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Group Divisible 3-Wise Balanced Designs: Theory and Applications

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可分组3-平衡设计：理论及应用

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摘　　要

可分组t-平衡设计在组合设计理论中有着极其重要的作用，并且被广泛应用于诸多领域。当$t = 2$时，可分组设计是当年组合设计理论奠基人Wilson和Hanani在构造成对平衡设计时所用的递归构造中不可缺少的组成部分。这些设计已被广泛研究。Hanani于1963年第一次提出了两类$t = 3$时的可分组设计，即烛台型设计和可分组3-设计。1994年，Hartman对$t = 3$时的可分组设计给出了更全面的解释说明，使其适用于推广的Wilson（和Hanani）基本构造，并用来构造3-平衡设计。其中可分组3-设计（下面称H-设计）在这个推广的基本构造中起到了重要的作用。

斯坦纳四元系是一类特殊的H-设计，有关斯坦纳四元系的研究可追溯到19世纪40年代。直到20世纪60年代，才由Hanani给出这类设计的存在性的两个完整证明。虽然Lenz（于1985年）和Hartman（于1994年）分别给出了它们的简化证明，但现已知的证明仍很繁琐。可分解的斯坦纳四元系，即每个组的大小都是1的可分解H-设计的存在性问题已经彻底解决。该工作是由Hartman，季利均和朱烈共同完成的，前后持续了二十年之久。到目前为止，可分解H-设计的一般存在性问题并没有新的结果。本文在第二、三章中不仅给出了斯坦纳四元系和可分解的斯坦纳四元系存在性的另一种证明，而且几乎彻底解决了可分解H-设计的存在性问题，并构造了一些型不一致H-设计的无穷类。由可分解H-设计的存在性结果，第二章还给出了另一类$t = 3$时的可分组设计，即可分解G-设计存在的充分必要条件，并顺便解决了最大可分解填充，最小可分解覆盖和一类一致可分解3-平衡设计的存在性问题，证明了这些设计存在的必要条件也是充分的。

作为3-平衡设计理论的应用，本文研究了组合群试和光纤网络领域中的两个公开问题。第四章彻底解决了由Jimbo等人提出的斯坦纳四元系的区组序列问题。该序列的元素和相邻并的集合所构成的码具有很好的纠错能力。在DNA实验室，这类序列被广泛应用于具有连续阳性显示的可纠错的组合群试中。第五章对于波分复用(WDM)光纤网络中最优容错路的设计进行了研究，成功地将最优容错路的设计问题转化为一类具有特殊性质的$\overrightarrow{P_3}$-设计的大集问题。

利用3-平衡设计理论和可划分烛台型设计，本章几乎解决了整个最优容错路设计问题的三分之一。

关键词： BSCU，烛台型设计，容错路，H-设计，H-标架，LELD，可分解，斯坦纳四元系

Abstract

Group divisible t-wise balanced designs are of utmost importance in combinatorial design theory, and have been widely used in many areas. For $t = 2$, group divisible designs were an essential ingredient in the recursive constructions used in the seminal works of Wilson and Hanani (two of the founders of combinatorial design theory), which established necessary and sufficient conditions for the existence of pairwise balanced designs. Much work has been done on such designs. For $t = 3$, two definitions for 3-analogues of group divisible designs – candelabra quadruple systems and group divisible 3-designs were first introduced by Hanani in 1963. In 1994, Hartman gave a more comprehensive account of the 3-analogues of group divisible designs, which were applicable for the generalization of Wilson's (and Hanani's) fundamental constructions to produce 3-wise balanced designs. In these 3-analogues of the fundamental constructions, group divisible 3-designs (called H-designs in the sequel) are also used as essential ingredients.

The research on Steiner quadruple systems – a special class of H-designs with each group of size one can be traced back to 1840s. The first and second complete proofs for the existence of such designs were given by Hanani in 1960s. All the existing proofs are rather cumbersome, even though simplified proofs have been given by Lenz in 1985 and by Hartman in 1994. For the existence of Steiner quadruple systems with resolvability, the known complete solution was obtained by a joint effort of Hartman, Ji and Zhu over twenty years long. As for the general existence of resolvable H-designs, however, not much is known. In Chapters 2 and 3 of this dissertation, not only do we provide alternative existence proofs for Steiner quadruple systems and resolvable Steiner quadruple systems, we give an almost complete solution to the general existence problem of resolvable H-designs and construct several infinite classes of nonuniform H-designs. As a consequence of the existence result of resolvable H-designs, we establish the necessary and sufficient conditions for the existence of resolvable G-designs in Chapter 2, which

is another kind of 3-analogue of group divisible designs. As a byproduct, we also show the existence of maximal resolvable packings of triples by quadruples, minimal resolvable coverings of triples by quadruples and a class of uniformly resolvable Steiner systems.

As applications of the theory of group divisible 3-wise balanced designs, two open problems in group testing and optical networks are also discussed. First, we give a complete solution to the problem posed by Jimbo et al. on the block sequences of Steiner quadruple systems with error correcting consecutive unions in Chapter 4. Such sequences are useful when considering the error detecting and correcting capability of combinatorial group testing for consecutive positives, which is essential in view of applications such as DNA library screening. Then in Chapter 5, we investigate the design of fault-tolerant routings with levelled minimum optical indices, which plays an important role in the wavelength division multiplexing optical networks. By introducing the new concept of a large set of even levelled $\overrightarrow{P_3}$-design, we solve nearly one-third of the existence problem for optimal routings with levelled minimum optical indices based on the theory of 3-wise balanced designs and partitionable candelabra systems.

Keywords: BSCU, candelabra systems, fault-tolerant routings, H-designs, H-frames, LELD, resolvable, Steiner quadruple systems

Contents

Chapter 1

Introduction

In this thesis, we focus on the construction of group divisible 3-wise balanced designs. We give an almost complete solution to the general existence problem of resolvable H-designs. Two applications of the theory of 3-wise balanced designs in group testing and optical networks are also discussed.

1.1 Background

A *t-wise balanced design* (tBD) of type t-(v, K, λ) is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of subsets of X (*blocks*) with the property that the size of every block is in the set K and every t-subset of X is contained in exactly λ blocks. A t-(v, K, λ) design is also denoted by $S_\lambda(t, K, v)$ or by $B_t[K, \lambda; v]$. If $K = \{k\}$, we simply write k for K and the tBD is called a *t-design*. If $\lambda = 1$, the notation $\mathrm{S}(t, K, v)$ is often used and the design is called a *Steiner system*.

A 2-wise balanced design is also called a *pairwise balanced design (PBD)*. A *group divisible design (GDD)* is a triple $(X, \mathcal{G}, \mathcal{B})$ with the property that $(X, \mathcal{G} \cup \mathcal{B})$ is a PBD and $\mathcal{G}$ is a partition of X into holes. PBDs and GDDs have been studied extensively [63], and they have been used to obtain constructions for various combinatorial designs, geometries, and orthogonal arrays. GDDs were an essential ingredient in the recursive constructions used in the seminal works of Wilson [67, 68] and Hanani [24, 26, 27] that established necessary and sufficient conditions for the existence of Steiner systems $\mathrm{S}(2, K, v)$.

An $\mathrm{S}(3, 4, v)$ is called a *Steiner quadruple system* of order v, denoted by $\mathrm{SQS}(v)$. The necessary conditions for the existence of an $\mathrm{SQS}(v)$ are that $v \equiv 2, 4 \pmod 6$ or $v = 1$. When $v < 4$, the system has no blocks, and when $v = 4$, it has one block. The smallest interesting system, SQS(8), was known to Kirkman [48] in 1847. The unique (up to isomorphism) SQS(10) was attributed to Barrau [4] as early as 1908 and to Richard Wilson in [12]. Several infinite families of quadruple systems were constructed by Kirkman [48] and by Carmichael [11]. The first complete proof for the existence of SQS(v) for all $v \equiv 2, 4 \pmod 6$ was given by Hanani [23] in 1960. The result is proved by induction using six recursive constructions together with explicit constructions of an SQS(14) and an SQS(38). In 1963, Hanani [25] gave a more sophisticated proof for the existence of SQS(v), where two definitions for 3-analogues of group divisible designs – candelabra quadruple systems and group divisible 3-designs (Hanani used quite different terminology) were first introduced. Apart from Hanani's two proofs, Hartman [31, 32, 34] and Lenz [51] used the existence of candelabra quadruple systems of type $(g^3 : s)$ with $s \in \{1, 2, 4, 8\}$ to give a purely tripling existence proof, which used only one type of construction and a small number of initial designs: SQS(v) with $v \in \{8, 10, 14\}$ and HQS($v : 8$) with $v \in \{26, 28, 32, 34, 38, 40\}$. For more information on Steiner quadruple systems, see the excellent survey paper by Hartman and Phelps [36].

In 1994, Hartman [34] gave a comprehensive account of the 3-analogues of group divisible designs, which were used for the generalization of Wilson's (and Hanani's) fundamental constructions to produce Steiner systems S$(3, K, v)$. In this 3-analogues of the fundamental constructions, the most difficult thing is to find appropriate definitions of "master" and "slave" 3-analogues of group divisible designs. One feasible definition for 3-analogues of group divisible design is: $(X, \mathcal{G}, \mathcal{B})$ is a *G-design*, if $(X, \mathcal{G} \cup \mathcal{B})$ is a 3-wise balanced design and $\mathcal{G}$ is a partition of X into holes. This definition relates to Mills' definition [55] of a G-design $G(m, r, k, 3)$ and Hanani's definition [25, Definition 2] of the systems $P_m[K, 1, v]$. A G-design tends to be used for "master" designs in the 3-analogues of the fundamental construction. Another important definition is: $(X, \mathcal{G}, \mathcal{B})$ is a *tripartite design* if $\mathcal{G}$ is a partition of X into holes and every 3-element transverse of $\mathcal{G}$ is contained in a unique block. Here a *transverse* of $\mathcal{G}$ is a subset of X intersects each hole in at most one point. This definition contains Hanani's definition [25, Definition 3] of the systems $P''_m[K, 1, v]$, which was introduced for the first time to construct Steiner quadruple systems SQS(v). It also relates to Hanani's definition [28] of the transversal 3-design $T_3[s, 1; r]$, Mills' definition [55] of H-designs $H(m, r, k, 3)$ and Lens' definition [52] of the divisible 3-design $D3D[k, r; v]$. A tripartite design tends to be appropriate for "slave" designs in the 3-analogues of the fundamental construction. Instead of the tripartite design, we use the more general definition of group divisible t-design which was first introduced by Hedvig Mohácsy and D.K. Ray-Chaudhuri in [58].

Let v be a non-negative integer, t be a positive integer and K be a set of positive integers. A *group divisible t-design* of order v with block sizes from K, denoted by GDD(t, K, v), is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of v elements (called *points*);

(2) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;

(3) $\mathcal{B}$ is a family of transverses (called *blocks*) of $\mathcal{G}$, each of cardinality from K;

(4) every t-element transverse T of $\mathcal{G}$ is contained in a unique block.

The *type* of the GDD(t, K, v) is defined as the list $(|G| | G \in \mathcal{G})$. If there are n_i groups of size g_i, $1 \le i \le r$, then we denote the group type by $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$. A GDD($t, K, v$) is called *uniform* if all groups have the same size. Note that when $t = 2$, the group divisible 2-design is the classical group divisible design. It is clear that an S(t, K, v) is a GDD(t, K, v) of type 1^v. Mills [57] used H(n, g, k, t) design to denote the GDD(t, k, ng) of type g^n. In the sequel of this thesis, we use H($g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$) to denote the GDD($3, 4, \sum n_i g_i$) of type $g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r}$ for short.

For the existence of uniform H-designs, Mills [57] in 1990, showed that for $n > 3$, $n \neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for $n = 5$, an H(g^5) exists if g is divisible by 4 or 6. Recently, Ji [42] improved these results by showing that an H(g^5) exists whenever g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$. We summarize the results as follows:

Theorem 1.1.1. ([42, 57]) *For $n > 3$ and $n \neq 5$, an $H(g^n)$ exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3. For $n = 5$, an $H(g^n)$ exists when g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$.*

For the nonuniform H-designs, however, not much is known yet. Recently, Lauinger et. al. [50] provided a table of existence results for H-designs of all types when the number of points $v \leq 24$. Keranen and Kreher [47] gave an investigation of H(g^4u^1) with five holes.

An H(g^n) is said to be *resolvable*, denoted by RH(g^n), if its block set $\mathcal{B}$ can be partitioned into *parallel classes* $P_1, P_2, \ldots, P_r$, each of which is a partition of the point set. In this case, we call $P_1|P_2|\ldots|P_r$ a *resolution* of $\mathcal{B}$.

When $g = 1$, an RH(1^n) is called a *resolvable Steiner quadruple system* of order n, denoted by RSQS(n). The necessary conditions for the existence of an RSQS(v) are that $v \equiv 4$ or $8 \pmod{12}$ or $v = 1$ or 2. In 1977, the only orders for which an RSQS(v) was known were $v = 2^n$, and the only recursive construction known was the doubling construction (i.e., a construction of an RSQS$(2v)$ from an RSQS(v)). In 1978, Booth [8] and Greenwell and Lindner [20] provided the first examples with v not a power of two by constructing an RSQS(20) and an RSQS(28). More examples were given by Hartman [29], where he constructed RSQS$(q+1)$ for all prime powers $q \equiv 7 \pmod{12}$ with $q \leq 379$, and RSQS$(4p)$ for $p \in \{19, 43, 127, 199, 223, 271, 1603\}$ [30].

The main recursive theorems for RSQS(v), i.e., two tripling constructions were provided by Hartman in [31, 33], both of which assume some subsystem structures on the input systems. Using the doubling and two tripling constructions together with a large number of initial designs, Hartman [33] proved by induction that the necessary condition $v \equiv 4$ or $8 \pmod{12}$ for the existence of a resolvable SQS(v) is also sufficient for all values of v, with 23 possible exceptions. These last 23 undecided orders were removed by Ji and Zhu [43] in 2005 by using resolvable H-designs and resolvable candelabra systems. They also constructed an RH(2^n) for each $n \in \{10, 14, 26, 146\}$ and showed the existence of RH(g^4) for all positive integers g. We summarize the results as follows:

Theorem 1.1.2. [33, 43] *There exists an RH(1^n) for each $n \equiv 4, 8 \pmod{12}$,* *an* $RH(2^n)$ *for each* $n \in \{10, 14, 26, 146\}$ *and an* $RH(g^4)$ *for all integer* $g > 0$.

Group divisible 3-wise balanced designs have attracted more researchers for its various applications in combinatorial group testing for consecutive positives [59], in the design of fault-tolerant routings in the context of optical networks [15] and in the construction of optimal constant weight codes [10].

Group testing was proposed by Dorfman [16] in 1940s to do large scale blood testing economically, and new applications of group testing have been found recently in the fields such as *DNA library screening*, being error-prone, in which it is desired to determine the set of all positive clones in an economical and correct way. In 1999, Colbourn [13] developed some strategy for group testing when the clones are *linearly ordered* and the positive clones form a *consecutive subset* of the set of all clones. Jimbo and his collaborators [60, 59, 61, 62] improved Colbourn's strategy by considering the error detecting and correcting capability of group testing which is essential in view of applications such as DNA library screening. Especially, Momihara and Jimbo [60, 59] suggested using a block sequence with consecutive unions having minimum distance d (BSCU$(t, k, v|d)$) to correct *false negative* or *false positive* clones in the pool outcomes. For more details of the progress we refer to [13, 17, 60, 59, 61, 62, 64] and references there in. In the cases of $d = 2$ and $d = 3$, systematic results about the existence of BSCU$(t, k, v|d)$s can be found in [62, 60]. For the case of $d = 4$, Momihara and Jimbo [59] recently showed the existence of a BSCU$(3, 4, v|4)$ for forty-seven small values $v \leq 500$.

The design of routings in optical networks has been a topic of considerable recent interest (see, for examples, [2, 5, 6, 7, 53]). In the model of *WDM optical networks*, namely, wavelength division multiplexing optical networks, routing nodes are joined by fiber-optic links, and each link can support some fixed number of wavelengths. Each routing path uses a particular wavelength, and two paths must use different wavelengths if they have common links. Most research concentrates on determining the minimum total number of wavelengths used in the network, which is related to two basic invariants – the *arc-forwarding* and *optical indices*. The f*-tolerant arc-forwarding* and f*-tolerant optical indices* were introduced by Maňuch and Stacho when they considered the fault-tolerant issues in [53]. The parameter f represents the number of faults that can be tolerated in the optical network. The design of fault-tolerant routings with levelled minimum optical indices plays an important role in the context of optical networks. However, not much is known for the existence of optimal routings with levelled minimum optical indices besides the results established by Dinitz, Ling and Stinson [15] via the partitionable Steiner quadruple systems approach.

Theorem 1.1.3. [15] *For each n, $5 \leq n \leq 8$, $n = 4^k$ or $n = 2(p^k + 1)$ with $p \in \{7, 31\}$, there exists an optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$ that has levelled minimum optical indices.*

1.2 Main Results

In this thesis, we give a near complete solution to the general existence problem of resolvable H-designs and construct several infinite classes of nonuniform H-designs. As applications, two problems in group testing and optical networks are also discussed.

In Chapter 2, we first describe several recursive constructions for resolvable H-designs based on the theory of uniformly resolvable candelabra systems and resolvable H-frames. In particular, we will introduce a simple but powerful construction–group halving construction, as well as the product construction, doubling construction and three tripling constructions. Combining several initial designs together with the recursive methods, we establish the main result as follows.

Theorem 1.2.1. *The necessary conditions $gn \equiv 0 \pmod 4$, $g(n-1)(n-2) \equiv 0 \pmod 3$ and $n \geq 4$ for the existence of a resolvable H-design of type g^n are sufficient for each $g \equiv 1, 2, 3, 5, 6, 7, 9, 10, 11 \pmod{12}$, are sufficient for each $g \equiv 4, 8 \pmod{12}$ with two possible exceptions $n = 73, 149$, and are sufficient for each $g \equiv 0 \pmod{12}$ with sixteen possible exceptions $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 213, 231, 243, 321\}$.*

As a corollary of the existence of RH(2^n)s, we provide an alternative existence proof for resolvable SQS(v)s, for which the known complete solution was obtained by a joint effort of Hartman [31, 33] and Ji and Zhu [43] over twenty years long. As consequences of the above result, we show the existence of several related designs.

Theorem 1.2.2. (i) *The necessary conditions $g = 1$ and $n \equiv 4$ or $8 \pmod{12}$, or g is even, $gn \equiv 0 \pmod 4$ and $g(n-1)(n-2) \equiv 0 \pmod 3$ for the existence of a resolvable G-design of type g^n are also sufficient.*

(ii) *A maximal resolvable packing (minimal resolvable covering) of triples by quadruples of order v with the number of blocks meeting the upper (lower) bound exists if and only if $v \equiv 0 \pmod 4$.*

(iii) *There exits a uniformly resolvable Steiner system $URS(3, \{3, 4\}, \{r_3, r_4\}, v)$ with $r_3 = 4$ if and only if $v \equiv 0 \pmod{12}$.*

All the known proofs for the existence of Steiner quadruple systems are rather cumbersome, even though simplified proofs have been given by Hartman [31, 32, 34], Lenz [51] et al., and much attention has been paid on the proofs of the existence of SQS(v). In Chapter 3, we mainly provide an alternative existence proof for Steiner quadruple systems by reestablishing the existence of H-designs of type 2^n based on the theory of candelabra systems and H-frames. By this approach, several new infinite classes of nonuniform H-designs of types $2^n u^1$ with $u = 4, 6, 8$ are also constructed.

Theorem 1.2.3. (i) *There exists an $H(2^n 4^1)$ if and only if $n \equiv 1 \pmod 3$ and $n \geq 4$.*

(ii) *There exists an $H(2^n 6^1)$ for each $n \equiv 1 \pmod 6$ and $n \geq 7$.*

(iii) *There exists an $H(2^n 8^1)$ for each $n \equiv 0, 1, 3, 6, 7, 12, 13, 16 \pmod{18}$, $n \geq 6$ except possibly for $n = 12, 34$.*

In Chapter 4, we give an application of the theory of 3-wise balanced design in the construction of block sequences of Steiner quadruple systems with consecutive unions having minimum distance 4, denoted by BSCU(v). The only orders for which a BSCU(v) was known were the forty-seven small values $v \leq 500$ established by Momihara and Jimbo [59]. By the theory of 3-wise balanced design, we completely determine the existence of BSCU(v)s as follows.

Theorem 1.2.4. *The necessary conditions for the existence of a BSCU(v), namely, $v \equiv 2, 4 \pmod 6$ and $v \geq 4$, are also sufficient with two exceptions $v = 8, 10$.*

In Chapter 5, we give another application of the theory of 3-wise balanced design in the design of fault-tolerant routings with levelled minimum optical indices. Not much is known for the existence of such routings besides the results established in Theorem 1.1.3 by Dinitz, Ling and Stinson [15]. By introducing the new concept of a large set of even levelled $\overrightarrow{P_3}$-design of order v and index 2 ($(v, \overrightarrow{P_3}, 2)$-LELD), and by the theory of 3-wise balanced designs and partitionable candelabra systems, the existence problem for an optimal, levelled $(v-2)$-fault tolerant routing with levelled minimum optical indices of the complete network with v nodes is solved nearly one-third.

Theorem 1.2.5. *For each positive integer n, $4 \leq n \leq 11$ or $n \geq 14$, $n \equiv k \pmod{144}$ with $k \in \{2, 6, 8, 11, 14, 18, 20, 22, 23, 30, 32, 34, 38, 44, 46, 47, 50, 54, 56, 59, 62, 66, 68, 70, 78, 80, 82, 83, 86, 92, 94, 95, 98, 102, 104, 110, 114, 116, 118, 119, 126, 128, 130, 131, 134, 140, 142\}$ and $n \neq 34, 50$, there exists an $(n, \overrightarrow{P_3}, 2)$-LELD and an optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$ that has levelled minimum optical indices.*

Chapter 2

Resolvable H-designs

In this chapter, we mainly investigate the general existence problem of resolvable H-designs and give a near complete solution to this problem. As applications, not only do we provide an alternative existence proof for resolvable SQS(v)s, we establish the existence results for several related designs, such as resolvable G-designs, maximal resolvable packings, minimal resolvable coverings of triples by quadruples and a class of uniformly resolvable Steiner systems.

2.1 Introduction

According to the necessary conditions for the existence of an RH(g^n), the general existence problem of RH(g^n) can be separated into the following six parts:

(i) $g \equiv 1, 5, 7, 11 \pmod{12}$ and $n \equiv 4, 8 \pmod{12}$,

(ii) $g \equiv 2, 10 \pmod{12}$ and $n \equiv 2, 4 \pmod{6}$,

(iii) $g \equiv 3, 9 \pmod{12}$ and $n \equiv 0 \pmod{4}$,

(iv) $g \equiv 4, 8 \pmod{12}$ and $n \equiv 1, 2 \pmod{3}$,

(v) $g \equiv 6 \pmod{12}$ and $n \equiv 0 \pmod{2}$,

(vi) $g \equiv 0 \pmod{12}$ and $n \in N$.

Theorem 2.1.1. (Weighting Construction) *If there exists an RH(g^n), then there is an RH($(mg)^n$) for any positive integer m.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given RH(g^n) with $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$ and a resolution of $\mathcal{B}$, $P(i)$, $1 \leq i \leq (n-1)(n-2)g^2/6$. For each positive integer m, we will construct an $\mathrm{RH}((mg)^n)$ on $X \times Z_m$ with groups $G_i \times Z_m$, $0 \le i \le n-1$ as follows.

For each block $B \in \mathcal{B}$, construct an $\mathrm{RH}(m^4)$ on $B \times Z_m$ with group set $\{x \times Z_m : x \in B\}$ and block set $\mathcal{A}_B$, which has a resolution $P_B(k)$, $1 \le k \le m^2$. Such a design exists by Theorem 1.1.2. Let $\mathcal{B}' = \cup_{B \in \mathcal{B}} \mathcal{A}_B$. Then $\mathcal{B}'$ is the block set of an $\mathrm{RH}((mg)^n)$, which has a resolution $Q_{i,k} = \cup_{B \in P(i)} P_B(k)$, where $1 \le i \le (n-1)(n-2)g^2/6$ and $1 \le k \le m^2$. □

By the Weighting Construction above, the whole existence problem of $\mathrm{RH}(g^n)$ depends on the solution of the following six cases:

(1) $g = 1$ and $n \equiv 4, 8 \pmod{12}$,

(2) $g = 2$ and $n \equiv 2, 4 \pmod{6}$,

(3) $g = 3$ and $n \equiv 0 \pmod{4}$,

(4) $g = 4$ and $n \equiv 1, 2 \pmod{3}$,

(5) $g = 6$ and $n \equiv 0 \pmod{2}$,

(6) $g = 12$ and $n \in N$.

Here, the only solved case is Case (1) by Theorem 1.1.2.

A regular graph (V, E) of degree k is said to have a *one-factorization* if the edge set E can be partitioned into k parts $E = F_1|F_2|\ldots|F_k$ so that each F_i is a partition of the vertex set V into pairs. The parts F_i are called *one-factors*. For all even integers n, the complete graph on n vertices K_n has a one-factorization.

Theorem 2.1.2. (Group Halving Construction) *If there exists an RH*$((2g)^n)$*, then there exists an RH*(g^{2n})*.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $\mathrm{RH}((2g)^n)$ with $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$. Therefore, gn is even. Halve each group G_i into G_{i0} and G_{i1}, $0 \le i \le n-1$. We will construct an $\mathrm{RH}(g^{2n})$ on the group set $\mathcal{G}' = \{G_{ij} |\ 0 \le i \le n-1, j = 0, 1\}$ as follows.

For each i, $0 \le i \le n-1$, let $\mathcal{F}^i = \{F_1^i, \ldots, F_g^i\}$ be a one-factorization of the bipartite graph on $G_{i0} \cup G_{i1}$. Let

$$\mathcal{A} = \{\{a,b,c,d\} : \{a,b\} \in F_j^i, \{c,d\} \in F_j^{i'}, 0 \le i, i' \le n-1, 1 \le j \le g\},$$

then $\mathcal{B}' = \mathcal{B} \cup \mathcal{A}$ is the block set of an H(g^{2n}) on the group set $\mathcal{G}'$. It remains to show that $\mathcal{A}$ can be partitioned into parallel classes.

For each j, $1 \le j \le g$, let

$$\begin{aligned} \mathcal{A}_j =& \{\{a,b,c,d\} : \{a,b\} \in F_j^i, \{c,d\} \in F_j^{i'}, 0 \le i, i' \le n-1\}, \text{ and} \\ \mathcal{A}'_j =& \{\{\{a,b\},\{c,d\}\} : \{a,b\} \in F_j^i, \{c,d\} \in F_j^{i'}, 0 \le i, i' \le n-1\}. \end{aligned}$$

If we regard each pair in $F_j^i, 0 \le i \le n-1$ as a vertex, we may construct a multi-partite complete graph Γ_j on the vertex set $X_j = \cup_{i=0}^{n-1} F_j^i$ with partite set $\{F_j^i : 0 \le i \le n-1\}$, where two different vertices connect if and only if they are from different factors F_j^i. Hence, $\mathcal{A}'_j$ is the edge set of Γ_j. That is to say we obtain a GDD$(2,2,gn)$ of type g^n on X_j with group set $\{F_j^i, 0 \le i \le n-1\}$ and block set $\mathcal{A}'_j$.

It is well-known that there always exists a resolvable GDD$(2,2,gn)$ of type g^n when gn is even (see [14]). Hence, we can partition the blocks $\mathcal{A}'_j$ of our resulting GDD$(2,2,gn)$ of type g^n into parallel classes on X_j. Therefore, $\mathcal{A}_j$ can also be partitioned in parallel classes of X. So does $\mathcal{A} = \cup_{1 \le j \le g} \mathcal{A}_j$. This completes the proof. □

By the Group Halving Construction above, the existence problems of the six cases have the following recursive relations.

$$RH(4^n) \Longrightarrow RH(2^{2n}) \Longrightarrow RH(1^{4n})$$

$$RH(12^n) \Longrightarrow RH(6^{2n}) \Longrightarrow RH(3^{4n})$$

Hence, the general existence problem of RH(g^n) depends on the following two cases:

(4) $g = 4$ and $n \equiv 1, 2 \pmod 3$,

(6) $g = 12$ and $n \in N$.

Moreover, the solution of Case (4) implies two-thirds of that of Case (6) by the Weighting Construction.

The remainder of this chapter is organized as follows. In Section 2.2, we will describe several recursive constructions for resolvable H-designs based on the theory of uniformly resolvable candelabra systems and resolvable H-frames. Combining several initial designs together with the recursive methods established in Section 2.2, we give an almost complete solution to the existence problem of RH(4^n) in Section 2.3. As consequences of this result, we show the necessary and sufficient conditions of RH(2^n), RH(6^n) and RH(3^n) successively in Section 2.4. Thus we provide an alternative existence proof for resolvable SQS(v)s. Furthermore, we show the existence of resolvable G-designs, maximal resolvable packings of triples by quadruples, minimal resolvable coverings of triples by quadruples as well as a class of uniformly resolvable Steiner systems. Finally in Section 2.5, combining the recursive methods established in Section 2.2 and the existence results of resolvable H-designs and resolvable G-designs established in Sections 2.3 and 2.4, we show the necessary conditions for the existence of RH(12^n) are also sufficient with sixteen possible exceptions.

2.2 Recursive Constructions

In this section, we shall describe several recursive constructions for resolvable H-designs. First, we need the following definitions and notation.

Let s be a non-negative integer. A *candelabra t-system* (or t-CS) of order v and block sizes from K, denoted by CS(t, K, v), is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements;

(2) S is an s-subset (called the *stem* of the *candelabra*) of X;

(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X \setminus S$, which partition $X \setminus S$;

(4) $\mathcal{A}$ is a collection of subsets of X, each of cardinality from K;

(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$, for all i, is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$, for any i, is contained in any block of $\mathcal{A}$.

The *group type* of a t-CS $(X, S, \mathcal{G}, \mathcal{A})$ is defined as the list $(|G||G \in \mathcal{G} : |S|)$. If a t-CS has n_i groups of size g_i, $1 \leq i \leq r$, and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ to denote the group type. A candelabra system with $t = 3$ and $K = \{4\}$ is called a *candelabra quadruple system* and denoted by CQS$(g_1^{n_1} g_2^{n_2} \cdots g_r^{n_r} : s)$.

A CS(t, K, v) $(X, S, \mathcal{G}, \mathcal{A})$ is said to be *resolvable*, denoted by RCS(t, K, v), if the block set $\mathcal{A}$ can be partitioned into several parts, each being a partition on X or a partition on $X \setminus (G \cup S)$ for some $G \in \mathcal{G}$ (called a *partial* parallel class). An RCS(t, K, v) is called *uniform*, denoted by URCS(t, K, v) if all the blocks in each resolution class have the same size. If $K = \{4\}$, it is denoted by RCQS, for which the number of parallel classes on X is $((\sum_{G \in \mathcal{G}} |G|)^2 - \sum_{G \in \mathcal{G}} |G|^2)/6$ and the number of partial parallel classes on $X \setminus (G \cup S)$ is $|G|(|G| + 2|S| - 3)/6$ for each $G \in \mathcal{G}$.

Theorem 2.2.1. [54] *For each integer $n \geq 2$, there exists an RCQS$(3^{(2^{2n}-1)/3} : 1)$.*

For non-negative integers q, g, k, and t, an $H(q, g, k, t)$ *frame* (as in [35]), denoted by HF(q, g, k, t), is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of qg points;

2. $\mathcal{G} = \{G_1, G_2, \ldots, G_q\}$ is an equipartition of X into q groups;

3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its size; and

4. $\mathcal{B}$ is a set of k-element transverses of $\mathcal{G}$ with the property that every t-element transverse of $\mathcal{G}$, which is not a t-element transverse of any hole $F_i \in \mathcal{F}$ is contained in precisely one block of $\mathcal{B}$, and no block contains a t-element transverse of any hole.

If an HF$(q, g, 4, 3)$ has n holes of size $m+s$, which intersect on a common hole of size s, then we denote such a design by HF$(m^n : s)$ with group size g, or shortly by HF$_g(m^n : s)$. If an HF$(q, g, 4, 3)$ has only one hole of size s, then we call it an *incomplete H-design* of type $(g^q : g^s)$, denoted by IH$(g^q : g^s)$.

An HF$_g(m^n : s)$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with $\mathcal{F} = \{F_i : 0 \le i \le n\}$ and F_0 the common hole of size s is said to be *resolvable*, denoted by RHF$_g(m^n : s)$, if its block set can be partitioned into $(nmg^2(m + 2s - 3) + n(n - 1)(mg)^2)/6$ parts with the following properties:

(1) For each hole F_i, $1 \le i \le n$, there are exactly $mg^2(m + 2s - 3)/6$ parts, each being a partition of $X \setminus (\bigcup_{G \in F_i} G)$;

(2) There are $n(n - 1)(mg)^2/6$ parts, each being a parallel class on X.

An IH$(g^{m+s} : g^s)$ $(X, \mathcal{G}, \mathcal{B}, F)$ with the only hole F of size s is said to be *resolvable*, denoted by IRH$(g^{m+s} : g^s)$, if its block set can be partitioned into $(m + s - 1)(m + s - 2)g^2/6$ parts, $(s - 1)(s - 2)g^2/6$ of which are partitions of $X \setminus (\bigcup_{G \in F} G)$, and $m(m + 2s - 3)g^2/6$ of which are parallel classes on X.

The construction given below is a generalization of the fundamental construction for 3-wise balanced designs.

Theorem 2.2.2. *Suppose that $(X, S, \Gamma, \mathcal{A})$ is a 3-CS$(m^n : s)$ and $\infty \in S$. Let $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ and $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. If there exists an HF$_g(t^{k_1 - 1} : a)$ for each $k_1 \in K_1$ and an H$((gt)^{k_2})$ for each $k_2 \in K_2$, then there exists an HF$_g((tm)^n : t(s-1)+a)$. Furthermore, if the 3-CS$(m^n : s)$ is uniformly resolvable, and each of HF$_g(t^{k_1 - 1} : a)$ and H$((gt)^{k_2})$ for $k_1 \in K_1$ and $k_2 \in K_2$ is resolvable, then the resultant HF$_g((tm)^n : t(s - 1) + a)$ is also resolvable.*

Proof. Suppose $(X, S, \Gamma, \mathcal{A})$ is the given URCS$(m^n : s)$, where $\Gamma = \{G_1, \dots, G_n\}$ and $\mathcal{A}$ has a resolution $\mathcal{A} = (\bigcup_{1 \le i \le n} \mathcal{Q}_i) \bigcup \mathcal{Q}$ with each member of $\mathcal{Q}_i$ being a partition of $X \setminus (G_i \cup S)$ and each member of $\mathcal{Q}$ being a partition of X. Define $G'_{x,j} = \{x\} \times \{j\} \times Z_g$. Let $X' = ((X \setminus \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$, $\mathcal{G}' = \{G'_{x,j} : x \in X \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$, $\mathcal{F} = \{F_i : 0 \le i \le n\}$, where $F_0 = \{G'_{x,j} : x \in S \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and $F_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F_0$ for $1 \le i \le n$. We will construct an $\mathrm{RHF}_g((tm)^n : t(s-1)+a)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}$.

For each $B \in \mathcal{A}$ and $\infty \in B$, construct an $\mathrm{RHF}_g(t^{|B|-1} : a)$ on $X'_B = ((B \setminus \{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$ with group set $\mathcal{G}'_B = \{G'_{x,j} : x \in B \setminus \{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and hole set $\mathcal{F}_B = \{F_x : x \in B\}$, where $F_x = \{G'_{x,j} : j \in Z_t\} \cup F_\infty$ with $F_\infty = \{G'_{\infty,j} : j \in Z_a\}$ being the common hole of size a. Denote its block set by $\mathcal{C}_B$, which has a resolution $\{\mathcal{C}_B(x,j) : x \in B \setminus \{\infty\}, 1 \le j \le tg^2(t+2a-3)/6\} \cup \{\mathcal{C}_B(l) : 1 \le l \le (|B|-1)(|B|-2)(tg)^2/6\}$ with each $\mathcal{C}_B(x,j)$ being a partition of $X'_B \setminus (\bigcup_{G \in F_x} G)$ and each $\mathcal{C}_B(l)$ being a parallel class on X'_B.

For each $B \in \mathcal{A}$ and $\infty \notin B$, construct an $\mathrm{RH}((gt)^{|B|})$ on $X'_B = B \times Z_t \times Z_g$ with group set $\mathcal{G}'_B = \{\{x\} \times Z_t \times Z_g : x \in B\}$ and block set $\mathcal{C}_B$, which can be partitioned into parallel classes $\mathcal{C}_B(l)$, $1 \le l \le (|B|-1)(|B|-2)(tg)^2/6$.

Then $\mathcal{A}' = \bigcup_{B \in \mathcal{A}} \mathcal{C}_B$ is the block set of the required design. We need to partition the blocks into resolution classes.

For each member $Q \in \mathcal{Q}_i$, $1 \le i \le n$, suppose its block size is k_Q. Then $P_Q(l) = \bigcup_{B \in Q} \mathcal{C}_B(l)$ is a partition of $X' \setminus (\bigcup_{G \in F_i} G)$ for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6$.

For each $x \in \bigcup_{G \in F_i} G$, $1 \le i \le n$, $P_{x,j} = \bigcup_{B \in \mathcal{A}, \infty \notin B} \mathcal{C}_B(x,j)$ is a partition of $X' \setminus (\bigcup_{G \in F_i} G)$ for $1 \le j \le tg^2(t+2a-3)/6$.

For each member $Q \in \mathcal{Q}$, suppose its block size is k_Q. Then $P'_Q(l) = \bigcup_{B \in Q} \mathcal{C}_B(l)$ is a partition of X' for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6$.

Thus we obtain an $\mathrm{RHF}_g((tm)^n : t(s-1)+a)$. □

Theorem 2.2.3. *Suppose that there exists an $RHF_g(m^n : s)$. If there exists an $IRH(g^{m+s} : g^s)$, then there exists an $IRH(g^{mn+s} : g^{m+s})$. Furthermore, if there is an $RH(g^{m+s})$, then there is an $RH(g^{mn+s})$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ be the given $\mathrm{RHF}_g(m^n : s)$, $\mathcal{F} = \{F_k : 0 \leq k \leq n\}$ and F_0 be the common hole of size s. Then the block set $\mathcal{B}$ has a partition $\{P(k,j) : 1 \leq k \leq n, 1 \leq j \leq mg^2(m+2s-3)/6\} \cup \{P'(i) : 1 \leq i \leq n(n-1)(mg)^2/6\}$ such that (1) for each pair (k,j), $1 \leq k \leq n$ and $1 \leq j \leq mg^2(m+2s-3)/6$, $P(k,j)$ is a partition of $X \setminus (\bigcup_{G \in F_k} G)$; (2) for each i, $1 \leq i \leq n(n-1)(mg)^2/6$, $P'(i)$ is a parallel class on X.

