (k + \lceil k/r \rceil, k, r)$ *with* $k > r \geq 1$, $d = 2$;
- $(n,k,r) = (3s+3, 2s+1, 2)$ *with* $s \geq 2$, $d = 3$;
- $(n,k,r) = (4s+3, 3s+1, 3)$ *with* $s \geq 2$, $d = 3$;
- $(n,k,r) = (2r+5, 2r+1, r)$ *with* $4 \leq r \leq 5$, $d = 3$;
- $(n,k,r) = (18, 13, 4)$ *with* $d = 3$;
- $(n,k,r) = (4s+4, 3s+2, 3)$ *with* $s \geq 2$, $d = 3$;
- $(n,k,r) = (r+4, r+1, r)$ *with* $2 \leq r \leq 15$, $d = 3$;
- $(n,k,r) = (r+5, r+2, r)$ *with* $3 \leq r \leq 15$, $d = 3$;
- $(n,k,r) = (r+6, r+3, r)$ *with* $6 \leq r \leq 15$, $d = 3$;
- $(n,k,r) = (4s+4, 3s+1, 3)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (5s+4, 4s+1, 4)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (6s+4, 5s+1, 5)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (25, 19, 6)$ *with* $d = 4$;
- $(n,k,r) = (5s+5, 4s+2, 4)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (6s+5, 5s+2, 5)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (19, 14, 6)$ *with* $d = 4$;
- $(n,k,r) = (6s+6, 5s+3, 5)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (2s+2, s, 1)$ *with* $s \geq 2$, $d = 4$;
- $(n,k,r) = (r+5, r+1, r)$ *with* $2 \leq r \leq 11$, $d = 4$;
- $(n,k,r) = (r+6, r+2, r)$ *with* $3 \leq r \leq 11$, $d = 4$;

- $(n, k, r) = (r + 7, r + 3, r)$ *with* $5 \le r \le 7$, $d = 4$;
- $(n, k, r) = (2r + 6, 2r + 1, r)$ *with* $2 \le r \le 7$, $d = 4$;
- $(n, k, r) = (2k + 4, k, 1)$ *with* $k = 2, 3$, $d = 6$;
- $(n, k, r) = (2k + 6, k, 1)$ *with* $k = 2, 3$, $d = 8$;
- $(n, k, r) = (n, k, k - 1)$ *with* $3 \le k \le 6$, $3 \le n - k \le 6$, $d = n - k$;
- $(n, k, r) = (3s + 6, 2s + 1, 2)$ *with* $s = 2, 3$, $d = 6$.

For each class of these parameters, we present explicit constructions. Moreover, we also obtain some new necessary conditions for the existence of optimal quaternary LRCs.

§ 6.3 k-optimal locally repairable codes

Given an $[n, k, d]$ linear code $\mathcal{C}$, the information rate $\frac{k}{n}$ quantifies its transformation or storage efficiency and the minimum distance d measures its error-correcting capacity. The main problem in coding theory is to find codes with large $\frac{k}{n}$ and large d. For LRCs in modern distributed storage system, many works concerning this problem has been done. In the first part of Chapter 5 and Section § 6.2 of this chapter, we've introduced the Singleton-type bound of d for LRCs ((r, δ)-LRCs) and some optimal constructions with respect to this bound. In this section, we concerns another type of bound of k that is dependent on the size of the alphabet and corresponding optimal constructions.

The first bound of this type is derived in [37], which is called Cadambe-Mazumdar (C-M) bound,

$$k \le \min_{t \in \mathbb{Z}^+} \left[tr + k_{\text{opt}}^{(q)}(n - (r + 1)t, d) \right], \tag{6.5}$$

where $k_{\text{opt}}^{(q)}(n, d)$ is the largest possible dimension of a code, for given field size q, code length n, minimum distance d, and t is a positive integer satisfying $t \le \min\{\lceil \frac{n}{r+1} \rceil, \lceil \frac{k}{r} \rceil\}$. This bound can be attained by binary simplex codes [37]. Later in [83, 94, 108, 151, 174, 179], optimal binary LRCs of different localities and distances are constructed via various techniques.

Recently, Agarwal et al. [2] derived a linear programming bound for LRCs under the setting of disjoint repair groups. Wang et al. [202] presented a sphere-packing bound for binary LRCs with disjoint repair groups and they also constructed a class of binary LRCs with disjoint repair groups achieving this bound. Later in [143], Ma and Ge proved an explicit bound for the dimension k of such codes, which can be viewed as a generalization of the bounds given in [94, 202, 210]. By several new constructions of binary LRCs based on weakly independent sets and partial spreads, they also showed that this bound is optimal.

With the same spirit, we extend Ma and Ge's upper bound for k to general (r, δ)-LRCs:

Theorem 6.6. *For any $[n, k, d; r, \delta]_q$ linear LRCs with disjoint repair groups, $n = (r + \delta - 1)\ell$. For any fixed $v \in [l]$, define*

$$P(v) = \sum_{\substack{0 \le i_1 + \cdots + i_v \le t \\ i_1, \ldots, i_v \in [\delta, r+\delta-1]}} \prod_{m=1}^{v} \left(\binom{r+\delta-1}{i_m} \sum_{j=0}^{i_m-\delta} (-1)^j \binom{i_m}{j} (q^{i_m+1-\delta-j} - 1) \right).$$

Then, we have

- *when $d = 2t + 1$,*

$$k \le \left\lfloor \frac{rn}{r+\delta-1} - \log_q \left(\sum_{v=1}^{l} P(v) \right) \right\rfloor. \tag{6.6}$$

- *when $d = 2t + 2$,*

$$k \le \left\lfloor \frac{rn}{r+\delta-1} - \log_q \left(\sum_{v=1}^{l} P(v) \right) + \frac{\sum_{v=1}^{l} P(v)}{\left\lfloor \frac{n(q-1)}{t+1} \right\rfloor} \right\rfloor. \tag{6.7}$$

Using partial spreads and covering Grassmanian codes, for certain parameters of l, d, r and δ, we can construct (r, δ)-LRCs of length $(r + \delta - 1)l$ with dimension k achieving the above upper bounds. This shows that these bounds are optimal to some extent. However, the range of parameters of such optimal (r, δ)-LRCs are limited and we are still trying to generalize our constructions to obtain infinite classes of optimal (r, δ)-LRCs achieving these bounds.