For $1 \leq k \leq n-1$, construct an $\mathrm{IRH}(g^{m+s} : g^s)$ on $\bigcup_{G \in F_k} G$ with group set F_k and hole F_0. Denote the set of blocks by $\mathcal{A}_k$. Then there are $(m+s-1)(m+s-2)g^2/6$ parts $Q(k,j)$, such that for $1 \leq j \leq m(m+2s-3)g^2/6$, $Q(k,j)$ is a partition of $\bigcup_{G \in F_k} G$; for $m(m+2s-3)g^2/6 < j \leq (m+s-1)(m+s-2)g^2/6$, each $Q(k,j)$ is a partition of $\bigcup_{G \in F_k \setminus F_0} G$. Then each $P(k,j) \cup Q(k,j)$ with $1 \leq k \leq n-1, 1 \leq j \leq mg^2(m+2s-3)/6$ forms a parallel class on X. Each $\cup_{1 \leq k \leq n-1} Q(k,j)$ with $m(m+2s-3)g^2/6 < j \leq (m+s-1)(m+s-2)g^2/6$ forms a partition of $X \setminus (\bigcup_{G \in F_n} G)$. So the resultant design is an $\mathrm{IRH}(g^{mn+s} : g^{m+s})$.

Furthermore, if we construct an $\mathrm{RH}(g^{m+s})$ on $\bigcup_{G \in F_n} G$ with group set F_n, then we obtain an $\mathrm{RH}(g^{mn+s})$. □

Theorem 2.2.4. (Product Construction) *If there exist both an RH(g^m) and an RH(g^n), then there exists an RH(g^{mn}) and an IRH($g^{mn} : g^n$).*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $\mathrm{RH}(g^m)$, where $\mathcal{G} = \{G_0, \ldots, G_{m-1}\}$. Applying Theorem 2.1.1, we construct an $\mathrm{RH}((ng)^m)$ on $X' = X \times Z_n$ with the group set $\mathcal{G}' = \{G'_i = G_i \times Z_n : 0 \leq i \leq m-1\}$ and block set $\mathcal{A}$.

For each i, $0 \leq i \leq m-1$, construct an $\mathrm{RH}(g^n)$ on $G_i \times Z_n$ with group set $\{G_i \times \{l\} : l \in Z_n\}$ and block set $\mathcal{C}_i$, which has a resolution $P_i(k)$, $1 \leq k \leq (n-1)(n-2)g^2/6$.

Since an $\mathrm{RH}(g^n)$ exists, gn is double even. For each i, $0 \leq i \leq m-1$, let $\mathcal{F}^i = \{F^i_1, \ldots, F^i_{g(n-1)}\}$ be a one-factorization of the complete multiple-graph on $G_i \times Z_n$ with n parts in $\{G_i \times \{l\} : l \in Z_n\}$. Let

$$\mathcal{D} = \{\{a,b,c,d\} : \{a,b\} \in F^i_j, \{c,d\} \in F^{i'}_j, 0 \leq i \neq i' \leq m-1, 1 \leq j \leq g(n-1)\},$$

then $\mathcal{B}' = \mathcal{A} \cup (\cup_{i=0}^{m-1} \mathcal{C}_i) \cup \mathcal{D}$ is the block set of an $\mathrm{H}(g^{mn})$ on the group set $\mathcal{G}'' = \{G_i \times \{l\} : l \in Z_n, 0 \leq i \leq m-1\}$. It is clear that $\cup_{i=0}^{m-1} \mathcal{C}_i$ has a resolution

$Q(k) = \cup_{i=0}^{m-1} P_i(k)$, $1 \le k \le (n-1)(n-2)g^2/6$. It remains to show that $\mathcal{D}$ can be partitioned into parallel classes.

For each j, $1 \le j \le g(n-1)$, let

$$\mathcal{D}_j = \{\{a,b,c,d\} : \{a,b\} \in F_j^i, \{c,d\} \in F_j^{i'}, 0 \le i < i' \le m-1\}, \text{ and}$$
$$D_j = \{\{\{a,b\},\{c,d\}\} : \{a,b\} \in F_j^i, \{c,d\} \in F_j^{i'}, 0 \le i < i' \le m-1\}.$$

If we regard each pair in $F_j^i, 0 \le i \le m-1$ as a vertex, we may construct a multi-partite complete graph Γ_j on the vertex set $X_j' = \cup_{i=0}^{m-1} F_j^i$ with partite set $\{F_j^i : 0 \le i \le m-1\}$, where two different vertices connect if and only if they are from different factors F_j^i. Hence, D_j is the edge set of Γ_j. That is to say we obtain a GDD$(2,2,gnm/2)$ of type $(gn/2)^m$ on X_j' with group set $\{F_j^i, 0 \le i \le m-1\}$ and block set D_j.

It is well-known that there always exists a resolvable GDD$(2,2,gnm/2)$ of type $(gn/2)^m$ when $gnm/2$ is even (see [14]). Hence, we can partition the block set D_j of our resulting GDD$(2,2,gnm/2)$ of type $(gn/2)^m$ into parallel classes on X_j'. Therefore, $\mathcal{D}_j$ can also be partitioned in parallel classes of X'. So does $\mathcal{D} = \cup_{1 \le j \le g(n-1)} \mathcal{D}_j$. Thus, the desired H$(g^{mn})$ is resolvable.

For each i, $0 \le i \le m-1$, $\mathcal{B}' \setminus \mathcal{C}_i$ is the block set of an incomplete design IRH$(g^{mn} : g^n)$ on X' with group set $\mathcal{G}''$ and hole set $\{G_i \times \{l\} : l \in Z_n\}$. □

With a similar proof to that of Theorem 2.2.4, we have the following theorem. Here, we just need to fill each hole with the trivial design RH(g^2).

Theorem 2.2.5. (Doubling Construction) *If there exists an RH(g^u), then there exists an RH(g^{2u}) and an IRH$(g^{2u} : g^u)$.*

Our first tripling construction given below is on resolvable H-frames, which is a generalization of the tripling construction for resolvable CQSs developed in [43].

Theorem 2.2.6. (Tripling Construction I) *Suppose there exists an RHF$_g(n^3 : s)$, then there exists an RHF$_g((3n)^3 : s)$.*

Proof. Start with a CQS$(3^3 : 1)$ (as in [43]) on $Z_9 \cup \{\infty\}$ with groups $G_i = \{i, i+3, i+6\}, 0 \le i \le 2$ and stem $\{\infty\}$, whose block set $\mathcal{B}$ is generated by the following 9 base blocks under the automorphism group $\langle(0\ 3\ 6)(1\ 4\ 7)(2\ 5\ 8)(\infty)\rangle$.

$$\begin{array}{llll} \mathcal{A}_\infty: & \{0,1,2,\infty\}, & \{0,4,8,\infty\}, & \{0,5,7,\infty\}, \\ \mathcal{A}_1: & \{1,3,2,6\}, & \{1,3,5,7\}, & \{2,6,5,7\}, \\ \mathcal{A}_2: & \{4,7,5,8\}, & \{3,6,5,8\}, & \{3,6,4,7\}. \end{array}$$

View each base block as an ordered quadruple given above so that each block $B \in \mathcal{B}$ is ordered.

Since an $\mathrm{RHF}_g(n^3 : s)$ exists, both gn and gs are even. We separate the proof into the following two cases:

Case (1): When g is even, we will construct an $\mathrm{RHF}_g((3n)^3 : s)$ on $X = (Z_9 \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with groups $G(x,j) = \{x\} \times Z_2 \times \{j, j+n, \ldots, j+(\frac{g}{2}-1)n\}$, $x \in Z_9$, $0 \le j \le n-1$, and $G(\infty, j) = \{\infty\} \times Z_2 \times \{j, j+s, \ldots, j+(\frac{g}{2}-1)s\}$, $0 \le j \le s-1$, and three holes $F_i = \{G(i,j), G(i+3,j), G(i+6,j) : 0 \le j \le n-1\} \cup S$, $0 \le i \le 2$, which intersect on a common hole $S = \{G(\infty,j) : 0 \le j \le s-1\}$.

For each block $B \in \mathcal{B}$ containing ∞, construct an $\mathrm{RHF}_g(n^3 : s)$ on $X_B = ((B \setminus \{\infty\}) \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with group set $\{G(x,j) : x \in B \setminus \{\infty\}, 0 \le j \le n-1\} \cup S$, three holes $\{G(x,j) : 0 \le j \le n-1\} \cup S$, $x \in B \setminus \{\infty\}$ and a common hole S. Denote its block set by $\mathcal{A}_B$, which has a resolution $\{P_B(x,l) : x \in B \setminus \{\infty\}, 1 \le l \le n(n+2s-3)g^2/6\} \cup \{P_B(r',r,h) : r', r \in Z_2, 1 \le h \le (gn)^2/4\}$ such that each $P_B(x,l)$ is a partition of $(B \setminus \{\infty, x\}) \times Z_2 \times Z_{gn/2}$ and each $P_B(r',r,h)$ is a parallel class on X_B.

For each block $B = \{a,b,c,d\} \in \mathcal{B}$ and $\infty \notin B$, we shall construct a special $\mathrm{H}((gn)^4)$ on $B \times Z_2 \times Z_{gn/2}$ with groups $\{x\} \times Z_2 \times Z_{gn/2}$, $x \in B$. Denote

$$C'_B(k,i,j) = \{(a,i), (b,i+k), (c,j), (d,j+k)\} \text{ and } \mathcal{C}'_B(k) = \{C'_B(k,i,j) : i,j \in Z_2\},$$

then $\mathcal{C}'_B = \mathcal{C}'_B(0) \cup \mathcal{C}'_B(1)$ is the block set of an $\mathrm{H}(2^4)$ on $B \times Z_2$. For each $A \in \mathcal{C}'_B$, construct an $\mathrm{RH}((gn/2)^4)$ on $A \times Z_{gn/2}$ with groups $\{a\} \times Z_{gn/2}, a \in A$. Denote its block set by $\mathcal{B}(A)$ and the $(gn)^2/4$ parallel classes by $P(A,h)$, $1 \le h \le (gn)^2/4$. Then, $\mathcal{C}_B = \cup_{A \in \mathcal{C}'_B} \mathcal{B}(A)$ is the block set of the desired $\mathrm{H}((gn)^4)$.

Let $\mathcal{D} = (\cup_{B \in \mathcal{B}, \infty \notin B} \mathcal{C}_B) \cup (\cup_{B \in \mathcal{B}, \infty \in B} \mathcal{A}_B)$. By Theorem 2.2.2, $\mathcal{D}$ is the block set of an $\mathrm{HF}_g(((3n)^3 : s))$. It remains to show the resolvability. This $\mathrm{HF}_g(((3n)^3 : s))$ should be partitioned into $9g^2n^2$ parallel classes on X and $g^2n(3n+2s-3)/2$ partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$ for each i, $0 \le i \le 2$.

For each i, $0 \le i \le 2$, let $P(i,x,l) = \cup_{B\in\mathcal{B},\{x,\infty\}\subset B} P_B(x,l)$, $1 \le l \le n(n+2s-3)g^2/6$, $x \in G_i$. Then each $P(i,x,l)$ is a partition of $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$. The other g^2n^2 partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$ can be obtained as follows. Denote the three base blocks of $\mathcal{A}_2$ by B_0, B_1, B_2 in order. For $0 \le i \le 2$, let $\mathcal{B}_i = \{3j + B_i : 0 \le j \le 2\}$, and for $r', r \in Z_2$, let $P(i,r',r) = \{C'_B(1,r',r) : B \in \mathcal{B}_i\}$. Then $P(i,r',r)$ is a partial class on $(Z_9 \setminus G_i) \times Z_2$. Note that for $0 \le i \le 2$, $\cup_{r',r\in Z_2} P(i,r',r) = \cup_{B\in\mathcal{B}_i} \mathcal{C}'_B(1)$. Let $P(i,r',r,h) = \cup_{A\in P(i,r',r)} P(A,h)$. Then, these $P(i,r',r,h)$s with $r', r \in Z_2$ and $1 \le h \le (gn)^2/4$ are g^2n^2 partial parallel classes on $(Z_9 \setminus G_i) \times Z_2 \times Z_{gn/2}$.

Now we give the required $9g^2n^2$ parallel classes on X. Denote the three base blocks of $\mathcal{A}_1$ by A_0, A_1, A_2 in order. Let $D_0 = A_0, D_1 = A_1+3 = \{4,6,8,1\}, D_2 = A_2+6 = \{8,3,2,4\}$. Let $\mathcal{A}(i,0)$ be as follows and $\mathcal{A}(i,j) = \{3j+B : B \in \mathcal{A}(i,0)\}$ for $0 \le j \le 2$.

$$\mathcal{A}(1,0) = \{\{0,4,8,\infty\}, A_0, A_1, A_2\},$$

$$\mathcal{A}(2,0) = \{\{0,1,2,\infty\}, B_0, B_1, B_2\},$$

$$\mathcal{A}(0,0) = \{\{0,5,7,\infty\}, D_0, D_1, D_2\}.$$

Let

$$\begin{aligned}
P'(1,j,r',r) =& \{C'_{A_0+3j}(0,r',r'+r), C'_{A_1+3j}(0,r'+1,r), C'_{A_2+3j}(0,r'+r+1,r+1)\},\\
P'(2,j,r',r) =& \{C'_{B_0+3j}(0,r'+r,r'), C'_{B_1+3j}(0,r,r'+1), C'_{B_2+3j}(0,r+1,r'+r+1)\},\\
P'(0,j,r',r) =& \{C'_{D_0+3j}(1,r',r'+r), C'_{D_1+3j}(1,r'+r+1,r'), C'_{D_2+3j}(1,r'+1,r'+r+1)\}.
\end{aligned}$$

Let $P'(i,j,r',r,h) = \cup_{A\in P'(i,j,r',r)} P(A,h)$ and $P''(i,j,r',r,h) = P_B(r',r,h) \cup P'(i,j,r',r,h)$, where $B \in \mathcal{A}(i,j)$ and $\infty \in B$. Then $P''(i,j,r',r,h)$ for $0 \le i,j \le 2$, $r',r \in Z_2, 1 \le h \le (gn)^2/4$ are the desired $9g^2n^2$ parallel classes on X.

So $\mathcal{D}$ has the resolution $\{P(i,x,l) : 0 \le i \le 2, x \in G_i, 1 \le l \le n(n+2s-3)g^2/6\} \cup \{P(i,r',r,h) : 0 \le i \le 2, r',r \in Z_2, 1 \le h \le (gn)^2/4\} \cup \{P''(i,j,r',r,h) : 0 \le i,j \le 2, r',r \in Z_2, 1 \le h \le (gn)^2/4\}$, and the $\mathrm{HF}_g(((3n)^3 : s))$ is resolvable.

Case (2): When g is odd, both n and s must be even, we will construct an $\mathrm{RHF}_g((3n)^3 : s)$ on X with groups $G'(x,k,j) = \{x\} \times \{k\} \times \{j, j+\frac{n}{2}, \ldots, j+(g-1)\frac{n}{2}\}$, $x \in Z_9$, $k \in Z_2$, $0 \le j \le \frac{n}{2}-1$, and $G'(\infty,k,j) = \{\infty\} \times \{k\} \times \{j, j+\frac{s}{2}, \ldots, j+(g-1)\frac{s}{2}\}$, $k \in Z_2$, $0 \le j \le \frac{s}{2}-1$, and three holes $F'_i = \{G'(i,k,j), G'(i+3,k,j), G'(i+6,k,j) : k \in Z_2, 0 \le j \le \frac{n}{2}-1\} \cup S'$, $0 \le i \le 2$, which intersect on a common hole $S' = \{G'(\infty,k,j) : k \in Z_2, 0 \le j \le \frac{s}{2}-1\}$.

For each block $B \in \mathcal{B}$ containing ∞, construct an $\mathrm{RHF}_g(n^3 : s)$ on $X_B = ((B \setminus \{\infty\}) \times Z_2 \times Z_{gn/2}) \cup (\{\infty\} \times Z_2 \times Z_{gs/2})$ with group set $\{G'(x,k,j) : x \in B \setminus \{\infty\}, k \in Z_2, 0 \leq j \leq \frac{n}{2} - 1\} \cup S'$, three holes $\{G'(x,k,j) : k \in Z_2, 0 \leq j \leq \frac{n}{2} - 1\} \cup S'$, $x \in B \setminus \{\infty\}$ and a common hole S'. Denote its block set by $\mathcal{A}_B$, which has a resolution $\{P_B(x,l) : x \in B \setminus \{\infty\}, 1 \leq l \leq n(n+2s-3)g^2/6\} \cup \{P_B(r', r, h) : r', r \in Z_2, 1 \leq h \leq (gn)^2/4\}$ such that each $P_B(x,l)$ is a partition of $(B \setminus \{\infty, x\}) \times Z_2 \times Z_{gn/2}$ and each $P_B(r', r, h)$ is a parallel class on X_B.

The remaining proof of this case is the same as that of Case (1). □

The construction of resolvable H-frames given below is used for our second tripling construction on resolvable H-designs. It is a generalization of that for resolvable Steiner quadruple systems proposed by Hartman in [31], which have played an important role in the construction of $\mathrm{RSQS}(v)$. The following notations are needed.

For $x \in Z_n$, we define $|x|$ by

$$|x| = \begin{cases} x, & \text{if } 0 \leq x \leq n/2, \\ -x, & \text{if } n/2 < x < n. \end{cases}$$

For $n \geq 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, define $\mathrm{G}(n, L)$ to be the regular graph with vertex set Z_n and edge set E given by $\{x, y\} \in E$ if and only if $|x - y| \in L$.

The following lemma is proved by Stern and Lenz in [65].

Lemma 2.2.7. *Let $L \subseteq \{1, 2, \ldots, n\}$. Then $G(2n, L)$ has a one-factorization if and only if $2n/gcd(j, 2n)$ is even for some $j \in L$.*

Let g be a positive integer and $m = lcm(g, 6)$. For non-negative integers n and s, a *B_g-pairing*, $B_g(n, s)$ consists of four subsets D, R_0, R_1, R_2 of Z_{mn+gs} and three subsets PR_0, PR_1, PR_2 of $Z_{mn+gs} \times Z_{mn+gs}$ with the following properties for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

(a) $|D| = gs$, $|R_i| = mn/3$,

(b) $D = -D$.

(2) *Partitioning conditions*

(a) PR_i is a partition of R_i into pairs, thus $|PR_i| = mn/6$,

(b) $Z_{mn+gs} = D|R_0|R_1|R_2$.

(3) *Pairing conditions*

Let $L_i = \{|x-y| : \{x,y\} \in PR_i\}$ and $N = \{mn/g + s, 2(mn/g + s), \ldots, \lfloor g/2 \rfloor (mn/g + s)\}$,

(a) $N \cap L_i = \emptyset$ and $(mn+gs)/2 \notin L_i$,

(b) $|L_i| = mn/6$,

(c) the complement G_i of the graph $G(mn+gs, L_i \cup N)$ has a one-factorization.

Suppose that $m = lcm(g, 12)$. Let $S_0, S_1, S_2, \overline{R}_0, \overline{R}_1, \overline{R}_2$ be subsets of Z_{mn+gs} and PS_0, PS_1, PS_2 be subsets of $Z_{mn+gs} \times Z_{mn+gs}$. A B_g-pairing $B_g(n,s)$ with $D, R_0, R_1, R_2, PR_0, PR_1$ and PR_2, is said to be *resolvable*, denoted by $RB_g(n,s)$, if the following properties are satisfied for each $i \in \{0,1,2\}$:

(1) *Cardinality and symmetry conditions*

(c) $|S_i| = mn/3$, $|\overline{R}_i| = mn/6$.

(2) *Partitioning conditions*

(c) PS_i is a partition of S_i into pairs, thus $|PS_i| = mn/6$,

(d) $Z_{mn+gs} = D|R_i|S_i|\overline{R}_{i+1}|(-\overline{R}_{i-1})$.

(3) *Pairing conditions*

Let $O_i = \{|x-y| : \{x,y\} \in PS_i\}$,

(d) $N \cap O_i = \emptyset$ and $(mn+gs)/2 \notin O_i$,

(e) $|O_i| = mn/6$, $L_i \cap O_i = \emptyset$, and all members of O_i are odd,

(f) the complement G'_i of the graph $G(mn+gs, L_i \cup O_i \cup N)$ has a one-factorization.

When $g=1$, a $B_g(n,s)$ is a simple pairing, and an $RB_g(n,s)$ is a resolvable pairing in [31], which are used to construct CQSs and RCQSs, respectively. The following theorem gives the relation between B_g-pairings and H-frames with group size g.

Theorem 2.2.8. *Suppose that $m = lcm(g,6)$ and there exists a $B_g(n,s)$. Then there exists an $HF_g((mn/g+s)^3 : s)$. Furthermore, if $m = lcm(g,12)$ and the $B_g(n,s)$ is resolvable, then the $HF_g((mn/g+s)^3 : s)$ is also resolvable. Moreover, if $k(mn/g+s) \in D$ for all k, $0 \le k \le g-1$, then the resultant $RHF_g((mn/g+s)^3 : s)$ has a sub-design $RH(g^4)$.*

Proof. Let $X = (Z_{mn+gs} \times Z_3) \cup \{\infty_0, \infty_1, \dots, \infty_{gs-1}\}$. Define the groups $G(i,j) = \{(k(mn/g+s)+i, j) : 0 \le k \le g-1\}$, $0 \le i \le mn/g+s-1, j \in \{0,1,2\}$, $G(\infty, j) = \{\infty_{ks+j} : 0 \le k \le g-1\}$, $0 \le j \le s-1$ and hole set $\mathcal{F} = \{F_0, F_1, F_2, S\}$ with $S = \{G(\infty,j) : 0 \le j \le s-1\}$ and $F_j = S \cup \{G(i,j) : 0 \le i \le mn/g+s-1\}$, $j = 0,1,2$.

For $i \in \{0,1,2\}$, let $D, R_i, \overline{R}_i, S_i, PR_i, PS_i$ be a resolvable B_g-pairing $RB_g(n,s)$. Let $F_i^{2k-1} | F_i^{2k}$ be a one-factorization of the graph $G(mn+gs, \{m\})$, where m is the k-th member of O_i for $1 \le k \le mn/6$. Let $F_i^{mn/3+1} | F_i^{mn/3+2} | \dots | F_i^{2mn/3+gs-g}$ be a one-factorization of the complement of the graph $G(mn+gs, L_i \cup O_i \cup N)$. Then it is natural that $F_i^1 | F_i^2 | \dots | F_i^{2mn/3+gs-g}$ is a one-factorization of the complement of the graph $G(mn+gs, L_i \cup N)$.

We construct an $\mathrm{HF}_g((mn/g+s)^3 : s)$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the block set $\mathcal{B}$ consisting of the following three parts:

(1) $\{\infty_j, (a,0), (b,1), (c,2)\}$, where $a+b+c \equiv d \pmod{mn+gs}$, d is the j-th member of D and $0 \le j < gs$.

(2) $\{(a+q,i), (a+t,i), (b,i+1), (c,i+2)\}$, where $a+b+c \equiv 0 \pmod{mn+gs}$, $\{q,t\} \in PR_i$ and $i \in \{0,1,2\}$.

(3) $\{(a,i),(b,i),(c,i+1),(d,i+1)\}$, where $\{a,b\} \in F_i^j$ and $\{c,d\} \in F_{i+1}^j$, $i \in \{0,1,2\}$ and $j = 1,2,\ldots,2mn/3+gs-g$.

Now, we partition them into (partial) parallel classes.

First, we give the partial parallel classes. Define $F_i^j = \{F_i^j(k) : 0 \le k \le (mn+gs)/2-1\}$. For each $i \in \{0,1,2\}$, the $(mn+gs)(mn+3gs-3g)/6$ partial parallel classes missing the hole F_i are defined as follows:

$$P_i(j,k) = \{\{(a,i+1),(b,i+1),(c,i+2),(d,i+2)\} : \{a,b\} = F_{i+1}^j(m),$$
$$\{c,d\} = F_{i+2}^j(m+k), 0 \le m \le (mn+gs)/2-1\},$$

where $mn/3+1 \le j \le 2mn/3+gs-g$ and $0 \le k \le (mn+gs)/2-1$. It is clear that each $P_i(j,k)$ forms a partition of $X \setminus (\bigcup_{G \in F_i} G)$.

Next, we give the $(mn+gs)^2$ complete parallel classes. For each (a,b,c) such that $a+b+c \equiv 0 \pmod{mn+gs}$, let $P(a,b,c)$ be comprised of gs blocks from Part (δ), $mn/2$ blocks from Part (ρ) and $mn/4$ blocks from Part (ϕ) as follows:

Part (δ): $\{\{\infty_j,(a+d,0),(b-d,1),(c+d,2)\} : d$ is the j-th member of $D, 0 \le j < gs\}$.

Part (ρ):

$$\{(a+q,0),(a+t,0),(b-u,1),(c+u,2)\} \text{ for } i=0,$$

$$\{(a+u,0),(b+q,1),(b+t,1),(c-u,2)\} \text{ for } i=1,$$

$$\{(a-u,0),(b+u,1),(c+q,2),(c+t,2)\} \text{ for } i=2,$$

where $\{q,t\}$ is the j-th pair in PR_i and u is the j-th member of $\overline{R}_i$, $1 \le j \le mn/6$.

Part (ϕ): To select the blocks of Part (ϕ), let $PA_i|PB_i$ be a partition of PS_i into parts of size $mn/12$. Then the blocks of Part (ϕ) are all those of the forms:

$$\{(a+s,0),(a+s',0),(b+t,1),(b+t',1)\},$$

$$\{(b+u,1),(b+u',1),(c+w,2),(c+w',2)\},$$

$$\{(c+y,2),(c+y',2),(a+z,0),(a+z',0)\},$$

where the pairs $\{s,s'\},\{t,t'\},\{u,u'\},\{w,w'\},\{y,y'\}$ and $\{z,z'\}$ are the j-th $(1 \le j \le mn/12)$ pairs selected from the sets PA_i, PB_i according to the parities of a,b and c, as follows:

(i) If a, b and c are all even, then $\{s, s'\} \in PA_0, \{t, t'\} \in PA_1, \{u, u'\} \in PB_1, \{w, w'\} \in PB_2, \{y, y'\} \in PA_2, \{z, z'\} \in PB_0$.

(ii) If just a is even, then $\{s, s'\} \in PB_0, \{t, t'\} \in PB_1, \{u, u'\} \in PA_1, \{w, w'\} \in PB_2, \{y, y'\} \in PA_2, \{z, z'\} \in PA_0$.

(iii) If just b is even, then $\{s, s'\} \in PB_0, \{t, t'\} \in PB_1, \{u, u'\} \in PA_1, \{w, w'\} \in PA_2, \{y, y'\} \in PB_2, \{z, z'\} \in PA_0$.

(iv) If just c is even, then $\{s, s'\} \in PA_0, \{t, t'\} \in PA_1, \{u, u'\} \in PB_1, \{w, w'\} \in PA_2, \{y, y'\} \in PB_2, \{z, z'\} \in PB_0$.

It is clear that each $P(a, b, c)$ forms a partition of X. Note that for all (a, b, c) such that $a+b+c \equiv 0 \pmod{mn+gs}$, the blocks of Part (ϕ) cover all the blocks of the form $\{(x, i+1), (y, i+1), (z, i+2), (w, i+2)\}$, where $\{x, y\} \in F_{i+1}^{j}$ and $\{z, w\} \in F_{i+2}^{j'}$ such that $1 \leq j, j' \leq mn/3$ and $\{j, j'\}$ is an appropriate pair, $i \in \{0, 1, 2\}$. Thus, the desired $\mathrm{HF}_g((mn/g+s)^3 : s)$ is resolvable.

Moreover, if $k(mn/g+s) \in D$ for each k, $0 \leq k \leq g-1$, without loss of generality we may assume $k(mn/g+s)$ is the (ks)th element of D. Let

$$\begin{aligned}\delta_0 = \{&\{\infty_{ks}, (a+d, 0), (b-d, 1), (c+d, 2)\} : a+b+c \equiv 0 \pmod{mn+gs},\\ &a, b, c \in \{i(mn/g+s) : 0 \leq i \leq g-1\}, d \text{ is the } (ks)\text{th member of } D,\\ &0 \leq k \leq g-1\}.\end{aligned}$$

Note that $\delta_0 \subset \delta$ and δ_0 forms the block set of an $\mathrm{RH}(g^4)$ with the group set $\{\{(k(mn/g+s), i) : 0 \leq k \leq g-1\} : i \in \{0, 1, 2\}\} \cup \{\{\infty_{ks} : 0 \leq k \leq g-1\}\}$ and parallel classes $\{\{\infty_{(i+j+k+l)s}, ((i+l)(mn/g+s), 0), ((j+l)(mn/g+s), 1), ((k+l)(mn/g+s), 2)\} : 0 \leq l \leq g-1\}$, $i+j+k \equiv 0 \pmod{g}$. Hence, the $\mathrm{RHF}_g((mn/g+s)^3 : s)$ contains a subdesign $\mathrm{RH}(g^4)$. □

2.3 Resolvable H-designs with Group Size 4

First, we give our second tripling construction for resolvable H-designs with groups size 4 by constructing resolvable B_4-pairings. In order to construct such structures, we describe a special class of B_g-pairings with extra properties. Suppose that D, R_i, PR_i, $i \in \{0,1,2\}$ form a $B_g(n,s)$ on Z_{mn+gs}. If there exist three subsets A_0, A_1, A_2 of Z_{mn+gs} and three subsets PA_0, PA_1, PA_2 of $Z_{mn+gs} \times Z_{mn+gs}$ satisfying the following conditions for each $i \in \{0,1,2\}$:

(1) $R_i = -R_i$, $A_i \subset R_i$, $|A_i| = mn/6$,

(2) PA_i is a partition of A_i into pairs. Let $O'_i = \{|x-y| : \{x,y\} \in PA_i\}$,

  (a) $|O'_i| = mn/12$, all O'_0, O'_1, O'_2 are disjoint and of odd members,

  (b) $(\cup_{i=0}^2 O'_i) \bigcap (N \bigcup (\cup_{i=0}^2 L_i)) = \emptyset$ and $(mn+gs)/2 \notin O'_i$,

then let

$$S_0 = A_1 \cup A_2,\ S_1 = A_0 \cup (-A_2),\ S_2 = (-A_0) \cup (-A_1),$$

$$PS_0 = PA_1 \cup PA_2,\ PS_1 = PA_0 \cup (-PA_2),\ PS_2 = (-PA_0) \cup (-PA_1),$$

$$\overline{R}_0 = -(R_0 \setminus A_0),\ \overline{R}_1 = R_1 \setminus A_1 \text{ and } \overline{R}_2 = -(R_2 \setminus A_2).$$

It is readily checked that $D, R_i, PR_i, S_i, PS_i, \overline{R}_i$, $i \in \{0,1,2\}$ form an $RB_g(n,s)$.

Now, we are in a position to construct $RB_4(n,s)$ for any $n \geq 0$ and $s \geq 1$. We list the components D, PR_i, PA_i, $i \in \{0,1,2\}$ for short or $D, PR_i, PS_i, \overline{R}_i$, $i \in \{0,1,2\}$ fully.

Lemma 2.3.1. *For each pair of integers $n \geq 0$ and $s \geq 1$, there exists an $RB_4(n,s)$.*

Proof. When $n = 0$, we take $D = Z_{4(3n+s)}$ and $R_i = S_i = \overline{R}_i = \emptyset$ for each $i \in \{0,1,2\}$. When $n > 0, s > 0$, the desired $RB_4(n,s)$ is constructed directly as follows:

(1) For s odd and n even, let

$D = \{(3n+s)j : 0 \leq j \leq 3\} \cup \{(3n+s)i + j : 0 \leq i \leq 3, 1 \leq j \leq (s-1)/2 \text{ or } 3n + (s-1)/2 + 1 \leq j \leq 3n + s - 1\}$,

$PR_0 = \{\{j, -j\} : (s-1)/2+1 \le j \le (s-1)/2+n$ or $(3n+s)+(s-1)/2+n+1 \le j \le (3n+s)+(s-1)/2+2n\}$,

$PR_1 = \{\{j, -j\} : (s-1)/2+2n+1 \le j \le (s-1)/2+3n$ or $(3n+s)+(s-1)/2+1 \le j \le (3n+s)+(s-1)/2+n\}$,

$PR_2 = \{\{j, -j\} : (s-1)/2+n+1 \le j \le (s-1)/2+2n$ or $(3n+s)+(s-1)/2+2n+1 \le j \le (3n+s)+(s-1)/2+3n\}$,

$PA_0 = \{\{(s-1)/2+j, 8n+3s-(s-1)/2-j\} : 1 \le j \le n\}$,

$PA_1 = \{\{(s-1)/2+2n+j, 4n+s+(s-1)/2-j\} : 1 \le j \le n-1\} \cup \{\{10n-(s-1)/2-1, 10n-(s-1)/2-2\}\}$,

$PA_2 = \{\{(s-1)/2+n+j, 6n+s+(s-1)/2+2-j\} : 2 \le j \le n\} \cup \{\{(s-1)/2+n+1, 11n+4s-(s-1)/2-2\}\}$.

(2) For s even and n even, let

$D = \{(3n+s)j, (3n+s)/2+(3n+s)j : 0 \le j \le 3\} \cup \{(3n+s)i+j : 0 \le i \le 3, 1 \le j \le (s-2)/2$ or $3n+s/2+1 \le j \le 3n+s-1\}$,

$PR_0 = \{\{j, -j\} : (s-2)/2+1 \le j \le (s-2)/2+n$ or $(s-2)/2+n+1 \le j \le 2n+(s-2)/2+1$ and $j \ne (3n+s)/2\}$,

$PR_1 = \{\{j, -j\} : 2n+(s-2)/2+2 \le j \le 3n+s/2$ or $3n+s+(s-2)/2+n+1 \le j \le 3n+s+(s-2)/2+2n+1$ and $j \ne 3n+s+(3n+s)/2\}$,

$PR_2 = \{\{j, -j\} : 3n+s+(s-2)/2+1 \le j \le 3n+s+(s-2)/2+n$ or $5n+s+(s-2)/2+2 \le j \le 6n+s+(s-2)/2\}$,

$PA_0 = \{\{(s-2)/2+j, (s-2)/2+2n+1-j\} : 1 \le j \le n$ and $j \ne n/2\} \cup \{\{11n+3s+(s+2)/2, 10n+(s+2)/2+3s-1\}\}$,

$PA_1 = \{\{(s-2)/2+2n+1+j, 8n+2s+(s+2)/2-j\} : 1 \le j \le n$ and $j \ne n/2\} \cup \{\{10n+3s+(s+2)/2-2, 3n+s+(s-2)/2+2n+1\}\}$,

$PA_2 = \{\{3n+s+(s-2)/2+j, 7n+2s+(s+2)/2-1-j\} : 1 \le j \le n\}$.

(3) For s even and n odd,

(3.1) $n \ge 3$ odd, let

$D = \{(3n+s)j : 0 \le j \le 3\} \cup \{(3n+s)i+j : 0 \le i \le 3, 1 \le j \le (s-2)/2$ or $3n+(s-2)/2+2 \le j \le 3n+s-1\} \cup \{\pm((s-2)/2+1), \pm(6n+s+(s-2)/2+1)\}$,

$PR_0 = \{\{j, -j\} : (s-2)/2+2 \le j \le (s-2)/2+n+1$ or $(s-2)/2+2n+2 \le j \le (s-2)/2+3n+1\}$,

$PR_1 = \{\{j, -j\} : (s-2)/2+n+2 \le j \le (s-2)/2+2n+1$ or $(5n+s)+(s-2)/2+1 \le j \le (5n+s)+(s-2)/2+n\}$,

$PR_2 = \{\{j, -j\} : 3n+s+(s-2)/2+1 \le j \le 3n+s+(s-2)/2+n$ or $3n+s+(s-2)/2+n+1 \le j \le 3n+s+(s-2)/2+2n\}$,

$PA_0 = \{\{(s-2)/2+2n+j, 10n+4s-(s-2)/2-1-j\} : 2 \le j \le n\} \cup \{(s-2)/2+3, (s-2)/2+3n+1\}$,

$PA_1 = \{\{(s-2)/2+n+j, 6n+s+(s-2)/2+2-j\} : 2 \le j \le n\} \cup \{\{5n+s+(s-2)/2+1, 11n+4s-(s-1)/2-2\}\}$,

$PA_2 = \{\{(s-2)/2+3n+s+j, 5n+s+(s-2)/2+1-j\} : 1 \le j \le n\}$.

(3.2) $n = 1$, let

$D = \{(3+s)j : 0 \le j \le 3\} \cup \{(3+s)i+j : 0 \le i \le 3, 1 \le j \le (s-2)/2$ or $3+(s-2)/2+2 \le j \le 3+s-1\} \cup \{\pm((s-2)/2+1), \pm((s-2)/2+2)\}$,

$PR_0 = \{\{j, -j\} : (s-2)/2+3 \le j \le (s-2)/2+4\}$,

$PR_1 = \{\{j, -j\} : 3+s+(s-2)/2+1 \le j \le 3+s+(s-2)/2+2\}$,

$PR_2 = \{\{j, -j\} : 3+s+(s-2)/2+3 \le j \le 3+s+(s-2)/2+4\}$,

$PA_0 = \{\{(s-2)/2+3, (s-2)/2+4\}\}$,

$PA_1 = \{\{3+s+(s-2)/2+2, 8+2s+(s+2)/2\}\}$,

$PA_2 = \{\{3+s+(s-2)/2+3, 5+2s+(s+2)/2\}\}$.

(4) For s odd and n odd,

(4.1) $s \ge 3$ odd and $n \ge 3$ odd, let

$D = \{(3n+s)j, (3n+s)/2+(3n+s)j : 0 \le j \le 3\} \cup \{(3n+s)i+j : 0 \le i \le 3, 1 \le j \le (s-3)/2$ or $3n+(s-3)/2+3 \le j \le 3n+s-1\} \cup \{\pm((s-3)/2+1), \pm(3n+s+3n+(s-3)/2+2)\}$,

$PR_0 = \{\{j, -j\} : (s-3)/2+2 \le j \le (s-3)/2+n+1$ or $(s-3)/2+n+2 \le j \le 2n+(s-3)/2+2$ and $j \ne (3n+s)/2\}$,

$PR_1 = \{\{j, -j\} : 2n+(s-3)/2+3 \le j \le 3n+(s-3)/2+2$ or $3n+s+(s-3)/2+n+1 \le j \le 3n+s+(s-3)/2+2n+1$ and $j \ne 3n+s+(3n+s)/2\}$,

$PR_2 = \{\{j, -j\} : 3n+s+(s-3)/2+1 \le j \le 3n+s+(s-3)/2+n$ or $5n+s+(s-3)/2+2 \le j \le 6n+s+(s-3)/2+1\}$,

$PA_0 = \{\{(s-3)/2+j, (s-3)/2+2n+3-j\} : 2 \le j \le n+1$ and $j \ne n+3-(n+3)/2\} \cup \{\{11n+3s+(s+3)/2-2, 10n+(s+3)/2+3s-2\}\}$,

$PA_1 = \{\{(s-3)/2+2n+2+j, 8n+2s+(s+3)/2-j\} : 1 \le j \le n$ and $j \ne (n-1)/2+2\} \cup \{\{10n+3s+(s+3)/2-4, 3n+s+(s-3)/2+2n+1\}\}$,

$PA_2 = \{\{3n+s+(s-3)/2+j, 7n+2s+(s+3)/2-1-j\} : 1 \le j \le n\}$.

(4.2) $s \ge 3$ odd and $n = 1$ odd, let

$D = \{(3+s)j, (3+s)/2+(3+s)j : 0 \le j \le 3\} \cup \{(3+s)i+j : 0 \le i \le 3, 1 \le j \le (s-3)/2$ or $3+(s+3)/2 \le j \le 3+s-1\} \cup \{\pm((s-3)/2+1), \pm((s-3)/2+2)\}$,

$PR_0 = \{\{j, -j\} : (s+3)/2+1 \le j \le (s+3)/2+2\}$,

$PR_1 = \{\{j, -j\} : 3+s+(s-3)/2+1 \le j \le 3+s+(s-3)/2+2\}$,

$PR_2 = \{\{j, -j\} : 3+s+(s+3)/2+1 \le j \le 3+s+(s+3)/2+1\}$,

$PA_0 = \{\{(s+3)/2+1, (s+3)/2+2\}\}$,
$PA_1 = \{\{3+s+(s-3)/2+1, 7+2s+(s+3)/2\}\}$,
$PA_2 = \{\{3+s+(s+3)/2+1, 7+2s+(s-3)/2\}\}$.

(4.3) For $s = 1$ and $n \geq 3$ odd, let

$D = \{(3n+1)i : 0 \leq i \leq 3\}$,
$PR_0 = \{\{j, -j\} : 1 \leq j \leq (n+1)/2$ or $(3n+1)/2+n+1 \leq j \leq 3n\} \cup \{\{j, -j-1\} : 3n+1+(n+1)/2+1 \leq j \leq 3n+1+(3n+1)/2-1\} \cup \{\{3n+1+(3n+1)/2, 3(3n+1)-(n+1)/2-1\}\}$,
$PR_1 = \{\{j, -j\} : 3n+1+1 \leq j \leq 3n+1+(n+1)/2$ or $3n+1+(3n+1)/2+n+1 \leq j \leq 2(3n+1)-1\} \cup \{\{j, -j-1\} : (n+1)/2+1 \leq j \leq (3n+1)/2-1\} \cup \{\{(3n+1)/2, 4(3n+1)-(n+1)/2-1\}\}$,
$PR_2 = \{\{j, -j\} : (3n+1)/2+1 \leq j \leq (3n+1)/2+n$ or $3n+1+(3n+1)/2+1 \leq j \leq 3n+1+(3n+1)/2+n\}$,
$PA_0 = \{\{j, -j-1\} : 3n+1+(n+1)/2+1 \leq j \leq 3n+1+(3n+1)/2-1\} \cup \{\{(n+1)/2-1, 4(3n+1)-(n+1)/2\}\}$,
$PA_1 = \{\{j, -j-1\} : (n+1)/2+1 \leq j \leq (3n+1)/2-1\} \cup \{\{3n+2, 3n+3\}\}$,
$PA_2 = \{\{(3n+1)/2+1+j, 3(3n+1)-(3n+1)/2-j\} : 1 \leq j \leq (n+1)/2\} \cup \{\{3n+1+(3n+1)/2+j, 3n+1+(3n+1)/2+n-j\} : 1 \leq j \leq (n-3)/2\} \cup V$, where $V = \{\{3n+1+(3n+1)/2+n, 3(3n+1)-(3n+1)/2-n+1\}\}$ for $n \geq 5$ and $V = \{\{17, 22\}\}$ when $n = 3$.

(4.4) For $s = 1$ and $n = 1$, let

$D = \{0, 1, 8, 15\}$,
$PR_0 = \{\{2, 3\}, \{4, 6\}\}$, $PR_1 = \{\{5, 11\}, \{9, 14\}\}$, $PR_2 = \{\{7, 13\}, \{10, 12\}\}$,
$PS_0 = \{\{7, 10\}, \{9, 14\}\}$, $PS_1 = \{\{6, 7\}, \{10, 13\}\}$, $PS_2 = \{\{2, 3\}, \{6, 9\}\}$,
$\overline{R}_0 = \{4, 14\}$, $\overline{R}_1 = \{5, 11\}$, $\overline{R}_2 = \{3, 4\}$. □

Combining Theorem 2.2.8 and Lemma 2.3.1, we obtain the following theorem.

Theorem 2.3.2. *Suppose that $n \geq 0$ and $s \geq 1$. There exists an $RHF_4((3n+s)^3 : s)$. When $(n, s) \neq (1, 1)$, the $RHF_4((3n+s)^3 : s)$ exists with a sub-design $RH(4^4)$.*

As a consequence of Theorem 2.3.2, we have our second tripling construction for resolvable H-designs with group size 4 as follows.

Corollary 2.3.3. (Tripling Construction II) *Let $n \equiv 2s \pmod 3$ and $s \geq 1$. If there exists an $IRH(4^n : 4^s)$, then there exist both an $IRH(4^{3n-2s} : 4^n)$ and an $IRH(4^{3n-2s} : 4^s)$. Furthermore, if there exists an $RH(4^n)$ or an $RH(4^s)$, then there exists an $RH(4^{3n-2s})$, as well as an $IRH(4^{3n-2s} : 4^4)$ when $(n, s) \neq (5, 1)$.*

The construction given below is a variation of the construction for resolvable candelabra quadruple systems in [33].

Theorem 2.3.4. *Suppose that* $n \geq 1$, $s \equiv 1, 2 \pmod 3$ *and* $3s \geq 5n$. *There exists an* $RHF_4((3n)^3 : s)$.

Proof. Let $n \geq 1$, $s \equiv 1, 2 \pmod 3$ and $3s \geq 5n$. Take $Y = \{\infty_0, \infty_1, \ldots, \infty_{4s-1}\}$ and let $X = (Z_{12n} \times Z_3) \cup Y$. We will construct an $\mathrm{RHF}_4((3n)^3 : s)$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with groups $G(i,j) = \{(i + 3kn, j) : k \in Z_4\}$, $i \in Z_{3n}$, $j \in Z_3$, and $G(\infty, j) = \{\infty_{sk+j} : k \in Z_4\}$, $0 \leq j \leq s-1$, and three holes $F_j = \{G(i,j) : i \in Z_{3n}\} \cup S$, $0 \leq j \leq 2$, which intersect on a common hole $S = \{G(\infty, j) : 0 \leq j \leq s-1\}$. In the sequel we shall write x_i for the ordered pair $(x, i) \in Z_{12n} \times Z_3$.

Let $h = (12n - 4s)/2$. Since $3s \geq 5n$, h is even and $h \leq 8n/3$. As in [33, Theorem 2.1], let

$$H_1^* = \{\{9n - i, 9n - 3 + i\} : 2 \leq i \leq 3n + 1, i \not\equiv 0 \pmod 3\}, \text{ and}$$

$$H_2^* = \{\{3n - i, 3n + i\} : 1 \leq i \leq 3n - 2, i \not\equiv 0 \pmod 3\}.$$

It is easy to check that $|H_1^*| = 2n$ and $|H_2^*| = 2n - 1$. Let H_i be any subset of H_i^* of cardinality $h/2$, $i = 1, 2$ and $H = H_1 \cup H_2$, which satisfies the following properties:

(1) $|H| = h = (12n - 4s)/2 \leq 8n/3$.

(2) The pairs in H are disjoint, i.e., $|\bigcup_{\{x,y\} \in H} \{x, y\}| = 2h$.

(3) Let $LH = \{|y - x| : \{x, y\} \in H\}$, then $|LH| = h$ and $LH \cap \{3, 6, \ldots, 6n\} = \emptyset$.

(4) The distances between members of H_1 are odd.

(5) $\{x, y\} \equiv \{1, 2\} \pmod 3$ for each $\{x, y\} \in H$.

Since $H_1 \subsetneq H_1^*$ and all distances between members of H_1^* are odd, the graph $G(12n, \{1, 2, \ldots, 6n\} \backslash (LH \bigcup \{3n, 6n\}))$ has a one-factorization $F_1|F_2|\ldots|F_{12n-2h-4}$

by Lemma 2.2.7. Let $F_{12n-2h-3}|F_{12n-2h-2}|F_{12n-2h-1}$ be a one-factorization of the graph $G(12n, \{3n, 6n\})$. Then it is natural that $F_1|F_2|\ldots|F_{12n-2h-1}$ is a one-factorization of the graph $G(12n, \{1, 2, \ldots, 6n\} \setminus LH)$. Using the above set of pairs H and the one-factorization of the graph $G(12n, \{1, 2, \ldots, 6n\} \setminus LH)$, Hartman [33, Theorem 2.1] constructed a resolvable RCQS$((12n)^3 : 4s)$ on X with group set $\{Z_{12n} \times \{i\} : i \in Z_3\}$ and stem Y, as well as the block set $\mathcal{B}'$ and its resolution $\mathcal{P}$ containing the following $6n(12n-2h-1)$ partitions of $Z_{12n} \times \{i+1, i+2\}$ for each $i \in Z_3$:

$$\begin{aligned} P_{i,u,k} = \{\{x_{i+1}, y_{i+1}, z_{i+2}, t_{i+2}\} : &\{x, y\} \text{ is the } m\text{th member of } F_u, \\ &\{z, t\} \text{ is the } (m+k)\text{th member of } F_u, m = 1, 2, \ldots, 6n\}, \end{aligned}$$

where $u = 1, 2, \ldots, 12n - 2h - 1$, and $k = 0, 1, \ldots, 6n - 1$.

For each $i \in Z_3$, let β_i be the union of partitions $P_{i,u,k}$ with $12n - 2h - 3 \leq u \leq 12n - 2h - 1$ and $0 \leq k \leq 6n - 1$. Then we have that $\mathcal{B} = \mathcal{B}' \setminus (\bigcup_{i \in Z_3} \beta_i)$ is the block set of the desired RHF$_4((3n)^3 : s)$ on X with group set $\mathcal{G}$ and hole set $\mathcal{F}$, where $\mathcal{B}$ has a resolution $\mathcal{P} \setminus \{P_{i,u,k} : 12n - 2h - 3 \leq u \leq 12n - 2h - 1, 0 \leq k \leq 6n - 1, i \in Z_3\}$. $\square$

As a consequence of Theorem 2.3.4, we have our third tripling construction as follows.

Corollary 2.3.5. (Tripling Construction III) *Let $n \equiv s \equiv 1$ or $2 \pmod 3$ and $14s \geq 5n$. If there exists an IRH$(4^n : 4^s)$, then there exists an IRH$(4^{3n-2s} : 4^n)$ and an IRH$(4^{3n-2s} : 4^s)$.*

In the sequel of this section, we shall establish the existence of resolvable H-designs with group size 4 by using the recursive constructions having been developed. The following initial designs are needed.

Lemma 2.3.6. *There exists an RH(4^5).*

Proof. Let the point set be Z_{20}, and the group set be $\{\{j, j+5, \ldots, j+15\} : j = 0, 1, \cdots, 4\}$. We list the base blocks as follows, which are developed by adding 2 modulo 20:

$\{3,4,7,11\}$	$\{6,10,17,18\}$	$\{0,2,9,13\}$	$\{5,8,19,1\}$	$\{12,14,15,16\}$
$\{8,9,16,17\}$	$\{11,19,0,2\}$	$\{18,4,12,15\}$	$\{3,6,10,14\}$	$\{1,5,7,13\}$
$\{1,7,9,18\}$	$\{11,13,14,15\}$	$\{19,5,6,12\}$	$\{8,10,2,4\}$	$\{16,17,0,3\}$
$\{1,12,18,19\}$				

Each of the first three rows forms a parallel class. The last block covers the four residues modulo 4, hence gives a parallel class by adding 4 modulo 20. $\square$

Lemma 2.3.7. *There exists an RH*(4^7).

Proof. Let the point set be Z_{28}, and the group set be $\{\{j, j+7, \ldots, j+21\} : j = 0, 1, \cdots, 6\}$. We list the base blocks as follows, each of which is developed by adding 2 modulo 28:

$\{3,7,11,23\}$	$\{27,9,15,17\}$	$\{13,14,19,1\}$	$\{2,4,10,20\}$	$\{18,22,26,6\}$
$\{24,0,5,16\}$	$\{8,12,21,25\}$			
$\{0,1,9,12\}$	$\{21,25,2,10\}$	$\{18,20,5,14\}$	$\{22,24,27,4\}$	$\{13,15,17,23\}$
$\{16,19,7,8\}$	$\{26,3,6,11\}$			
$\{3,6,18,21\}$	$\{8,9,19,24\}$	$\{20,5,7,11\}$	$\{10,15,16,0\}$	$\{4,14,17,1\}$
$\{25,2,12,13\}$	$\{22,23,26,27\}$			
$\{2,4,6,24\}$	$\{1,5,7,23\}$	$\{9,12,20,21\}$	$\{16,18,22,0\}$	$\{3,8,11,13\}$
$\{10,15,19,27\}$	$\{14,17,25,26\}$			
$\{3,4,7,8\}$	$\{21,26,2,13\}$	$\{22,23,24,25\}$	$\{14,17,20,11\}$	$\{18,19,0,9\}$
$\{27,1,10,12\}$	$\{16,5,6,15\}$			
$\{2,7,13,24\}$	$\{5,7,22,24\}$	$\{7,12,13,18\}$	$\{12,18,21,27\}$	$\{13,15,16,18\}$

The seven blocks in the ith and $(i+1)$th rows form a parallel class for each $i = 1, 3, 5, 7, 9$. Each block of the last row covers the four residues modulo 4, hence gives a parallel class by adding 4 modulo 28. $\square$

Lemma 2.3.8. *There exists an RHF*$_4(3^5 : 2)$.

Proof. We first construct an HF$_2(3^5 : 2)$ on $Z_{30} \cup \{\infty_0, \ldots, \infty_3\}$, with groups $G'_j = \{j, j+15\}$, $j = 0, 1, \cdots, 14$, $G'_{\infty_i} = \{\infty_i, \infty_{i+2}\}$, $i = 0, 1$, five holes $F'_i = \{G'_i, G'_{i+5}, G'_{i+10}\} \cup S'$, $i = 0, 1, \ldots, 4$ and a common hole $S' = \{G'_{\infty_0}, G'_{\infty_1}\}$. We list below the set of base blocks $\mathcal{B}' = \Delta \cup \Theta$, which will be is developed under the automorphism group $\langle \alpha' \rangle$, where $\alpha' = (0\ 1\ 2\ 3 \ldots 28\ 29)(\infty_0)(\infty_1)(\infty_2)(\infty_3)$.

$\Delta :$	$\{0,1,13,22\}$	$\{0,3,4,7\}$	$\{0,14,16,27\}$	$\{0,6,18,19\}$
	$\{0,3,6,24\}$	$\{0,19,21,22\}$	$\{0,1,2,8\}$	$\{0,11,19,27\}$

	$\{0,2,29,\infty_0\}$	$\{0,4,22,\infty_0\}$	$\{0,7,16,\infty_0\}$	$\{0,6,17,\infty_0\}$
	$\{0,3,12,\infty_1\}$	$\{0,2,24,\infty_1\}$	$\{0,16,29,\infty_1\}$	$\{0,4,11,\infty_1\}$
	$\{0,19,28,\infty_2\}$	$\{0,13,27,\infty_2\}$	$\{0,8,26,\infty_2\}$	$\{0,6,7,\infty_2\}$
	$\{0,3,9,\infty_3\}$	$\{0,22,29,\infty_3\}$	$\{0,14,26,\infty_3\}$	$\{0,11,13,\infty_3\}$
Θ :	$\{0,2,18,28\}$	$\{0,5,14,18\}$	$\{0,1,14,19\}$	$\{0,2,25,27\}$
	$\{0,3,8,25\}$	$\{0,7,12,28\}$	$\{0,7,14,25\}$	$\{0,1,6,25\}$
	$\{0,10,19,26\}$	$\{0,9,10,29\}$	$\{0,12,20,22\}$	$\{0,6,16,22\}$
	$\{0,3,20,23\}$	$\{0,21,25,26\}$	$\{0,7,17,24\}$	$\{0,10,21,28\}$
	$\{0,20,24,26\}$	$\{0,13,17,21\}$		

For each block $B=\{a,b,c,d\}\in\mathcal{B}'$, construct an RH$(2^4)$ with group set $\{\{x,x'\}:x\in B\}$, where $x'=x+30$ when $x\in Z_{30}$ or $x'=\infty_{i+4}$ when $x=\infty_i$, and block set $\mathcal{A}_B$ having a resolution $P_B(1)=\{\{a,b,c,d\},\{a',b',c',d'\}\}$, $P_B(2)=\{\{a,b,c',d'\},\{a',b',c,d\}\}$, $P_B(3)=\{\{a,b',c,d'\},\{a',b,c',d\}\}$, $P_B(4)=\{\{a,b',c',d\},\{a',b,c,d'\}\}$. Let $\mathcal{B}=\cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an HF$_4(3^5:2)$ on $X=Z_{60}\cup\{\infty_0,\ldots,\infty_7\}$ with the group set $G_j=\{j+15k:0\le k\le 3\}$, $j=0,1,\cdots,14$, $G_{\infty_i}=\{\infty_{i+2k}:0\le k\le 3\}$, $i=0,1$, five holes $F_i=\{G_i,G_{i+5},G_{i+10}\}\cup S$, $i\in Z_5$, a common hole $S=\{G_{\infty_0},G_{\infty_1}\}$ and an automorphism group $\langle\alpha\rangle$, where $\alpha=(0\ 1\ 2\ 3\ldots 28\ 29)(30\ 31\ 32\ 33\ldots 58\ 59)(\infty_0)\ldots(\infty_7)$. Now, we need to give the resolution. The design should contain 16×30 parallel classes on X and 8×4 partial parallel classes on $X\setminus(\cup_{G\in F_i}G)$ for each $i\in Z_5$.

Note that each block $B\in\Delta$ covers all but one, say j, distinct residues modulo 5. Then for each $i\in\{1,2,3,4\}$ and a fixed $s\in Z_5$, $P_B(i)$ gives a partial parallel class on $X\setminus(\cup_{G\in F_{j+s}}G)$ when developed by the automorphisms $\{\alpha^{5k+s}:k\in Z_6\}$. That is, $\cup_{B\in\Delta}\mathcal{A}_B$ gives 32 partial parallel classes on $X\setminus(\cup_{G\in F_i}G)$ for each $i\in Z_5$ when developed under $\langle\alpha\rangle$.

Then we shift each block $B\in\cup_{B\in\Theta}\mathcal{A}_B$ by a suitable automorphism $\alpha_B\in\langle\alpha\rangle$. The result is listed below, where the blocks in each of the four consecutive rows, namely the ith, $(i+1)$th, $(i+2)$th and $(i+3)$th rows for $i\in\{4k+1:k=0,1,\ldots,15\}$, form a parallel class.

$\{1,37,38,\infty_2\}$	$\{3,44,46,\infty_3\}$	$\{6,40,28,\infty_4\}$	$\{34,7,16,\infty_5\}$	$\{32,8,9,\infty_6\}$
$\{30,11,13,\infty_7\}$	$\{31,36,45,49\}$	$\{35,10,19,53\}$	$\{15,20,59,33\}$	$\{57,29,52,24\}$
$\{18,51,26,43\}$	$\{47,50,25,12\}$	$\{55,27,54,\infty_0\}$	$\{17,48,23,42\}$	$\{4,41,21,58\}$
$\{39,22,56,0\}$	$\{2,5,14,\infty_1\}$			
$\{32,6,24,\infty_4\}$	$\{2,5,44,\infty_5\}$	$\{3,9,40,\infty_6\}$	$\{8,19,51,\infty_7\}$	$\{35,10,49,23\}$

$\{11,46,55,29\}$	$\{33,38,17,21\}$	$\{57,28,41,16\}$	$\{12,13,48,37\}$	$\{59,39,18,25\}$
$\{36,43,20,1\}$	$\{0,4,22,\infty_0\}$	$\{56,30,7,\infty_1\}$	$\{26,45,54,\infty_2\}$	$\{50,27,34,15\}$
$\{52,53,58,47\}$	$\{31,42,14,\infty_3\}$			
$\{55,26,9,44\}$	$\{10,12,5,7\}$	$\{54,57,14,17\}$	$\{13,49,30,\infty_0\}$	$\{4,46,24,56\}$
$\{37,43,23,29\}$	$\{52,25,31,\infty_3\}$	$\{45,48,27,\infty_1\}$	$\{33,39,50,\infty_4\}$	$\{8,20,28,0\}$
$\{19,35,18,\infty_5\}$	$\{51,59,47,\infty_6\}$	$\{2,40,58,\infty_2\}$	$\{42,15,21,\infty_7\}$	$\{36,38,1,3\}$
$\{34,11,16,32\}$	$\{22,53,6,41\}$			
$\{1,15,27,\infty_3\}$	$\{32,39,48,\infty_4\}$	$\{34,36,58,\infty_5\}$	$\{40,59,38,\infty_6\}$	$\{51,35,47,\infty_7\}$
$\{2,3,16,21\}$	$\{10,41,54,29\}$	$\{17,49,12,44\}$	$\{53,25,18,50\}$	$\{56,5,6,55\}$
$\{7,20,24,28\}$	$\{9,52,26,30\}$	$\{14,46,43,\infty_0\}$	$\{0,4,11,\infty_1\}$	$\{45,22,57,13\}$
$\{23,33,42,19\}$	$\{31,37,8,\infty_2\}$			
$\{45,28,12,\infty_6\}$	$\{52,58,38,44\}$	$\{39,25,8,\infty_5\}$	$\{43,19,29,35\}$	$\{4,41,16,32\}$
$\{21,27,7,13\}$	$\{53,0,10,47\}$	$\{33,9,50,\infty_0\}$	$\{15,22,59,40\}$	$\{55,17,54,\infty_3\}$
$\{1,34,51,24\}$	$\{23,56,31,18\}$	$\{49,5,48,\infty_1\}$	$\{3,46,30,\infty_2\}$	$\{42,14,11,\infty_4\}$
$\{57,6,37,26\}$	$\{36,20,2,\infty_7\}$			
$\{51,58,33,49\}$	$\{50,42,19,\infty_3\}$	$\{8,40,2,\infty_5\}$	$\{20,57,36,\infty_0\}$	$\{48,24,34,10\}$
$\{31,11,22,59\}$	$\{3,6,53,56\}$	$\{43,46,52,\infty_7\}$	$\{4,44,23,30\}$	$\{0,37,14,55\}$
$\{17,25,13,\infty_2\}$	$\{27,29,21,\infty_1\}$	$\{32,9,18,\infty_4\}$	$\{16,54,12,\infty_6\}$	$\{1,38,45,26\}$
$\{7,47,28,35\}$	$\{15,5,39,41\}$			
$\{17,20,25,12\}$	$\{2,53,27,58\}$	$\{30,34,22,\infty_0\}$	$\{44,47,23,\infty_3\}$	$\{37,9,1,\infty_5\}$
$\{18,21,56,43\}$	$\{3,46,50,24\}$	$\{26,59,16,49\}$	$\{5,36,41,0\}$	$\{38,45,55,32\}$
$\{48,39,13,14\}$	$\{6,19,33,\infty_6\}$	$\{28,31,40,\infty_1\}$	$\{10,29,8,\infty_2\}$	$\{52,54,51,\infty_4\}$
$\{15,57,35,7\}$	$\{42,4,11,\infty_7\}$			
$\{24,1,40,\infty_4\}$	$\{10,30,4,36\}$	$\{33,6,53,26\}$	$\{11,0,39,\infty_6\}$	$\{59,32,49,52\}$
$\{54,8,50,\infty_3\}$	$\{16,18,41,43\}$	$\{56,42,55,\infty_5\}$	$\{46,48,15,\infty_0\}$	$\{47,21,58,\infty_1\}$
$\{5,38,14,\infty_7\}$	$\{19,22,9,12\}$	$\{7,28,2,3\}$	$\{20,27,37,44\}$	$\{57,17,51,23\}$
$\{35,25,29,31\}$	$\{45,34,13,\infty_2\}$			
$\{0,7,12,28\}$	$\{23,55,47,\infty_1\}$	$\{18,58,39,16\}$	$\{34,44,53,30\}$	$\{20,33,17,\infty_6\}$
$\{41,21,32,9\}$	$\{31,11,50,27\}$	$\{24,4,45,52\}$	$\{19,29,8,15\}$	$\{22,36,48,\infty_3\}$
$\{40,42,35,37\}$	$\{13,26,10,\infty_2\}$	$\{57,3,14,\infty_4\}$	$\{25,59,6,\infty_5\}$	$\{2,54,1,\infty_7\}$
$\{43,46,51,38\}$	$\{49,56,5,\infty_0\}$			
$\{16,18,40,\infty_5\}$	$\{23,9,22,\infty_1\}$	$\{49,56,1,17\}$	$\{41,54,8,\infty_2\}$	$\{31,5,53,\infty_0\}$
$\{10,2,39,\infty_7\}$	$\{55,3,21,\infty_6\}$	$\{52,4,12,44\}$	$\{45,57,35,37\}$	$\{15,28,32,36\}$
$\{51,42,46,47\}$	$\{14,6,13,\infty_3\}$	$\{27,59,26,\infty_4\}$	$\{48,19,24,43\}$	$\{38,20,58,0\}$
$\{34,25,29,30\}$	$\{50,33,7,11\}$			
$\{36,25,4,\infty_6\}$	$\{8,11,17,\infty_3\}$	$\{30,50,54,56\}$	$\{21,58,38,15\}$	$\{27,29,26,\infty_0\}$
$\{47,19,41,\infty_1\}$	$\{28,34,44,20\}$	$\{32,45,49,53\}$	$\{43,35,42,\infty_7\}$	$\{40,23,37,\infty_2\}$
$\{24,31,10,\infty_4\}$	$\{51,22,57,16\}$	$\{0,6,46,52\}$	$\{48,1,5,39\}$	$\{33,7,14,\infty_5\}$
$\{59,9,18,55\}$	$\{3,12,13,2\}$			
$\{27,36,37,26\}$	$\{23,30,35,21\}$	$\{39,47,5,\infty_2\}$	$\{46,32,15,\infty_1\}$	$\{53,54,29,18\}$
$\{57,34,41,52\}$	$\{59,43,25,\infty_3\}$	$\{16,2,45,\infty_5\}$	$\{40,50,1,8\}$	$\{19,55,6,\infty_4\}$
$\{7,17,56,33\}$	$\{3,9,20,\infty_0\}$	$\{31,44,58,\infty_6\}$	$\{10,13,49,\infty_7\}$	$\{4,24,28,0\}$

$\{48,38,12,14\}$	$\{42,51,22,11\}$			
$\{0,34,52,\infty_0\}$	$\{30,3,42,\infty_1\}$	$\{1,7,8,\infty_2\}$	$\{2,13,15,\infty_3\}$	$\{31,35,53,\infty_4\}$
$\{33,36,45,\infty_5\}$	$\{32,38,39,\infty_6\}$	$\{44,55,57,\infty_7\}$	$\{24,26,12,22\}$	$\{18,50,6,46\}$
$\{56,28,14,54\}$	$\{41,43,29,9\}$	$\{19,21,37,47\}$	$\{49,58,59,48\}$	$\{11,20,51,40\}$
$\{25,5,16,23\}$	$\{10,17,27,4\}$			
$\{0,7,16,\infty_0\}$	$\{30,32,24,\infty_1\}$	$\{33,9,40,\infty_2\}$	$\{31,12,44,\infty_3\}$	$\{1,5,53,\infty_4\}$
$\{2,35,14,\infty_5\}$	$\{3,39,10,\infty_6\}$	$\{15,56,28,\infty_7\}$	$\{50,52,38,48\}$	$\{55,27,43,23\}$
$\{19,51,37,17\}$	$\{8,13,22,26\}$	$\{6,41,20,54\}$	$\{58,59,42,47\}$	$\{4,11,18,29\}$
$\{36,46,57,34\}$	$\{25,45,49,21\}$			
$\{30,7,46,\infty_0\}$	$\{0,34,41,\infty_1\}$	$\{31,20,59,\infty_2\}$	$\{3,36,42,\infty_3\}$	$\{2,8,49,\infty_4\}$
$\{43,47,54,\infty_5\}$	$\{1,9,57,\infty_6\}$	$\{29,13,55,\infty_7\}$	$\{38,39,22,27\}$	$\{50,23,28,45\}$
$\{37,14,21,32\}$	$\{24,33,4,53\}$	$\{10,11,16,5\}$	$\{6,18,56,58\}$	$\{19,40,44,15\}$
$\{26,17,51,52\}$	$\{48,25,35,12\}$			
$\{30,36,17,\infty_0\}$	$\{0,46,59,\infty_1\}$	$\{51,29,47,\infty_2\}$	$\{5,57,34,\infty_3\}$	$\{11,13,40,\infty_4\}$
$\{3,7,44,\infty_5\}$	$\{16,35,14,\infty_6\}$	$\{23,37,19,\infty_7\}$	$\{24,25,38,43\}$	$\{1,33,56,28\}$
$\{53,26,31,18\}$	$\{27,4,39,55\}$	$\{48,21,8,41\}$	$\{50,32,10,12\}$	$\{6,42,22,58\}$
$\{54,15,49,20\}$	$\{45,52,2,9\}$			

$\square$

As a corollary of the Tripling Construction II, we obtain

Theorem 2.3.9. *If there exists a constant* $M \geq 6$*, such that for every* $n \equiv 1,2 \pmod 3$ *in the range* $M \leq n < 3M$*, there exists an IRH*$(4^n : 4^{17})$*, then for every* $n \equiv 1,2 \pmod 3$ *and* $n \geq M$*, there exists an IRH*$(4^n : 4^{17})$*.*

Proof. First, we claim that there exists an IRH$(4^{17} : 4^s)$ for each $s \in \{1,2,4,5,7\}$. Applying the Tripling Construction II with $(n,s) = (7,2)$ and an RH(4^7) in Lemma 2.3.7, we obtain an RH(4^{17}), an IRH$(4^{17} : 4^4)$ and an IRH$(4^{17} : 4^7)$. An IRH$(4^{17} : 4^5)$ can be constructed by applying Theorem 2.2.3 with an RHF$_4(3^5 : 2)$ in Lemma 2.3.8 and an RH(4^5) in Lemma 2.3.6. The designs with a hole of sizes 1 or 2 are actually an RH(4^{17}).

The above statement yields that the existence of an IRH$(4^n : 4^{17})$ implies the existence of an IRH$(4^n : 4^s)$ for all $s \in \{1,2,4,5,7,17\}$. The proof proceeds by induction. Let $n \geq 3M$ and $n \equiv 1,2 \pmod 3$. Assume that for each n', $M \leq n' < n$, $n' \equiv 1,2 \pmod 3$, there exists an IRH$(4^{n'} : 4^{17})$. Write $n = 3m - 2s$, where $s = 7,5,1,17,4,2$ when $n \equiv 1,2,4,5,7,8 \pmod 9$, respectively. It is easy to check that $M \leq m < n$ and $m \equiv 1,2 \pmod 3$. Applying the Tripling Construction II, the conclusion then follows. $\square$

Lemma 2.3.10. *For each integer* $n \equiv 1, 2 \pmod{3}$, $n \geq 4$ *and* $n \notin \{73, 149, 181, 599\}$, *there exists an RH*$(4^n)$.

Proof. Let L be the list of pairs (n, s) such that an IRH$(4^n : 4^s)$ is known. For every two pairs (n, s) and (n', s'), define $(n, s) \prec (n', s')$ if $n < n'$ or, $n = n'$ and $s < s'$. We will compute the output of the Tripling Constructions I, II and III, the Doubling Construction and the Product Construction by a computer programme, which involves the following steps:

Step 1: Initialize L. Let $L = \{(4,1), (4,2), (5,1), (5,2), (7,1), (7,2), (13,1), (13,2), (13,5), (19,1), (19,2), (41,1), (41,2)\}$. The designs with 13 groups can be constructed by applying Tripling Construction II with $(n, s) = (5, 1)$. The designs with 19 or 41 groups are constructed in the Appendix. Sort L in ascending order. Let (n, s) be the smallest pair in L.

Step 2: Check whether (n, s) satisfies the Tripling Construction II's condition, i.e., $n \equiv 2s \pmod 3$ and $(n, s) \neq (5, 1)$. If not, go to Step 3. If yes, update L by adding pairs $(3n - 2s, n)$, $(3n - 2s, 4)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Sort the updated L in ascending order, then go to Step 4.

Step 3: Check whether $n - s \equiv 0 \pmod 3$. If not, go to Step 4. If yes, write $n - s = 3^x \cdot t$, such that $t > s$ and $3 \nmid t$, or $s < t < 3s$ and $3 | t$. Check whether $(t + s, s)$ satisfies the Tripling Construction II's condition, i.e., $t + s \equiv 2s \pmod 3$ and $(t + s, s) \neq (5, 1)$, or the Tripling Construction III's condition, i.e., $t \equiv 0 \pmod 3$ and $9s \geq 5t$. If yes, update L by adding pairs $(3n - 2s, n)$ and $(3n-2s, k)$ for all k such that $(n, k) \in L$. Furthermore, add $(3n-2s, 4)$ into L if $(t+s, s)$ satisfies the Tripling Construction II's condition. Sort the updated L in ascending order, then go to Step 4.

Step 4: Apply the Doubling Construction and the Product Construction. Update L by adding the pair $(2n, k)$ for all k such that $(n, k) \in L$. For each m such that $(m, 1) \in L$, update L by adding pairs (mn, n), (mn, m) and (mn, k) for all k such that (n, k) or $(m, k) \in L$. Sort the updated L in ascending order. Let (n, s) be the next smallest pair in the updated L, then go to Step 2.

The programme was run with $n < 2000$ and $s \leq 64$, and produced two results as follows:

Result 1: For each $n \equiv 1, 2 \pmod 3$ and $4 \le n < 1285$, there exists an RH(4^n) with four possible exceptions $\{73, 149, 181, 599\}$.

Result 2: There exists an IRH($4^n : 4^{17}$) for all $n \equiv 1, 2 \pmod 3$ and $1285 \le n < 3855$.

By Theorem 2.3.9, there exists an IRH($4^n : 4^{17}$) for all $n \equiv 1, 2 \pmod 3$ and $n \ge 1285$. Hence there exists an RH(4^n) by Theorem 2.2.3. This completes the proof. □

Lemma 2.3.11. *There exists an RH(4^n) for each $n \in \{181, 599\}$.*

Proof. For $n = 181$, there exists an RCQS($1^{15} : 1$) obtained from an RSQS(16). By Theorem 2.3.2, there exists an RHF$_4$($4^3 : 1$), thus an RHF$_4$($12^3 : 1$) exists by Tripling Construction I. Applying Theorem 2.2.2 with an RH(48^4) and an RCQS($1^{15} : 1$), we get an RHF$_4$($12^{15} : 1$). Then applying Theorem 2.2.3 with an RH(4^{13}), we obtain an RH(4^{181}).

For $n = 599$, there exists an RCQS($1^7 : 1$) obtained from an RSQS(8). By Theorem 2.3.2, there exists an RHF$_4$($85^3 : 4$). Applying Theorem 2.2.2 with the RCQS($1^7 : 1$), the RHF$_4$($85^3 : 4$) and an RH(340^4), we get an RHF$_4$($85^7 : 4$). Applying Theorem 2.2.3 with an IRH($4^{89} : 4^4$) gives the desired RH(4^{599}). Here, the input IRH($4^{89} : 4^4$) can be constructed by applying Tripling Construction II with $(n, s) = (31, 2)$ and an RH(4^{31}). □

Combining Lemmas 2.3.10 and 2.3.11, we obtain the main result in this section.

Theorem 2.3.12. *The necessary conditions $n \equiv 1$ or $2 \pmod 3$ and $n \ge 4$ for the existence of an RH(4^n) are sufficient except possibly for $n \in \{73, 149\}$.*

2.4 Applications of the Existence of RH(4^n)

In this section, we give several applications of the existence result of resolvable H-designs of group size 4 stated in Theorem 2.3.12.

Firstly, we establish the necessary and sufficient conditions for the existence of resolvable H-designs of group size 2. As a corollary of Theorem 2.3.12, we have the following result by the Group Halving Construction.

Lemma 2.4.1. *There exists an RH*(2^n) *for each* $n \equiv 2, 4 \pmod 6$ *and* $n \notin \{146, 298\}$.

Lemma 2.4.2. *There exists an RH*(6^6).

Proof. Let the point set be Z_{36} and the group set be $\{\{j, j+6, \ldots, j+30\} : j = 0, 1, \cdots, 5\}$. We list the base blocks of an RH(6^6) as follows:

Part 1:	$\{7, 9, 14, 16\}$,	$\{5, 10, 27, 32\}$,	$\{1, 3, 12, 14\}$,
	$\{2, 5, 16, 19\}$,	$\{1, 11, 20, 30\}$,	$\{1, 4, 11, 14\}$.
Part 2:	$\{0, 2, 4, 9\}$,	$\{14, 21, 22, 30\}$,	$\{6, 11, 34, 3\}$,
	$\{18, 31, 5, 10\}$,	$\{12, 16, 26, 1\}$,	$\{19, 20, 23, 24\}$,
	$\{25, 32, 33, 35\}$,	$\{7, 27, 28, 29\}$,	$\{8, 13, 15, 17\}$.
Part 3:	$\{11, 16, 27, 32\}$,	$\{1, 14, 17, 34\}$,	$\{2, 13, 21, 30\}$,
	$\{5, 9, 24, 28\}$,	$\{19, 23, 4, 8\}$,	$\{35, 7, 12, 22\}$,
	$\{18, 20, 3, 10\}$,	$\{26, 31, 6, 15\}$,	$\{0, 25, 29, 33\}$.
Part 4:	$\{18, 32, 33, 34\}$,	$\{1, 5, 16, 27\}$,	$\{19, 29, 9, 10\}$,
	$\{14, 22, 23, 0\}$,	$\{20, 21, 24, 25\}$,	$\{12, 17, 3, 7\}$,
	$\{2, 11, 15, 31\}$,	$\{8, 30, 35, 4\}$,	$\{6, 13, 26, 28\}$.
Part 5:	$\{9, 22, 23, 31\}$,	$\{17, 25, 26, 33\}$,	$\{29, 0, 16, 20\}$,
	$\{4, 7, 12, 15\}$,	$\{14, 21, 35, 1\}$,	$\{8, 10, 11, 18\}$,
	$\{5, 13, 24, 32\}$,	$\{2, 27, 30, 34\}$,	$\{19, 28, 3, 6\}$.
Part 6:	$\{0, 2, 5, 15\}$,	$\{9, 30, 4, 7\}$,	$\{29, 31, 12, 14\}$,
	$\{23, 27, 16, 20\}$,	$\{3, 10, 25, 32\}$,	$\{24, 26, 35, 1\}$,
	$\{11, 13, 21, 28\}$,	$\{6, 19, 22, 33\}$,	$\{17, 18, 34, 8\}$.
Part 7:	$\{1, 10, 14, 33\}$,	$\{4, 11, 19, 32\}$,	$\{31, 9, 12, 26\}$,
	$\{13, 15, 16, 18\}$,	$\{0, 5, 20, 21\}$,	$\{24, 25, 34, 35\}$,
	$\{2, 17, 27, 28\}$,	$\{22, 23, 3, 6\}$,	$\{7, 8, 29, 30\}$.

Here, the base blocks are developed by adding 2 modulo 36. The elements of each block in Part 1 cover the residues modulo 4, hence each block in Part 1 gives a parallel class when developed by adding 4 modulo 36. The elements of blocks in each of the other parts are different. Hence, each of these parts forms a parallel class. □

Lemma 2.4.3. *There exists an RH*(2^{146}) *and an RH*(2^{298}).