References

[1] N. Q. A, L. Györfi and J. L. Massey, "Constructions of binary constant-weight cyclic codes and cyclically permutable codes," *IEEE Trans. Inform. Theory*, vol. 38, no. 3, pp. 940–949, 1992.

[2] A. Agarwal, A. Barg, S. Hu, A. Mazumdar and I. Tamo, "Combinatorial Alphabet-Dependent Bounds for Locally Recoverable Codes," *IEEE Trans. Inform. Theory*, vol. 64, no. 5, pp. 3481–3492, 2018.

[3] E. Agrell, A. Vardy, and K. Zeger, "Upper Bounds for Constant-Weight Codes," *IEEE Trans. Inform. Theory*, vol. 46, no. 3, pp. 2373–2395, 2000.

[4] R. Ahlswede and G. O. H. Katona, "Contributions to the geometry of Hamming spaces", *Discrete Math.*, vol. 17, no. 1, pp. 1–22, 1977.

[5] M. Ajtai, J. Komlós, J. Pintz, J. Spencer, and E. Szemerédi, "Extremal uncrowded hypergraphs," *J. Comb. Theory, Ser. A*, vol. 32, no. 3, pp. 321–335, 1982.

[6] M. Ajtai, J. Komlós, and E. Szemerédi, "A note on Ramsey numbers," *J. Comb. Theory, Ser. A*, vol. 29, no. 3, pp. 354–360, 1980.

[7] M. O. Albertson and K. L. Collins, "Homomorphisms of 3-chromatic graphs", *Discrete Math.*, vol. 54, no. 2, pp. 127–132, 1985.

[8] N. Alon, "An extremal problem for sets with applications to graph theory", *J. Comb. Theory, Ser. A*, vol. 40, no. 1, pp. 82–89, 1985.

[9] N. Alon, "Combinatorial nullstellensatz," *Combin. Probab. Comput.*, vol. 8, no. 1–2, pp. 7–29, 1999.

[10] N. Alon and G. Kalai, "A simple proof of the upper bound theorem", *European J. Combin.*, vol. 6, no. 3, pp. 211–214, 1985.

[11] N. Alon and A. Shapira, "On an extremal hypergraph problem of Brown, Erdős and Sós," *Combinatorica*, vol. 26, no. 6, pp. 627–645, 2006.

[12] N. Alon and J. H. Spencer, "The Probabilistic Method, Third Edition", *Wiley-Interscience series in discrete mathematics and optimization.* Wiley, 2008.

[13] I. Althöfer and T. Sillke, "An "average distance" inequality for large subsets of the cube", *J. Comb. Theory, Ser. B*, vol. 56, no. 2, pp. 296–301, 1992.

[14] L. Babai and P. Frankl, "Linear Algebra Methods in Combinatorics With Applications to Geometry and Computer Science", 1992.

[15] S. Balaji, M. N. Krishnan, M. Vajha, V. Ramkumar, B. Sasidharan, and P. V. Kumar, "Erasure coding for distributed storage: an overview", *Science China Information Sciences*, vol. 61, no. 10, pp. 100301, 2018.

[16] S. B. Balaji, K. P. Prasanth, and P. V. Kumar, "Binary codes with locality for multiple erasures having short block length," in *IEEE International Symposium on Information Theory*, pp. 655–659, 2016.

[17] S. Ballentine, A. Barg, and S. Vlădu. "Codes with hierarchical locality from covering maps of curves", *IEEE Trans. Inform. Theory*, vol. 65, no. 10, pp. 6056–6071, 2019.

[18] J. Balogh, S. Das, M. Delcourt, H. Liu, and M. Sharifzadeh, "Intersecting families of discrete structures are typically trivial", *J. Comb. Theory, Ser. A*, vol. 132, pp. 224 – 245, 2015.

[19] J. Balogh, S. Das, H. Liu, M. Sharifzadeh, and T. Tran, "Structure and supersaturation for intersecting families", *Electron. J. Combin.*, vol. 26, no. 2, Paper 2.34, 38, 2019.

[20] A. Barg, I. Tamo, and S. Vlăduţ, "Locally recoverable codes on algebraic curves," *IEEE Trans. Inform. Theory*, vol. 63, no. 8, pp. 4928–4939, 2017.

[21] F. A. Berend, "On Sets of Integers Which Contain No Three Terms in Arithmetical Progression," *Proc. Nat. Acad. Sci. U. S. A.*, vol. 32, pp. 561–563, 1946.

[22] A. Bernstein, "Maximally connected arrays on the n-cube", *SIAM J. Appl. Math.*, vol. 15， pp. 1485–1489, 1967.

[23] N. Biggs, "Intersection matrices for linear graphs", in *Combinatorial Mathematics and its Applications*, pages 15–23, Academic Press, London, 1971.

[24] N. Biggs, "Algebraic graph theory", *Cambridge University Press*, London, 1974.

[25] M. Blaum and J. Bruck, "Coding for tolerance and detection of skew in parallel asynchronous communications," *IEEE Trans. Inform. Theory*, vol. 46, no. 7, pp. 2329–2335, 2000.

[26] A. Blokhuis, "Solution of an extremal problem for sets using resultants of polynomials", *Combinatorica*, vol. 10, no. 4, pp. 393–396, 1990.

[27] A. Blokhuis, A. E. Brouwer, A. Chowdhury, P. Frankl, T. Mussche, B. Patkós, and T. Szönyi, "A Hilton-Milner Theorem for Vector Spaces", *Electron. J. Combin.*, vol. 17, no. 1, Paper 71, 12, 2010.

[28] B. Bollobás, "On generalized graphs", *Acta Math. Acad. Sci. Hungar*, vol. 16, pp. 447–452, 1965.

[29] B. Bollobás, "Combinatorics: Set Systems, Hypergraphs, Families of Vectors, and Combinatorial Probability", Cambridge University Press, Cambridge, 1986 (English).

[30] P. Borg, "Intersecting and cross-intersecting families of labeled sets", *Electron. J. Combin.*, vol. 15, no. 1, 2008.

[31] P. Borg. "A short proof of a cross-intersection theorem of Hilton", *Discrete Math.*, vol. 309, no. 14, pp. 4750–4753, 2009.

[32] P. Borg, "Cross-intersecting families of permutations", *J. Comb. Theory, Ser. A*, vol. 117, no. 4, pp. 483–487, 2010.

[33] P. Borg and I. Leader, "Multiple cross-intersecting families of signed sets", *J. Comb. Theory, Ser. A*, vol. 117, no. 5, pp. 583–588, 2010.

[34] W. Brown, P. Erdős, and V. Sós. "Some extremal problems on r-graphs". in *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich, 1971)*, pages 53–63, 1973.

[35] C. Bujtás and Z. Tuza. "Turán numbers and batch codes", *Discrete Appl. Math.*, vol. 186, pp. 45–55, 2015.

[36] K. A. Bush, W. T. Federer, H. Pesotan, and D. Raghavarao, "New combinatorial designs and their application to group testing," *J. Statist. Plann. and Inference*, vol. 10, no. 3, pp. 335–343, 1984.

[37] V. Cadambe and A. Mazumdar, "Bounds on the size of locally recoverable codes," *IEEE Trans. Inform. Theory*, vol. 61, no. 11, pp. 5787–5794, 2015.

[38] H. Cai, M. Cheng, C. Fan, and X. Tang. "Optimal locally repairable systematic codes based on packings", *IEEE Trans. Commun.*, vol. 67, no. 1, pp. 39–49, 2019.

[39] H. Cai, Y. Miao, M. Schwartz, and X. Tang. "On optimal locally repairable codes with super-linear length", *IEEE Trans. Inform. Theory*, vol. 66, no. 8, pp. 4853–4868, 2020.

[40] H. Cai and M. Schwartz, "On optimal locally repairable codes and generalized sector-disk codes", *IEEE Trans. Inform. Theory*, online, 2020.

[41] P. J. Cameron and C. Y. Ku, "Intersecting families of permutations", *European J. Combin.*, vol. 24, no. 7, pp. 881–890, 2003.

[42] B. Chen, S. Xia, J. Hao, and F. Fu, "Constructions of optimal cyclic (r, δ) locally repairable codes", *IEEE Trans. Inform. Theory*, vol. 64, no. 4, pp. 2499–2511, 2018.

[43] M. Chen, C. Huang, and J. Li, "On the maximally recoverable property for multi-protectiongroup codes," in *IEEE International Symposium on Information Theory*, pp. 486–490, 2007.

[44] A. Chowdhury and B. Patkós, "Shadows and intersections in vector spaces", *J. Comb. Theory, Ser. A*, vol. 117, pp. 1095–1106, 2010.

[45] F. R. K. Chung, J. A. Salehi, and V. K. Wei, "Optical orthogonal codes: Design, analysis, and applications," *IEEE Trans. Inform. Theory*, vol. 35, no. 3, pp. 595–604, 1989.

[46] G. D. Cohen and G. Zemor, "Intersecting codes and independent families," *IEEE Trans. Inform. Theory*, vol. 40, no. 6, pp. 1872–1881, 1994.

[47] S. Das, W. Gan, and B. Sudakov, "The minimum number of disjoint pairs in set systems and related problems", *Combinatorica*, vol. 36, no. 6, pp. 623–660, 2016.

[48] S. Das and T. Tran, "Removal and stability for Erdős-Ko-Rado", *SIAM J. Discrete Math.*, vol. 30, no. 2, pp. 1102–1114, 2016.

[49] P. Delsarte, "An algebraic approach to the association schemes of coding theory", *Philips Res. Rep. Suppl.*, (10):vi+97, 1973.

[50] A. G. Dimakis, P. B. Godfrey, M. Wainwright, and K. Ramchandran, "Network coding for distributed storage systems," in *INFOCOM, 2007 Proceedings IEEE*, pp. 2000–2008, 2007.

[51] R. Duke, H. Lefmann, and V. Rödl, "On uncrowded hypergraphs," *Random Structures & Algorithms*, vol. 6, no. 2-3, pp. 209–212, 1995.

[52] A. G. Dyachkov and V. V. Rykov, "Bounds on the length of disjunctive codes," *Problemy Peredachi Informatsii*, vol. 18, no. 3, pp. 7–13, 1982. [In Russian]

[53] D. Ellis, "Stability for t-intersecting families of permutations", *J. Comb. Theory, Ser. A*, vol. 118, no. 1, pp. 208–227, 2011.

[54] D. Ellis, "Setwise intersecting families of permutations", *J. Comb. Theory, Ser. A*, vol. 119, no. 4, pp. 825–849, 2012.

[55] D. Ellis, "Forbidding just one intersection, for permutations", *J. Comb. Theory, Ser. A*, vol. 126, pp. 136–165, 2014.

[56] D. Ellis, Y. Filmus, and E. Friedgut, "A quasi-stability result for dictatorships in S_n", *Combinatorica*, vol. 35, no. 5, pp. 573–618, 2015.

[57] D. Ellis, Y. Filmus, and E. Friedgut, "Low-degree Boolean functions on S_n, with an application to isoperimetry", *Forum Math. Sigma*, 5:e23, 46, 2017.

[58] D. Ellis, E. Friedgut, and H. Pilpel, "Intersecting families of permutations", *J. Amer. Math. Soc.*, vol. 24, no. 3, pp. 649–682, 2011.

[59] D. Ellis, N. Keller, and N. Lifshitz, "On the structure of subsets of the discrete cube with small edge boundary", *Discrete Anal.*, Paper No.9, 29pp, 2018.

[60] D. Ellis, N. Keller, and N. Lifshitz, "On a biased edge isoperimetric inequality for the discrete cube", *J. Comb. Theory, Ser. A*, vol. 163, pp. 118–162, 2019.

[61] P. Erdős, P. Frankl, and Z. Füredi, "Families of finite sets in which no set is covered by the union of two others," *J. Comb. Theory, Ser. A*, vol. 33, no. 2, pp. 158–166, 1982.

[62] P. Erdős, P. Frankl, and Z. Füredi, "Families of finite sets in which no set is covered by the union of r others," *Isreal J. Math.*, vol. 51, no. 1-2, pp. 75–89, 1985.

[63] P. Erdős, P. Frankl, and V. Rödl, "The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent," *Graphs Combin.*, vol. 2, no. 2, pp. 113–121, 1986.

[64] P. Erdős, C. Ko, and R. Rado, "Intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser. (2)*, vol. 12, pp. 313–320, 1961.

[65] T. Ernvall, T. Westerbäck, R. Freij-Hollanti, and C. Hollanti, "Constructions and properties of linear locally repairable codes", *IEEE Trans. Inform. Theory*, vol. 62, no. 3, pp. 1129–1143, 2016.