Proof. An RH(2^{146}) was constructed in [43]. For RH(2^{298}), there exists an RHF$_2$(1^3 : 1) which is actually an RH(2^4). By the Tripling Construction I, there is an $\mathrm{RHF}_2(9^3:1)$ and an $\mathrm{RHF}_2(27^3:1)$. Applying Theorem 2.2.2 with an RCQS$(3^5:1)$ from Theorem 2.2.1, an $\mathrm{RHF}_2(9^3:1)$ and an RH(18^4), we get an $\mathrm{RHF}_2(27^5:1)$.

Now we need a URCS$(3,\{4,6\},12)$ of type $(1^{11}:1)$ which is constructed as follows. Let X_i, $i=1,2$, be two disjoint point sets of size 6. Let $\mathcal{B}=\{\{a,b,c,d\}:\{a,b\}\in F_j^1,\{c,d\}\in F_j^2, 0\le j\le 4\}\cup\{X_i:i=1,2\}$, where $\{F_j^i:0\le j\le 4\}$ is a one-factorization of the complete graph on X_i, $i=1,2$. Then $\mathcal{B}$ is the block set of a URCS$(3,\{4,6\},12)$ of type $(1^{11}:1)$ on $X_1\cup X_2$ with any point as a stem. Here, $\mathcal{B}$ has a resolution $\{P_{j,k}:0\le j\le 4, 0\le k\le 2\}\cup\{Q\}$, where $P_{j,k}=\{\{a,b,c,d\}:\{a,b\}$ is the mth member of $F_j^1,\{c,d\}$ is the $(m+k)$th member of $F_j^2, 1\le m\le 3\}$ and $Q=\{X_i:i=1,2\}$.

Applying Theorem 2.2.2 with the above URCS$(3,\{4,6\},12)$, an $\mathrm{RHF}_2(27^{k-1}:1)$ and an RH(54^k) for $k\in\{4,6\}$, we get an $\mathrm{RHF}_2(27^{11}:1)$. Applying Theorem 2.2.3 with an RH(2^{28}), we get an RH(2^{298}). Here, the input RH(54^6) can be obtained from an RH(6^6) in Lemma 2.4.2 by applying the Weighting Construction with $m=9$. □

Combining Lemmas 2.4.1 and 2.4.3, we obtain

Theorem 2.4.4. *The necessary conditions $n\equiv 2$ or $4 \pmod 6$ and $n\ge 4$ for the existence of an RH(2^n) are also sufficient.*

As a consequence of Theorem 2.4.4, we have the following corollary by the Group Halving Construction.

Corollary 2.4.5. *The necessary condition $v\equiv 4$ or $8 \pmod{12}$ for the existence of an RSQS(v) is also sufficient.*

Note that the proof of Corollary 2.4.5 is independent of the existence of RSQSs but individual designs of small orders. Hence, we provide an alternative existence proof for resolvable SQS(v)s. The existence problem for such designs is a challenging one in combinatorial designs theory. A complete solution was obtained by a joint effort of Hartman [31, 33] and Ji and Zhu [43] over twenty years long. This new proof is beneficial not only from the tripling constructions, but also from the Group Halving Construction developed in Section 2.2.

Secondly, we establish the necessary and sufficient conditions for the existence of resolvable H-designs of group sizes 6 and 3. The following construction for resolvable H-designs is similar as but much stronger than the Product Construction in Theorem 2.2.4.

Lemma 2.4.6. *Suppose that there exist both an RH(g^{2u}) and an RH(g^{2t}). Then there exists an RH(g^{2ut}).*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given RH(g^{2u}), where $\mathcal{G} = \{G_0, \ldots, G_{2u-1}\}$. Let $\mathcal{F} = \{F_1, \ldots, F_{2u-1}\}$ be a one-factorization of the complete graph on Z_{2u}. Applying Lemma 2.1.1, we construct an RH($(tg)^{2u}$) on $X' = X \times Z_t$ with the group set $\mathcal{G}' = \{G'_i = G_i \times Z_t : 0 \le i \le 2u-1\}$ and a resolution of the block set $\mathcal{A}$, $P_1 \mid P_2 \mid \cdots \mid P_s$, where $s = (2u-1)(2u-2)(tg)^2/6$.

Since an RH(g^{2t}) exists, gt is even. For each i, $0 \le i \le 2u-1$, let $\mathcal{F}^i = \{F^i_1, \ldots, F^i_{g(t-1)}\}$ be a one-factorization of the complete multiple-graph on $G_i \times Z_t$ with t parts $\{G_i \times \{l\} : l \in Z_t\}$. For any $\{a, b\} \in F^x_m$ and $\{c, d\} \in F^y_m$, construct a block $\{a, b, c, d\}$, where $1 \le m \le g(t-1)$ and $\{x, y\} \in F_n$ with $2 \le n \le 2u-1$. Denote the set of all these blocks by $\mathcal{A}'$. Here, for any fixed r, $0 \le r \le tg/2 - 1$, the blocks $\{a, b, c, d\}$ with $\{a, b\}$ being the k-th edge of F^x_m, $\{c, d\}$ being the $(k+r)$-th edge of F^y_m with $1 \le k \le tg/2$ form a partition of the set $G'_x \cup G'_y$. Hence, for each $\{x, y\} \in F_n$, we can obtain $g(t-1) \cdot tg/2$ parts each of which partitions $G'_x \cup G'_y$. In total, we can get $(2u-2) \cdot g(t-1) \cdot tg/2$ parallel classes.

For $1 \le k \le u$, let the k-th edge of F_1 be $\{x, y\}$. Construct an RH(g^{2t}) on $G'_x \cup G'_y$ with group set $\{G_x \times \{l\}, G_y \times \{l\} : l \in Z_t\}$. Denote its block set by $\mathcal{C}_k$, which can be partitioned into parallel classes $Q(k, 1), \ldots, Q(k, (2t-1)(2t-2)g^2/6)$. Let $\mathcal{C} = \cup_{1 \le k \le u} \mathcal{C}_k$. Here, for each fixed j, $1 \le j \le (2t-1)(2t-2)g^2/6$, $\cup_{1 \le k \le u} Q(k, j)$ forms a parallel class.

Let $\mathcal{G}'' = \{G_i \times \{l\} : 0 \le i \le 2u-1, l \in Z_t\}$, it is easy to check that $(X', \mathcal{G}'', \mathcal{A} \cup \mathcal{A}' \cup \mathcal{C})$ is an H(g^{2ut}). By the construction, the number of parallel classes is $(2u-1)(2u-2)(tg)^2/6 + (2u-2) \cdot g(t-1) \cdot tg/2 + (2t-1)(2t-2)g^2/6 = (2ut-1)(2ut-2)g^2/6$. Hence, the resultant H-design is resolvable. $\square$

Theorem 2.4.7. *There exists an RH(6^n) for each $n \equiv 0 \pmod 2$ and $n \ge 2$.*

Proof. For each $n \equiv 2$ or $4 \pmod 6$ and $n \geq 4$, there exists an RH(6^n) by the Weighting Construction with an RH(2^n) from Theorem 2.4.4 and $m = 3$.

For $n = 6$, there exists an RH(6^6) from Lemma 2.4.2. For each $n = 6h$ and $h \geq 2$, the proof proceeds by induction. Assume that for each $n' \equiv 0 \pmod 6$ and $n' < n$, there exists an RH$(6^{n'})$. Thus there exists an RH(6^k) for each $k \equiv 0 \pmod 2$ and $k < n$. By Lemma 2.4.6, an RH(6^n) exists since there exists an RH(6^6) and an RH(6^{2h}). □

As a corollary of Theorem 2.4.7, we have the following result by the Group Halving Construction.

Theorem 2.4.8. *The necessary conditions $n \equiv 0 \pmod 4$ and $n \geq 4$ for the existence of an RH(3^n) are also sufficient.*

Thirdly, we completely determine the existence of resolvable G-designs.

A *G-design* of order v with block sizes from K, denoted by $G(t, K, v)$, is a triple $(X, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements;

(3) $\mathcal{G} = \{G_1, G_2, \dots\}$ is a set of nonempty subsets of X, which partition X;

(4) $\mathcal{A}$ is a family of subsets of X, each of cardinality from K;

(5) every t-subset T of X with $|T \cap G_i| < t$, for all i, is contained in a unique block, and no t-subset of G_i, for any i, is contained in any block.

The *type* of the G(t, K, v) is defined as the list $(|G||G \in \mathcal{G})$. In this chapter, we denote a G$(3, \{4\}, v)$ of type g^n by G(g^n) for short. Recently, Zhuralev et al. [69] investigated the existence of such designs (called *group divisible Steiner quadruple systems* as in [69]). A table was provided that includes existence results when the number of points is not more than 24. They also proved the following theorem in [69].

Theorem 2.4.9. *There exists a $G(g^n)$ if and only if $g = 1$ and $n \equiv 2$ or 4 (mod 6), or g is even and $g(n-1)(n-2) \equiv 0 \pmod 3$.*

A $G(g^n)$ is said to be *resolvable*, denoted by $RG(g^n)$, if its block set can be partitioned into parallel classes. It is clear that the necessary conditions for the existence of an $RG(g^n)$ are $g = 1$ and $n \equiv 4$ or $8 \pmod{12}$, or g is even, $gn \equiv 0 \pmod 4$ and $g(n-1)(n-2) \equiv 0 \pmod 3$.

Lemma 2.4.10. *If there exists an $RH(g^{2t})$ with g even, then there exist both an $RG((2g)^t)$ and an $RG(g^{2t})$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $RH(g^{2t})$, where $\mathcal{G} = \{G_0, \dots, G_{2t-1}\}$. Let $\mathcal{F} = \{F_1, \dots, F_{2t-1}\}$ be a one-factorization of the complete graph on Z_{2t}.

For $0 \leq i \leq 2t-1$, let $\mathcal{F}^i = \{F_1^i, \dots, F_{g-1}^i\}$ be a one-factorization of the complete graph on G_i. Let $F_j^i = \{f_j^i(0), \dots, f_j^i(g/2-1)\}$. For all $1 \leq n \leq 2t-2$, $1 \leq j \leq g-1$, $0 \leq k \leq g/2-1$, it is easy to see that

$$\{f_j^x(l) \cup f_j^y(l+k) : 0 \leq l \leq g/2-1, \{x, y\} \in F_n\}$$

is a partition of X. Denote the set of all these blocks by $\mathcal{A}$.

Then it is easy to check that $(X, \mathcal{G}', \mathcal{A} \cup \mathcal{B})$ is an $RG((2g)^t)$ with group set $\mathcal{G}' = \{G_x \cup G_y : \{x, y\} \in F_{2t-1}\}$.

Furthermore, if we adjoin the parallel classes formed by $\{f_j^x(l) \cup f_j^y(l+k) : 0 \leq l \leq g/2-1, \{x, y\} \in F_n\}$ into the $RG((2g)^t)$, where $n = 2t-1$, $1 \leq j \leq g-1$, $0 \leq k \leq g/2-1$, then we obtain an $RG(g^{2t})$ with group set $\mathcal{G}$. $\square$

Lemma 2.4.11. *If there exists an $RG(g^n)$, then there exists an $RG((2mg)^n)$ for any positive integer m.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $RG(g^n)$ with $\mathcal{G} = \{G_1, G_2, \dots, G_n\}$ and a resolution of $\mathcal{B}$, P_i, $1 \leq i \leq r$, where $r = ((gn-1)(gn-2) - (g-1)(g-2))/6$. Let $X' = X \times Z_{2m}$ and $G_k' = G_k \times Z_{2m}$, $1 \leq k \leq n$. We will construct an $RG((2mg)^n)$ on X' with group set $\mathcal{G}' = \{G_k' : 1 \leq k \leq n\}$.

For each block $B \in \mathcal{B}$, construct an $RH((2m)^4)$ on $B \times Z_{2m}$ with group set $\{x \times Z_{2m} : x \in B\}$ and block set $\mathcal{A}_B$ having resolution classes $P_B(j)$, $1 \leq j \leq (2m)^2$.

Let Γ be a multi-partite complete graph on the vertex set X with partite set $\mathcal{G}$. Denote its edge set by E. Then E is the block set of a GDD$(2, 2, gn)$ of type g^n on X with group set $\mathcal{G}$. Since an RG(g^n) exists, gn is even. There exists a resolvable GDD$(2, 2, gn)$ of type g^n by [14], i.e., E has a resolution $\{Q_i : 1 \leq i \leq g(n-1)\}$ on X.

For each $x \in X$, let $\mathcal{F}^x = \{F_1^x, \ldots, F_{2m-1}^x\}$ be a one-factorization of the complete graph on $x \times Z_{2m}$. For each edge $\{x, y\} \in E$, let

$$\mathcal{E}_{\{x,y\}} = \{\{a, b, c, d\} : \{a, b\} \in F_k^x, \{c, d\} \in F_k^y, 1 \leq k \leq 2m-1\}.$$

Then $\mathcal{C} = (\bigcup_{B\in\mathcal{B}} \mathcal{A}_B) \bigcup (\bigcup_{\{x,y\}\in E} \mathcal{E}_{\{x,y\}})$ is the block set of the required G$((2mg)^n)$. We need to give its required resolution classes.

For each P_i, $1 \leq i \leq r$, $P'_{i,j} = \bigcup_{B\in P_i} P_B(j)$ is a parallel class of X', where $1 \leq j \leq (2m)^2$.

For each Q_i, $1 \leq i \leq g(n-1)$, and for each pair of k, l with $1 \leq k \leq 2m-1$ and $0 \leq l \leq m-1$,

$$Q'_{i,k,l} = \bigcup_{\{x,y\}\in Q_i} \{\{a, b, c, d\} : \text{where } \{a, b\} \text{ is the } j\text{th member of } F_k^x \text{ and } \{c, d\} \text{ is the } (j+l)\text{th member of } F_k^y, 1 \leq j \leq m\}$$

is a parallel class of X'.

Thus we obtain an RG$((2mg)^n)$. □

Theorem 2.4.12. *The necessary conditions $g = 1$ and $n \equiv 4$ or $8 \pmod{12}$, or g is even, $gn \equiv 0 \pmod 4$ and $g(n-1)(n-2) \equiv 0 \pmod 3$ for the existence of an RG(g^n) are also sufficient.*

Proof. According to the necessary conditions for the existence of an RG(g^n), we partition the parameters into seven classes as follows:

(1) $g = 1$ and $n \equiv 4, 8 \pmod{12}$,

(2) $g \equiv 2 \pmod{12}$ and $n \equiv 2, 4 \pmod 6$,

(3) $g \equiv 4 \pmod{12}$ and $n \equiv 1, 2 \pmod 3$,

(4) $g \equiv 6 \pmod{12}$ and $n \equiv 0 \pmod{2}$,

(5) $g \equiv 8 \pmod{12}$ and $n \equiv 1, 2 \pmod{3}$,

(6) $g \equiv 10 \pmod{12}$ and $n \equiv 2, 4 \pmod{6}$,

(7) $g \equiv 0 \pmod{12}$ and $n \in N$.

For Case (1), an RG(1^n) is actually an RSQS(n), whose existence has been solved completely [33, 43]. For Cases (2), (4) and (6), an RG(g^n) can be obtained by applying Lemma 2.4.10 with an RH(g^n). For Cases (3), (5) and (7), we continue to partition them into two subcases (A) $g \equiv 4, 20, 12 \pmod{24}$ and (B) $g \equiv 16, 8, 0 \pmod{24}$. For Subcase (A), an RG(g^n) can be obtained by applying Lemma 2.4.10 with an RH($(g/2)^{2n}$). For Subcase (B), the existence of an RG(g^n) can be obtained by applying Lemma 2.4.11 with an RG(4^n) or an RG(12^n). □

Finally, we give two applications of the existence result of resolvable G-designs.

A *packing quadruple system* (*covering quadruple system, respectively*) of order v, denoted by PQS(v) (**C**QS(v)) is a pair $(X, \mathcal{B})$, where X is a set of cardinality n and $\mathcal{B}$ is a set of 4-subsets of X such that every 3-subset of X is contained in at most one (at least one) block of $\mathcal{B}$. Note that we use a bold black letter "**C**" in the notation "**C**QS" to distinguish it from the one of candelabra quadruple system.

A PQS(v) (**C**QS(v)) $(X, \mathcal{B})$ is called *maximal* (*minimal*), denoted by MPQS(v) (M**C**QS(v)), if there does not exist any PQS(v) (**C**QS(v)) $(X, \mathcal{A})$ with $|\mathcal{A}| > |\mathcal{B}|$ ($|\mathcal{A}| < |\mathcal{B}|$). We denote by $p(v)$ ($c(v)$) the number of blocks in an MPQS(v) (M**C**QS(v)).

The Johnson bound [44] $j(v)$ for the packing numbers is given by

$$p(v) \leq j(v) = \begin{cases} \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \left\lfloor \frac{v-2}{2} \right\rfloor \right\rfloor \right\rfloor, & v \not\equiv 0 \pmod{6}, \\ \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \left\lfloor \frac{v-2}{2} \right\rfloor \right\rfloor - 1 \right\rfloor, & v \equiv 0 \pmod{6}. \end{cases}$$

Here, $\lfloor x \rfloor$ denotes the largest integer not greater than x.

When $v \equiv 2,4 \pmod 6$, Hanani [23] showed that $p(v) = j(v)$ by constructing an SQS(v). Deleting a point and all blocks containing it from an SQS$(v+1)$ yields that $p(v) = j(v)$ for $v \equiv 1,3 \pmod 6$. Brouwer [9] showed that $p(v) = j(v)$ for all $v \equiv 0 \pmod 6$. Recently, Ji [40] showed that the last packing number for $v \equiv 5 \pmod 6$ is equal to Johnson bound with 21 undecided values $v = 6k+5, k \in \{m : m \text{ is odd}, 3 \leq m \leq 35, m \neq 17, 21\} \cup \{45, 47, 75, 77, 79, 159\}$.

The Schönheim bound [51] $s(v)$ for the covering numbers is given by

$$c(v) \geq s(v) = \lceil \frac{v}{4} \lceil \frac{v-1}{3} \lceil \frac{v-2}{2} \rceil \rceil \rceil.$$

Here, $\lceil x \rceil$ denotes the smallest integer not less than x.

Mills [55] has shown that $c(v) = s(v)$ for all $v \not\equiv 7 \pmod{12}$. Kalbfleisch and Stanton [45] and Swift [66] have noted that $c(7) = s(7) + 1$. Mills [56] has also proved that $c(499) = s(499)$. Hartman et al. [35] have shown that $c(v) = s(v)$ for all $v \geq 52423$. Recently, Ji [41] proved that $c(v) = s(v)$ for all $v \equiv 7 \pmod{12}$ with an exception $v = 7$ and possible exceptions of $v = 12k + 7, k \in \{1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 16, 21, 23, 25, 29\}$.

A PQS(v) (**C**QS(v)) is called *resolvable*, denoted by RPQS(v) (R**C**QS(v)), if its block set can be partitioned into parallel classes.

An RPQS(v) (R**C**QS(v)) $(X, \mathcal{B})$ is called *maximal* (*minimal*), denoted by MRPQS(v) (MR**C**QS(v)), if there does not exist any RPQS(v) (R**C**QS(v)) $(X, \mathcal{A})$ with $|\mathcal{A}| > |\mathcal{B}|$ ($|\mathcal{A}| < |\mathcal{B}|$). It is easy to see that the necessary condition for the existence of an MRPQS(v) (MR**C**QS(v)) is $v \equiv 0 \pmod 4$.

Maximal resolvable packings and minimal resolvable coverings with strength $t = 2$ are fundamental problems in combinatorial designs theory (see, for examples, [18, 49]). It is nature and interesting to consider the corresponding problems for strength $t = 3$. For the existence of MRPQS(v) and MR**C**QS(v), we need only to consider the case $v \equiv 0 \pmod{12}$, since an RSQS(v) is simply both an MRPQS(v) and an MR**C**QS(v). Now, we focus on the investigation of the existence of MRPQS(v) and MR**C**QS(v) with $v = 12t$ for all $t \geq 1$. Denote by $p'(v)$ ($c'(v)$) the number of blocks in an MRPQS(v) (MR**C**QS(v)). Since $v = 12t$, it is easy to check that $p'(v) \leq 3t(24t^2 - 6t - 1)$ and $c'(v) \geq 3t(24t^2 - 6t + 1)$. In the sequel, when we talk about an MRPQS(v) (MR**C**QS(v)) we will mean the RPQS(v) (R**C**QS(v)) with the number of blocks meeting the previous upper (lower) bound for $p'(v)$ ($c'(v)$).

Lemma 2.4.13. *There exist both an MRPQS*(12) *and an MR**C**QS*(12).

Proof. It is easy to check that $p'(12) \leq 51$ and $c'(12) \geq 57$.

Let $X = Z_6 \times Z_2$ with two subsets $A = Z_6 \times \{0\}$ and $B = Z_6 \times \{1\}$. It is easy to check that $F_1 = \{\{0,1\},\{2,4\},\{3,5\}\}$, $F_2 = \{\{4,5\},\{0,2\},\{1,3\}\}$, $F_3 = \{\{0,3\},\{2,5\},\{1,4\}\}$, $F_4 = \{\{2,3\},\{0,4\},\{1,5\}\}$ and $F_5 = \{\{0,5\},\{1,2\},\{3,4\}\}$ form a one-factorization of the complete graph on Z_6. Let $f^j_{i,k} = \{(x,j),(y,j)\}$, where $\{x,y\}$ is the k-th member of F_i, $1 \leq k \leq 3$, $1 \leq i \leq 5$ and $j \in Z_2$. Then $\{F^j_i = \{f^j_{i,k} : 1 \leq k \leq 3\} : 1 \leq i \leq 5\}$ is a one-factorization of the complete graph on $Z_6 \times \{j\}$ for each $j \in Z_2$. We will construct an MR**C**QS(12) and an MRPQS(12) on X as follows.

For MRPQS(12), $\{f^0_{i,k} \cup f^1_{i,k+l} : 1 \leq k \leq 3\}$ with $(i,l) \in (\{2,3,4,5\} \times Z_3) \cup (\{1\} \times (Z_3 \setminus \{0\}))$ are the first 14 parallel classes. Next, $\{f^0_{1,s} \cup f^0_{1,1+s}, f^1_{1,s} \cup f^1_{1,1+s}, f^0_{1,2+s} \cup f^1_{1,2+s}\}$ for $s \in Z_3$ are the last 3 parallel classes. It is clear that all the blocks in these 17 parallel classes form an MRPQS(12).

For MR**C**QS(12), $\{f^0_{i,k} \cup f^1_{i,k+l} : 1 \leq k \leq 3\}$ with $(i,l) \in (\{3,5\} \times Z_3) \cup (\{1,2,4\} \times (Z_3 \setminus \{0\}))$ are the first 12 parallel classes. Next, $\{f^0_{i,1} \cup f^1_{i,1} : i = 1,2,4\}$ and $\{f^0_{i,1} \cup f^0_{i,1+l}, f^1_{i,1} \cup f^1_{i,1+l}, f^0_{i,1+l'} \cup f^1_{i,1+l'}\}$ for $i \in \{1,2,4\}$ and $(l,l') \in \{(1,2),(2,1)\}$ are the last 7 parallel classes. It is clear that all the blocks in these 19 parallel classes form an MR**C**QS(12). □

Now, we give a complete solution to the existence problem of MRPQSs and MR**C**QSs as follows.

Theorem 2.4.14. *An MRPQS*(v) (*MR**C**QS*(v)) *with the number of blocks meeting the upper* (*lower*) *bound exists if and only if* $v \equiv 0 \pmod 4$.

Proof. Start from an RG(12^t) based on X with $|X| = 12t$. It is easy to check that an RG(12^t) contains $18t(4t+3)(t-1)$ blocks. Adjoining these $18t(4t+3)(t-1)$

blocks with t disjoint MRPQS(12)s based on the t different groups of the RG(12^t), we obtain $3t(24t^2-6t-1)$ blocks which cover the triples of X at most once. Hence, we have an MRPQS($12t$). Similarly, we can obtain an MR**C**QS($12t$). $\square$

An $S(t,K,v)$ $(X,\mathcal{B})$ is said to be *resolvable*, denoted by $RS(t,K,v)$, if the block set $\mathcal{B}$ can be partitioned into parallel classes. A parallel class is *uniform* if all blocks in the parallel class have the same size. A *uniformly resolvable Steiner system*, URS(t,K,R,v), is an $RS(t,K,v)$ such that all the blocks in each parallel class have the same size, where R is a multiset with $|R|=|K|$ and for each $k\in K$ there corresponds a positive $r_k\in R$ such that there are exactly r_k parallel classes of size k.

When $t=2$, much work has been done on uniformly resolvable pairwise balanced designs, see [1]. However, for $t>2$, not much is known for uniformly resolvable t-wise balanced design. For each $v\equiv 4,8 \pmod{12}$, an RSQS(v) is actually a URS$(3,\{4,k\},\{r_4,r_k\},v)$ with $r_k=0$ for any $k\neq 4$. So it is interesting to investigate the existence of a URS$(3,\{4,k\},\{r_4,r_k\},v)$ with $r_k>0$, where the smallest nontrivial case is $k=3$. First, we give the necessary conditions for the existence of a URS$(3,\{3,4\},\{r_3,r_4\},v)$.

Lemma 2.4.15. *If there exists a URS*$(3,\{3,4\},\{r_3,r_4\},v)$ *with* $r_3>0$*, then* $v\equiv 0 \pmod{12}$ *and* $r_3\equiv 1 \pmod 3$.

Proof. If there exists a URS$(3,\{3,4\},\{r_3,r_4\},v)$, then $v\equiv 0\pmod{12}$ since v must be divided by both 3 and 4. Let $v=12k$, $k\geq 1$. Since there are r_3 parallel classes of size 3,

$$r_4=\frac{\binom{3}{12k}-(12k/3)\times r_3\times\binom{3}{3}}{\binom{3}{4}\times(12k/4)}=\frac{72k^2-18k+1-r_3}{3}.$$

Since r_4 is an integer, we have $r_3\equiv 1 \pmod 3$. $\square$

Lemma 2.4.16. *For each positive integer* n*, there does not exist a URS*$(3,\{3,4\},$ $\{r_3,r_4\},12n)$ *with* $r_3=1$.

Proof. Suppose that there exists a URS$(3, \{3, 4\}, \{r_3, r_4\}, 12n)$ with $r_3 = 1$. Regarding each block in the parallel class of size 3 as a group, we get an RG(3^{4n}), which leads to a contradiction by Theorem 2.4.12. □

Lemma 2.4.17. *There exists a URS*$(3, \{3, 4\}, \{r_3, r_4\}, 12)$ *with* $r_3 = 4$ *and* $r_4 = 17$.

Proof. We will construct a URS$(3, \{3, 4\}, \{r_3, r_4\}, 12)$ with $r_3 = 4$ on $X = \{a, b, c\} \times Z_2 \times Z_2$. For convenience, we write xij for the ordered triple $(x, i, j) \in X$. Arrange the points of X in the following array.

$$\begin{array}{llcll} a00 & a10 & \qquad & a01 & a11 \\ b00 & b10 & & b01 & b11 \\ c00 & c10 & & c01 & c11 \end{array}$$

Take the four parallel classes with blocks of size three below:

$$\begin{aligned} T_1 &= \{\{aij, bij, cij\} : i \in Z_2, j \in Z_2\}, \\ T_2 &= \{\{aij, bij, c(i+1)j\} : i \in Z_2, j \in Z_2\}, \\ T_3 &= \{\{aij, b(i+1)j, cij\} : i \in Z_2, j \in Z_2\}, \\ T_4 &= \{\{aij, b(i+1)j, c(i+1)j\} : i \in Z_2, j \in Z_2\}. \end{aligned}$$

Let F_i, $i = 1, 2, \ldots, 5$ be a one-factorization of the complete graph on the vertex set $\{a, b, c\} \times Z_2$ and f_i^j be the jth member of F_i. Without loss of generality, we may assume $F_0 = \{\{a0, a1\}, \{b0, b1\}, \{c0, c1\}\}$.

The first five parallel classes with blocks of size four are given below:

$$\begin{aligned} Q_1 &= \{\{a0j, a1j, b0j, b1j\} : j \in Z_2\} \cup \{\{c00, c10, c01, c11\}\}, \\ Q_2 &= \{\{a0j, a1j, c0j, c1j\} : j \in Z_2\} \cup \{\{b00, b10, b01, b11\}\}, \\ Q_3 &= \{\{b0j, b1j, c0j, c1j\} : j \in Z_2\} \cup \{\{a00, a10, a01, a11\}\}, \\ Q_4 &= \{\{a00, a10, b01, b11\}, \{b00, b10, c01, c11\}, \{c00, c10, a01, a11\}\}, \\ Q_5 &= \{\{a00, a10, c01, c11\}, \{b00, b10, a01, a11\}, \{c00, c10, b01, b11\}\}. \end{aligned}$$

The remaining twelve parallel classes with blocks of size four $P_{i,m}$ with $i = 2, 3, 4, 5$ and $m = 0, 1, 2$ are obtained as follows:

$$P_{i,m} = \{\ \{f_i^j \times \{0\}, f_i^{j+m} \times \{1\}\} : 1 \le j \le 3\}.$$

Then $(\cup_{1\le i\le 4}T_i)\bigcup(\cup_{1\le i\le 5}Q_i)\bigcup(\cup_{2\le i\le 5, 0\le m\le 2}P_{i,m})$ is the block set of a URS$(3, \{3,4\}, \{r_3, r_4\}, 12)$ with $r_3 = 4$ and $r_4 = 17$. □

Theorem 2.4.18. *There exits a URS*$(3, \{3,4\}, \{r_3, r_4\}, v)$ *with* $r_3 = 4$ *if and only if* $v \equiv 0 \pmod{12}$.

Proof. For each $v = 12t$, start from an RG(12^t) on X with group set $\mathcal{G}$ and block set $\mathcal{B}$. For each group $G \in \mathcal{G}$, construct a URS$(3, \{3,4\}, \{r_3, r_4\}, 12)$ with $r_3 = 4$ on G with block set $\mathcal{B}_G$. Then $\mathcal{B} \cup (\cup_{G\in\mathcal{G}}\mathcal{B}_G)$ is the block set of a URS$(3, \{3,4\}, \{r_3, r_4\}, v)$ with $r_3 = 4$. □

2.5 Resolvable H-designs with Group Size 12

As in Section 2.3, we first give our second tripling construction for resolvable H-designs with groups size 12 by constructing resolvable B_{12}-pairings. Combining this tripling construction together with the Product Construction and the existence result of resolvable H-designs and resolvable G-designs stated in Sections 2.3 and 2.4, we give a near complete solution to the existence problem of resolvable H-designs with group size 12. Finally, a main result of this chapter is given to close this section.

Lemma 2.5.1. *There exists an* $RB_{12}(n, s)$.

Proof. When $n = 0$, we take $D = Z_{12n+12s}$ and $R_i = S_i = \overline{R}_i = \emptyset$. When $n > 0, s > 0$, the desired $RB_{12}(n, s)$ is constructed directly as follows:

(1) For s odd and n even, let

$D = \{(n+s)j : 0 \le j \le 11\} \cup \{(n+s)i + j : 0 \le i \le 11, n/2+1 \le j \le n/2+s-1\}$,

$PR_0 = \{\{j, -j\} : 1 \le j \le n/2$ or $5(n+s)+1 \le j \le 5(n+s)+n/2$ or $n+s+1 \le j \le n+s+n/2$ or $n+s+n/2+s \le j \le 2(n+s)-1\}$,

$PR_1 = \{\{j, -j\} : 2(n+s)+1 \le j \le 2(n+s)+n/2$ or $2(n+s)+n/2+s \le j \le 3(n+s)-1$ or $4(n+s)+1 \le j \le 4(n+s)+n/2$ or $4(n+s)+n/2+s \le j \le 5(n+s)-1\}$,

$PR_2 = \{\{j, -j\} : 3(n+s)+1 \le j \le 3(n+s)+n/2$ or $3(n+s)+n/2+s \le j \le 4(n+s)-1$ or $n/2+s \le j \le n+s-1$ or $5(n+s)+n/2+s \le j \le 6(n+s)-1\}$.

$PA_0 = \{\{j, 7(n+s)-j\} : 1 \le j \le n/2\} \cup \{\{5(n+s)+j, 2(n+s)-j\} : 1 \le j \le n/2\}$,

$PA_1 = \{\{2(n+s)+j, 5(n+s)-j\} : 1 \le j \le n/2\} \cup \{\{4(n+s)+j, 3(n+s)-j\} : 1 \le j \le n/2\}$,

$PA_2 = \{\{3(n+s)+j, 4(n+s)-j\} : 1 \le j \le n/2\} \cup \{\{n+s-j, 8(n+s)+j : 1 \le j \le n/2\}\}$.

(2) For s even and n odd,

(2.1) $n \ge 3$, let

$D = \{(n+s)j : 0 \le j \le 11\} \cup \{(n+s)i + j : 0 \le i \le 11, 1 \le j \le (s-2)/2 \text{ or } n+s-(s-2)/2 \le j \le n+s-1\} \cup \{(n+s)i + (s-2)/2 + 1, (n+s)i' - (s-2)/2 - 1 : i = 0,1,2,6,7,8, i' = 4,5,6,10,11,12\}$,

$PR_0 = \{\{j, -j\} : s/2+1 \le j \le n+s/2 \text{ or } 3(n+s)+s/2 \le j \le 3(n+s)+n+s/2-1\}$,

$PR_1 = \{\{j, -j\} : n+s+s/2+1 \le j \le n+s+n+s/2 \text{ or } 5(n+s)+s/2 \le j \le 5(n+s)+n+s/2-1\}$,

$PR_2 = \{\{j, -j\} : 2(n+s)+s/2+1 \le j \le 2(n+s)+n+s/2 \text{ or } 4(n+s)+s/2 \le j \le 4(n+s)+n+s/2-1\}$.

$PA_0 = \{\{s/2+j, n+s/2-j\} : 1 \le j \le (n-1)/2\} \cup \{\{3(n+s)+s/2-1+j, 9(n+s)-s/2-j\} : 1 \le j \le (n-1)/2\} \cup \{\{n+s/2, 3(n+s)+n+s/2-2\}\}$,

$PA_1 = \{\{(n+s)+s/2+j, 5(n+s)+n+s/2-j\} : 1 \le j \le n\}$,

$PA_2 = \{\{2(n+s)+s/2+j, 4(n+s)+n+s/2-j\} : 1 \le j \le n\}$.

(2.2) $n = 1$, let

$D = Z_{12(s+1)} \setminus \{\pm((n+s)i + s/2 + 1) : 0 \le i \le 5\}$,

$PR_0 = \{\{s/2+1, 11(s+1)+s/2\}, \{s+1+s/2+1, 10(s+1)+s/2\}\}$,

$PR_1 = \{\{2(s+1)+s/2+1, 9(s+1)+s/2\}, \{3(s+1+s/2+1, 8(s+1)+s/2\}\}$,

$PR_2 = \{\{4(s+1)+s/2+1, 7(s+1)+s/2\}, \{5(s+1)+s/2+1, 6(s+1)+s/2\}\}$.

$PA_0 = \{\{s/2+1, 10(s+1)+s/2\}\}$,

$PA_1 = \{\{2(s+1)+s/2+1, 8(s+1)+s/2\}\}$,

$PA_2 = \{\{4(s+1)+s/2+1, 6(s+1)+s/2\}\}$.

(3) For s even and n even,

(3.1) $n \ge 4$, let

$D = \{(n+s)j : 0 \le j \le 11\} \cup \{(n+s)i + j : 0 \le i \le 11, n/2+1 \le j \le n/2+s-1\}$,

$PR_0 = \{\{j, -j\} : 1 \le j \le n/2 \text{ or } n/2+s \le j \le n+s-1 \text{ or } 2(n+s)+1 \le j \le 2(n+s)+n/2 \text{ or } 2(n+s)+n/2+s \le j \le 3(n+s)-1\}$,

$PR_1 = \{\{j, -j\} : n+s+1 \le j \le n+s+n/2 \text{ or } n+s+n/2+s \le j \le 2(n+s)-1 \text{ or } 5(n+s)+1 \le j \le 5(n+s)+n/2 \text{ or } 5(n+s)+n/2+s \le j \le 6(n+s)-1\}$,

$PR_2 = \{\{j, -j\} : 3(n+s)+1 \le j \le 3(n+s)+n/2 \text{ or } 3(n+s)+n/2+s \le j \le 4(n+s)-1 \text{ or } 4(n+s)+1 \le j \le 4(n+s)+n/2 \text{ or } 4(n+s)+n/2+s \le j \le 5(n+s)-1\}$.

$PA_0 = \{\{j, 12(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{2(n+s)+j, 10(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{n/2, 11(n+s)+n/2-1\}, \{2(n+s)+n/2, 9(n+s)+n/2-1\}\}$,

$PA_1 = \{\{(n+s)+j, 11(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{5(n+s)+j, 7(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{(n+s)+n/2, 10(n+s)+n/2-1\}, \{5(n+s)+n/2, 6(n+s)+n/2-1\}\}$,

$PA_2 = \{\{3(n+s)+j, 9(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{4(n+s)+j, 8(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{3(n+s)+n/2, 8(n+s)+n/2-1\}, \{4(n+s)+n/2, 7(n+s)+n/2-1\}\}$.

(3.2) $s > 2$ even and $n = 2$, let

$D = \{(2+s)i+j : 0 \le i \le 2 \text{ or } 9 \le i \le 11, 0 \le j \le s+1\} \cup \{(2+s)i+j : 3 \le i \le 8, 0 \le j \le s/2-2 \text{ or } j = s/2+1 \text{ or } s/2+4 \le j \le s+1\}$,

$PR_0 = \{\{j, -j\} : 3(s+2)+s/2-1 \le j \le 3(s+2)+s/2 \text{ or } 3(s+2)+s/2+2 \le j \le 3(s+2)+s/2+3\}$,

$PR_1 = \{\{j, -j\} : 4(s+2)+s/2-1 \le j \le 4(s+2)+s/2 \text{ or } 4(s+2)+s/2+2 \le j \le 4(s+2)+s/2+3\}$,

$PR_2 = \{\{j, -j\} : 5(s+2)+s/2-1 \le j \le 5(s+2)+s/2 \text{ or } 5(s+2)+s/2+2 \le j \le 5(s+2)+s/2+3\}$.

$PA_0 = \{\{3(s+2)+s/2-1, 3(s+2)+s/2+2\}, \{3(s+2)+s/2, 9(s+2)-s/2-3\}\}$,

$PA_1 = \{\{4(s+2)+s/2-1, 8(s+2)-s/2-2\}, \{4(s+2)+s/2+2, 8(s+2)-s/2-3\}\}$,

$PA_2 = \{\{5(s+2)+s/2, 7(s+2)-s/2+1\}, \{5(s+2)+s/2+2, 5(s+2)+s/2+3\}\}$.

(3.3) For $s = 2$ and $n = 2$, let

$D = \{4i+j : 0 \le i \le 2 \text{ or } 9 \le i \le 11, 0 \le j \le 3\} \cup \{24\} \setminus \{36\}$,

$PR_0 = \{\{12, 34\}, \{13, 36\}, \{14, 35\}, \{15, 33\}\}$,

$PR_1 = \{\{16, 30\}, \{17, 32\}, \{18, 31\}, \{19, 29\}\}$,

$PR_2 = \{\{20, 26\}, \{21, 28\}, \{22, 27\}, \{23, 25\}\}$.

$PA_0 = \{\{13, 36\}, \{14, 35\}\}$,

$PA_1 = \{\{17, 32\}, \{18, 31\}\}$,

$PA_2 = \{\{21, 28\}, \{22, 27\}\}$.