[66] A. Ferber, G. McKinley, and W. Samotij, "Supersaturated sparse graphs and hypergraphs," *International Mathematics Research Notices*, pp. 1–25, 2018.

[67] M. Forbes and S. Yekhanin, "On the locality of codeword symbols in non-linear codes," *Discrete Math.*, vol. 324, pp. 78–84, 2014.

[68] J. S. Frame, G. B. Robinson, and R. M. Thrall, "The Hook graphs of S_n", *Canad. J. Math.*, vol. 6, pp. 316–324, 1954.

[69] P. Frankl. "An extremal problem for two families of sets", *European J. Combin.*, vol. 3, no. 2, pp. 125–127, 1982.

[70] P. Frankl, "An Erdős-Ko-Rado theorem for direct products", *European J. Combin.*, vol. 17, no. 8, pp. 727–730, 1996.

[71] P. Frankl and M. Deza, "On the maximum number of permutations with given maximal or minimal distance", *J. Comb. Theory, Ser. A*, vol. 22, no. 3, pp. 352–360, 1977.

[72] P. Frankl and Z. Füredi, "On hypergraphs without two edges intersecting in a given number of vertices", *J. Comb. Theory, Ser. A*, vol. 36, no. 2, pp. 230–236, 1984.

[73] P. Frankl, Y. Kohayakawa, and V. Rödl, "A note on supersaturated set systems", *European J. Combin.*, vol. 51, pp. 190–199, 2016.

[74] P. Frankl and A. Kupavskii, "Uniform s-cross-intersecting families", *Combin. Probab. Comput.*, vol. 26, no. 4, pp. 517 – -524, 2017.

[75] P. Frankl, K. Ota, and N. Tokushige, "Exponents of uniform l-systems", *J. Comb. Theory, Ser. A*, vol. 75, no. 1, pp. 23–43, 1996.

[76] P. Frankl and N. Tokushige, "Some best possible inequalities concerning cross-intersecting families", *J. Comb. Theory, Ser. A*, vol. 61, no. 1, pp. 87–97, 1992.

[77] P. Frankl and N. Tokushige, "Invitation to intersection problems for finite sets", *J. Comb. Theory, Ser. A*, vol. 144, pp. 157–211, 2016.

[78] P. Frankl and R. Wilson, "The Erdős-Ko-Rado theorem for vector spaces", *J. Comb. Theory, Ser. A*, vol. 43, pp. 228–236, 1986.

[79] Y. Fujiwara and C. J. Colbourn, "A Combinatorial Approach to X-Tolerant Compaction Circuits," *IEEE Trans. Inform. Theory*, vol. 56, no. 7, pp. 3196–3206, 2010.

[80] Z. Füredi, "Geometrical solution of an intersection problem for two hypergraphs", *European J. Combin.*, vol. 5, no. 2, pp. 133–136, 1984.

[81] Z. Füredi, "On r-cover-free families," *J. Comb. Theory, Ser. A*, vol. 73, no. 1, pp. 172–173, 1996.

[82] F.-W. Fu, T. Kløve, and S.-Y. Shen, "On the Hamming distance between two i.i.d. random n-tuples over a finite set", *IEEE Trans. Inform. Theory*, vol. 45, no. 2, pp. 803–807, 1999.

[83] Q. Fu, R. Li, L. Guo, and L. Lv, "Locality of optimal binary codes," *Finite Fields Appl.*, vol. 48, pp. 371–394, 2017.

[84] F.-W. Fu, V. K. Wei, and R. W. Yeung, "On the minimum average distance of binary codes: linear programming approach", *Discrete Appl. Math.*, vol. 111, no. 3, pp. 263–281, 2001.

[85] V. Gandikota, E. Grigorescu, C. Thomas, and M. Zhu, "Maximally recoverable codes: The bounded case," in *55th Annual Allerton Conference on Communication, Control, and Computing*, pp. 1115–1122, 2017.

[86] G. Ge and C. Shangguan, "Sparse hypergraphs: new bounds and constructions," *J. Comb. Theory, Ser. B*, vol. 147, pp. 96–132, 2021.

[87] X. B. Geng, J. Wang, and H. J. Zhang, "Structure of independent sets in direct products of some vertex-transitive graphs", *Acta. Math. Sin.-English Ser.*, vol. 28, no. 4, pp. 697–706, 2012.

[88] D. Gerbner, B. Keszegh, A. Methuku, D. T. Nagy, B. Patkós, C. Tompkins, and C. Xiao, "Set systems related to a house allocation problem", *Discrete Math.*, vol. 343, no. 7, 111886, 2020.

[89] A. Girão and R. Snyder, "Disjoint pairs in set systems with restricted intersection", *European J. Combin.*, vol. 83, no. 102998, 13pp, 2020.

[90] C. Godsil and K. Meagher, "Erdős-Ko-Rado theorems: algebraic approaches", *Cambridge University Press*, 2016.

[91] P. Gopalan, C. Huang, B. Jenkins, and S. Yekhanin, "Explicit maximally recoverable codes with locality," *IEEE Trans. Inform. Theory*, vol. 60, no. 9, pp. 5245–5256, 2014.

[92] P. Gopalan, C. Huang, H. Simitci, and S. Yekhanin, "On the locality of codeword symbols," *IEEE Trans. Inform. Theory*, vol. 58, no. 11, pp. 6925–6934, 2012.

[93] P. Gopalan, G. Hu, S. Kopparty, S. Saraf, C. Wang, and S. Yekhanin, "Maximally recoverable codes for grid-like topologies," in *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 2092–2108, 2017.

[94] S. Goparaju and R. Calderbank, "Binary cyclic codes that are locally repairable," in *IEEE International Symposium on Information Theory*, pp. 676–680, 2014.

[95] S. Gopi, V. Guruswami, and S. Yekhanin, "Maximally recoverable LRCs: A field size lower bound and constructions for few heavy parities", *IEEE Trans. Inform. Theory*, vol. 66, no. 10, pp 6066–6083, 2020.

[96] S. Gopi and V. Guruswami, "Improved Maximally Recoverable LRCs using Skew Polynomials", *Electron. Colloquium Comput. Complex.*, vol. 28, pp. 25, 2021.

[97] J. R. Griggs, J. Stahl, and W. T. Trotter, "A Sperner theorem on unrelated chains of subsets", *J. Comb. Theory, Ser. A*, vol. 36, no. 1, pp. 124–127, 1984.

[98] V. Guruswami, C. Xing, and C. Yuan, "How long can optimal locally repairable codes be?", *IEEE Trans. Inform. Theory*, vol. 65, no. 6, pp. 3662–3670, 2019.

[99] V. Guruswami, L. Jin, and C. Xing, "Constructions of Maximally Recoverable Local Reconstruction Codes via Function Fields", *IEEE Trans. Inform. Theory*, vol. 66, no. 10, pp. 6133–6143, 2020.

[100] J. Hao, S. Xia, and B. Chen, "Some results on optimal locally repairable codes," in *IEEE International Symposium on Information Theory*, pp. 440–444, 2016.

[101] J. Hao, S. Xia, and B. Chen, "On optimal ternary locally repairable codes," in *IEEE International Symposium on Information Theory*, pp. 171–175, 2017.

[102] L. Harper, "Optimal assignments of numbers to vertices", *SIAM J. Appl. Math.*, vol. 12, pp. 131–135, 1964.

[103] L. Harper, "A note on the edges of the n-cube", *Discrete Math.*, vol. 14, pp. 157–163, 1976.

[104] A. Hilton, "An intersection theorem for a collection of families of subsets of a finite set", *J. London Math. Soc.*, vol. 15, no. 2, pp. 369–376, 1977.

[105] A. J. W. Hilton and E. C. Milner, "Some intersection theorems for systems of finite sets", *Quart. J. Math. Oxford Ser. (2)*, vol. 18, pp. 369–384, 1967.

[106] H. Hollmann, "Design of test sequences for VLSI self-testing using LFSR," *IEEE Trans. Inform. Theory*, vol. 36, no. 32, pp. 386–392, 1990.

[107] C. Huang, H. Simitci, Y. Xu, A. Ogus, B. Calder, P. Gopalan, J. Li, and S. Yekhanin, "Erasure coding in windows azure storage", in *Proceedings of the 2012 USENIX Conference on Annual Technical Conference*, USENIX ATC' 12, page 2, USA, 2012. USENIX Association.

[108] P. Huang, E. Yaakobi, H. Uchikawa, and P. H. Siegel, "Binary linear locally repairable codes," *IEEE Trans. Inform. Theory*, vol. 62, no. 11, pp. 6268–6283, 2016.

[109] F. K. Hwang and V. T. Sós, "Non-adaptive hypergeometric group testing," *Studia Sci. Math. Hungar.*, vol. 22, no. 1, pp. 257–263, 1987.

[110] N. Jacobson, "Basic Algebra, volume I", *W.H. Freeman and Company*, New York, second edition, 1985.

[111] G. D. James, "The representation theory of the symmetric groups", *Springer*, Berlin, 1978.

[112] L. Jin, "Explicit construction of optimal locally recoverable codes of distance 5 and 6 via binary constant weight codes", *IEEE Trans. Inform. Theory*, vol. 65, no. 8, pp. 4658–4663, 2019.

[113] L. Jin, L. Ma, and C. Xing, "Construction of optimal locally repairable codes via automorphism groups of rational function fields", *IEEE Trans. Inform. Theory*, vol. 66, no. 1, pp. 210–221, 2020.

[114] G. Kalai, "Weakly saturated graphs are rigid", *Convexity and graph theory (Jerusalem, 1981)*, pp. 189–190, 1984.

[115] G. Kalai, "Intersection patterns of convex sets", *Israel J. Math.*, vol. 48, no. 2–3, pp. 161–174, 1984.

[116] D. Kang, J. Kim, and Y. Kim. "On the Erdős-Ko-Rado theorem and the Bollobás theorem for t-intersecting families", *European J. Combin.*, vol. 47, pp. 68–74, 2015.

[117] D. Kane, S. Lovett, and S. Rao, "The independence number of the Birkhoff polytope graph, and applications to maximally recoverable codes," in *58th IEEE Symposium on Foundations of Computer Science (FOCS)*, pp. 252–259, 2017.

[118] Z. Király, Z. L. Nagy, D. Pálvölgyi, and M. Visontai, "On families of weakly cross-intersecting set-pairs", *Fund. Inform.*, vol. 117, no. 1–4, pp. 189–198, 2012.

[119] G. Katona, "Solution of a problem of A. Ehrenfeucht and J. Mycielski", *J. Comb. Theory, Ser. A*, vol. 17, pp. 265–266, 1974.

[120] G. O. H. Katona, "A general 2-part Erdős-Ko-Rado theorem", *Opuscula Math.*, vol. 37, no. 4, pp. 577–588, 2017.

[121] G. O. H. Katona, G. Y. Katona, and Z. Katona, "Most probably intersecting families of subsets", *Combin. Probab. Comput.*, vol 21, no. 1-2, pp. 219–227, 2012.

[122] W. H. Kautz and R. Singleton, "Nonrandom binary superimposed codes," *IEEE Trans. Inform. Theory*, vol. 10, no. 4, pp. 363–377, 1964.

[123] P. Keevash, "Hypergraph Turán problems," in *Surveys in combinatorics 2011, volume 392 of London Math. Soc. Lecture Note Ser.*, pages 83–139, Cambridge Univ. Press, Cambridge, 2011.

[124] X. Kong and G. Ge, "On an inverse problem of the Erdős-Ko-Rado type theorems", *arXiv:2004.01529v1*, 2020.

[125] A. Kostochka, D. Mubayi, and J. Verstraëte, "On independent sets in hypergraphs," *Random Structures & Algorithms*, vol. 44, no. 2, pp. 224–239, 2014.

[126] C. Y. Ku, T. Lau, and K. B. Wong, "Cayley graph on symmetric group generated by elements fixing k points", *Linear Algebra Appl.*, vol. 471, pp. 405–426, 2015.

[127] C. Y. Ku, T. Lau, and K. B. Wong, "On the partition associated to the smallest eigenvalues of the k-point fixing graph", *European J. Combin.*, vol. 63, pp. 70–94, 2017.

[128] M. Kwan, B. Sudakov, and P. Vieira, "Non-trivially intersecting multi-part families", *J. Comb. Theory, Ser. A*, vol. 156, pp. 44–60, 2018.

[129] B. Larose and C. Malvenuto, "Stable sets of maximal size in Kneser-type graphs", *European J. Combin.*, vol. 25, no. 5, pp. 657–673, 2004.

[130] F. Lazebnik and J. Verstraete, "On hypergraphs of girth five," *Electron. J. Combin.*, vol. 10, no. R25, pp. 1–15, 2003.