(4) For s odd and n odd,

(4.1) $s \ge 3$ odd and $n \ge 5$ odd, let

$D = \{(n+s)j, (n+s)j+(n+s)/2 : 0 \le j \le 11\} \cup \{(n+s)i+j : 0 \le i \le 11, 1 \le j \le (s-3)/2 \text{ or } n+s-(s-3)/2 \le j \le n+s-1\} \cup \{(n+s)i+(s-3)/2+1, (n+s)i'-(s-3)/2-1 : i = 0, 1, 2, 6, 7, 8, i' = 4, 5, 6, 10, 11, 12\}$,

$PR_0 = \{\{j, -j\} : (s-3)/2+2 \le j \le (n+s)/2-1 \text{ or } (n+s)/2+1 \le j \le n+s-(s-3)/2-1 \text{ or } 3(n+s)+(s-3)/2+1 \le j \le 3(n+s)+(n+s)/2-1 \text{ or } 3(n+s)+(n+s)/2+1 \le j \le 4(n+s)-(s-3)/2-2\}$,

$PR_1 = \{\{j, -j\} : n+s+(s-3)/2+2 \le j \le n+s+(n+s)/2-1$ or $n+s+(n+s)/2+1 \le j \le 2(n+s)-(s-3)/2-1$ or $5(n+s)+(s-3)/2+1 \le j \le 5(n+s)+(n+s)/2-1$ or $5(n+s)+(n+s)/2+1 \le j \le 6(n+s)-(s-3)/2-2\}$,

$PR_2 = \{\{j, -j\} : 2(n+s)+(s-3)/2+2 \le j \le 2(n+s)+(n+s)/2-1$ or $2(n+s)+(n+s)/2+1 \le j \le 3(n+s)-(s-3)/2-1$ or $4(n+s)+(s-3)/2+1 \le j \le 4(n+s)+(n+s)/2-1$ or $4(n+s)+(n+s)/2+1 \le j \le 5(n+s)-(s-3)/2-2\}$.

$PA_0 = \{\{(s-3)/2+1+j, 12(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{3(n+s)+(s-3)/2+j, 9(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{(n+s)/2-1, 12(n+s)-(n+s)/2-2\}, \{3(n+s)+(n+s)/2-1, 9(n+s)-(n+s)/2-2\}\}$,

$PA_1 = \{\{(n+s)+(s-3)/2+1+j, 11(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{5(n+s)+(s-3)/2+j, 7(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{(n+s)+(n+s)/2-1, 11(n+s)-(n+s)/2-2\}, \{5(n+s)+(n+s)/2-1, 7(n+s)-(n+s)/2-2\}\}$,

$PA_2 = \{\{2(n+s)+(s-3)/2+1+j, 10(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{4(n+s)+(s-3)/2+j, 8(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{2(n+s)+(n+s)/2-1, 10(n+s)-(n+s)/2-2\}, \{4(n+s)+(n+s)/2-1, 8(n+s)-(n+s)/2-2\}\}$.

(4.2) $s = 1$ and $n \equiv 1 \pmod 4$ and $n \ge 5$, let

$D = \{(n+1)i : 0 \le i \le 11\}$,

$PR_0 = \{\{j, -j-1\} : 1 \le j \le (n+1)/2-1$ or $(n+1)+1 \le j \le (n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)/2+1 \le j \le n$ or $(n+1)+(n+1)/2+1 \le j \le 2(n+1)-1\} \cup \{\{(n+1)/2, 12(n+1)-1\}, \{(n+1)+(n+1)/2, 11(n+1)-1\}\}$,

$PR_1 = \{\{j, -j-1\} : 2(n+1)+1 \le j \le 2(n+1)+(n+1)/2-1$ or $4(n+1)+1 \le j \le 4(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 2(n+1)+(n+1)/2+1 \le j \le 3(n+1)-1$ or $4(n+1)+(n+1)/2+1 \le j \le 5(n+1)-1\} \cup \{\{2(n+1)+(n+1)/2, 10(n+1)-1\}, \{4(n+1)+(n+1)/2, 8(n+1)-1\}\}$,

$PR_2 = \{\{j, -j-1\} : 3(n+1)+1 \le j \le 3(n+1)+(n+1)/2-1$ or $5(n+1)+1 \le j \le 5(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 3(n+1)+(n+1)/2+1 \le j \le 4(n+1)-1$ or $5(n+1)+(n+1)/2+1 \le j \le 6(n+1)-1\} \cup \{\{3(n+1)+(n+1)/2, 9(n+1)-1\}, \{5(n+1)+(n+1)/2, 7(n+1)-1\}\}$.

$PA_0 = \{\{j, -j-1\} : 1 \le j \le (n+1)/2-1$ or $(n+1)+1 \le j \le (n+1)+(n+1)/2-1\} \cup \{\{n-1, n\}\}$,

$PA_1 = \{\{j, -j-1\} : 2(n+1)+1 \le j \le 2(n+1)+(n+1)/2-1$ or $4(n+1)+1 \le j \le 4(n+1)+(n+1)/2-1\} \cup \{\{2(n+1)+(n+1/2+1, 4(n+1)+(n+1)/2\}\}$,

$PA_2 = \{\{j, -j-1\} : 3(n+1)+1 \le j \le 3(n+1)+(n+1)/2-1$ or $5(n+1)+1 \le j \le 5(n+1)+(n+1)/2-1\} \cup \{\{3(n+1)+(n+1)/2, 5(n+1)+(n+1)/2+1\}\}$.

(4.3) $s = 1$ and $n \equiv 3 \pmod 4$ and $n \ge 5$, let

$D = \{(n+1)i : 0 \le i \le 11\}$,

$PR_0 = \{\{j, -j-1\} : 2 \le j \le (n+1)/2-1$ or $3(n+1)+2 \le j \le 3(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)/2+1 \le j \le n$ or $3(n+1)+(n+1)/2+1 \le j \le 4(n+1)-1$ or $j = 1, 3(n+1)+1\} \cup \{\{(n+1)/2, 12(n+1)-2\}, \{3(n+1)+(n+1)/2, 9(n+1)-2\}\}$,

$PR_1 = \{\{j, -j-1\} : (n+1)+2 \leq j \leq (n+1)+(n+1)/2-1$ or $5(n+1)+2 \leq j \leq 5(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)+(n+1)/2+1 \leq j \leq 2(n+1)-1$ or $5(n+1)+(n+1)/2+1 \leq j \leq 6(n+1)-1\} \cup \{\{(n+1)+(n+1)/2, 11(n+1)-2\}, \{5(n+1)+(n+1)/2, 7(n+1)-2\}\}$,

$PR_2 = \{\{j, -j-1\} : 2(n+1)+2 \leq j \leq 2(n+1)+(n+1)/2-1$ or $4(n+1)+2 \leq j \leq 4(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 2(n+1)+(n+1)/2+1 \leq j \leq 3(n+1)-1$ or $4(n+1)+(n+1)/2+1 \leq j \leq 5(n+1)-1\} \cup \{\{2(n+1)+(n+1)/2, 10(n+1)-2\}, \{4(n+1)+(n+1)/2, 8(n+1)-2\}\}$.

$PA_0 = \{\{j, -j-1\} : 2 \leq j \leq (n+1)/2-1$ or $3(n+1)+2 \leq j \leq 3(n+1)+(n+1)/2-1\} \cup \{\{n-6+J, n-j\} : 0 \leq j \leq 1\} \cup \{\{1, 4(n+1)-2\}\}$,

$PA_1 = \{\{j, -j-1\} : (n+1)+2 \leq j \leq (n+1)+(n+1)/2-1$ or $5(n+1)+2 \leq j \leq 5(n+1)+(n+1)/2-1\} \cup \{\{(n+1)+(n+1)/2+j, 5(n+1)+(n+1)/2+3-j\} : 0 \leq j \leq 2\}$,

$PA_2 = \{\{j, -j-1\} : 2(n+1)+2 \leq j \leq 2(n+1)+(n+1)/2-1$ or $4(n+1)+2 \leq j \leq 4(n+1)+(n+1)/2-1\} \cup \{\{2(n+1)+(n+1)/2+j, 4(n+1)+(n+1)/2+3-j\} : 0 \leq j \leq 2\}$.

(4.4) For $s > 1$ odd and $n = 3$, let

$D = \{(n+s)j : 0 \leq j \leq 11\} \cup \{(n+s)i+j : 0 \leq i \leq 11, 3 \leq j \leq s\} \cup \{(n+s)i+j : (i,j) \in \{(0,1), (0,2), (1,1), (1,2), (2,1), (2,2), (9, s+1), (9, s+2), (10, s+1), (10, s+2), (11, s+1), (11, s+2)\}\}$,

$PR_0 = \{\{j, -j\} : s+1 \leq j \leq s+2$ or $3(3+s)+1 \leq j \leq 3(n+s)+2$ or $3(3+s)+s+1 \leq j \leq 3(n+s)+s+2\}$,

$PR_1 = \{\{j, -j\} : (3+s)+s+1 \leq j \leq (3+s)+s+2$ or $5(3+s)+1 \leq j \leq 5(n+s)+2$ or $5(3+s)+s+1 \leq j \leq 5(n+s)+s+2\}$,

$PR_2 = \{\{j, -j\} : 2(3+s)+s+1 \leq j \leq 2(3+s)+s+2$ or $4(3+s)+1 \leq j \leq 4(n+s)+2$ or $4(3+s)+s+1 \leq j \leq 4(n+s)+s+2\}$.

$PA_0 = \{\{s+1, 3(s+3)+s+2\}, \{s+2, 3(s+3)+s+1\}, \{3(s+3)+1, 3(s+3)+2\}\}$,

$PA_1 = \{\{(s+3)+s+1, 5(s+3)+s+2\}, \{(s+3)+s+2, 5(s+3)+s+1\}, \{5(s+3)+1, 10(s+3)+2\}\}$,

$PA_2 = \{\{2(s+3)+s+1, 4(s+3)+s+2\}, \{2(s+3)+s+2, 4(s+3)+s+1\}, \{4(s+3)+2, 9(s+3)+1\}\}$.

(4.5) For $s = 1$ and $n = 3$, let

$D = \{4i+j : 0 \leq i \leq 2$ or $9 \leq i \leq 11, j = 0, 2\} \cup \{24\} \setminus \{36\}$,

$PR_0 = \{\{1, 47\}, \{3, 45\}, \{12, 34\}, \{13, 36\}, \{14, 35\}, \{15, 33\}\}$,

$PR_1 = \{\{9, 39\}, \{11, 37\}, \{16, 30\}, \{17, 32\}, \{18, 31\}, \{19, 29\}\}$,

$PR_2 = \{\{5, 43\}, \{7, 41\}, \{20, 26\}, \{21, 28\}, \{22, 27\}, \{23, 25\}\}$.

$PA_0 = \{\{13, 36\}, \{14, 35\}, \{3, 12\}\}$,

$PA_1 = \{\{17, 32\}, \{18, 31\}, \{11, 30\}\}$,

$PA_2 = \{\{21, 28\}, \{22, 27\}, \{20, 23\}\}$.

(4.6) For $s > 3$ odd and $n = 1$, let

$$D = Z_{12(s+1)} \setminus \{\pm((n+s)i + j) : 0 \leq i \leq 2, 1 \leq j \leq 2\},$$
$$PR_0 = \{\{1, 11(s+1) + s\}, \{2, 11(s+1) + s - 1\}\},$$
$$PR_1 = \{\{(s+1) + 1, 10(s+1) + s\}, \{(s+1) + 2, 10(s+1) + s - 1\}\},$$
$$PR_2 = \{\{2(s+1) + 1, 9(s+1) + s\}, \{2(s+1) + 2, 9(s+1) + s - 1\}\}.$$
$$PA_0 = \{\{1, 2\}\},$$
$$PA_1 = \{\{(s+1) + 1, 10(s+1) + s - 1\}\},$$
$$PA_2 = \{\{2(s+1) + 1, 9(s+1) + s - 1\}\}.$$

(4.7) For $s = 3$ and $n = 1$, let

$$D = Z_{48} \setminus \{1, 2, 5, 6, 9, 10, 38, 39, 42, 43, 46, 47\},$$
$$PR_0 = \{\{5, 6\}, \{10, 39\}\},$$
$$PR_1 = \{\{42, 43\}, \{9, 46\}\},$$
$$PR_2 = \{\{2, 47\}, \{1, 38\}\},$$
$$PS_0 = \{\{9, 46\}, \{2, 47\}\},$$
$$PS_1 = \{\{5, 38\}, \{1, 10\}\},$$
$$PS_2 = \{\{6, 39\}, \{42, 43\}\},$$
$$\overline{R}_0 = \{9, 46\},$$
$$\overline{R}_1 = \{38, 43\},$$
$$\overline{R}_2 = \{6, 47\}.$$

(4.8) For $s = 1$ and $n = 1$, let

$$D = \{0, 1, 2, 3, 4, 5, 12, 19, 20, 21, 22, 23\},$$
$$PR_0 = \{\{6, 7\}, \{8, 11\}\},$$
$$PR_1 = \{\{9, 10\}, \{13, 18\}\},$$
$$PR_2 = \{\{14, 17\}, \{15, 16\}\},$$
$$PS_0 = \{\{9, 16\}, \{10, 15\}\},$$
$$PS_1 = \{\{8, 15\}, \{11, 14\}\},$$
$$PS_2 = \{\{6, 13\}, \{9, 18\}\},$$
$$\overline{R}_0 = \{7, 8\},$$
$$\overline{R}_1 = \{13, 14\},$$
$$\overline{R}_2 = \{6, 7\}.$$

□

Combining Theorem 2.2.8 and Lemma 2.5.1, we obtain the following theorem.

Theorem 2.5.2. *Suppose that $n \geq 0$ and $s \geq 1$. There exists an $RHF_{12}((n+s)^3 : s)$. When $(n, s) \notin \{(1, 1), (2, 2), (3, 1)\}$, the $RHF_{12}((n+s)^3 : s)$ exists with a subdesign $RH(12^4)$.*

As a consequence of Theorem 2.5.2, we have our second tripling construction for resolvable H-designs with group size 12 as follows.

Corollary 2.5.3. (Tripling construction II) *Let $n \geq 0$ and $s \geq 1$. If there exists an IRH$(12^n : 12^s)$, then there exist both an IRH$(12^{3n-2s} : 12^n)$ and an IRH$(12^{3n-2s} : 12^s)$. Furthermore, if there exists an RH(12^n) or an RH(12^s), then there exists an RH(12^{3n-2s}), as well as an IRH$(12^{3n-2s} : 12^4)$ when $(n, s) \notin \{(3, 1), (5, 1), (6, 2)\}$.*

Lemma 2.5.4. *There exists an RH(12^n) for $n \equiv 0, 1, 2, 4, 5 \pmod 6$ and $n \geq 4$.*

Proof. For each $n \equiv 1$ or $2 \pmod 3$, $n \geq 4$ and $n \notin \{73, 149\}$, an RH(12^n) can be obtained by applying the Weighting Construction with an RH(4^n) in Theorem 2.3.12 and $m = 3$. For each $n \equiv 0 \pmod 6$ and $n \geq 4$, an RH(12^n) can be obtained by applying the Weighting Construction with an RH(6^n) in Theorem 2.4.7 and $m = 2$.

For the design RH(12^{73}), it can be constructed by applying Tripling Construction II with $(n, s) = (25, 1)$. For the design RH(12^{149}), it can be obtained by applying Tripling Construction II with $(n, s) = (51, 2)$. Here, the IRH$(12^{51} : 12^2)$ exists by Tripling Construction II with $(n, s) = (23, 9)$ and the IRH$(12^{23} : 12^9)$ exists by Tripling Construction II with $(n, s) = (9, 2)$, where an RH(12^9) is constructed in the Appendix. □

Lemma 2.5.5. *There exists an RHF$_{12}(3^5 : 2)$ and an IRH$(12^n : 12^s)$ for each $(n, s) \in \{(13, 5), (17, 7)\}$.*

Proof. For the existence of an RHF$_{12}(3^5 : 2)$, start from an RHF$_4(3^5 : 2)$ on X with group set $\mathcal{G}$, hole set $\mathcal{F}$ and block set $\mathcal{B}$. Such a design exists by Lemma 2.3.8. Let $X' = X \times Z_3$, $\mathcal{G}' = \{G \times Z_3 : G \in \mathcal{G}\}$ and $\mathcal{F}' = \{\{G \times Z_3 : G \in F\} : F \in \mathcal{F}\}$. For each block $B \in \mathcal{B}$, construct an RH(3^4) on $B \times Z_3$ with block set $\mathcal{A}_B$. Then $\bigcup_{B \in \mathcal{B}} \mathcal{A}_B$ is the block set of an RHF$_{12}(3^5 : 2)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}'$.

An IRH$(12^{13} : 12^5)$ and an IRH$(12^{17} : 12^7)$ can be obtained by applying the Tripling Construction II with $(n, s) = (5, 1)$ and $(7, 2)$, respectively. □

Lemma 2.5.6. *There exists an IRH*$(12^n : 12^s)$ *for all* $n \equiv 35 \pmod{36}$ *and* $s \in \{1, 2, 4, 5, 6, 7, 11, 17\}$.

Proof. For each $n = 36m - 1$, $m \geq 1$, start from an RG(6^{2m}) in Theorem 2.4.12. Considering each group as a block of size 6 and taking any fixed point as a stem, we get a URCS$(3, \{4, 6\}, 12m)$ of type $(1^{12m-1} : 1)$. Applying Theorem 2.2.2 with an RHF$_{12}(3^{k-1} : 2)$ and an RH(36^k) with $k \in \{4, 6\}$, we get an RHF$_{12}(3^{12m-1} : 2)$. Applying Theorem 2.2.3 with an RH(12^5), we get an RH(12^n) and an IRH$(12^n : 12^5)$. Here, the input designs RHF$_{12}(3^{k-1} : 2)$ with $k \in \{4, 6\}$ are from Theorem 2.5.2 and Lemma 2.5.5, respectively. The designs with a hole of sizes 1 or 2 are actually an RH(12^n). The designs with a hole of sizes 11 or 17 exist since we input an RHF$_{12}(3^{k-1} : 2)$ with $k \in \{4, 6\}$ respectively when applying Theorem 2.2.2. The design with a hole size 7 exists since there exists an IRH$(12^{17} : 12^7)$ by Lemma 2.5.5. The designs with a hole of sizes $k = 4$ or 6 exist since the input designs RH(36^k) exist with a subdesign RH(12^k). □

As a corollary of the Tripling Construction II, we obtain

Theorem 2.5.7. *If there exists a constant* $M \geq 7$*, such that for any odd integer* n *in the range* $M \leq n < 3M$*, there exists an IRH*$(12^n : 12^6)$*, then for all odd integer* $n \geq M$*, there exists an IRH*$(12^n : 12^6)$*.*

Proof. It is clear that the existence of an IRH$(12^n : 12^6)$ implies the existence of an IRH$(12^n : 12^s)$ for all $s \in \{1, 2, 6\}$, since there exists an RH(12^6) obtained by the Weighting Construction with an RH(6^6). The proof proceeds by induction. Let n be an odd integer and $n \geq 3M$. Assume that for all odd n' in the range $M \leq n' < n$, there exists an IRH$(12^{n'} : 12^6)$. Write $n = 3m - 2 \cdot s$, where $s = 1, 6, 2$ when $n \equiv 1, 3, 5 \pmod 6$, respectively. It is simple to check that m is odd and $M \leq m < n$. Then applying Tripling Construction II gives the conclusion. □

Lemma 2.5.8. *For each odd integer* $n \geq 5$ *and* $n \notin \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 189, 213, 231, 243, 321, 681\}$*, there exists an RH*$(12^n)$*.*

Proof. As in Lemma 2.3.10, let L be the poset of pairs (n, s) such that an IRH$(12^n : 12^s)$ is known. We will compute the output of the Tripling Construction II, the Doubling Construction and the Product Construction by a computer programme, which involves the following steps:

Step 1: Initialize L. Let $L = \{(5,1),(5,2),(7,1),(7,2),(9,1),(9,2),(13,1),(13,2),(13,5)\} \cup \{(n,s) : \text{RH}(12^n : 12^s) \text{ in Lemma 2.5.6}\}$. Sort L in ascending order. Let (n,s) be the smallest pair in L.

Step 2: Check whether (n,s) satisfies the Tripling Construction II's condition, i.e., $(n,s) \notin \{(3,1),(5,1),(6,2)\}$. If not, go to Step 3. If yes, update L by adding pairs $(3n-2s,n)$, $(3n-2s,4)$ and $(3n-2s,k)$ for all k such that $(n,k) \in L$. Sort the updated L in ascending order, then go to Step 3.

Step 3: Apply the Doubling Construction and the Product Construction. Update L by adding the pair $(2n,k)$ for all k such that $(n,k) \in L$. For each m such that $(m,1) \in L$, update L by adding pairs (mn,n), (mn,m) and (mn,k) for all k such that (n,k) or $(m,k) \in L$. Sort the updated L in ascending order. Let (n,s) be the next smallest pair in the updated L, then go to Step 2.

The programme was run with $n < 2000$ and $s \leq 64$, and produced two results as follows:

Result 1: For all odd n and $4 \leq n < 1102$, there exists an RH(12^n) with eighteen possible exceptions $n \in \{15,21,27,33,39,69,75,87,105,111,129,147,189,213,231,243,321,681\}$.

Result 2: There exists an IRH$(12^n : 12^6)$ for all odd n in the range $1102 \leq n < 3306$.

By Theorem 2.5.7, there exists an IRH$(12^n : 12^6)$ for all odd $n \geq 1102$. Hence there exists an RH(12^n) by Theorem 2.2.3. This completes the proof. □

Lemma 2.5.9. *There exists an RH*(12^n) *for each* $n \in \{189, 681\}$.

Proof. For $n = 189$, start from an RCQS$(3^5 : 1)$. Applying Theorem 2.2.2 with an RHF$_{12}(12^3 : 9)$ and an RH(144^4), we get an RHF$_{12}(36^5 : 9)$. Applying Theorem 2.2.3 with an IRH$(12^{45} : 12^9)$, we get the desired RH(12^{189}). Here, the RHF$_{12}(12^3 : 9)$ exists by Theorem 2.5.2. The IRH$(12^{45} : 12^9)$ can be obtained by applying the Product Construction with an RH(12^5) and an RH(12^9).

For $n = 681$, start from an RCQS$(1^7 : 1)$. Applying Theorem 2.2.2 with an RHF$_{12}(97^3 : 2)$ from Theorem 2.5.2 and an RH$((12 \times 97)^4)$, we get an RHF$_{12}(97^7 : 2)$. Applying Theorem 2.2.3 with an RH(12^{99}), we get an RH(12^{681}). □

Combining Lemmas 2.5.4, 2.5.8 and 2.5.9, we obtain the main result in this section.

Theorem 2.5.10. *The necessary conditions for the existence of RH(12^n) are sufficient except possibly with* $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 213, 231, 243, 321\}$.

Combining Theorems 1.1.2, 2.3.12, 2.4.4, 2.4.7, 2.4.8 and 2.5.10, we have the general existence result of resolvable H-designs as follows, which is the main result of this chapter.

Theorem 2.5.11. *The necessary conditions* $gn \equiv 0 \pmod 4$, $g(n-1)(n-2) \equiv 0 \pmod 3$ *and* $n \geq 4$ *for the existence of a resolvable H-design of type* g^n *are sufficient for each* $g \equiv 1, 2, 3, 5, 6, 7, 9, 10, 11 \pmod{12}$, *are sufficient for each* $g \equiv 4, 8 \pmod{12}$ *with two possible exceptions* $n = 73, 149$, *and are sufficient for each* $g \equiv 0 \pmod{12}$ *with sixteen possible exceptions* $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 213, 231, 243, 321\}$.

Chapter 3

A New Existence Proof for Steiner Quadruple Systems

The purpose of this chapter is to provide an alternative existence proof for Steiner quadruple systems via H-designs of type 2^n. However, the existing proof for the existence of H(2^n), which is the main context of Mills' paper in 1990, is based on the existence result of Steiner quadruple systems. In this chapter, by using the theory of candelabra systems and H-frames, we give a new existence proof for H-designs of type 2^n independent of the existence result of Steiner quadruple systems. As an application of this approach, several new infinite classes of nonuniform H-designs of types $2^n u^1$ with $u = 4, 6, 8$ are also constructed.

3.1 Introduction

The necessary conditions for the existence of an SQS(v) are that $v \equiv 2, 4 \pmod 6$ or $v = 1$. When $v < 4$, the systems have no blocks, and when $v = 4$, it has one block. The smallest interesting system, SQS(8), was known to Kirkman [48] in 1847. The unique (up to isomorphism) SQS(10) was attributed to Barrau [4] as early as 1908 and to Richard Wilson in [12]. Several infinite families of quadruple systems were constructed by Kirkman [48] and by Carmichael [11]. The first complete proof for the existence of SQS(v) was given by Hanani [23] in 1960.

Theorem 3.1.1. *There exists an SQS(v) for all $v \equiv 2, 4 \pmod 6$.*

This result is proved by induction using six recursive constructions together with explicit constructions of an SQS(14) and an SQS(38). Hanani also gave a more sophisticated proof of the existence theorem for SQS(v) in [25], which relies on the construction of 3-wise balanced designs and 3-analogs of group divisible designs. Apart from Hanani's two proofs, Hartman [31, 32, 34] and Lenz [51] used the existence of candelabra quadruple systems of type $(g^3 : s)$ with $s \in \{1, 2, 4, 8\}$ to give a purely tripling existence proof, which used only one type of construction and a small number of initial designs: SQS(v) with $v \in \{8, 10, 14\}$ and HQS(v : 8) with $v \in \{26, 28, 32, 34, 38, 40\}$.

It is easy to see that the existence of an H(2^n) implies that of an SQS($2n$) by combining every two groups of the H(2^n) to form a quadruple as a new block. However, the existing proof for the existence of H(2^n), which is the main context of Mills' paper [57], is based on the existence result of Steiner quadruple systems. The purpose of this chapter is to provide an alternative existence proof for Steiner quadruple systems via H-designs of type 2^n. By using the theory of candelabra systems and H-frames, we give a new existence proof for H-designs of type 2^n independent of the existence result of Steiner quadruple systems. As an application of this approach, several new infinite classes of nonuniform H-designs of types $2^n u^1$ with $u = 4, 6, 8$ are also constructed.

3.2 Recursive Constructions

In this section, we shall describe several recursive constructions for H-designs from candelabra systems and H-frames.

The following is a construction for 3-CSs which is a special case of the fundamental construction of Hartman [34].

Theorem 3.2.1. *Suppose that $(X, \mathcal{A})$ is an $S(t, K', v)$ and $\infty \in X$. Let $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ and $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. If there exists a $CS(3, K, t(k_1 - 1) + a)$ of type $(t^{k_1-1} : a)$ for each $k_1 \in K_1$ and a $GDD(3, K, tk_2)$ of type t^{k_2} for each $k_2 \in K_2$, then there exists a $CS(3, K, t(v - 1) + a)$ of type $(t^{v-1} : a)$.*

Now we give two tripling constructions and a doubling construction for H(2^n). The two tripling constructions are variations of those for SQS(v) proposed by Hartman in [31] and [32], which will play a similar role with that of the tripling constructions of Hartman [31, 32, 34] and Lenz [51] to deal with SQS(v).

First, recall that we have defined a B_g-pairing with components D, R_i, PR_i ($i \in \{0,1,2\}$) in Section 2.2, which is called a *simple pairing* in [31] when $g = 1$. In the sequel of this chapter, we denote a $B_1(n,s)$ by $P(n,s)$ as in [31].

Theorem 3.2.2. *For each pair of integers $n \geq 0$ and $s \geq 1$, there exists a simple pairing $P(n,2s)$ with the extra property that $\{0, 3n+s\} \subset D$ and $G_i = G(6n+2s, \{1,2,\ldots,3n+s\} \setminus L_i)$ has a one-factorization with $\{\{k, k+3n+s\} : 0 \leq k \leq 3n+s-1\}$ as one of the one-factors for each $i \in \{0,1,2\}$.*

Proof. For each pair of integers $n \geq 0$ and $s \geq 1$, a $P(n,2s)$ was constructed in [31, Theorem 3.3]. It is easy to check that $\{0, 3n+s\} \subset D$. The lengths L_i of all $P(n,2s)$s for each $i \in \{0,1,2\}$ are listed below:

Case (a) $s = 1$ and n even, or $s \geq 2$.

$$L_0 = \{2j : 0 < j \leq \lfloor n/2 \rfloor \text{ or } n < j \leq n + \lceil n/2 \rceil\},$$

$$L_1 = \{2j : \lfloor n/2 \rfloor < j \leq n + \lfloor n/2 \rfloor\},$$

$$L_2 = \{2j : 0 < j \leq n\}.$$

Case (b) $n = 2k+1$, $k \geq 0$ and $s = 1$.

$$L_0 = \{2j : 0 < j \leq k, 2k < j \leq 3k+1\},$$

$$L_1 = \{2j : k < j \leq 3k\} \cup \{1\},$$

$$L_2 = \{2j : 0 < j \leq 2k\} \cup \{1\}.$$

Let $G_i' = G(6n+2s, \{1,2,\ldots,3n+s\} \setminus (L_i \cup \{3n+s\}))$, $i \in \{0,1,2\}$. By Lemma 2.2.7, each of G_i' and G$(6n+2s, \{3n+s\})$ has a one-factorization. Hence, G_i has a one-factorization with $\{\{k, k+3n+s\} : 0 \leq k \leq 3n+s-1\}$ as one of the one-factors for each $i \in \{0,1,2\}$. $\square$

Example 3.2.3. [31] *Let $n = 1$ and $s = 1$. Construct a $P(1,2)$ on Z_8 as follows:*

$$D = \{0,4\}, PR_0 = \{\{3,5\}\}, PR_1 = \{\{1,2\}\}, PR_2 = \{\{6,7\}\}.$$

Note that each of the graphs $G_0 = G(8, \{1,3,4\})$, $G_1 = G(8, \{2,3,4\})$ and $G_2 = G(8, \{1,3,4\})$ has a one-factorization with $\{\{k, k+4\} : 0 \leq k \leq 3\}$ as one of the one-factors.

Theorem 3.2.4. *There exists an* $HF_2((3n+s)^3 : s)$ *with a subdesign* $H(2^4)$ *for each pair of integers* $n \geq 0$ *and* $s \geq 1$.

Proof. By Theorem 3.2.2, for each pair of integers $n \geq 0$ and $s \geq 1$, there is a simple pairing $P(n, 2s)$: D, R_i, PR_i, such that $\{0, 3n+s\} \subset D$ and G_i has a one-factorization $F_i^{(1)}|F_i^{(2)}|\ldots|F_i^{(4n+2s-1)}$ with $F_i^{(1)} = \{\{k, k+3n+s\} : 0 \leq k \leq 3n+s-1\}$ for each $i \in \{0,1,2\}$. Using this simple pairing, Hartman [31, Theorem 3.4] constructed a CQS$((6n+2s)^3 : 2s)$ on the point set $X = \{a_i : a \in Z_{6n+2s}, i \in \{0,1,2\}\} \cup \{\infty_1, \infty_2, \ldots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n+2s}\} : i \in \{0,1,2\}\}$ and a stem $\{\infty_1, \infty_2, \ldots, \infty_{2s}\}$, as well as the block set $\mathcal{B}$ consisting of the following three parts:

$$\begin{aligned}
\delta = \{&\{\infty_j, (a+d)_0, (b-d)_1, (c+d)_2\} :\ a+b+c \equiv 0 \pmod{6n+2s}, \\
&d \text{ is the } j\text{th member of } D, 1 \leq j \leq 2s\}, \\
\rho = \{&\{(a+q)_i, (a+t)_i, b_{i+1}, c_{i+2}\} : a+b+c \equiv 0 \pmod{6n+2s}, \\
&\{q,t\} \in PR_i, i \in \{0,1,2\}\}, \text{and} \\
\phi = \{&\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a,b\} \in F_i^{(k)}, \{c,d\} \in F_{i+1}^{(k)}, 1 \leq k \leq 4n+2s-1, \\
&i \in \{0,1,2\}\}.
\end{aligned}$$

Let

$$\phi_1 = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a,b\} \in F_i^{(1)}, \{c,d\} \in F_{i+1}^{(1)}, i \in \{0,1,2\}\}.$$

The desired HF$_2((3n+s)^3 : s)$ will be on X with the group set $\mathcal{G} = \{\{k_i, (k+3n+s)_i\} : 0 \leq k \leq 3n+s-1, i \in \{0,1,2\}\} \cup \{\{\infty_i, \infty_{i+s}\} : 1 \leq i \leq s\}$, three holes $\{\{k_i, (k+3n+s)_i\} : 0 \leq k \leq 3n+s-1\} \cup \mathcal{F}_0$, $i \in \{0,1,2\}$ and a common hole $\mathcal{F}_0 = \{\{\infty_i, \infty_{i+s}\} : 1 \leq i \leq s\}$, as well as the block set $\mathcal{B} \setminus \phi_1$.

Since $\{0, 3n+s\} \subset D$, without loss of generality we may assume 0, $3n+s$ are respectively the first and $(s+1)$th elements of D. Let

$$\begin{aligned}
\delta_0 = \{&\{\infty_j, (a+d)_0, (b-d)_1, (c+d)_2\} : a+b+c \equiv 0 \pmod{6n+2s}, \\
&a, b, c \in \{0, 3n+s\}, d \text{ is the } j\text{th member of } D \text{ and } j = 1 \text{ or } s+1\}.
\end{aligned}$$

Note that $\delta_0 \subset \delta$ and δ_0 forms the block set of an H(2^4) with the group set $\{\{0_i, (3n+s)_i\} : i \in \{0,1,2\}\} \cup \{\{\infty_1, \infty_{1+s}\}\}$. Hence, the above HF$_2((3n+s)^3 : s)$ contains a subdesign H(2^4). □

Example 3.2.3 (continued): Using the foregoing $P(1,2)$, we may construct a CQS$(8^3 : 2)$ on the point set $X = \{a_i : a \in Z_8, i \in \{0,1,2\}\} \cup \{\infty_1, \infty_2\}$ with three groups $\{\{a_i : a \in Z_8\} : i \in \{0,1,2\}\}$ and a stem $\{\infty_1, \infty_2\}$, as well as the block set $\mathcal{B}$ consisting of the following three sets:

$$\delta = \{\{\infty_1, a_0, b_1, c_2\}, \{\infty_2, (a+4)_0, (b-4)_1, (c+4)_2\} : \ a+b+c \equiv 0 \pmod 8\},$$

$$\rho = \{\{(a+3)_0, (a+5)_0, b_1, c_2\}, \{(a+1)_1, (a+2)_1, b_2, c_0\},$$
$$\{(a+6)_2, (a+7)_2, b_0, c_1\} : a+b+c \equiv 0 \pmod 8\}, \text{and}$$

$$\phi = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a,b\} \in F_i^{(k)}, \{c,d\} \in F_{i+1}^{(k)}, 1 \le k \le 5, i \in \{0,1,2\}\}.$$

Here, $F_i^{(1)}|F_i^{(2)}|\ldots|F_i^{(5)}$ is a one-factorization of G_i with $F_i^{(1)} = \{\{k, k+4\} : 0 \le k \le 3\}$ for each $i \in \{0,1,2\}$. Let $\phi_1 = \{\{k_i, (k+4)_i, k'_{i+1}, (k'+4)_{i+1}\} : 0 \le k, k' \le 3, i \in \{0,1,2\}\} \subset \phi$. The block set $(\delta \cup \rho \cup \phi) \setminus \phi_1$ forms an HF$_2(4^3 : 1)$ on X with the group set $\{\{k_i, (k+4)_i\} : 0 \le k \le 3, i \in \{0,1,2\}\} \cup \{\{\infty_1, \infty_2\}\}$, three holes $\{\{k_i, (k+4)_i\} : 0 \le k \le 3\} \cup \mathcal{F}_0$, $i \in \{0,1,2\}$ and a common hole $\mathcal{F}_0 = \{\{\infty_1, \infty_2\}\}$. Furthermore, as a subset of δ, $\delta_0 = \{\{\infty_1, a_0, b_1, c_2\}, \{\infty_2, (a+4)_0, (b-4)_1, (c+4)_2\} : \ a, b, c \in \{0,4\}, a+b+c \equiv 0 \pmod 8\}$ forms an H(2^4) with group set $\{\{0_i, 4_i\} : i \in \{0,1,2\}\} \cup \{\{\infty_1, \infty_2\}\}$. □

As a consequence of Theorem 3.2.4, we have our first tripling construction as follows.

Corollary 3.2.5. (Tripling Construction I) *Let $n \equiv 2s \pmod 3$ and $s \ge 1$. If there exists an IH$(2^n : 2^s)$, then there exist both an IH$(2^{3n-2s} : 2^n)$ and an IH$(2^{3n-2s} : 2^s)$.*

Theorem 3.2.6. *There exists an HF$_2((3n)^3 : s)$ for each pair of integers n, s such that $3n \ge s \ge 0$.*

Proof. For each pair of integers n, s such that $3n \ge s \ge 0$ and $(n,s) \ne (1,1)$, the proof is similar to that of Theorem 3.2.4. We may start from a particular CQS$((6n)^3 : 2s)$ and partition the points of each group into disjoint pairs. Then, we can remove the blocks formed by all these pairs from different groups. Such a CQS$((6n)^3 : 2s)$ was constructed by Hartman in [32, Section 4] on $X = \{a_i : a \in Z_{6n}, i \in \{0,1,2\}\} \cup \{\infty_1, \infty_2, \ldots, \infty_{2s}\}$ with three groups $\{\{a_i : a \in Z_{6n}\} :$

$i \in \{0,1,2\}\}$ and stem $\{\infty_1, \infty_2, \ldots, \infty_{2s}\}$, as well as the block set $\mathcal{B}$ containing the following blocks:

$$\phi = \{\{a_i, b_i, c_{i+1}, d_{i+1}\} : \{a,b\} \in F_i^{(k)}, \{c,d\} \in F_{i+1}^{(k)}, 1 \le k \le 6n-1-2r-2h, \\ i \in \{0,1,2\}\},$$

where $F_i^{(1)}, F_i^{(2)}, \ldots, F_i^{(6n-1-2r-2h)}$ are different disjoint partitions of pairs of Z_{6n} for each $i \in \{0,1,2\}$ and r, h are non-negative integers such that $6n = 2s+2h+6r$.

An $\mathrm{HF}_2(3^3 : 1)$ can be constructed by applying Theorem 2.2.2 with a $\mathrm{CQS}(3^3 : 1)$ in [23] and an $\mathrm{H}(2^4)$. □

As a consequence of Theorem 3.2.6, we have our second tripling construction as follows.

Corollary 3.2.7. (Tripling Construction II) *Let $n \equiv s \pmod 3$ and $s \ge 0$. If there exists an IH$(2^n : 2^s)$, then there exists an IH$(2^{3n-2s} : 2^n)$ and an IH$(2^{3n-2s} : 2^s)$.*

Theorem 3.2.8. (Doubling Construction) *If there exists an $H(2^n)$, then there exists an $H(2^{2n})$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given $\mathrm{H}(2^n)$. Let $\mathcal{F} = \{F_1, \ldots, F_{2(n-1)}\}$ be a one-factorization of the multi-partite complete graph on X with partite set $\mathcal{G}$. The desired $\mathrm{H}(2^{2n})$ is based on $X \times \{0,1\}$ with $2n$ groups $G \times \{i\}$, $G \in \mathcal{G}$ and $i \in \{0,1\}$. The block set is $\mathcal{A} = (\mathcal{B} \times \{0,1\}) \cup \mathcal{C}$, where $\mathcal{C} = \{\{(a,0),(b,0),(c,1),(d,1)\} : \{a,b\} \in F_i, \{c,d\} \in F_i, 1 \le i \le 2(n-1)\}$. □

The following two constructions are modifications of the filling holes construction.