[131] A. Lempel and M. Cohn, "Design of universal test sequences for VLSI," *IEEE Trans. Inform. Theory*, vol. 31, no. 1, pp. 10–17, 1985.

[132] X. Li, L. Ma, and C. Xing, "Optimal locally repairable codes via elliptic curves", *IEEE Trans. Inform. Theory*, vol. 65, no. 1, pp. 108–117, 2019.

[133] R. Lidl and N. Niederreiter, *Finite Fields*. Cambridge Univ. Press, Cambridge, 1983.

[134] J. Lindsey, "Assignment of numbers to vertices", *Amer. Math. Monthly*, vol. 71, pp. 508–516, 1964.

[135] L. Lovász, "Flats in matroids and geometric graphs", *Combinatorial surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977)*, pp. 45–86, 1977.

[136] L. Lovász, *Kneser's conjecture, chromatic number, and homotopy*, *J. Comb. Theory, Ser. A*, vol. 25, no. 3, pp. 319–324, 1978.

[137] L. Lovász, "Topological and algebraic methods in graph theory", *Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977)*, pp. 1–14, Academic Press, New York-London, 1979.

[138] L. Lovász, "Combinatorial Problems and Exercises", second ed., North-Holland Publishing Co., Amsterdam, 1993.

[139] S. S. Lumetta and S. Mitra, "X-codes: Theory and Applications of Unknowable Inputs", Center for Reliable and High-Performance Computing, Univ. Illinois at Urbana Champaign, 2003, Tech. Rep. CRHC-03-08 (also UILU-ENG-03-2217).

[140] S. S. Lumetta and S. Mitra, "X-codes: Error control with unknowable inputs," in *Proc. IEEE Int. Symp. Inf. Theory*, p. 102, 2003.

[141] G. Luo and X. Cao, "Optimal cyclic codes with hierarchical locality", *IEEE Trans. Commun.*, vol. 68, no. 6, pp. 3302–3310, 2020.

[142] Y. Luo, C. Xing, and C. Yuan, "Optimal locally repairable codes of distance 3 and 4 via cyclic codes", *IEEE Trans. Inform. Theory*, vol. 65, no. 2, pp. 1048–1053, 2019.

[143] J. Ma and G. Ge, "Optimal Binary Linear Locally Repairable Codes with Disjoint Repair Groups," *SIAM J. Discrete Math.*, vol. 33, no. 4, pp.2509–2529, 2019.

[144] F. J. MacWilliams and N. J. A. Sloane, "The theory of error-correcting codes", *North-Holland*, Amsterdam, 1977.

[145] E. J. McCluskey, D. Burek, B. Koenemann, S. Mitra, J. H. Patel, J. Rajski, and J. A. Waicukauski, "Test Data Compression," *IEEE Design & Test of Computers*, vol. 20, no. 2, pp. 76–87, 2003.

[146] S. P. Mishchenko, "Lower bounds on the dimensions of irreducible representations of symmetric groups and of the exponents of the exponential of varieties of Lie algebras", *Mat. Sb.*, vol. 187, pp. 83–94, 1996.

[147] S. Mitra and K. S. Kim, "X-compact: An efficient response compaction technique," *IEEE Trans. Comput.-Aided Design Integr. Circuits Syst.*, vol. 23, no. 3, pp. 421–432, 2004.

[148] S. Mitra, S. S. Lumetta, M. Mitzenmacher, and N. Patil, "X-tolerant test response compaction," *IEEE Design & Test of Computers*, vol. 22, no. 6, pp. 566–574, 2005.

[149] B. Mounits, "Lower bounds on the minimum average distance of binary codes", *Discrete Math.*, vol. 308, no. 24, pp. 6241–6253, 2008.

[150] D. Mubayi and V. Rödl, "Specified intersections", *Trans. Amer. Math. Soc.*, vol. 366, no. 1, pp. 491–504, 2014.

[151] M. Y. Nam and H. Y. Song, "Binary locally repairable codes with minimum distance at least six based on partial t -spreads," *IEEE Communications Letters*, vol. 21, no. 8, pp. 1683–1686, 2017.

[152] B. Newton and B. Benesh, "A classification of certain maximal subgroups of symmetric groups", *Journal of Algebra*, vol. 304, no. 2, pp. 1108–1113, 2006.

[153] F. Oggier and A. Datta, "Self-repairing homomorphic codes for distributed storage systems," in *INFOCOM, 2011 Proceedings IEEE*, pp. 1215–1223, 2011.

[154] L. Pamies-Juarez, H. D. L. Hollmann, and F. Oggier, "Locally repairable codes with multiple repair alternatives," in *IEEE International Symposium on Information Theory*, pp. 892–896, 2013.

[155] D. S. Papailiopoulos and A. G. Dimakis, "Locally repairable codes," in *IEEE International Symposium on Information Theory*, pp. 2771–2775, 2012.

[156] D. S. Papailiopoulos, J. Luo, A. G. Dimakis, C. Huang, and J. Li, "Simple regenerating codes: Network coding for cloud storage," in *INFOCOM, 2012 Proceedings IEEE*, pp. 2801–2805, 2012.

[157] J. H. Patel, S. S. Lumetta, and S. M. Reddy, "Application of Saluja-Karpovsky compactors to test responses with many unknowns," in *Proc.21st IEEE VLSI Test Symp.*, pp. 107–112, 2003.

[158] B. Patkós, "On the general position problem on Kneser graphs", *arXiv:1903.08056v2*, 2019.

[159] J. S. Plank and M. Blaum, "Sector-disk (SD) erasure codes for mixed failure modes in RAID systems", *ACM Trans. Storage*, vol. 10, no. 1, pp. 4:1–4:17, 2014.

[160] N. Prakash, G. M. Kamath, V. Lalitha, and P. V. Kumar, "Optimal linear codes with a local-error-correction property," in *IEEE International Symposium on Information Theory*, pp. 2776–2780, 2012.

[161] N. Prakash, V. Lalitha, and P. V. Kumar, "Codes with locality for two erasures," in *IEEE International Symposium on Information Theory*, pp. 1962–1966, 2014.

[162] N. Prakash and M. Médard, "Communication Cost for Updating Linear Functions When Message Updates are Sparse: Connections to Maximally Recoverable Codes", *IEEE Trans. Inform. Theory*, vol. 64, no. 12, pp. 7557–7576, 2018.