Lemma 3.2.9. *Suppose that there exists an HF$_g(m^n : s)$,*

1. *if there exists an IH$(g^m : g^s)$, then there exists an IH$(g^{mn+s} : g^{m+s})$. Furthermore, if there is an H(g^{m+s}), then there is an H(g^{mn+s});*

2. *if there exists an H$(g^{m+\epsilon}(gs-g\epsilon)^1)$ with $\epsilon = 0$ or 1, then there exists an H$(g^{mn+\epsilon}(gs-g\epsilon)^1)$.*

The following recursive construction for nonuniform H-designs was first given in [50].

Lemma 3.2.10. [50] *Let mn be even. If there exists an $H((mn)^r(s+t)^1)$ and an $H(m^n s^1 t^1)$, then there exists an $H(m^{rn} s^1 t^1)$.*

3.3 Alternative Existence Proof for H(2^n)

In this section, we give an alternative existence proof for H(2^n) with $n \equiv 1, 2 \pmod 3$ and $n \neq 5$, which is mainly based on the recursive constructions listed in Section 3.2. The proof is independent of the existence result of Steiner quadruple systems. Hence, we also give a new proof for the existence of SQS(v) in the meantime. First, we need the following initial ingredient designs.

Lemma 3.3.1. [25, 55, 57] *There exists an $H(2^k)$ for each $k \in \{7, 11, 13\}$, an $H(6^k)$ for each $k \in \{4, 6\}$ and an $IH(2^{11} : 2^5)$.*

Proof. An H(2^7) can be found in [25]. An H(2^{11}), an H(2^{13}) and an IH$(2^{11} : 2^5)$ were constructed by Mills in [57]. An H(6^k) for each $k \in \{4, 6\}$ exists by [55, Lemma 7] . □

Lemma 3.3.2. *There exists an $H(2^{25})$.*

Proof. We construct an H(2^{25}) on $X = Z_{25} \times Z_2$ with the group set $\mathcal{G} = \{G_i = \{(i, 0), (i, 1)\} : i \in Z_{25}\}$. The block set consists of the following quadruples with $m \in Z_{25}$, $a \in Z_2$ and $b \in Z_2$.

(m, a)	$(m+5, b)$	$(m+7, b+1)$	$(m+12, a+b+1)$
(m, a)	$(m+2, b)$	$(m+3, b+1)$	$(m+5, a+b)$
(m, a)	$(m+3, b)$	$(m+15, a)$	$(m+18, b)$
(m, a)	$(m+8, b)$	$(m+15, a+b)$	$(m+23, a+1)$
(m, a)	$(m+7, b)$	$(m+20, a+b)$	$(m+22, b+1)$
(m, a)	$(m+10, a)$	$(m+16, b)$	$(m+23, a+b+1)$
(m, a)	$(m+2, a)$	$(m+10, b)$	$(m+20, a+b)$
(m, a)	$(m+6, b)$	$(m+16, b)$	$(m+18, a+b)$
(m, a)	$(m+2, b)$	$(m+3, b)$	$(m+9, a+b+1)$
(m, a)	$(m+10, a+1)$	$(m+13, b)$	$(m+19, a+b+1)$
(m, a)	$(m+2, b)$	$(m+15, a+1)$	$(m+22, a+b)$

(m, a)	$(m+10, a+1)$	$(m+16, b)$	$(m+22, a+b+1)$
(m, a)	$(m+8, b)$	$(m+10, b)$	$(m+15, a+b+1)$
(m, a)	$(m+1, a+1)$	$(m+7, b)$	$(m+19, a+b)$
(m, a)	$(m+16, b)$	$(m+22, a+b)$	$(m+23, a+b)$
(m, a)	$(m+14, a)$	$(m+23, b)$	$(m+24, a+b)$
(m, a)	$(m+3, b)$	$(m+11, b)$	$(m+22, a+b+1)$
(m, a)	$(m+8, a)$	$(m+11, b)$	$(m+14, a+b)$
(m, a)	$(m+17, b)$	$(m+19, a)$	$(m+23, b)$
(m, a)	$(m+6, b)$	$(m+14, b+1)$	$(m+17, a+1)$
(m, a)	$(m+1, a+1)$	$(m+9, b)$	$(m+18, a+b)$
(m, a)	$(m+9, b)$	$(m+23, b)$	$(m+24, a+b+1)$
(m, a)	$(m+7, b)$	$(m+8, b)$	$(m+24, a)$
(m, a)	$(m+14, a+1)$	$(m+15, b)$	$(m+24, a+b+1)$
(m, a)	$(m+7, b)$	$(m+16, a+b+1)$	$(m+24, a+1)$
(m, a)	$(m+2, b)$	$(m+4, a+b)$	$(m+21, b+1)$
(m, a)	$(m+1, b)$	$(m+12, b)$	$(m+14, a)$
(m, a)	$(m+2, b)$	$(m+11, a+1)$	$(m+13, b+1)$
(m, a)	$(m+1, b)$	$(m+12, b+1)$	$(m+13, a+b+1)$
(m, a)	$(m+4, b)$	$(m+21, a+b)$	$(m+23, b+1)$
(m, a)	$(m+3, a)$	$(m+4, b)$	$(m+7, b)$
(m, a)	$(m+3, a+1)$	$(m+4, b)$	$(m+7, b+1)$
(m, a)	$(m+5, a)$	$(m+11, b)$	$(m+16, b)$
(m, a)	$(m+5, a+1)$	$(m+11, b)$	$(m+16, b+1)$
(m, a)	$(m+13, a)$	$(m+17, b)$	$(m+22, a+b+1)$
(m, a)	$(m+4, b)$	$(m+12, a+1)$	$(m+17, b+1)$
(m, a)	$(m+6, b)$	$(m+10, b+1)$	$(m+21, a+1)$
(m, a)	$(m+13, a)$	$(m+16, b)$	$(m+21, a+b+1)$
(m, a)	$(m+5, b)$	$(m+20, a+b)$	$(m+24, a+b)$
(m, a)	$(m+4, b)$	$(m+13, a+1)$	$(m+16, b+1)$
(m, a)	$(m+4, a)$	$(m+14, b)$	$(m+18, b)$
(m, a)	$(m+4, b)$	$(m+9, a+b)$	$(m+13, a)$
(m, a)	$(m+1, b)$	$(m+5, b+1)$	$(m+6, a+b+1)$
(m, a)	$(m+7, b)$	$(m+14, a+b+1)$	$(m+21, a+1)$
(m, a)	$(m+15, b)$	$(m+21, a+b+1)$	$(m+21, a)$
(m, a)	$(m+5, b)$	$(m+17, b)$	$(m+22, a+b)$

□

The following lemma is useful for us to unify the proofs following-up, which also provides another proof for the existence of $S(3, \{4, 6\}, v)$ with some small initial ingredients.

Lemma 3.3.3. *For each integer $n \geq 3$, there exists a CS$(3, \{4, 6\}, 2n+2)$ of* *type* $(2^{n-2\epsilon}4^{\epsilon}:2)$ *with* $\epsilon \in \{0,1\}$.

Proof. For each integer $n \geq 3$, it is sufficient to prove that there exists an $\mathrm{S}(3,\{4,6\},2n+2)$ $(X,\mathcal{A})$ such that the design has two particular points $\{x,y\} \subset X$ with at most one block of size 6 containing both of them.

For $n = 3,4$, the conclusion is true since an SQS$(2n+2)$ exists. For $n = 5$, there exists an $\mathrm{S}(3,\{4,6\},12)$ with two disjoint blocks of size 6 partitioning the point set, which can be obtained from a GDD$(3,\{4,6\},12)$ of type 2^6 [25, Lemma 1].

For $n > 5$, assume that the conclusion is true for each i, $3 < i < n$. The proof proceeds by induction.

Firstly, suppose that there exists an $\mathrm{S}(3,\{4,6\},n+1)$ $(X,\mathcal{A})$ with two particular points $\{x,y\} \subset X$, such that there is at most one block of size 6 containing $\{x,y\}$. Let $\mathcal{F} = \{F_1,\dots,F_n\}$ be a one-factorization of the complete graph on X. Construct an $\mathrm{S}(3,\{4,6\},2n+2)$ on $X \times \{0,1\}$ with block set $\mathcal{B} = (\mathcal{A} \times \{0,1\}) \cup \mathcal{C}$, where $\mathcal{C} = \{\{(a,0),(b,0),(c,1),(d,1)\} : \{a,b\} \in F_i, \{c,d\} \in F_i, 1 \leq i \leq n\}$. It is not difficult to check that there is at most one block of size 6 in $\mathcal{B}$ containing $\{(x,0),(y,0)\}$.

Secondly, suppose that there exists an $\mathrm{S}(3,\{4,6\},n+2)$ $(X,\mathcal{A})$ with two particular points $\{x,y\} \subset X$, such that there is at most one block of size 6 containing $\{x,y\}$. Take a point $\infty \in X \setminus \{x,y\}$ and let $X' = (X \setminus \{\infty\}) \times \{0,1\}$. For each block $A \in \mathcal{A}$ containing ∞, construct a CS$(3,\{4,6\},2|A|-2)$ of type $(2^{|A|-1}:0)$ on $(A \setminus \{\infty\}) \times \{0,1\}$. For each block A not containing ∞, construct a GDD$(3,\{4,6\},2|A|)$ of type $2^{|A|}$ on $A \times \{0,1\}$. When $|A| = 6$, let $A \times \{0\}$ and $A \times \{1\}$ be the two special blocks of size 6 of the input GDD$(3,\{4,6\},12)$ of type 2^6. By Theorem 3.2.1, we get a CS$(3,\{4,6\},2n+2)$ of type $(2^{n+1}:0)$, which is actually an $\mathrm{S}(3,\{4,6\},2n+2)$ on X'. Here, the input CS$(3,\{4,6\},6)$ of type $(2^3:0)$ contains only one block of size 6. The input CS$(3,\{4,6\},10)$ of type $(2^5:0)$ is actually an SQS(10) which contains only blocks of size 4. Take the two points $\{(x,0),(y,1)\}$ into consideration. If $\{\infty,x,y\}$ determine a block of size 6 in $\mathcal{A}$, then there is no block of size 6 containing $\{(x,0),(y,1)\}$. If $\{\infty,x,y\}$ determine a block of size 4 in $\mathcal{A}$, then there is only one block of size 6 containing

$\{(x, 0), (y, 1)\}$. □

Lemma 3.3.4. *There exists an $H(2^n)$ for each $n \equiv 5 \pmod 6$, $n \geq 11$ and an $IH(2^n : 2^4)$ for each $n \equiv 5 \pmod 6$, $n \geq 17$.*

Proof. For $n = 11$, an $\mathrm{H}(2^{11})$ exists by Lemma 3.3.1. For $n = 17$, there exists an $\mathrm{HF}_2(5^3 : 2)$ with a subdesign $\mathrm{H}(2^4)$ by Theorem 3.2.4. Applying Lemma 3.2.9 with an $\mathrm{H}(2^7)$ from Lemma 3.3.1, we obtain an $\mathrm{H}(2^{17})$ and an $\mathrm{IH}(2^{17} : 2^4)$.

For each $n = 6m + 5$, $m \geq 3$, there exists a $\mathrm{CS}(3, \{4, 6\}, 2m + 2)$ of type $(2^{m-2\epsilon}4^{\epsilon} : 2)$ with $\epsilon \in \{0, 1\}$ by Lemma 3.3.3. Apply Theorem 2.2.2 with an $\mathrm{HF}_2(3^{k-1} : 2)$ and an $\mathrm{H}(6^k)$ for $k \in \{4, 6\}$ to obtain an $\mathrm{HF}_2(6^{m-2\epsilon}12^{\epsilon} : 5)$. Applying Lemma 3.2.9 with an $\mathrm{IH}(2^{11} : 2^5)$, an $\mathrm{H}(2^{11})$ or an $\mathrm{H}(2^{17})$, we get an $\mathrm{H}(2^{6m+5})$. Here, the input $\mathrm{HF}_2(3^{k-1} : 2)$ comes from Theorem 3.2.6 or Lemma 2.3.8, and the other ingredients are from Lemma 3.3.1. Since there exists an $\mathrm{H}(6^4)$ with a subdesign $\mathrm{H}(2^4)$, the resulting $\mathrm{H}(2^n)$ has a subdesign $\mathrm{H}(2^4)$. □

Lemma 3.3.5. *There exists an $H(2^n)$ for each $n \equiv 7, 13 \pmod{18}$ and $n \geq 7$.*

Proof. For each $n = 18k + 7$ and $k \geq 2$, we obtain an $\mathrm{IH}(2^n : 2^4)$ by applying Corollary 3.2.5 with an $\mathrm{IH}(2^{6k+5} : 2^4)$ from Lemma 3.3.4. Applying Lemma 3.2.9 with an $\mathrm{H}(2^4)$, we obtain an $\mathrm{H}(2^n)$. For $n = 7, 25$, the design exists by Lemmas 3.3.1 or 3.3.2.

For each $n = 18k + 13$ and $k \geq 1$, there is an $\mathrm{H}(2^n)$ by applying Corollary 3.2.5 with an $\mathrm{IH}(2^{6k+5} : 2^1)$ from Lemma 3.3.4. For $n = 13$, the design exists by Lemma 3.3.1. □

Lemma 3.3.6. *There exists an $H(2^n)$ for each $n \equiv 1 \pmod{18}$.*

Proof. For each $n = 18k + 1$ and $k \geq 1$, the proof proceeds by induction. For $k = 1$, an $\mathrm{H}(2^{19})$ exists by applying Corollary 3.2.7 with an $\mathrm{IH}(2^7 : 2^1)$. When $k > 1$, suppose that there exists an $\mathrm{H}(2^{18i+1})$ for each $i < k$. By Lemma 3.3.5, we have that an $\mathrm{H}(2^{6j+1})$ exists for all $j < 3k$. Applying Corollary 3.2.7 with an $\mathrm{IH}(2^{6k+1} : 2^1)$, we get an $\mathrm{H}(2^{18k+1})$. □

Theorem 3.3.7. *There exists an $H(2^n)$ for each $n \equiv 1, 2 \pmod 3$ and $n \neq 5$.*

Proof. Combining Lemmas 3.3.4–3.3.6, we obtain an $\mathrm{H}(2^n)$ for each $n \equiv 1, 5 \pmod 6$ and $n \neq 5$. By Theorem 3.2.8, we obtain an $\mathrm{H}(2^m)$ for each $m \equiv 2, 4 \pmod 6$ and $m \neq 10$. An $\mathrm{H}(2^{10})$ can be obtained by applying Corollary 3.2.7 with an $\mathrm{IH}(2^4 : 2^1)$. □

As a consequence of Theorem 3.3.7, we have the following corollary.

Corollary 3.3.8. *There exists an SQS(v) for each $v \equiv 2, 4 \pmod 6$.*

Proof. The existence of SQS(v) with small orders of $v = 4, 8, 10$ was mentioned in Section 3.1. Combining every two groups of an $\mathrm{H}(2^n)$ to form a quadruple as a new block, we get an SQS($2n$) for each $n \equiv 1, 2 \pmod 3$ and $n \geq 7$. □

3.4 Existence of $\mathbf{H}(2^n u^1)$ with $u = 4, 6, 8$

For the existence of nonuniform H-designs, Lauinger et al. [50] developed a computational method and several recursive constructions for constructing H-designs (called *transverse Steiner quadruple systems* as in [50]). They also provided an existence table for H-designs with the number of points $v \leq 24$, where the following small designs are needed.

Lemma 3.4.1. [50] *There exists an $H(2^n 4^1)$ for each $n \in \{4, 7, 10\}$ and an $H(2^n 8^1)$ for each $n \in \{6, 7\}$.*

Recently, Keranen et al. [47] settled the existence problem on H-designs of type g^4u^1 except when $g \equiv u \equiv 2 \pmod 4$ and all except 40 parameter situations when $g \equiv u + 2 \equiv 0 \pmod 4$. In this section, we will establish several new existence results for $\mathrm{H}(2^n u^1)$ with $u = 4, 6, 8$ by using the theory of candelabra systems and H-frames. By [47, Theorem 3.1], we have the following the necessary conditions.

Lemma 3.4.2. *If there exists an $H(2^n 4^1)$, then $n \equiv 1 \pmod 3$ and $n \geq 4$; if there exists an $H(2^n 6^1)$, then $n \equiv 1 \pmod 3$ and $n \geq 7$; if there exists an $H(2^n 8^1)$, then $n \equiv 0, 1 \pmod 3$ and $n \geq 6$.*

Lemma 3.4.3. [55] *There exists a $CQS(6^n : 0)$ for each $n \geq 0$.*

Combining the existence results of SQS(v) and CQS($6^n : 0$), we have a more strong result than that in Lemma 3.3.3.

Lemma 3.4.4. *For each integer $n \geq 3$, there exists a CS$(3, \{4, 6\}, 2n + 2)$ of type $(2^n : 2)$.*

Proof. For each $n = 3m + 0$, $3m + 1$, $m \geq 1$, there exists a CQS($2^n : 2$) obtained from an SQS($2n+2$). For each $n = 3m+2$, $m \geq 1$, there exists a CS$(3, \{4, 6\}, 2n+2)$ of type $(2^n : 2)$ obtained from a CQS($6^{(n+1)/3} : 0$) by taking two points from two distinct groups as stem points. □

First, we give a complete solution to the existence of H(2^n4^1) as follows.

Theorem 3.4.5. *There exists an H(2^n4^1) if and only if $n \equiv 1 \pmod 3$ and $n \geq 4$.*

Proof. For each $n = 3k + 1$ and $n \geq 10$, there exists an H(6^{k+1}) by Theorem 1.1.1. Applying Lemma 3.2.10 with $m = 2$, $n = 3$, $s = 2$, $t = 4$ and an H(2^44^1), we get an H(2^n4^1). For $n = 4, 7$, the desired designs exist by Lemma 3.4.1. □

Lemma 3.4.6. *There exists an HF$_2(3^5 : 1)$.*

Proof. The desired design is obtained by applying Theorem 2.2.2 with a CQS($3^5 : 1$) [3] and an H(2^4). □

Lemma 3.4.7. *There exists an H(2^76^1).*

Proof. We construct an H(2^76^1) on Z_{20} with group set $\mathcal{G} = \{\{i, i+7\} : 0 \leq i \leq 6\} \cup \{\{14, 15, 16, 17, 18, 19\}\}$. We list below the base blocks, which are developed under the automorphism group $G = \langle(0\ 1\ 2)(3\ 4\ 5)(6)(7\ 8\ 9)(10\ 11\ 12)(13)(14)(15)(16)(17)(18)(19)\rangle$.

$\{0,4,13,18\}$	$\{1,2,7,15\}$	$\{2,3,4,15\}$	$\{3,9,13,18\}$	$\{1,9,13,17\}$	$\{5,6,11,14\}$
$\{0,3,8,18\}$	$\{4,7,10,15\}$	$\{1,4,12,14\}$	$\{0,1,4,17\}$	$\{0,2,5,14\}$	$\{0,6,9,15\}$
$\{3,4,7,17\}$	$\{3,11,12,18\}$	$\{1,7,9,18\}$	$\{2,10,12,14\}$	$\{0,4,6,19\}$	$\{7,10,13,17\}$
$\{0,5,6,18\}$	$\{0,3,5,17\}$	$\{1,4,5,7\}$	$\{4,5,9,17\}$	$\{4,6,7,18\}$	$\{8,9,12,16\}$
$\{2,8,12,15\}$	$\{1,4,10,13\}$	$\{7,11,13,16\}$	$\{4,6,8,12\}$	$\{0,9,12,18\}$	$\{0,1,12,19\}$
$\{1,10,12,15\}$	$\{3,7,8,14\}$	$\{1,2,11,19\}$	$\{3,5,13,16\}$	$\{0,1,5,11\}$	$\{0,1,13,16\}$

$\{0,3,6,14\}$	$\{8,10,11,14\}$	$\{2,4,7,14\}$	$\{7,8,12,18\}$	$\{3,7,12,19\}$	$\{3,7,9,15\}$
$\{3,8,12,17\}$	$\{0,6,8,11\}$	$\{2,8,10,18\}$	$\{1,9,10,11\}$	$\{5,7,8,10\}$	$\{0,4,8,14\}$
$\{1,11,13,15\}$	$\{5,6,7,15\}$	$\{3,5,11,15\}$	$\{5,7,13,14\}$	$\{1,3,11,17\}$	$\{6,7,8,17\}$
$\{1,12,13,18\}$	$\{6,7,12,14\}$	$\{1,3,7,13\}$	$\{2,6,11,15\}$	$\{3,11,13,19\}$	$\{7,12,13,15\}$
$\{7,10,11,18\}$	$\{0,6,10,17\}$	$\{6,10,11,19\}$	$\{1,5,10,16\}$	$\{0,1,2,6\}$	$\{3,4,5,6\}$
$\{7,8,9,13\}$	$\{10,11,12,13\}$				

□

Lemma 3.4.8. *There exists an H$(2^{13}6^1)$.*

Proof. We construct an H$(2^{13}6^1)$ on $Z_{26} \cup \{\infty_0, \dots, \infty_5\}$ with group set $\mathcal{G} = \{\{i, i+13\} : 0 \le i \le 12\} \cup \{\{\infty_0, \dots, \infty_5\}\}$. We list below the base blocks, which are developed under the cyclic group Z_{26}:

$\{0,15,19,\infty_0\}$	$\{0,8,20,\infty_0\}$	$\{0,9,25,\infty_0\}$	$\{0,3,24,\infty_0\}$
$\{0,6,11,\infty_1\}$	$\{0,10,24,\infty_1\}$	$\{0,17,18,\infty_1\}$	$\{0,4,23,\infty_1\}$
$\{0,12,15,\infty_2\}$	$\{0,1,18,\infty_2\}$	$\{0,6,22,\infty_2\}$	$\{0,19,24,\infty_2\}$
$\{0,20,25,\infty_3\}$	$\{0,4,14,\infty_3\}$	$\{0,2,9,\infty_3\}$	$\{0,15,23,\infty_3\}$
$\{0,20,22,\infty_4\}$	$\{0,9,12,\infty_4\}$	$\{0,1,8,\infty_4\}$	$\{0,5,16,\infty_4\}$
$\{0,1,15,\infty_5\}$	$\{0,17,21,\infty_5\}$	$\{0,18,20,\infty_5\}$	$\{0,3,19,\infty_5\}$
$\{0,10,14,18\}$	$\{0,5,9,10\}$	$\{0,2,7,14\}$	$\{0,2,16,19\}$
$\{0,19,23,25\}$	$\{0,22,23,24\}$	$\{0,3,9,21\}$	$\{0,6,15,20\}$
$\{0,11,22,25\}$	$\{0,9,11,19\}$	$\{0,4,19,20\}$	$\{0,3,5,23\}$
$\{0,4,18,25\}$	$\{0,8,17,23\}$	$\{0,8,10,19\}$	$\{0,1,6,16\}$

□

Theorem 3.4.9. *There exists an H(2^n6^1) for each $n \equiv 1 \pmod 6$ and $n \ge 7$.*

Proof. For $n = 7, 13$, the desired designs are constructed in Lemmas 3.4.7 and 3.4.8, respectively. For each $n = 6m+1$ with $m \ge 3$, there exists a CS$(3, \{4,6\}, (n+5)/3)$ of type $(2^{(n-1)/6} : 2)$. Apply Theorem 2.2.2 with an HF$_2(3^{k-1} : 1)$ and an H(6^k) for $k \in \{4,6\}$ to obtain an HF$_2(6^{(n-1)/6} : 4)$. Applying Lemma 3.2.9 with an H(2^76^1), we get an H(2^n6^1). Here, all the small ingredient designs are from Theorem 3.2.6, Lemmas 3.3.1 and 3.4.6. □

Lemma 3.4.10. *There exists an H$(2^{13}8^1)$.*

Proof. We construct the desired H$(2^{13}8^1)$ on Z_{34} with group set $\mathcal{G} = \{\{i, i+13\} : 0 \le i \le 12\} \cup \{\{26,27,28,29,30,31,32,33\}\}$. We list below the base blocks, which are developed under the following automorphism group:

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25)$$
$$(26\ 27)(28\ 29)(30\ 31)(32\ 33)\rangle$$

$\{1,4,7,26\}$	$\{2,4,20,27\}$	$\{8,11,16,23\}$	$\{0,6,11,21\}$	$\{9,16,24,31\}$	$\{1,3,21,32\}$
$\{0,15,24,31\}$	$\{10,11,21,27\}$	$\{15,18,25,31\}$	$\{9,21,23,24\}$	$\{6,9,11,33\}$	$\{11,18,20,26\}$
$\{0,14,18,30\}$	$\{6,14,17,32\}$	$\{7,9,11,18\}$	$\{0,7,11,26\}$	$\{6,9,17,23\}$	$\{3,6,22,24\}$
$\{8,20,24,32\}$	$\{9,10,12,30\}$	$\{10,21,25,29\}$	$\{6,16,20,23\}$	$\{9,10,24,33\}$	$\{5,7,13,29\}$
$\{2,9,10,14\}$	$\{13,14,20,32\}$	$\{0,12,17,27\}$	$\{7,10,11,28\}$	$\{3,8,20,31\}$	$\{10,13,19,28\}$
$\{15,16,21,31\}$	$\{14,19,23,32\}$	$\{3,7,8,27\}$	$\{13,19,23,30\}$	$\{3,12,13,21\}$	$\{0,8,20,26\}$
$\{0,10,19,33\}$	$\{16,17,20,24\}$	$\{0,1,20,25\}$	$\{1,2,19,28\}$	$\{2,7,12,28\}$	$\{1,2,11,17\}$
$\{1,13,20,29\}$	$\{0,14,24,29\}$	$\{0,2,6,12\}$	$\{0,17,21,23\}$		

□

Lemma 3.4.11. *There exists an $H(2^n 8^1)$ for each $n \equiv 0, 1 \pmod 6$, $n \geq 6$ and $n \neq 12$.*

Proof. For $n = 6, 7, 13$, the desired designs come from Lemmas 3.4.1 and 3.4.10, respectively. For each $n = 6m + s$, $s \in \{0, 1\}$, $m \geq 3$, there exists a CS$(3, \{4, 6\}, (n-s+6)/3)$ of type $(2^{(n-s)/6} : 2)$. Apply Theorem 2.2.2 with an HF$_2(3^{k-1} : s+1)$ and an H(6^k) for $k \in \{4, 6\}$ to obtain an HF$_2(6^{(n-s)/6} : s + 4)$. Applying Lemma 3.2.9 with an H$(2^{6+s}8^1)$, we get an H$(2^n 8^1)$. Here, the input designs are from Theorem 3.2.6, Lemmas 3.3.1, 2.3.8, 3.4.1 and 3.4.6. □

Lemma 3.4.12. *There exists an $H(2^n 8^1)$ for each $n \equiv 3, 16 \pmod{18}$, $n \geq 16$ and $n \neq 34$.*

Proof. For each $n = 18k + 3$ with $k \geq 1$, there is an HF$_2((3(2k-1)+4)^3 : 4)$ by Theorem 3.2.4. By applying Lemma 3.2.9 with an H$(2^{6k+1}8^1)$ from Lemma 3.4.11, we get an H$(2^n 8^1)$.

For each $n = 18k + 16$ with $k \geq 0$ and $k \neq 1$, there is an HF$_2((6k+5)^3 : 5)$ by Theorem 3.2.4. Applying Lemma 3.2.9 with an H$(2^{6k+6}8^1)$ from Lemma 3.4.11, we get an H$(2^n 8^1)$. □

Combining Lemmas 3.4.11 and 3.4.12, we obtain

Theorem 3.4.13. *There exists an $H(2^n 8^1)$ for each $n \equiv 0, 1, 3, 6, 7, 12, 13, 16 \pmod{18}$, $n \geq 6$ except possibly for $n = 12, 34$.*

Chapter 4

Block Sequences of Steiner Quadruple Systems with Error Correcting Consecutive Unions

Motivated by applications in combinatorial group testing for consecutive positives, we investigate a block sequence of a maximum packing MP(t, k, v) which contains the blocks exactly once such that the collection of all blocks together with all unions of two consecutive blocks of this sequence forms an error correcting code with minimum distance d. Such a sequence is usually called a block sequence with consecutive unions having minimum distance d, and denoted by BSCU$(t, k, v|d)$. In this chapter, we show that the necessary conditions for the existence of BSCU$(3, 4, v|4)$s of Steiner quadruple systems, namely, $v \equiv 2, 4 \pmod 6$ and $v \geq 4$, are also sufficient except $v = 8, 10$.

4.1 Introduction

Let V be a finite set of v element and let $\mathcal{X}$ be a collection of k-subsets of V with $|\mathcal{X}| = m$. Let $S = [x_0, x_1, \ldots, x_{m-1}]$ be a sequence of the elements in $\mathcal{X}$. The indices of the elements x_i of S are considered modulo m. Define $y_i = x_i \cup x_{i+1}$ for each i, $0 \leq i \leq m-1$. The sequence S is called a *cyclic sequence of $\mathcal{X}$ with consecutive unions having minimum distance d*, denoted as CSCU$(k, v|d)$, if $C = \{x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}\}$ has minimum distance d. Note that the distance between any two sets x and y is defined as $d(x, y) = |(x \cup y) \setminus (x \cap y)|$. Furthermore, a CSCU$(k, v|d)$ is said to be *maximal* if the number of elements in $\mathcal{X}$ is maximum for given k, v and d, denoted as MCSCU$(k, v|d)$.

The concept of an MCSCU is motivated by the applications in *combinatorial group testing for consecutive positives.* Group testing was proposed by Dorfman

[16] in 1940s to do large scale blood testing economically, and new applications of group testing have been found recently in the fields such as *DNA library screening*, being error-prone, in which it is desired to determine the set of all clones containing a specific sequence of nucleotides in an economical and correct way. A clone is *positive* if it contains the specific sequence, and *negative* otherwise. One chooses arbitrarily a subset of clones called a *group* or *pool*, and test all clones in the pool in one stroke by some chemical analysis. The pool is *positive* when it contains at least one positive clone, and *negative* otherwise. Colbourn [13] developed some strategy for group testing when the clones are *linearly ordered* and the positive clones form a *consecutive subset* of the set of all clones, the typical example being the problem of locating a *sequence-tagged site* (or STS) among ordered clones. Jimbo and his collaborators [60, 59, 61, 62] improved Colbourn's strategy by considering the error detecting and correcting capability of group testing which is essential in view of applications such as DNA library screening. Especially, Momihara and Jimbo [60, 59] suggested using MCSCUs of a combinatorial structure called *t-packings* to correct *false negative* or *false positive* clones in the pool outcomes. For more details of such applications we refer to [13, 17, 60, 59, 61, 62, 64] and references there in.

A *t-packing* of order v, block size k, briefly $\mathrm{P}(t, k, v)$, is an ordered pair $(V, \mathcal{B})$, where V is a finite set of v elements called *points*, and $\mathcal{B}$ is a set of k-subsets of V called *blocks*, such that each t-tuple of distinct points of V is contained in at most one block of $\mathcal{B}$. In particular, a $\mathrm{P}(t, k, v)$ is said to be *maximal*, denoted $\mathrm{MP}(t, k, v)$, if the number of blocks is maximum for given t, k and v. For $t = 3$ and $k = 4$, an $\mathrm{MP}(t, k, v)$ is denoted by $\mathrm{MPQS}(v)$ which has been described in Section 2.4.

It is known (see [60]) that a $\mathrm{CSCU}(k, v|d)$ of $\mathcal{B}$ is maximal if $\mathcal{B}$ is the block set of an $\mathrm{MP}(\lfloor k - d/2 \rfloor + 1, k, v)$. A $\mathrm{CSCU}(k, v|d)$ of $\mathcal{B}$ which is the block set of an $\mathrm{MP}(t, k, v)$ is also called a *block sequence of $\mathcal{B}$ with consecutive unions having minimum distance d*, briefly $\mathrm{BSCU}(t, k, v|d)$.

In the case of $d = 2$, Müller and Jimbo [62] showed that there exists a $\mathrm{BSCU}(k, k, v|2)$ for every $v \geq v_k$ for the following pairs of parameters k and

v_k: $(k, v_k) = (2,6),(3,8),(4,11),(5,12),(6,17)$ and $(7,19)$, without introducing the notion of block sequences of t-packings. In the case of $d = 3$, Momihara and Jimbo [60] showed the existence of a BSCU$(2,3,v|3)$ for every $v \geq 10$. For the case of $d = 4$, it is clear that a BSCU$(3,4,v|4)$ forms an MCSCU$(4,v|4)$. Momihara and Jimbo [59] recently showed the existence of a BSCU$(3,4,v|4)$ for forty-seven small values $v \leq 500$ using the following two constructions.

Theorem 4.1.1. ([59]) Let v be an integer satisfying $v \equiv 2,4 \pmod 6$ and $v \geq 14$.

(1) If there exists a BSCU$(3,4,v|4)$, then there exists a BSCU$(3,4,2v|4)$ which contains a sub-BSCU$(3,4,v|4)$.

(2) If there exists a BSCU$(3,4,v|4)$, then there exists a BSCU$(3,4,3v-2|4)$ which contains a sub-BSCU$(3,4,v|4)$.

It is not difficult to see ([59]) that if there exists a BSCU$(3,4,v|4)$, then every two consecutive blocks must be disjoint. Furthermore, there does not exist a BSCU$(3,4,v|4)$ for $v \leq 11$ except for $v = 4$, in which there is only one block. We call such a BSCU$(3,4,4|4)$ *trivial*.

In this chapter, we write BSCU$(3,4,v|4)$ of the block sets of Steiner quadruple systems as BSCU(v) for brevity. The necessary conditions for the existence of a BSCU(v) are $v \equiv 2,4 \pmod 6$ and $v \geq 4$. In the following sections, we will prove that the above necessary conditions are also sufficient except $v = 8,10$. Our main tools are the recursive constructions used in the 3-design theory (see [34, 37, 38] for the detailed information).

4.2 Recursive Constructions

A *holey quadruple system* of order v with a hole of order s, denoted by HSQS$(v : s)$, is a triple $(X, S, \mathcal{A})$ where X is a set of v elements (called *points*), S is an s-subset of X, and $\mathcal{A}$ is a collection of 4-subsets (called *blocks*) of X such that every 3-subset T of X with $T \not\subseteq S$ is contained in a unique block of $\mathcal{A}$ and no 3-subset of S is contained in any block of $\mathcal{A}$.

Let $(X, S, \mathcal{G}, \mathcal{A})$ be a CS$(3, K, v)$ of type $(g_1^{n_1} g_2^{n_2} \dots g_r^{n_r} : s)$ with $S = \{\infty_1, \infty_2, \dots, \infty_s\}$, where $s \geq 1$. For $1 \leq i \leq s$, let $\mathcal{B}_i = \{A \setminus \{\infty_i\} \mid \infty_i \in A \in \mathcal{A}\}$ and $\mathcal{T} = \{A \in \mathcal{A} \mid A \cap S = \emptyset\}$. Then the $(s+3)$-tuple $(X \setminus S, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2, \dots, \mathcal{B}_s, \mathcal{T})$ is called an *s-fan design*. If block sizes of $\mathcal{B}_i$, $1 \leq i \leq s$, and $\mathcal{T}$ are from K_i and $K_{\mathcal{T}}$, respectively, then the s-fan design is denoted by s-FG$(3, (K_1, \dots, K_s, K_{\mathcal{T}}), \sum_{i=1}^{r} n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \dots g_r^{n_r}$.

A CSCU$(4, v|4)$ of $\mathcal{B}$ which is the block set of an H$(g_1^{n_1} g_2^{n_2} \dots g_r^{n_r})$ will be denoted by CSCU-GDD$(g_1^{n_1} g_2^{n_2} \dots g_r^{n_r})$ in this chapter. Similarly, we can define CSCU-HSQS, CSCU-CQS, etc.

Now we apply the fundamental constructions in the 3-design theory, where "filling in holes" and "weighting method" are always useful (see [34]). First, we may think of one CSCU (the master design) as a cycle which can be cut off at any place. Next, we view the sequence of the other cut-off CSCU (the sub-design) as a segment, and insert it into some cut place of the master design to form a bigger cycle. Then we calculate the number of the places in the master design where the obtained bigger cycle is also a CSCU. If this number is positive, then the construction succeeds. We explain it in detail as follows.

For any k-subset sequence $S = [x_0, x_1, \dots, x_{m-1}]$ with length m, define

$$\sigma^j(S) = [x_j, x_{j+1}, \dots, x_{m-1}, x_0, \dots, x_{j-1}],$$

$$\overline{S} = \{x_0 \cup x_1, x_1 \cup x_2, \dots, x_{m-2} \cup x_{m-1}\},$$

and

$$\widehat{S} = \{x_0 \cup x_1, x_1 \cup x_2, \dots, x_{m-2} \cup x_{m-1}, x_{m-1} \cup x_0\}.$$

Let U, V be two finite sets with $|U| = u$ and $|V| = v$, where U is not necessarily disjoint with V. Let $S = [b_0, b_1, \dots, b_{p-1}]$ be a CSCU$(4, u|4)$ of $\mathcal{B}$ which is a collection of 4-subsets of U with $p = |\mathcal{B}|$, and $T = [a_0, a_1, \dots, a_{q-1}]$ be a CSCU$(4, v|4)$ of $\mathcal{A}$ which is a collection of 4-subsets of V with $q = |\mathcal{A}|$. It is clear that $|b \cap b'| \leq 2$ and $|a \cap a'| \leq 2$ for any distinct $b, b' \in \mathcal{B}$ and $a, a' \in \mathcal{A}$. We may assume that for any $b \in \mathcal{B}$, we always have $|b \cap V| \leq 2$. Then for any $a \in \mathcal{A}$ and $b \in \mathcal{B}$, we always have $|b \cap a| \leq 2$. We view S as a cycle, cut T between a_0 and a_{q-1} and keep the order fixed. We insert $T = [a_0, a_1, \dots, a_{q-1}]$

into S between b_{i-1} and b_i for some i, $0 \leq i \leq p-1$, and denote the bigger cycle $[a_0, a_1, \ldots, a_{q-1}, b_i, b_{i+1}, \ldots, b_{i-1}]$ by $S_i = [T, \sigma^i(S)]$. Let $M = \{i \mid S_i$ is a CSCU$(4, w|4)$ of $\mathcal{B} \cup \mathcal{A}$, $0 \leq i \leq p-1\}$, where $w = |U \cup V|$, and let $|M| = m$. If $m > 0$, then we obtain a bigger CSCU$(4, w|4)$ from the two small CSCUs. Next, we estimate the value of m.

Let $C = \mathcal{A} \cup \mathcal{B} \cup \overline{T} \cup \overline{\sigma^i(S)} \cup D$, where $D = \{a_0 \cup b_{i-1}, a_{q-1} \cup b_i\}$. We check the distance between any two elements of C. First, we consider the case that $a_0 \cap b_{i-1} = \emptyset$ and $a_{q-1} \cap b_i = \emptyset$. In this case, we have the following conclusions:

Since T is a CSCU of $\mathcal{A}$, we have

Case (1): $d(a, a') \geq 4$ for any $a, a' \in \mathcal{A}$;

Case (2): $d(c, c') \geq 4$ for any $c, c' \in \overline{T}$;

Case (3): $d(a, c) \geq 4$ for any $a \in \mathcal{A}$ and $c \in \overline{T}$.

Since S is a CSCU of $\mathcal{B}$, we have

Case (4): $d(b, b') \geq 4$ for any $b, b' \in \mathcal{B}$;

Case (5): $d(c, c') \geq 4$ for any $c, c' \in \overline{\sigma^i(S)}$;

Case (6): $d(b, c) \geq 4$ for any $b \in \mathcal{B}$ and $c \in \overline{\sigma^i(S)}$.

Since $|a_0 \cap b_i| \leq 2$, $|a_{q-1} \cap b_{i-1}| \leq 2$ and $b_{i-1} \cap b_i = \emptyset$, we know that $d(b_{i-1}, b_i) = 8$, $d(a_0 \cup b_{i-1}, b_i) \geq 6$, and also

Case (7): $d(a_0 \cup b_{i-1}, a_{q-1} \cup b_i) \geq 4$.

Since $|a \cap b| \leq 2$, we have

Case (8): $d(a, b) \geq 4$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$.

Since $|a| = 4$, $|b| = 4$ and $|c| = 8$ for any $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \overline{T} \cup \overline{\sigma^i(S)} \cup D$, we have

Case (9): $d(a, c) \geq 4$ for any $a \in \mathcal{A}$ and $c \in \overline{\sigma^i(S)} \cup D$;

Case (10): $d(b, c) \geq 4$ for any $b \in \mathcal{B}$ and $c \in \overline{T} \cup D$.

Since $|b \cap V| \leq 2$ for any $b \in \mathcal{B}$, we have

Case (11): $d(c, c') \geq 4$ for $c \in \overline{T}$ and $c' \in \overline{\sigma^i(S)} \cup D$.

Under the assumption that $a_0 \cap b_{i-1} = \emptyset$ and $a_{q-1} \cap b_i = \emptyset$, we still need to consider the values of $d(c, c')$ for any $c \in \overline{\sigma^i(S)}$ and $c' \in D$. Note that we should also check the distance between any two elements of C in the case that $a_0 \cap b_{i-1} \neq \emptyset$ or $a_{q-1} \cap b_i \neq \emptyset$.

Let $N(a_{q-1}) = \{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k \neq \emptyset$ or $d(a_{q-1} \cup b_k, c) < 4$ for some $c \in \widehat{S}\}$ and $n(a_{q-1}) = |N(a_{q-1})|$. Also let $\alpha(a_{q-1}) = |\{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k \neq \emptyset\}|$. Then $n(a_{q-1}) = \alpha(a_{q-1}) + |\{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k = \emptyset$ and $d(a_{q-1} \cup b_k, c) < 4$ for some $c \in \widehat{S}\}|$. In order to estimate $n(a_{q-1})$, we consider the case that $a_{q-1} \cap b_k = \emptyset$. It is clear that for any index $0 \leq l \leq p-1$, $|(b_l \cup b_{l+1}) \cap a_{q-1}| \leq 4$.

If there exists an index l such that $|(b_l \cup b_{l+1}) \cap a_{q-1}| = 4$, i.e., $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 2$, then if $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$, we should have $|(b_l \cup b_{l+1}) \cap b_k| \geq 3$. In the case that $\mathcal{B}$ is the block set of some 3-packing of order u, there is at most one such k that $|(b_l \cup b_{l+1}) \cap b_k| = 4$, that is, $b_k = (b_l \cup b_{l+1}) \setminus a_{q-1}$, or there are at most 4 such k that $|(b_l \cup b_{l+1}) \cap b_k| = 3$, that is, b_k are obtained by choosing any three points from the four points in $(b_l \cup b_{l+1}) \setminus a_{q-1}$ and the other one from the other points of U, which implies that $|\{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k = \emptyset$ and $d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| \leq 4$.

If there exists an index l such that $|(b_l \cup b_{l+1}) \cap a_{q-1}| = 3$, i.e., $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 1$, or $|b_l \cap a_{q-1}| = 1$ and $|b_{l+1} \cap a_{q-1}| = 2$, then we should have $|(b_l \cup b_{l+1}) \cap b_k| = 4$ if $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$. In the case that $\mathcal{B}$ is the block set of some 3-packing of order u, there is at most one such k that $|(b_l \cup b_{l+1}) \cap b_k| = 4$, that is, b_k is obtained by choosing four points from $(b_l \cup b_{l+1}) \setminus a_{q-1}$. If there is another k' such that $|(b_l \cup b_{l+1}) \cap b_{k'}| = 4$, then $|b_k \cap b_{k'}| \geq 3$ because $|(b_l \cup b_{l+1}) \setminus a_{q-1}| = 5$, which leads to a contradiction. In this case, we have $|\{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k = \emptyset$ and $d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| \leq 1$.

If $|(b_l \cup b_{l+1}) \cap a_{q-1}| \leq 2$, then we can easily check that there is no such k that $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$, that is, $|\{k \mid 0 \leq k \leq p-1,\ a_{q-1} \cap b_k = \emptyset$ and $d(a_{q-1} \cup b_k, b_l \cup b_{l+1}) < 4\}| = 0$.

Therefore, if we define $\gamma(a_{q-1}) = |\{l \mid 0 \leq l \leq p-1,\ |(b_l \cup b_{l+1}) \cap a_{q-1}| = 4\}|$ and $\delta(a_{q-1}) = |\{l \mid 0 \leq l \leq p-1,\ |(b_l \cup b_{l+1}) \cap a_{q-1}| = 3\}|$, then under the condition that $a_{q-1} \cap b_k = \emptyset$, there are at most $4\gamma(a_{q-1}) + \delta(a_{q-1})$ such k that $d(b_l \cup b_{l+1}, a_{q-1} \cup b_k) < 4$. So we have $n(a_{q-1}) \leq \alpha(a_{q-1}) + 4\gamma(a_{q-1}) + \delta(a_{q-1})$.

From the definition of $\gamma(a_{q-1})$, we know that the existence of one such index l in $\gamma(a_{q-1})$ would imply both $|b_l \cap a_{q-1}| = 2$ and $|b_{l+1} \cap a_{q-1}| = 2$. Also, from the definition of $\delta(a_{q-1})$, the existence of one such index l in $\delta(a_{q-1})$ would imply $|b_l \cap a_{q-1}| = 2$ or $|b_{l+1} \cap a_{q-1}| = 2$, but not both. Keeping in mind the possibilities of occurrences of consecutive blocks in $\Xi = \{k \mid 0 \leq k \leq p-1,\ |b_k \cap a_{q-1}| = 2\}$ and one block in Ξ followed by one block in $\{k \mid 0 \leq k \leq p-1,\ |b_k \cap a_{q-1}| = 1\}$, we can know that these would imply $2\gamma(a_{q-1}) + \delta(a_{q-1}) < 2\beta(a_{q-1})$, since γ and δ are mutually exclusive, where $\beta(a_{q-1}) = |\Xi|$.

Similarly, we can analyze the set $N(a_0) = \{k \mid 0 \leq k \leq p-1,\ a_0 \cap b_{k-1} \neq \emptyset$ or $d(a_0 \cup b_{k-1}, c) < 4$ for some $c \in \widehat{S}\}$, where $n(a_0) = |N(a_0)|$.

Then from the definitions of M, $N(a_0)$ and $N(a_{q-1})$, we immediately have that $M \supseteq Z_p \backslash (N(a_0) \cup N(a_{q-1}))$ and $m \geq p - n(a_0) - n(a_{q-1}) + |N(a_0) \cap N(a_{q-1})| \geq p - n(a_0) - n(a_{q-1}) + |E|$, where $E \subseteq N(a_0) \cap N(a_{q-1})$.

Theorem 4.2.1. *Suppose that there are both a CSCU-HSQS($u : v$) and a BSCU(v). Then there is a BSCU(u) when $u \geq 44$ and $u > v$.*

Proof. Let $S = [b_0, b_1, \ldots, b_{p-1}]$ be a CSCU-HSQS($u : v$) on U and $T = [a_0, a_1, \ldots, a_{q-1}]$ be a BSCU(v) on V with $V \subset U$. By the definition of an HSQS($u : v$), we know that for any of its blocks, say b, we always have $|b \cap V| \leq 2$. We view S as a cycle, cut T between a_0 and a_{q-1} and keep the order fixed. Next, insert $T = [a_0, a_1, \ldots, a_{q-1}]$ into S between b_{i-1} and b_i for some i, $0 \leq i \leq p-1$, and denote the resultant cycle $[a_0, a_1, \ldots, a_{q-1}, b_i, b_{i+1}, \ldots, b_{i-1}]$ by $S_i = [T, \sigma^i(S)]$. Using the same notation as above, we prove the theorem as follows.

Since T is a BSCU(v), we have $a_{q-1} \cap a_0 = \emptyset$ since they are consecutive. From the balanced property of t-designs, we also have $n(a_0) = n(a_{q-1})$. Then $m \geq p - n(a_0) - n(a_{q-1}) \geq p - 2(\alpha(a_{q-1}) + 4\gamma(a_{q-1}) + \delta(a_{q-1})) > p - 2\alpha(a_{q-1}) - 8\beta(a_{q-1})$. Here $p = u(u-1)(u-2)/24 - v(v-1)(v-2)/24$, $\alpha(a_{q-1}) = 2(u-1)(u-2)/3 - 3(u-2) - 2(v-1)(v-2)/3 + 3(v-2)$ and $\beta(a_{q-1}) = 3(u-v)$. Then we have $m > p - 2\alpha(a_{q-1}) - 8\beta(a_{q-1}) > 0$ when $u \geq 44$ and $u > v$. This means that there is a BSCU(u) when $u \geq 44$ and $u > v$. $\square$

Theorem 4.2.2. *Suppose that there are both a CSCU-CQS($m^n : s$) and a CSCU-HSQS($m+s : s$). Then there are both a CSCU-HSQS($mn+s : m+s$) and a CSCU-HSQS($mn+s : s$) when $mn \geq 44$, $m+2s \geq 5$ and $m \geq 2$.*

Proof. Let $(X, S, \{G_1, \ldots, G_n\}, \mathcal{B})$ be the CQS($m^n : s$). Then we construct an HSQS($m+s : s$) on $S \cup G_k$, $1 \leq k \leq n$, with S as the hole to obtain the desired HSQS($mn+s : m+s$) (or HSQS($mn+s : s$), respectively). Let $S_0 = [b_0, b_1, \ldots, b_{p-1}]$ be a CSCU-CQS($m^n : s$) and $T_k = [a_0^k, a_1^k, \ldots, a_{q-1}^k]$ be a CSCU-HSQS($m+s : s$) on $S \cup G_k$. Note that for each block $b \in \mathcal{B}$, we have $|b \cap (S \cup G_k)| \leq 2$. View S_0 as a cycle, and cut each T_k between a_0^k and a_{q-1}^k. Then we insert each T_k into S_0 between b_{i_k-1} and b_{i_k} one by one. Here, we require that $i_k \neq i_{k'}$ if $k \neq k'$.

Using the same notation, we have that $p = m^2 n(n-1)(m+mn+3s-3)/24$. By counting the number r_x of blocks in $\mathcal{B}$ containing a point $x \in X$, and the assumption that $m+2s \geq 5$, we know that $r_x \leq m(n-1)(mn+m+2s-3)/6$. By counting the number $r_{x,y}$ of blocks in $\mathcal{B}$ containing a pair of distinct points $\{x, y\}$ of X, and the assumption that $m \geq 2$, we also know that $r_{x,y} \geq m(n-1)/2$. Then we have $\max\{\alpha(a_0^k), \alpha(a_{q-1}^k)\} \leq \alpha = 4 \times m(n-1)(mn+m+2s-3)/6 - 6 \times m(n-1)/2$ and $\max\{\beta(a_0^k), \beta(a_{q-1}^k)\} \leq \beta = 6 \times m(n-1)/2$ for any $1 \leq k \leq n$.

First, since $m_1 \geq p - \alpha(a_0^1) - 4\beta(a_0^1) - \alpha(a_{q-1}^1) - 4\beta(a_{q-1}^1) \geq p - 2\alpha - 8\beta \geq 1$, there exists one i_1, $0 \leq i_1 \leq p-1$, such that $S_1 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots]$ is a CSCU. Here, S_1 is obtained by inserting T_1 into S_0 between b_{i_1-1} and b_{i_1}.

Next, we want to insert T_2 into S_1 between b_{i_2-1} and b_{i_2}, where $0 \leq i_2 \leq p-1$ and $i_2 \neq i_1$, so that $S_2 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots, b_{i_2-1}, T_2, b_{i_2}, \ldots]$ is a CSCU. Since $|b \cap (S \cup G_2)| \leq 2$ for each block $b \in T_1 \cup \mathcal{B}$, in order to estimate m_2, the number of the suitable places that we can properly inset T_2 into S_1, we only need to compute the numbers of the consecutive unions $c \in \widehat{S_1} = \overline{\sigma^{i_1}(S_0)} \cup \{a_{q-1}^1 \cup b_{i_1}, a_0^1 \cup b_{i_1-1}\} \cup \overline{T_1}$ such that $|c \cap a_j^2| = 3$ and 4, $j = 0, q-1$, respectively, for the reason that $m_2 \geq p' - n'(a_0^2) - n'(a_{q-1}^2) \geq p - 1 - (\alpha'(a_0^2) + 4\gamma'(a_0^2) + \delta'(a_0^2)) - (\alpha'(a_{q-1}^2) + 4\gamma'(a_{q-1}^2) + \delta'(a_{q-1}^2))$, where $\alpha'(a_j^2) = |\{k \mid 0 \leq k \leq p-1,\ b_k \cap a_j^2 \neq \emptyset\}| = \alpha(a_j^2)$, $\gamma'(a_j^2) = |\{l \mid |(c_l \cup c_{l+1}) \cap a_j^2| = 4\}|$, $\delta'(a_j^2) = |\{l \mid |(c_l \cup c_{l+1}) \cap a_j^2| = 3\}|$ for $j = 0$ and $q-1$, and $c_l \cup c_{l+1} \in \widehat{S_1}$. It is easy to know that there are no such unions in $\overline{T_1}$. We then consider the unions in $\{a^1_{q-1} \cup b_{i_1}, a^1_0 \cup b_{i_1-1}\} \cup \overline{\sigma^{i_1}(S_0)}$. For the unions in $\overline{\sigma^{i_1}(S_0)}$, we know that $4\gamma(a^2_j) + \delta(a^2_j) < 4\beta(a^2_j)$ holds for $j = 0, q-1$. For the unions in $\{a^1_{q-1} \cup b_{i_1}, a^1_0 \cup b_{i_1-1}\}$, since $|(a^1_0 \cup a^1_{q-1}) \cap a^2_j| \leq 2$ for $j = 0, q-1$, and $a^1_0 \cap a^1_{q-1} = \emptyset$, we know that the only possible cases are the following: (1) both $a^1_{q-1} \cup b_{i_1}$ and $a^1_0 \cup b_{i_1-1}$ intersect a^2_j at 3 elements; (2) exactly one of $a^1_{q-1} \cup b_{i_1}$ and $a^1_0 \cup b_{i_1-1}$ intersects a^2_j at 4 elements, and the other at less than 3 points; (3) exactly one of $a^1_{q-1} \cup b_{i_1}$ and $a^1_0 \cup b_{i_1-1}$ intersects a^2_j at 3 elements, and the other at less than 3 points. In any case, we have $m_2 \geq p - 1 - 2\alpha - 8\beta - 2 \times 4 \geq 1$. This means that there exists at least one such index $i_2 \neq i_1$ so that $S_2 = [\ldots, b_{i_1-1}, T_1, b_{i_1}, \ldots, b_{i_2-1}, T_2, b_{i_2}, \ldots]$ is a CSCU, where S_2 is obtained by inserting T_2 into S_1 between b_{i_2-1} and b_{i_2}.

Suppose we have inserted T_k into S_{k-1} for $k = 1, \ldots, n-1$, and let S_k denote the obtained CSCU. We want to insert T_{k+1} into S_k between $b_{i_{k+1}-1}$ and $b_{i_{k+1}}$ where $0 \leq i_{k+1} \leq p-1$ and $i_{k+1} \neq i_l$ for any $1 \leq l \leq k$. Similarly, we only need to care about the unions in $\widehat{S_0} \cup \{a^1_{q-1} \cup b_{i_1}, \ldots, a^k_{q-1} \cup b_{i_k}, a^1_0 \cup b_{i_1-1}, \ldots, a^k_0 \cup b_{i_k-1}\}$. Then we have $m_{k+1} \geq p-k-2\alpha-8\beta-8k$. It is easy to check that $m_k \geq 1$ for any $1 \leq k \leq n$. So there exist n distinct indices $0 \leq i_1, i_2, \ldots, i_n \leq p-1$ such that when we insert each T_k into S_{k-1} between b_{i_k-1} and b_{i_k}, the obtained sequence is a CSCU-HSQS$(mn + s : s)$ when $1 \leq k \leq n$, or a CSCU-HSQS$(mn + s : m + s)$ when $1 \leq k \leq n-1$. □

For a CQS $(X, S, \mathcal{G}, \mathcal{B})$, we may view S as a special group, that is, let $S \in \mathcal{G}$, and we will write CQS $(X, \mathcal{G}, \mathcal{B})$ for convenience. If a block of size k intersects each group in at most one point, we say it is *k-partite* (see [34]). For any design $(X, \mathcal{G}, \mathcal{B})$, H-design or CQS, let P be a permutation on X. For each $G \in \mathcal{G}$, if $P(G) = G$, then the design $(X, \mathcal{G}, P(\mathcal{B}))$ is isomorphic to $(X, \mathcal{G}, \mathcal{B})$. For a point $x \in X$, denote by $\overline{G}_x$ the group containing x. For a block $B \in \mathcal{B}$, let $P_B = \{\prod_{x \in B}(x\ y) \mid y \in \overline{G}_x$ and $(x\ y)$ is a transposition$\}$. Note that each permutation in P_B permutes each point of B to a point in the same group and leaves any other point invariant.

Theorem 4.2.3. *Let $(X, \mathcal{G}, \mathcal{B}_1, \ldots, \mathcal{B}_e, \mathcal{T})$ be an e-FG$(3, (K_1, \ldots, K_e, K_{\mathcal{T}}), v)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$. Suppose that there exist a CQS$(m^{k_1} : s_1)$ for any $k_1 \in K_1$, an*

$H(m^{k_i}s_i^1)$ for any $k_i \in K_i$ with $2 \le i \le e$, and an $H(m^k)$ for any $k \in K_{\mathcal{T}}$. Then there exists a CQS $((mg_1)^{n_1}(mg_2)^{n_2}\ldots(mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$. Furthermore, if

(1) *the block set of each ingredient design can be arranged into a CSCU, and for any $A \in \mathcal{B}_1$, the ingredient CQS$(m^{|A|} : s_1)$ contains a 4-partite block,*

(2) *the master e-fan design has two disjoint blocks $b, b' \in \mathcal{T}$ if $e = 0$, or $b \in \mathcal{T}$ and $b' \in \mathcal{B}_1$ if $e \neq 0$,*

then there exists a CSCU-CQS $((mg_1)^{n_1}(mg_2)^{n_2}\ldots(mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$ when $m \ge \max\{5, s_i \mid 1 \le i \le e\}$ and $s_i \neq 1$ for each $1 \le i \le e$.

Proof. Let $I_l = \{0, 1, \ldots, l-1\}$ for any positive integer l and $I_0 = \emptyset$. Denote $G_x = \{x\} \times I_m$ for $x \in X$, and $S_j = \{\infty_j\} \times I_{s_j}$ for $1 \le j \le e$, where $\{\infty_j \mid 1 \le j \le e\} \cap X = \emptyset$. We construct the desired design on $X' = (X \times I_m) \cup S$ with the group set $\mathcal{G}' = \{G \times I_m \mid G \in \mathcal{G}\}$ and the stem $S = S_1 \cup S_2 \cup \ldots \cup S_e$. Clearly, $(X \times I_m) \cap S = \emptyset$.

For each block $A \in \mathcal{B}_1$, construct a CSCU-CQS$(m^{|A|} : s_1)$ on $X_A = (A \times I_m) \cup S_1$ having $\{G_x \mid x \in A\}$ as its group set, S_1 as its stem, and $\mathcal{A}_A$ as its block set. Denote $G_A = \{G_x \mid x \in A\} \cup \{S_1\}$.

For each block $A \in \mathcal{B}_j$, $2 \le j \le e$, construct a CSCU-GDD$(m^{|A|}s_j^1)$ on $X_A = (A \times I_m) \cup S_j$ having $G_A = \{G_x \mid x \in A\} \cup \{S_j\}$ as its group set and $\mathcal{A}_A$ as its block set.

For each block $A \in \mathcal{T}$, construct a CSCU-GDD$(m^{|A|})$ on $X_A = A \times I_m$ having $G_A = \{G_x \mid x \in A\}$ as its group set and $\mathcal{A}_A$ as its block set.

Let $\mathcal{B} = (\cup_{1\le i\le e}\mathcal{B}_i) \cup \mathcal{T}$. Then $\cup_{A\in\mathcal{B}}\mathcal{A}_A$ is the block set of a CQS $((mg_1)^{n_1}(mg_2)^{n_2}\ldots(mg_r)^{n_r} : \sum_{1\le i\le e} s_i)$. We try to find a CSCU of $\cup_{A\in\mathcal{B}}\mathcal{A}_A$.

First, by our assumption, when $e \neq 0$, we can arrange $\mathcal{B}$ into a sequence $\mathcal{S}' = [b_0, b_1, \ldots, b_{p-1}]$ where the blocks of $\mathcal{B}_1$ are consecutive with $b_{p-2} \in \mathcal{B}_1$ being the tail-end, $b_{p-1} \in \mathcal{T}$, and $b_{p-2} \cap b_{p-1} = \emptyset$; when $e = 0$, we simply let $b_{p-2} \cap b_{p-1} = \emptyset$. Next, we replace each block b_i by a cut CSCU $T_i = [a_0^i, a_1^i, \ldots, a_{q_i-1}^i]$ of $\mathcal{A}_{b_i}$, where a_0^i and $a_{q_i-1}^i$ are the two ends, and $q_i = |\mathcal{A}_{b_i}|$, $0 \le i \le p-1$. By the hypothesis and the definition of an H-design, without loss of generality, we may assume that a_0^i intersects each group in G_A in at most one point. Now we have the following Claim.

Claim: There exists a set of permutations $\{\sigma_k \in P_{a_0^k} \mid 0 \leq k \leq p-1\}$ such that in the cyclic sequence $\mathcal{S} = [\sigma_0(T_0), \sigma_1(T_1), \ldots, \sigma_{p-1}(T_{p-1})]$, we have $\sigma_{k-1}(a_{q_{k-1}-1}^{k-1}) \cap \sigma_k(a_0^k) = \emptyset$ and $d(\sigma_{l-1}(a_{q_{l-1}-1}^{l-1}) \cup \sigma_l(a_0^l), \sigma_{k-1}(a_{q_{k-1}-1}^{k-1}) \cup \sigma_k(a_0^k)) \geq 4$ for any $0 \leq k, l \leq p-1$ and $|k-l| \geq 2$.

We use a recursive method to prove this claim. Denote $\Gamma_0 = \{\sigma \in P_{a_0^0} \mid a_{q_{p-1}-1}^{p-1} \cap \sigma(a_0^0) = \emptyset\} \subseteq P_{a_0^0}$. From the assumptions on a_0^i and b_{p-1}, we know that $|a_0^0 \cap a_{q_{p-1}-1}^{p-1}| \leq 2$. We consider all possible intersections of a_0^0 and $a_{q_{p-1}-1}^{p-1}$. Let $a_0^0 = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $a_{q_{p-1}-1}^{p-1} = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. We first consider the case that $x_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$. If $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 0$, then $|\Gamma_0| = m^4$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 1$, then $|\Gamma_0| = (m-1)m^3$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 2$, then $|\Gamma_0| = (m-1)^2 m^2$. Next we consider the case that $x_i = \infty_j$ for a unique $1 \leq i \leq 4$ and a unique $1 \leq j \leq e$, which implies that $s_j \geq 2$. If $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 0$, then $|\Gamma_0| = s_j m^3$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 1$, then $|\Gamma_0| = s_j(m-1)m^2$; if $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}| = 2$, then $|\Gamma_0| = s_j(m-1)^2 m$. So we know that $|\Gamma_0| \geq \min\{m^4, (m-1)m^3, (m-1)^2 m^2, s_j m^3, s_j(m-1)m^2, s_j(m-1)^2 m \mid s_j \geq 2\} \geq 1$. Choose $\sigma_0 \in \Gamma_0$ and let $\mathcal{S}_0 = \langle \sigma_0(T_0) \rangle$ be a non-cyclic sequence of $\sigma_0(T_0)$, that is, $\langle \sigma_0(T_0) \rangle$ is exactly the same as $[\sigma_0(T_o)]$ except that $\sigma(a_0^0)$ is not considered as a successor of $\sigma(a_{q_0-1}^0)$.

Similarly, we denote $\Gamma_1 = \{\sigma \in P_{a_0^1} \mid \sigma_0(a_{q_0-1}^0) \cap \sigma(a_0^1) = \emptyset\} \subseteq P_{a_0^1}$. Again, we consider all possible intersections of a_0^1 and $\sigma(a_{q_0-1}^0)$. Let $a_0^1 = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $\sigma_0(a_{q_0-1}^0) = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and all y_i are distinct, then $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-1)^2 m^2$ or $s_j m^3$ or $s_j(m-1)m^2$ or $s_j(m-1)^2 m$, with $s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and exactly two of y_i are equal, then $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-2)m^3$ or $(m-1)^2 m^2$ or $(m-2)(m-1)m^2$ or $s_j m^3$ or $s_j(m-1)m^2$ or $s_j(m-2)m^2$ or $s_j(m-1)^2 m$ or $s_j(m-2)(m-1)m$, with

$s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i \neq \infty_j$ for any $1 \leq i \leq 4$ and $1 \leq j \leq e$, and $y_{i_1} = y_{i_2}$, $y_{i_3} = y_{i_4}$, but these two values are not the same, then $|\Gamma_1| = m^4$ or $(m-2)m^3$ or $(m-2)^2m^2$ or s_jm^3 or $s_j(m-2)m^2$ or $s_j(m-2)^2m$, with $s_j \geq 2$ and $1 \leq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. If $y_i = \infty_j$ for a unique $1 \leq i \leq 4$ and a unique $1 \leq j \leq e$, then all y_i should be distinct, and $|\Gamma_1| = m^4$ or $(m-1)m^3$ or $(m-1)^2m^2$ or s_im^3 or $s_i(m-1)m^2$ or $(s_j-1)m^3$ or $s_i(m-1)^2m$ or $(s_j-1)(m-1)m^2$, with $s_i \geq 2$, $s_j \geq 2$ and $1 \leq \ i \neq j \leq e$, depending on whether $|\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\}|$ is equal to 0 or 1 or 2, and whether $|\{x_1, x_2, x_3, x_4\} \cap \{\infty_j \mid 1 \leq j \leq e\}|$ is equal to 0 or 1. In any case, we know that $|\Gamma_1| \geq 1$. Let $\mathcal{S}_1 = \langle \sigma_0(T_0), \sigma_1(T_1) \rangle$, where $\sigma_1 \in \Gamma_1$.

Suppose that there exists a set of permutations $\{\sigma_k \in P_{a_0^k} \mid 0 \leq k \leq i-1 < p-2\}$ such that $\sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cap \sigma_k(a_0^k) = \emptyset$ and $d(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l), \sigma_{k-1}(a^{k-1}_{q_{k-1}-1}) \cup \sigma_k(a_0^k)) \geq 4$ for any $0 \leq k, l \leq i-1$ and $|k-l| \geq 2$. Let $\mathcal{S}_{i-1} = \langle \sigma_0(T_0), \sigma_1(T_1), \ldots, \sigma_{i-1}(T_{i-1}) \rangle$.

For $k = i$, we try to find a permutation $\sigma_i \in P_{a_0^i}$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a_0^i) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a_0^i), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l)) \geq 4$ for any $0 \leq l < i-1$.

Let $a_0^i = \{(x_1, l_1), (x_2, l_2), (x_3, l_3), (x_4, l_4)\}$ and $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) = \{(y_1, l_1'), (y_2, l_2'), (y_3, l_3'), (y_4, l_4')\}$. Denote $\Gamma_i = \{\sigma \in P_{a_0^i} \mid \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma(a_0^i) = \emptyset\} \subseteq P_{a_0^i}$. We first divide the problem into two possible cases.

(a): Suppose that $\{x_1, x_2, x_3, x_4\} \cap \{y_1, y_2, y_3, y_4\} = \{\infty_j\}$ for some j, $1 \leq j \leq e$. Then $s_j \geq 2$. For convenience, let $x_4 = y_4 = \infty_j$. Then $b_i, b_{i-1} \in \mathcal{B}_j$ and $|b_i \cap b_{i-1}| \leq 1$. In a similar way to the above analysis, we can prove that $|\Gamma_i| \geq (s_j-1)(m-1)m^2 \geq 1$. Now we choose $\sigma_{i,0} \in \Gamma_i$, which satisfies that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_{i,0}(a_0^i) = \emptyset$. If there exists an index l, $0 \leq l < i-1$, such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a_0^i), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a_0^l)) < 4$, then exactly one of the two blocks $\{b_{l-1}, b_l\}$ belongs to $\mathcal{B}_j$. The reason is explained below. If $b_{l-1}, b_l \in \mathcal{B}_j$, then since $b_{i-1}, b_i \in \mathcal{B}_j$, we know that $|(b_i \cup b_{i-1}) \cap (b_{l-1} \cup b_l)| \leq |b_i \cap b_{l-1}| + |b_{i-1} \cap$

$b_{l-1}|+|b_i \cap b_l|+|b_{i-1} \cap b_l| \leq 4$, and hence $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \leq 6$, which is impossible, for in this case we would have $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$. On the other hand, if $b_{l-1}, b_l \notin \mathcal{B}_j$, then $(\infty_j, l''_4) \notin \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)$ for any $l''_4 \in I_{s_j}$, so $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \leq 6$, which is again impossible. Then there are two cases to be considered: $b_{l-1} \in \mathcal{B}_j$, $b_l \notin \mathcal{B}_j$; and $b_l \in \mathcal{B}_j$, $b_{l-1} \notin \mathcal{B}_j$. We first assume that $b_{l-1} \in \mathcal{B}_j$ and $b_l \notin \mathcal{B}_j$. Then clearly $\sigma_l(a^l_0) \cap S_j = \emptyset$. Since we have supposed that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$, i.e., $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)))| \geq 7$, we should have that one of $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cap (\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0))|$ and $|\sigma_l(a^l_0) \cap (\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0))|$ equals 4 and the other at least 3. Since $b_{l-1} \in \mathcal{B}_j$, then $|b_{l-1} \cap b_i| \leq 1$ and $|b_{l-1} \cap b_{i-1}| \leq 1$, i.e., $|(b_{l-1} \cup \{\infty_j\}) \cap (b_i \cup b_{i-1} \cup \{\infty_j\})| \leq 3$, which implies that $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cap (\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0))| \leq 3$. Therefore it is necessary that $|\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cap (\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0))| = 3$ and $|\sigma_l(a^l_0) \cap (\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0))| = 4$. Then we can let $\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) = \{(x_1, \sigma_{i,0}(l_1)), (y_1, l'_1), (\infty_j, \diamondsuit), (\star, *)\}$ and $\sigma_l(a^l_0) = \{(x_2, \sigma_{i,0}(l_2)), (x_3, \sigma_{i,0}(l_3)), (y_2, l'_2), (y_3, l'_3)\}$, where $\diamondsuit \in \{\sigma_{i,0}(l_4), l'_4\}$. If $\diamondsuit = \sigma_{i,0}(l_4)$, no permutation $\sigma \in \Gamma_i$ satisfies that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$ except $\sigma = \sigma_{i,0}$. If $\diamondsuit = l'_4$, then all the permutations $\sigma_{i,1} \in \Gamma_i$ which change $(\infty_j, \sigma_{i,0}(l_4))$ to every element in $\{\infty_j\} \times (Z_{s_j} \setminus \{l'_4\})$ and fix the other three points in $\sigma_{i,0}(a^i_0)$ satisfy that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,1}(\sigma_{i,0}(a^i_0))) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)))| \geq 7$. So for such a pair (b_{l-1}, b_l), there are at most $(s_j - 1)$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. Now we compute the number of such pairs (b_{l-1}, b_l), or equivalently, the number of such b_l. There are $\binom{6}{3} - 2 = 18$ triples in $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ excluding the two triples $\{x_1, x_2, x_3\}$ and $\{y_1, y_2, y_3\}$. Since each triple occurs in exactly one block of $\mathcal{B}$, each block of $\mathcal{B}$ contains exactly 4 triples, $|b_l \cap \{x_1, x_2, x_3\}| = 2$, and $|b_l \cap \{y_1, y_2, y_3\}| = 2$, we know that there are at most $\lfloor 18/4 \rfloor = 4$ such b_l. From the assumption, we have $|\Gamma_i| \geq (m-1)m^2(s_j - 1) > 4(s_j - 1)$, which implies that there exists at least one permutation $\sigma_i \in \Gamma_i$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a^i_0) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $0 \leq l < i-1$. For the case that $b_l \in \mathcal{B}_j$ and $b_{l-1} \notin \mathcal{B}_j$, we can also prove, in the same fashion as above, that the same assertion holds.

(b): Suppose that $\{x_1,x_2,x_3,x_4\}\cap\{y_1,y_2,y_3,y_4\}\neq\{\infty_j\}$ for any j, $1\leq j\leq e$. We further divide this case into two possible sub-cases.

$(b.1)$: $y_1=y_2=x_1$ and $y_3=x_2$, i.e., $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})=\{(x_1,l'_1),(x_1,l'_2),(x_2,l'_3),(y_4,l'_4)\}$. If $y_4=x_2$, then $|\Gamma_i|\geq\min\{(m-2)^2m^2,(m-2)^2ms_j\mid s_j\geq 2\}\geq 1$. If $y_4\neq x_2$, then $|\Gamma_i|\geq\min\{(m-2)(m-1)m^2,(m-2)(m-1)ms_j\mid s_j\geq 2\}\geq 1$. In any case, we know that $|\Gamma_i|\geq 1$. Assume that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cap\sigma_{i,0}(a^i_0)=\emptyset$. We now prove that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\geq 4$ for $0\leq l\leq i-1$. If $\{x_1,x_2\}\subset b_l$, then $|b_l\cap(b_{i-1}\cup b_i)|=2$. Since a^l_0 is 4-partite, we know that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap\sigma_l(a^l_0)|\leq 2$, which makes $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\geq 4$. If $\{x_1,x_2\}\subset b_{l-1}$, then since $y_1=y_2=x_1$, we know that $b_{i-1}\in\mathcal{B}_1$ and thus $b_{l-1}\notin\mathcal{B}_1$, implying that $a^{l-1}_{q_{l-1}-1}$ is a block in some ingredient H-design and therefore is 4-partite. So $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})|\leq 2$, which ensures that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\geq 4$. We still need to consider the case when $\{x_1,x_2\}\not\subset b_{l-1}$ and $\{x_1,x_2\}\not\subset b_l$. If $y_4=x_2$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap\sigma_l(a^l_0)|\leq 2$, which makes again $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\geq 4$. If $y_4\neq x_2$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap\sigma_l(a^l_0)|\leq 3$ and $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})|\leq 3$, which also ensures that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))\geq 4$.

$(b.2)$: All cases except $(b.1)$, that is, $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\in\{\{(x_1,l'_1),(x_1,l'_2),(y_3,l'_3),(y_4,l'_4)\},\{(x_1,l'_1),(x_2,l'_2),(y_3,l'_3),(y_4,l'_4)\},\{(x_1,l'_1),(y_2,l'_2),(y_3,l'_3),(y_4,l'_4)\},\{(y_1,l'_1),(y_2,l'_2),(y_3,l'_3),(y_4,l'_4)\}\}$, where $y_i\neq x_j$ for any i and j. A tedious calculation shows that $|\Gamma_i|\geq\min\{(m-2)m^3,(m-2)m^2s_j,(m-1)^2m^2,(m-1)^2ms_j,(m-1)m^3,(m-1)m^2s_j,m^4,m^3s_j\mid s_j\geq 2\}\geq 2(m-2)m^2\geq 1$. Choose $\sigma_{i,0}\in\Gamma_i$. Then $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cap\sigma_{i,0}(a^i_0)=\emptyset$. If there exists an index l, $0\leq l<i-1$, such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))<4$, then $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|\geq 7$. Let $R=\{r\subset X'\mid r\subset\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0),r\not\subset\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}),r\not\subset\sigma_{i,0}(a^i_0)\text{ and }|r|=3\}$, then $|R|\leq\binom{4}{2}\times\binom{4}{1}\times 2=48$. Suppose that there are t_1 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|=8$ and t_2 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1})\cup\sigma_{i,0}(a^i_0))\cap(\sigma_{l-1}(a^{l-1}_{q_{l-1}-1})\cup\sigma_l(a^l_0))|=7$. For the former case, each block in $\{\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}),\sigma_l(a^l_0)\}$ contains 4 triples from R. Even if there is one point in one of these two blocks with its second component being changed, we still have $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$, that is, $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)), (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. So there are at most $\max\{4(m-1)+1, 3(m-1)+(s_j-1)+1 \mid s_j \geq 2\} = 4m-3$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. For the latter case, one block in $\{\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}), \sigma_l(a^l_0)\}$ contains 4 triples and the other one contains only one triple from R. Even if the second component of the uncommon point is changed in the block which contains only one triple from R, we still have $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$, that is, $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)), (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. So there are at most $\max\{m, s_j \mid s_j \geq 2\} = m$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. Here, $8t_1 + 5t_2 \leq |R| \leq 48$. Since $t_1 \leq |R|/8 \leq 6$, we have $t_1(4m-3) + t_2 m \leq t_1(4m-3) + (48-8t_1)m/5 = t_1(2.4m-3) + 9.6m \leq 6(2.4m-3) + 9.6m = 24m-18$. From the assumption that $m \geq 5$, we have $2(m-2)m^2 > 24m-18$, which implies that there exists at least one permutation $\sigma_i \in \Gamma_i$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a^i_0) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $0 \leq l < i-1$.

Then we set $\mathcal{S}_i = \langle \mathcal{S}_{i-1}, \sigma_i(T_i) \rangle$ for $1 \leq i \leq p-2$.