[163] K. V. Rashmi, N. B. Shah, and P. V. Kumar, "Optimal exact-regenerating codes for distributed storage at the MSR and MBR points via a product-matrix construction," *IEEE Trans. Inform. Theory*, vol. 57, no. 68, pp. 5227–5239, 2011.

[164] A. S. Rawat, D. S. Papailiopoulos, A. G. Dimakis, and S. Vishwanath, "Locality and availability in distributed storage", *IEEE Trans. Inform. Theory*, vol. 62, no. 8, pp. 4481–4493, 2016.

[165] R. Rockafellar, "Convex Analysis", *Princeton Landmarks in Mathematics and Physics*, Princeton University Press, 1970.

[166] V. Rödl, "On a packing and covering problem," *European J. Combin.*, vol. 5, no. 1, pp. 69–78, 1985.

[167] P. A. Russell, "Compressions and probably intersecting families", *Combin. Probab. Comput.*, vol. 21, no. 1-2, pp. 301–313, 2012.

[168] A. Sali, "Some intersection theorems", *Combinatorica*, vol. 12, no. 3, pp. 351–361, 1992.

[169] K. K. Saluja and M. Karpovsky, "Testing computer hardware through data compression in space and time," in *Proc. Int. Test Conf.*, pp.83–88, 1983.

[170] B. Sasidharan, G. K. Agarwal, and P. V. Kumar, "Codes with hierarchical locality", in *IEEE International Symposium on Information Theory*, pages 1257–1261, 2015.

[171] M. Sathiamoorthy, M. Asteris, D. Papailiopoulos, A. G. Dimakis, R. Vadali, S. Chen, and D. Borthakur, "XORing elephants: Novel erasure codes for big data", *Proc. VLDB Endow.*, vol. 6, no. 5, pp. 325–336, Mar. 2013.

[172] A. Scott and E. Wilmer, "Combinatorics in the exterior algebra and the Bollobás two families theorem", *arXiv:1907.06019v2*, 2019.

[173] G. Seroussi and N. H. Bshouty, "Vector sets for exhaustive testing of logic circuits," *IEEE Trans. Inform. Theory*, vol. 34, no. 3, pp. 513–522, 1988.

[174] M. Shahabinejad, M. Khabbazian, and M. Ardakani, "A class of binary locally repairable codes," *IEEE Trans. Commun.*, vol. 64, no. 8, pp. 3182–3193, 2016.

[175] C. Shangguan and I. Tamo, "Sparse hypergraphs with applications to coding theory," *SIAM J. Discrete Math.*, vol. 34, no. 3, pp. 1493–1504, 2020.

[176] D. Shivakrishna, V. Arvind Rameshwar, V. Lalitha, and Birenjith Sasidharan, "On Maximally Recoverable Codes for Product Topologies," *arXiv1801.03379*, 2018.

[177] N. Silberstein, T. Etzion, and M. Schwartz, "Locality and availability of array codes constructed from subspaces", *IEEE Trans. Inform. Theory*, vol. 65, no. 5, pp. 2648–2660, 2019.

[178] N. Silberstein, A. S. Rawat, O. Koyluoglu, and S. Vishwanath, "Optimal locally repairable codes via rank-metric codes," in *IEEE International Symposium on Information Theory*, pp. 1819–1823, 2013.

[179] N. Silberstein and A. Zeh, "Optimal binary locally repairable codes via anticodes," in *IEEE International Symposium on Information Theory*, pp. 1247–1251, 2015.

[180] H. S. Snevily, "A sharp bound for the number of sets that pairwise intersect at k positive values", *Combinatorica*, vol. 23, no. 3, pp. 527–533, 2003.

[181] W. Song, K. Cai, C. Yuen, K. Cai, and G. Han, "On sequential locally repairable codes," *IEEE Trans. Inform. Theory*, 2017. [Online].

[182] W. Song, S. H. Dau, C. Yuen, and T. J. Li, "Optimal locally repairable linear codes," *IEEE Journal on Selected Areas in Communications*, vol. 32, no. 5, pp. 1019–1036, 2014.

[183] D. R. Stinson and R. Wei, "Some new upper bounds for cover-free families," *J. Comb. Theory, Ser. A*, vol. 90, no. 1, pp. 224–234, 2000.

[184] B. Sudakov, "Recent developments in extremal combinatorics: Ramsey and Turán type problems," in *Proceedings of the International Congress of Mathematicians*, Volume IV, pages 2579–2606, Hindustan Book Agency, New Delhi, 2010.

[185] M. Subramanian, W. Lloyd, S. Roy, C. Hill, E. Lin, W. Liu, S. Pan, S. Shankar, V. Sivakumar, L. Tang, and S. Kumar. "f4: Facebook's warm BLOB storage system," in *11th USENIX Symposium on Operating Systems Design and Implementation (OSDI)*, pages 383–398, 2014.

[186] I. Tamo and A. Barg, "A family of optimal locally recoverable codes", *IEEE Trans. Inform. Theory*, vol. 60, no. 8, pp. 4661–4676, 2014.

[187] I. Tamo, A. Barg, and A. Frolov, "Bounds on the parameters of locally recoverable codes", *IEEE Trans. Inform. Theory*, vol. 62, no. 6, pp. 3070–3083, 2016.

[188] I. Tamo, D. S. Papailiopoulos, and A. G. Dimakis, "Optimal locally repairable codes and connections to matroid theory", *IEEE Trans. Inform. Theory*, vol. 62, no. 12, pp. 6661–6671, 2016.

[189] H. Tanaka, "Classification of subsets with minimal width and dual width in grassmann, bilinear forms and dual polar graphs", *J. Comb. Theory, Ser. A*, vol. 113, no. 5, pp. 903–910, 2006.

[190] J. Talbot, "A new Bollobás-type inequality and applications to t-intersecting families of sets", *Discrete Math.*, vol. 285, no. 1–3, pp. 349–353, 2004.

[191] T. G. Tarján, "Complexity of lattice-configurations", *Studia Sci. Math. Hungar.*, vol. 10, no. 1–2, pp. 203–211, 1975.

[192] Y. Tsuboda and Y. Fujiwara, "Bounds and polynomial-time construction algorithm for X-codes of constat weight three," in *Proc. IEEE Int. Symp. Inf. Theory*, pp. 2515–2519, 2018.

[193] Y. Tsuboda ,Y. Fujiwara, H. Ando, and P. Vandendriessche, "Bounds on Separating Redundancy of Linear Codes and Rates of X-Codes," *IEEE Trans. Inform. Theory*, vol. 64, no. 12, pp. 7577–7593, 2018.

[194] Zs. Tuza, "Inequalities for two-set systems with prescribed intersections", *Graphs Combin.*, vol. 3, no. 1, pp. 75–80, 1987.

[195] Zs. Tuza, "Applications of the set-pair method in extremal hypergraph theory", *Extremal problems for finite sets (Visegr'ad, 1991), Bolyai Soc. Math. Stud.*, vol. 3, János Bolyai Math. Soc., Budapest, pp. 479–514, 1994.

[196] Zs. Tuza, "Applications of the set-pair method in extremal problems. II", *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993), Bolyai Soc. Math. Stud.*, vol. 2, János Bolyai Math. Soc., Budapest, pp. 459–490, 1996.

[197] X. M. Wang and Y. X. Yang, "On the undetected error probability of nonlinear binary constant weight codes," *IEEE Trans. Commun.*, vol. 42, no. 7, pp. 2390–2394, 1994.

[198] J. Wang and H. Zhang, "Cross-intersecting families and primitivity of symmetric systems", *J. Comb. Theory, Ser. A*, vol. 118, no. 2, pp. 455–462, 2011.

[199] J. Wang and H. Zhang, "Nontrivial independent sets of bipartite graphs and cross-intersecting families" *J. Comb. Theory, Ser. A*, vol. 120, no. 1, pp. 129–141, 2013.

[200] A. Wang and Z. Zhang, "Repair locality with multiple erasure tolerance", *IEEE Trans. Inform. Theory*, vol. 60, no. 11, pp. 6979–6987, 2014.

[201] A. Wang and Z. Zhang, "An integer programming-based bound for locally repairable codes," *IEEE Trans. Inform. Theory*, vol. 61, no. 10, pp. 5280–5294, 2015.

[202] A. Wang, Z. Zhang and D. Lin, "Bounds for Binary Linear Locally Repairable Codes via a Sphere-Packing Approach," *IEEE Trans. Inform. Theory*, vol. 65, no. 7, pp. 4167–4179, 2019.

[203] P. Wohl and L. Huisman, "Analysis and design of optimal combinational compactors," in *Proc. 21st IEEE VLSI Test Symp.*, pp. 101–106, 2003.

[204] Y. Wu, A. Dimakis, and K. Ramchandran, "Deterministic regenerating codes for distributed storage," in *Proceedings of Allerton Conference on Control, Computing and Communication*, pp. 1–5, 2007.

[205] Y. Wu and A. Dimakis, "Reducing repair traffic for erasure coding-based storage via interference alignment," in *IEEE International Symposium on Information Theory*, pp. 2276–2280, 2009.

[206] S. Xia and F. Fu, "On the average Hamming distance for binary codes", *Discrete Appl. Math.*, vol. 89, no. 1-3, pp. 269–276, 1998.

[207] S.-T. Xia, F.-W. Fu, and Y. Jiang, "On the minimum average distance of binary constant weight codes", *Discrete Math.*, vol. 308, no. 17, pp. 3847–3859, 2008.

[208] C. Xing and C. Yuan, "Construction of optimal locally recoverable codes and connection with hypergraph", in *46th International Colloquium on Automata, Languages, and Programming*, ICALP 2019, July 9-12, 2019, Patras, Greece, volume 132 of LIPIcs, pages 98:1 – 98:13. Schloss Dagstuhl - Leibniz-Zentrum f ¨ ur Informatik, 2019.

[209] L. Yu and V. Y. F. Tan, "An improved linear programming bound on the average distance of a binary code", *arXiv:1910.09416v1*, 2019.

[210] A. Zeh and E. Yaakobi, "Optimal linear and cyclic locally repairable codes over small fields," in *2015 IEEE Information Theory Workshop (ITW)*, pp. 1–5, 2015.

[211] H. Zhang, "Primitivity and independent sets in direct products of vertex-transitive graphs", *Journal of Graph Theory*, vol. 67, no. 3, pp. 218–225, 2011.

[212] H. Zhang, "Independent sets in direct products of vertex-transitive graphs", *J. Comb. Theory, Ser. B*, vo. 102, no. 3, pp. 832–838, 2012.

[213] G. Zhang, "A new construction of optimal (r, δ) locally recoverable codes", *IEEE Communications Letters*, vol. 24, no. 9, pp. 1852–1856, 2020.

[214] G. Zhang and H. Liu, "Constructions of optimal codes with hierarchical locality", *IEEE Trans. Inform. Theory*, vol. 66, no. 12, pp. 7333–7340, 2020.

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Research Results and Published Academic Papers

1. **Xiangliang Kong**, Yuanxiao Xi, and Gennian Ge, "Multi-part cross-intersecting families", submitted to *Journal of Algebraic Combinatorics.*

2. **Xiangliang Kong**, Jingxue Ma and Gennian Ge, "New Bounds on the Field Size for Maximally Recoverable Codes Instantiating Grid-like Topologies", to appear in *Journal of Algebraic Combinatorics*, 2021.

3. **Xiangliang Kong**, Xin Wang, and Gennian Ge, "New bounds and constructions for constant weighted X-codes", *IEEE Transactions on Information Theory*, vol. 67, no. 4, pp. 2181–2191, 2021.

4. Wenjun Yu, **Xiangliang Kong**, Yuanxiao Xi, Xiande Zhang, and Gennian Ge, "Bollobás type theorems for hemi-bundled two families", submitted to *European Journal of Combinatorics* (under second round review).

5. **Xiangliang Kong** and Gennian Ge, "On an inverse problem of the Erdős-Ko-Rado type theorems", submitted to *Journal of Combinatorial Theory, Series A.*

6. **Xiangliang Kong**, Yuanxiao Xi, Binchen Qian, and Gennian Ge, "Inverse problems of the Erdős-Ko-Rado type theorems for families of vector spaces and permutations", to appear in *SCIENCE CHINA Mathematics.*

7. **Xiangliang Kong**, Xin Wang, and Gennian Ge, "New constructions of optimal locally repairable codes with super-linear length", submitted to *IEEE Transactions on Information Theory.*

8. Yuanxiao Xi, **Xiangliang Kong**, and Gennian Ge, "Optimal Quaternary Locally Repairable Codes Attaining the Singleton-like Bound", preprint.

9. Yuanxiao Xi, **Xiangliang Kong**, and Gennian Ge, "k-optimal linear locally repairable codes with disjoint repair groups", in preparation.