When $k = p-1$, we want to find a permutation $\sigma_{p-1} \in P_{a^{p-1}_0}$ such that $\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0) = \emptyset$, $\sigma_{p-1}(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}$ and $d(\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $1 \leq l < p-2$. By assumption, $b_{p-2} \cap b_{p-1} = \emptyset$. This implies that $\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma_{p-1}(a^{p-1}_0) = \emptyset$ for any $\sigma_{p-1} \in P_{a^{p-1}_0}$. Denote $\Gamma_{p-1} = \{\sigma \in P_{a^{p-1}_0} \mid \sigma(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}\}$. Since $b_{p-1} \in \mathcal{T}$ is replaced by the cut CSCU $T_{p-1} = [a^{p-1}_0, a^{p-1}_1, \ldots, a^{p-1}_{q_{p-1}-1}]$, then, as we said in Section 4.1, we have that $a^{p-1}_0 \cap a^{p-1}_{q_{p-1}-1} = \emptyset$. This, together with $\sigma(a^{p-1}_{q_{p-1}-1}) = a^{p-1}_{q_{p-1}-1}$, shows that $|\Gamma_{p-1}| \geq (m-1)^4 \geq 1$. Similar to the proof in $(b.2)$, we can prove that there are at most $(24m-18)$ $\sigma \in \Gamma_i$ such that $d(\sigma_{p-2}(a^{p-2}_{q_{p-2}-1}) \cap \sigma(a^{p-1}_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$ for some l, $1 \leq l < p-2$. From the assumption, we have $(m-1)^4 > 24m-18$. Thus we have proved the existence of σ_{p-1}.

Now we have finished the proof for the claim. For convenience, we use T_k to denote $\sigma_k(T_k)$ for $0 \leq k \leq p-1$. Then $\mathcal{S} = [T_0, T_1, \ldots, T_{p-1}]$ satisfies the conditions that $a^{k-1}_{q_{k-1}-1} \cap a^k_0 = \emptyset$ and $d(a^{l-1}_{q_{l-1}-1} \cup a^l_0, a^{k-1}_{q_{k-1}-1} \cup a^k_0) \geq 4$ for any $0 \leq k, l \leq p-1$ and $|k-l| \geq 2$. Next, we will prove $\mathcal{S}$ is actually a CSCU.

To do this, we should check the distance between any two elements of $C = (\bigcup_{i=0}^{p-1} \mathcal{A}_{b_i}) \cup (\bigcup_{i=0}^{p-1} \overline{T_i}) \cup (\bigcup_{i=0}^{p-1} \{a_{q_{i-1}-1}^{i-1} \cup a_0^i\})$. Elements of C are classified into three types.

Type I: $a \in \mathcal{A}_{b_i}$ for some i, $0 \le i \le p-1$. If $b_i \in \mathcal{B}_1$, we say that a belongs to Type $\mathrm{I_{CQS}}$, otherwise, a belongs to Type $\mathrm{I_H}$.

Type II: $c \in \overline{T_i}$ for some i, $0 \le i \le p-1$. If $b_i \in \mathcal{B}_1$, we say that c belongs to Type $\mathrm{II_{CQS}}$, otherwise, c belongs to Type $\mathrm{II_H}$.

Type III: $c \in \bigcup_{i=0}^{p-1} \{a_{q_{i-1}-1}^{i-1} \cup a_0^i\}$.

Since the resultant design is a CQS, we easily know the following

Case (1): $d(a, a') \ge 4$ for any two distinct a, a' from Type I;

Case (2): $d(a, c) \ge 4$ for any a from Type I and c from Type II, III, respectively.

Since each T_i, $0 \le i \le p-1$, is a CSCU, we have

Case (3): $d(c, c') \ge 4$ for any $c, c' \in \overline{T_i}$, $0 \le i \le p-1$, from Type II.

For each $a \in \mathcal{A}_{b_i}$ from Type $\mathrm{I_H}$, $|a \cap X_{b_j}| \le 2$ when $a \notin \mathcal{A}_{b_j}$, so

Case (4): $d(c, c') \ge 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_H}$ and $c' \in \overline{T_j}$ from Type II.

For each $a \in \mathcal{A}_{b_i}$ from Type $\mathrm{I_{CQS}}$, we know that $b_i \in \mathcal{B}_1$ and $|a \cap X_{b_j}| \le 2$ when $b_j \in \mathcal{B}_1$ and $a \notin \mathcal{A}_{b_j}$, so

Case (5): $d(c, c') \ge 4$ for any c, c' from Type $\mathrm{II_{CQS}}$.

If $b_i \notin \mathcal{B}_1$, then by the definition of an H-design and our special arrangement of $\mathcal{B}$ into $\mathcal{S}' = [b_0, b_1, \ldots, b_{p-1}]$, we know that $|a \cap a'| \le 1$ for any $a \in \mathcal{A}_{b_i}$ and $a' \in \mathcal{A}_{b_{i-1}}$, so

Case (6): $d(c, a_{q_{i-1}-1}^{i-1} \cup a_0^i) \ge 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_H}$.

If $b_i \in \mathcal{B}_1$, then $|a \cap X_{b_i}| \le 2$ for any $a \notin \mathcal{A}_{b_i}$, so

Case (7): $d(c, a_{q_{i-1}-1}^{i-1} \cup a_0^i) \ge 4$ for any $c \in \overline{T_i}$ from Type $\mathrm{II_{CQS}}$.

Since a_0^j is 4-partite, we know that $|a_0^j \cap X_{b_i}| \le 2$ for any $1 \le i \ne j \le p-1$, and then

Case (8): $d(c, a_{q_{j-1}-1}^{j-1} \cup a_0^j) \ge 4$ for any $c \in \overline{T_i}$ and $1 \le i \ne j \le p-1$.

Since $a_0^i \cap a_{q_i-1}^i = \emptyset$ and $|a_{q_{i-1}-1}^{i-1} \cap a_{q_i-1}^i| \le 2$, we have

Case (9): $d(a^{i-1}_{q_{i-1}-1} \cup a^i_0, a^i_{q_i-1} \cup a^{i+1}_0) \geq 4$, $0 \leq i \leq p-1$.

From the property of $\mathcal{S}$, we know that

Case (10): $d(a^{i-1}_{q_{i-1}-1} \cup a^i_0, a^{l-1}_{q_{l-1}-1} \cup a^l_0) \geq 4$ for $|i-l| \geq 2$.

Then we have proved that $\mathcal{S}$ is in fact a CSCU. $\square$

Theorem 4.2.4. *In Theorem 4.2.3, if we change the condition (1) to be*

(1′) *the block set of each ingredient design can be arranged into a CSCU with two consecutive 4-partite blocks,*

then there exists a CSCU-CQS $((mg_1)^{n_1}(mg_2)^{n_2}\ldots(mg_r)^{n_r} : \sum_{1\leq i\leq e} s_i)$ *when* $m \geq \max\{4, s_i \mid 1 \leq i \leq e\}$ *and* $s_i \neq 1$ *for each* $1 \leq i \leq e$.

Proof. Since the proof is similar to that of Theorem 4.2.3, we will look at only those places which are different from Theorem 4.2.3.

First, without loss of generality, we may assume that both a^i_0 and $a^i_{q_i-1}$ of $T_i = [a^i_0, a^i_1, \ldots, a^i_{q_i-1}]$ are 4-partite for any i, $0 \leq i \leq p-1$.

Remember that in the proof of Theorem 4.2.3, we need $m = 5$ only in Case $(b.2)$. So we can omit the proof for all cases except for $(b.2)$. We divide Case $(b.2)$ into two sub-cases.

$(b.2.1)$: If $x_i = \infty_j$ for some i and j, $1 \leq i \leq 4$, $1 \leq j \leq e$, then $|\Gamma_i| \geq \min\{(m-1)^2 m s_j, (m-1)m^2 s_j, m^3 s_j \mid s_j \geq 2\} = (m-1)^2 m s_j \geq 1$. Assume that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_{i,0}(a^i_0) = \emptyset$. If there exists an index l, $0 \leq l < i-1$, such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$, then, as we knew already in Case $(b.2)$ of Theorem 4.2.3, $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| \geq 7$. Again, let $R = \{r \subset X' \mid r \subset \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0), r \not\subset \sigma_{i-1}(a^{i-1}_{q_{i-1}-1}), r \not\subset \sigma_{i,0}(a^i_0)$ and $|r| = 3\}$, and then we know that $|R| \leq \binom{4}{2} \times \binom{4}{1} \times 2 = 48$. Suppose again that there are t_1 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| = 8$ and t_2 l's such that $|(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_{i,0}(a^i_0)) \cap (\sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0))| = 7$. For the former case, just as in Case $(b.2)$ of Theorem 4.2.3, we can prove that there are at most $3(m-1)+(s_j-1)+1 = 3m+s_j-3$ $\sigma \in \Gamma_i$ such that $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. Similarly, for the latter case, we can prove that there are at most $\max\{m, s_j\} = m$ $\sigma \in \Gamma_i$ such that

$d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) < 4$. Since $8t_1 + 5t_2 \leq |R| \leq 48$, we have $t_1 \leq |R|/8 \leq 6$, and $t_1(3m+s_j-3)+t_2m \leq t_1(3m+s_j-3)+(48-8t_1)m/5 = t_1(1.4m + s_j - 3) + 9.6m \leq 6(1.4m + s_j - 3) + 9.6m = 18m + 6s_j - 18$. From the assumptions that $m \geq 4$ and $s_j \geq 2$, we have $(m-2)m^2 s_j > 18m + 6s_j - 18$, which implies that there exists at least one permutation $\sigma_i \in \Gamma_i$ such that $\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cap \sigma_i(a^i_0) = \emptyset$ and $d(\sigma_{i-1}(a^{i-1}_{q_{i-1}-1}) \cup \sigma_i(a^i_0), \sigma_{l-1}(a^{l-1}_{q_{l-1}-1}) \cup \sigma_l(a^l_0)) \geq 4$ for any $0 \leq l < i-1$.

$(b.2.2)$: If $x_i \neq \infty_j$ for any i and j, $1 \leq i \leq 4$, $1 \leq j \leq e$, then the proof is exactly the same as that for $(b.2.1)$ except that s_j is replaced by m. □

Corollary 4.2.5. *Let $m \geq 4$ and $gn \geq 16$. Assume that there exists an $H(g^n)$. If there exists a CSCU-GDD(m^4), then there exists a CSCU-GDD$((mg)^n)$.*

Proof. In an H(g^n), the number of blocks is $\lambda_0 = \frac{g^3n(n-1)(n-2)}{24}$, the number of blocks containing one point is $\lambda_1 = \frac{g^2(n-1)(n-2)}{6}$, and the number of blocks containing two distinct points is $\lambda_2 = \frac{g(n-2)}{2}$. There exist two disjoint blocks if and only if $\lambda_0 > \binom{4}{1}(\lambda_1 - 1) - \binom{4}{2}(\lambda_2 - 1) + 1$. This inequality is satisfied provided that $gn \geq 16$. Then apply Theorem 4.2.4 with $e = 0$. □

Theorem 4.2.6. *There exists a CSCU-GDD(g^4) for any $g \geq 5$.*

Proof. Let $X = Z_4 \times Z_g$. We build an H(g^4) on X with the group set $\mathcal{G} = \{\{i\} \times Z_g \mid i \in Z_4\}$ and the block set $\mathcal{B} = \{\alpha(i,j,k) = \{(0,i),(1,i+j),(2,k),(3,k+j)\} \mid i,j,k \in Z_g\}$. Let $T(j,k) = \langle \alpha(0,j,k), \alpha(1,j,k+1), \ldots, \alpha(g-1,j,k+g-1)\rangle$, $T_j = \langle T(j,j), T(j,j+1), \ldots, T(j,j-1)\rangle$ and $S = [T_0, T_1, \ldots, T_{g-1}]$. It is clear that $S = \mathcal{B}$ if we view S as a block set. We will check that S is in fact a CSCU.

It is easy to check that any two consecutive blocks in S are disjoint and $d(\alpha(i,j,k), \alpha(i',j',k')) \geq 4$ for any distinct (i,j,k) and (i',j',k'). Let c_t be the union of two consecutive blocks. Then $d(\alpha(i,j,k), c_t) \geq 4$ for any $c_t \in \widehat{S}$. Thus we only need to consider the distance between any two unions. We part the unions into the following three types.

Type I: $c_1(i,j,k) = \alpha(i,j,k) \cup \alpha(i+1,j,k+1), 0 \leq i \leq g-2, 0 \leq j,k \leq g-1$.

Type II: $c_2(j,k) = \alpha(g-1,j,k-2) \cup \alpha(0,j,k), 0 \leq j,k \leq g-1$.

Type III: $c_3(j) = \alpha(g-1, j-1, j-3) \cup \alpha(0, j, j), 0 \le j \le g-1$.

We should check that any two unions from these three types have distance more than or equal to 4. Let n_q be the number of points in $c_1(i,j,k) \cap c_1(i',j',k')$ with the first coordinate being q, where $q \in Z_4$. Then $n_q \le 2$ for any $q \in Z_4$. If there are at least two n_q's of the $c_1(i,j,k) \cap c_1(i',j',k')$ having value no more than 1, then $|c_1(i,j,k) \cap c_1(i',j',k')| \le 6$, which means that $d(c_1(i,j,k), c_1(i',j',k')) \ge 4$.

Case (a): Two unions from Type I, say $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_1(i',j',k') = \{(0,i'),(0,i'+1),(1,i'+j'),(1,i'+j'+1),(2,k'),(2,k'+1),(3,k'+j'),(3,k'+j'+1)\}$. We will show that $|c_1(i,j,k) \cap c_1(i',j',k')| \le 6$ for any distinct (i,j,k) and (i',j',k'). Note the fact that if $l \ne l'$ and $g \ge 5$, then $|\{l, l+1\} \cap \{l', l'+1\}| \le 1$. Since each of the three parameters $\{i,j,k\}$ is related to two different first coordinates, it is easy to check that at least two of the n_q's having value no more than 1. The details are listed below.

1). When $i \ne i'$, $j \ne j'$ and $k \ne k'$, we have $n_0 \le 1$ and $n_2 \le 1$.

2). When $i \ne i'$, $j \ne j'$ and $k = k'$, then $k+j \ne k'+j'$, so $n_0 \le 1$ and $n_3 \le 1$.

3). When $i \ne i'$, $j = j'$ and $k \ne k'$, we have $n_0 \le 1$ and $n_2 \le 1$.

4). When $i \ne i'$, $j = j'$ and $k = k'$, then $i+j \ne i'+j'$, so $n_0 \le 1$ and $n_1 \le 1$.

5). When $i = i'$, $j \ne j'$ and $k \ne k'$, then $i+j \ne i'+j'$, so $n_1 \le 1$ and $n_2 \le 1$.

6). When $i = i'$, $j \ne j'$ and $k = k'$, then $i+j \ne i'+j'$ and $k+j \ne k'+j'$, so $n_1 \le 1$ and $n_3 \le 1$.

7). When $i = i'$, $j = j'$ and $k \ne k'$, then $k+j \ne k'+j'$, so $n_2 \le 1$ and $n_3 \le 1$.

Case (b): Two unions from Type I and Type II respectively, say $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_2(j',k') = \{(0,g-1),(0,0),(1,j'-1),(1,j'),(2,k'-2),(2,k'),(3,k'+j'-2),(3,k'+j')\}$. Since $0 \le i \le g-2$ in $c_1(i,j,k)$, we know that $n_0 \le 1$. Since $g \ge 5$, we have $|\{k,k+1\} \cap \{k'-2,k'\}| \le 1$, i.e., $n_2 \le 1$. Then $d(c_1(i,j,k), c_2(j',k')) \ge 4$.

Case (c): Two unions from Type I and Type III respectively, say $c_1(i,j,k) = \{(0,i),(0,i+1),(1,i+j),(1,i+j+1),(2,k),(2,k+1),(3,k+j),(3,k+j+1)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$. Since $0 \le i \le g-2$ and $g \ge 5$, in a similar way, we can know that $n_0 \le 1$ and $n_1 \le 1$.

Case (d): Two unions from Type II, say $c_2(j,k) = \{(0,g-1),(0,0),(1,j-1),(1,j),(2,k-2),(2,k),(3,k+j-2),(3,k+j)\}$ and $c_2(j',k') = \{(0,g-1),(0,0),(1,j'-1),(1,j'),(2,k'-2),(2,k'),(3,k'+j'-2),(3,k'+j')\}$, where (j,k) and (j',k') are distinct. Similar to Case (a), we can show that there are at least two n_q's having value no more than 1.

Case (e): Two unions from Type II and Type III respectively, say $c_2(j,k) = \{(0,g-1),(0,0),(1,j-1),(1,j),(2,k-2),(2,k),(3,k+j-2),(3,k+j)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$. It is readily checked that at least one of the following assertions holds:

(e.1): $n_1 \le 1$ and $n_2 \le 1$;

(e.2): $n_1 \le 1$ and $n_3 \le 1$.

Case (f): Two unions from Type III, say $c_3(j) = \{(0,g-1),(0,0),(1,j-2),(1,j),(2,j-3),(2,j),(3,2j-4),(3,2j)\}$ and $c_3(j') = \{(0,g-1),(0,0),(1,j'-2),(1,j'),(2,j'-3),(2,j'),(3,2j'-4),(3,2j')\}$, where $j \ne j'$. Similar to Case (a), we can prove that there are at least two n_q's having value no more than 1. $\square$

4.3 Direct Constructions

In this section, we directly construct some small CSCUs which will be used in the recursive constructions. In order to save space, we list only a few examples. The interested reader is referred to the authors or to the Web site [70] for a copy of the detailed cyclic sequences of blocks.

Lemma 4.3.1. *There exists a CSCU-CQS($g^n : s$) for each* $(g,n,s) \in \{(4,4,2), (4,4,4),(6,3,2),(6,3,4),(6,5,2),(6,5,4),(8,3,2),(12,3,2),(12,3,4),(12,4,2), (12,4,4)\}$.

Proof. We only list the sequence of a CSCU-CQS($4^4 : 2$) on the point set $X = Z_{18}$, with the group set $\mathcal{G} = \{\{i, 4+i, 8+i, 12+i\} \mid i \in Z_4\}$ and the stem $S = \{16, 17\}$.

S= [{6,7,12,17}, {4,10,11,16},{12,14,15,17},{2,5,7,16},{1,6,8,10},{2,7,9,17}, {5,10,15,16},
{0,1,2,14},{4,5,6,10},{1,3,14,16},{0,6,9,10}, {1,2,15,17},{3,6,9,16},{1,7,10,17},
{9,11,14,16},{3,5,6,17}, {2,4,13,14},{1,6,11,16},{3,13,14,17},{1,7,11,12},{2,8,9,14},
{0,7,11,13},{2,5,12,14},{4,6,7,8},{0,3,5,15},{6,10,12,13}, {1,3,4,15},{5,7,8,11},
{0,2,3,12},{4,7,9,11},{3,8,13,15}, {0,10,11,12},{2,5,9,15},{3,4,8,10},{0,6,12,15},
{2,4,8,11}, {3,9,12,15},{1,2,7,13},{4,8,14,15},{1,3,6,13},{2,4,9,10}, {0,7,12,14},
{5,6,9,11},{1,10,13,15},{3,5,9,14},{0,2,10,13}, {5,6,8,14},{0,1,3,7},{2,6,8,13},
{5,7,9,10},{1,11,13,14}, {2,5,8,10},{3,11,12,13},{0,2,5,6},{3,7,8,9},{1,4,10,14},
{2,6,9,12},{3,4,5,11},{1,2,10,12},{0,3,9,11},{1,2,4,6}, {8,10,13,14},{3,5,7,12},
{0,6,13,14},{1,3,8,11},{0,5,10,14}, {3,4,7,13},{9,10,12,14},{4,5,7,15},{1,6,12,14},
{8,9,11,15}, {0,3,4,6},{7,12,13,15},{4,6,9,14},{5,11,12,15},{0,3,8,14}, {4,11,13,15},
{0,7,8,10},{1,9,14,15},{0,2,4,7},{3,6,8,12}, {0,1,11,15},{2,7,8,12},{0,4,11,14},
{1,7,8,15},{3,4,12,14}, {0,7,9,15},{4,6,11,12},{1,3,9,10},{8,11,12,14},{1,2,3,5},
{0,4,10,15},{1,6,7,9},{0,2,8,15},{4,7,10,12},{1,2,9,11}, {8,10,12,15},{2,3,9,13},
{0,6,8,11},{5,13,14,15}, {2,3,6,7},{1,5,10,11},{0,4,9,13},{10,11,14,15},{5,6,7,13},
{2,4,12,15},{9,10,11,13},{1,5,8,12},{7,9,13,14},{0,1,4,5}, {2,6,11,15},{3,7,10,14},
{2,5,11,13},{1,4,9,12},{2,3,14,15}, {0,5,8,13},{2,7,10,15},{0,1,12,13},{6,7,14,15},
{4,5,8,9}, {2,7,11,14},{1,5,6,15},{8,9,12,13},{3,6,11,14},{4,5,12,13}, {6,7,10,11},
{0,1,8,9},{3,6,10,15},{1,5,7,14},{6,9,13,15}, {2,3,10,11},{4,5,14,16},{2,12,13,17},
{0,1,10,16},{2,4,5,17}, {8,9,10,16},{0,1,6,17},{3,5,10,13},{6,8,9,17},{12,13,14,16},
{5,8,15,17},{1,4,7,16},{0,5,9,12},{1,4,8,13},{5,10,12,17}, {0,2,9,16},{4,10,13,17},
{1,2,8,16},{0,9,14,17},{4,6,13,16}, {1,8,14,17},{5,6,12,16},{0,13,15,17},{7,9,12,16},
{1,4,11,17}, {0,3,13,16},{9,11,12,17},{3,5,8,16},{0,2,11,17},{1,12,15,16}, {3,4,9,17},
{0,6,7,16},{1,3,12,17},{0,5,11,16},{7,8,13,17}, {4,9,15,16},{0,5,7,17},{6,8,15,16},
{0,3,10,17},{8,11,13,16}, {4,6,15,17},{7,8,14,16},{6,11,13,17},{3,10,12,16},{4,7,14,17},
{2,11,12,16},{9,10,15,17},{2,3,4,16},{8,10,11,17},{0,14,15,16}, {2,3,8,17},{7,10,13,16},
{5,11,14,17},{2,13,15,16}]. □

Lemma 4.3.2. *There exists a CSCU-GDD(g^u) for each $(g,u) \in \{(3,4), (4,4), (4,5), (6,5), (6,6)\}$.*

Proof. We only list two examples here. First, we list the sequence of a CSCU-GDD(3^4) on the point set $X = Z_{12}$ with the group set $\mathcal{G} = \{\{i, 4+i, 8+i\} \mid i \in Z_4\}$.

S= [{0,1,2,3},{4,5,6,7},{8,9,10,11},{0,1,6,7},{4,5,10,11},{8,9,2,3}, {0,1,10,11},
{4,5,2,3},{8,9,6,7},{0,5,10,3},{4,9,2,7},{8,1,6,11}, {0,5,2,7},{4,9,6,11},
{8,1,10,3},{0,5,6,11},{4,9,10,3},{8,1,2,7}, {0,9,6,3},{4,1,10,7},{8,5,2,11},
{0,9,10,7},{4,1,2,11},{8,5,6,3}, {0,9,2,11},{4,1,6,3},{8,5,10,7}].

Next, we list the sequence of a CSCU-GDD(4^4) on the point set $X = Z_{16}$ with the group set $\mathcal{G} = \{\{i, 4+i, 8+i, 12+i\} \mid i \in Z_4\}$.

S= [{0,1,2,3},{4,5,6,7},{8,9,10,11},{12,13,14,15},{0,1,6,7}, {4,5,10,11},{8,9,14,15}, {2,3,12,13},{0,1,10,11},{4,5,14,15}, {2,3,8,9},{6,7,12,13},{2,3,4,5},{0,1,14,15}, {6,7,8,9}, {10,11,12,13},{0,2,5,7},{4,6,9,11},{8,10,13,15},{1,3,12,14},{0,5,6,11}, {4,9,10,15},{3,8,13,14},{1,2,7,12},{0,5,10,15}, {3,4,9,14},{2,7,8,13},{1,6,11,12}, {2,4,7,9},{6,8,11,13}, {1,10,12,15},{0,3,5,14},{4,6,13,15},{0,2,9,11},{1,3,8,10}, {5,7,12,14},{0,6,9,15},{3,4,10,13},{1,7,8,14},{2,5,11,12}, {0,3,9,10},{4,7,13,14}, {1,2,8,11},{5,6,12,15},{2,4,11,13}, {1,6,8,15},{3,5,10,12},{0,7,9,14},{1,3,4,6}, {0,2,13,15}, {5,7,8,10},{9,11,12,14},{0,3,6,13},{1,4,7,10},{5,8,11,14},{0,7,10,13}, {5,9,12,15},{1,4,11,14},{2,5,8,15},{3,6,9,12}, {1,2,4,15},{0,11,13,14},{3,5,6,8}, {7,9,10,12}]. □

Lemma 4.3.3. *There exists a CSCU-HSQS*$(v : s)$ *for each* $(v, s) \in \{(16, 4), (20, 8), (22, 10), (26, 10)\}$.

Proof. Here we only list the sequence of a CSCU-HSQS$(16 : 4)$ on the point set $X = Z_{16}$ with the hole set $\{0, 1, 2, 3\}$.

S= [{3,4,11,12},{0,1,6,7},{8,9,10,11},{0,1,4,5},{2,3,6,7}, {8,9,12,13},{0,2,4,6}, {8,9,14,15},{0,2,5,7},{8,10,12,14}, {0,3,4,7},{8,10,13,15},{0,3,5,6},{8,11,12,15}, {4,5,6,7}, {8,11,13,14},{2,3,4,5},{12,13,14,15},{1,3,5,7},{10,11,14,15}, {1,3,4,6}, {10,11,12,13},{1,2,5,6},{9,11,13,15},{1,2,4,7}, {9,11,12,14},{0,1,10,15},{2,7,8,9}, {0,1,11,14},{2,7,10,15}, {0,1,12,13},{2,7,11,14},{9,10,12,15},{3,6,11,14},{0,1,8,9}, {3,6,10,15},{2,7,12,13},{4,5,10,15},{3,6,8,9},{0,2,12,15}, {4,5,8,9},{3,6,12,13}, {4,5,11,14},{1,3,12,15},{9,10,13,14}, {5,6,12,15},{0,2,8,10},{4,5,12,13},{0,2,9,11}, {1,3,8,10}, {0,2,13,14},{1,3,9,11},{4,7,8,10},{1,3,13,14},{4,7,9,11}, {5,6,8,10}, {2,4,9,13},{0,3,8,11},{4,7,13,14},{1,5,10,12}, {0,3,9,13},{4,7,12,15},{5,6,13,14}, {0,3,10,12},{2,4,8,11}, {0,3,14,15},{5,6,9,11},{2,4,10,12},{1,5,9,13},{0,4,8,12}, {1,5,14,15},{6,7,10,12},{2,4,14,15},{1,5,8,11},{6,7,9,13}, {0,4,10,14},{6,7,8,11}, {0,4,9,15},{2,6,11,13},{3,5,8,12}, {0,4,11,13},{6,7,14,15},{3,5,11,13},{2,6,10,14}, {3,5,9,15}, {1,7,8,12},{3,5,10,14},{2,6,8,12},{1,7,10,14},{2,6,9,15}, {0,5,8,13}, {1,7,9,15},{0,5,12,14},{4,6,11,15},{0,5,9,10}, {1,7,11,13},{4,6,12,14},{3,7,8,13}, {0,5,11,15},{4,6,8,13}, {3,7,11,15},{4,6,9,10},{3,7,12,14},{1,2,8,13},{3,7,9,10}, {1,2,12,14},{0,6,13,15},{1,2,9,10},{0,6,8,14},{1,2,11,15}, {0,6,9,12},{5,7,8,14}, {1,4,13,15},{0,6,10,11},{5,7,13,15}, {1,4,9,12},{5,7,10,11},{1,4,8,14},{5,7,9,12}, {2,3,8,14}, {1,4,10,11},{2,3,13,15},{0,7,9,14},{2,3,10,11},{0,7,8,15}, {1,6,9,14}, {0,7,11,12},{1,6,8,15},{2,5,10,13},{3,4,8,15}, {1,6,10,13},{2,3,9,12},{0,7,10,13}, {2,5,9,14},{3,4,10,13}, {2,5,11,12},{3,4,9,14},{1,6,11,12},{2,5,8,15}]. □

Lemma 4.3.4. *There exists a BSCU*(v) *for each* $v \in \{20, 22, 26, 34, 38\}$.

Proof. Here we only show the existence of a BSCU(20). Take $X = Z_{19} \cup \{\infty\}$ as the point set. Let

A= [{0,4,5,6},{2,7,12,14},{1,3,4,9},{0,8,14,17},{∞,1,7,9}, {5,11,18,2}, {0,1,3,7},{2,6,11,17}, {∞,1,10,13},{14,16,18,2},{7,15,0,6},{1,8,9,12}, {5,6,13,15}, {1,2,12,17},{∞,8,9,13}],

and $S = [A, A+1, A+2, \ldots, A+18]$, where additions are taken modulo 19. Then S is the required BSCU(20). □

4.4 Existence of BSCUs

First, we denote the set of all positive integers v such that an S(t, K, v) exists is denoted by $B_t(K)$.

Theorem 4.4.1. ([38]) $B_3(\{4,5,6\}) = \{v > 0 \mid v \equiv 0, 1, 2 \pmod 4$ and $v \neq 9, 13\}$.

Lemma 4.4.2. ([59]) There exists a BSCU(v) for $v \in \{14, 16, 32, 46, 56\}$.

Lemma 4.4.3. *If there exists a CSCU-GDD(g^n), then there exists a CSCU-GDD$((mg)^n)$ for any integer $m \geq 3$.*

Proof. Combining Lemma 4.3.2 with Theorem 4.2.6, we know that there exists a CSCU-GDD(m^4) for any $m \geq 3$.

Let $S = [b_0, \ldots, b_{q-1}]$ be the CSCU-GDD(g^n). For any $b_i \in S$, there is a CSCU-GDD(m^4), denoted S_i, on the point set $b_i \times I_m$ for any integer $m \geq 3$. Let $S' = [S_0, \ldots, S_{q-1}]$. Then it is easy to check that S' is a CSCU-GDD$((mg)^n)$. □

Lemma 4.4.4. *There exists a CSCU-CQS$(12^n : s)$ for each $s \in \{8, 10\}$ and $n \geq 4$.*

Proof. For each $n \equiv 0, 1 \pmod 3$ and $n \geq 4$, there exists an $S(3, 4, 2n+2)$. Deleting two points from this design yields a 2-FG$(3, (\{3\}, \{3\}, \{4\}), 2n)$ of type 2^n. By counting the numbers of blocks in the $S(3, 4, 2n+2)$ containing t, where $t = 0, 1, 2$, common points, we can know that in the 2-FG$(3, (\{3\}, \{3\}, \{4\}), 2n)$

of type 2^n, when $n \geq 4$, there exist two disjoint blocks with one of size 4 and the other of size 3. For each $s \in \{8, 10\}$, applying Theorem 4.2.3 with a CSCU-CQS$(6^3 : s-6)$ and a CSCU-GDD(6^4), we obtain a CSCU-CQS$(12^n : s)$. Here, the ingredient designs come from Theorem 4.2.6 and Lemma 4.3.1.

For any $n \equiv 2 \pmod 3$ and $n \geq 5$, there is a CQS$(6^{\frac{n+1}{3}} : 0)$ by Lemma 3.4.3. For $n = 5, 8, 11$, it can be checked from the detailed construction in [55] for each of these CQS$(6^{\frac{n+1}{3}} : 0)$ that there exist two disjoint blocks a and b intersecting two groups, say g_1 and g_2, in two points, respectively. So there are two points $y \in g_1$ and $z \in g_2$ not covered by a and b. Choose $x \in a \cap g_2$ and delete x, y. Then we obtain a 2-FG$(3, (\{3,5\}, \{3,5\}, \{4,6\}), 10)$ of type 2^n with two disjoint blocks $a \setminus \{x\} \in \mathcal{B}_1$ and $b \in \mathcal{T}$. For $n \geq 14$, let x, y be two points from different groups g_x, g_y, respectively, and g be a group disjoint from a block containing x, y. By deleting x and y, we obtain a 2-FG$(3, (\{3,5\}, \{3,5\}, \{4,6\}), 2n)$ of type 2^n with two disjoint blocks $g_x \setminus \{x\} \in \mathcal{B}_1$ and $g \in \mathcal{T}$. Then for each $s \in \{8, 10\}$, by applying Theorem 4.2.3 with a CSCU-CQS$(6^3 : s-6)$, a CSCU-CQS$(6^5 : s-6)$, a CSCU-GDD(6^4) and a CSCU-GDD(6^6), we obtain a CSCU-CQS$(12^n : s)$, where the ingredient designs come from Theorem 4.2.6, Lemmas 4.3.1 and 4.3.2. □

Lemma 4.4.5. *There exists a BSCU(v) for each $v \equiv 8, 10 \pmod{12}$ and $v \geq 12$.*

Proof. For each $v \in \{20, 22, 32, 34, 46\}$, there is a BSCU$(v)$ by Lemmas 4.3.4 and 4.4.2. For $v = 44$, there is a BSCU(v) by applying Theorem 4.1.1.(1) with a BSCU(22).

For each $v \equiv 8, 10 \pmod{12}$ and $v \geq 56$, there is a CSCU-CQS$(12^n : s)$ where $v = 12n + s$, $n \geq 4$ and $s \in \{8, 10\}$ by Lemma 4.4.4. Then by applying Theorem 4.2.2 with a CSCU-HSQS$(12+s : s)$, we obtain a CSCU-HSQS$(12n+s : 12+s)$, and furthermore, by applying Theorem 4.2.1 with a BSCU$(12+s)$, we obtain a BSCU$(12n+s)$, where the ingredient CSCU-HSQSs come from Lemma 4.3.3. □

Lemma 4.4.6. *There exists a CSCU-GDD(12^u) for each $u \in \{5, 6\}$.*

Proof. From Lemma 4.3.2, we know that there exists a CSCU-GDD(4^5). Applying Lemma 4.4.3 with $m = 3$, we obtain a CSCU-GDD(12^5).

From Theorem 1.1.1, we know that there exists an H(3^6). Applying Corollary 4.2.5 with a CSCU-GDD(4^4) from Lemma 4.3.2, we obtain a CSCU-GDD(12^6). □

Lemma 4.4.7. *There exists a CSCU-CQS*($12^n : s$) *for each* $n \in \{5, 8\}$ *and* $s \in \{2, 4\}$.

Proof. For each $n \in \{5, 8\}$, there is an S$(3, 5, 3n+2)$ in [28]. Deleting two points gives a 2-FG$(3, (\{4\}, \{4\}, \{5\}), 3n)$ of type 3^n, which is also a 1-FG$(3, (\{4\}, \{4, 5\}), 3n)$ of type 3^n. By counting the numbers of blocks in the $S(3, 5, 3n+2)$ containing t, where $t = 0, 1, 2$, common points, we can know that in the 2-FG$(3, (\{4\}, \{4\}, \{5\}), 3n)$ of type 3^n, when $n = 5, 8$, there exist two disjoint blocks with one of size 5 and the other of size 4. For each $s \in \{2, 4\}$, by applying Theorem 4.2.4 with a CSCU-CQS($4^4 : s$), a CSCU-GDD(4^4) and a CSCU-GDD(4^5), which come from Lemmas 4.3.1 and 4.3.2, we obtain a CSCU-CQS($12^n : s$). □

Lemma 4.4.8. *There exists a CSCU-CQS*($12^n : s$) *for* $s \in \{2, 4\}$ *and* $n \equiv 0, 1, 3 \pmod 4$*,* $n \geq 7$*,* $n \neq 8, 12$*.*

Proof. For each $n \equiv 0, 1, 3 \pmod 4$, $n \geq 7$ and $n \neq 8, 12$, there exists an $S(3, \{4, 5, 6\}, n+1)$ $(X, \mathcal{B})$ by Theorem 4.4.1. Let x, y be two points of X, and $b_1, b_2, \ldots, b_w$ be the blocks in $\mathcal{B}$ containing both x and y. Then $\{b_1 \setminus \{x, y\}, b_2 \setminus \{x, y\}, \ldots, b_w \setminus \{x, y\}\}$ is a partition of $X \setminus \{x, y\}$, and $2 \leq |b_i \setminus \{x, y\}| \leq 4$ for $i = 1, 2, \ldots, w$. Let $u \in b_1 \setminus \{x, y\}$, $v \in b_2 \setminus \{x, y\}$, and b be a block containing both u and v. If $w \geq 7$, which would happen if $n \geq 27$, then there must exist one $b_i \setminus \{x, y\}$, say $i = i_0$, which is disjoint with b. Deleting u from this 3-BD yields a 1-FG$(3, (\{3, 4, 5\}, \{4, 5, 6\}), n)$ of type 1^n with two disjoint blocks $b \setminus \{u\} \in \mathcal{B}_1$ and $b_{i_0} \in \mathcal{T}$. For each $n \equiv 1, 3 \pmod 6$, there exists an $S(3, 4, n+1)$. By counting the numbers of blocks in the $S(3, 4, n+1)$ containing t, where $t = 0, 1, 2$, common points, we can know that there exist two disjoint blocks b, b' when $n \geq 7$. Deleting one point $x \in b$ from this 3-BD yields a 1-FG$(3, (\{3\}, \{4\}), n)$ of type 1^n with two disjoint blocks $b \setminus \{x\} \in \mathcal{B}_1$ and $b' \in \mathcal{T}$. For $n = 16$, there exists an S$(3, 5, 17)$ from [28]. By the same method as above, we know that there exists a 1-FG$(3, (\{4\}, \{5\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. For $n = 20, 24$, there exist an S$(3, 6, 22)$ and an S$(3, 6, 26)$ from [28]. In a similar fashion, we can prove the existence of two disjoint blocks in each of these two Steiner systems. Deleting two points from one of these two disjoint blocks yields a 1-FG$(3, (\{4, 5\}, \{5, 6\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. For $n = 11, 17, 23$, just as in the proof of Lemma 4.4.4, we can know that there exist two disjoint blocks in the CQS$(6^{\frac{n+1}{6}} : 0)$. Deleting one point from one of these two disjoint blocks yields a 1-FG$(3, (\{3, 5\}, \{4, 6\}), n)$ of type 1^n with two disjoint blocks, one in $\mathcal{B}_1$ and the other in $\mathcal{T}$. Now for each $s \in \{2, 4\}$, by applying Theorem 4.2.3 with a CSCU-CQS$(12^h : s)$ and a CSCU-GDD(12^{h+1}) for each $h \in \{3, 4, 5\}$, we obtain a CSCU-CQS$(12^n : s)$. Here, the ingredient designs come from Theorem 4.2.6, Lemmas 4.3.1, 4.4.6 and 4.4.7. □

Lemma 4.4.9. *There exists a BSCU*$(12n+s)$ *for* $s \in \{2, 4\}$, $n \equiv 0, 1, 3 \pmod 4$, $n \geq 4$ *and* $n \neq 12$.

Proof. For each $n \equiv 0, 1, 3 \pmod 4$, $n \geq 4$ and $n \neq 12$, there exists a CSCU-CQS$(12^n : s)$ for $s \in \{2, 4\}$ by Lemmas 4.3.1, 4.4.7 and 4.4.8. Then by applying Theorem 4.2.2 with a CSCU-HSQS$(12 + s : s)$ and Theorem 4.2.1 with a BSCU$(12 + s)$, we obtain a BSCU$(12n + s)$. Here, the ingredient designs come from Theorem 4.1.1, Lemmas 4.3.3 and 4.4.2, where the BSCU(14) in Theorem 4.1.1 is actually a CSCU-HSQS$(12 + 2 : 2)$. □

Lemma 4.4.10. *There exists a BSCU*$(48n + 26)$ *for any* $n \geq 0$.

Proof. A BSCU(26) was shown in Lemma 4.3.4. For each integer $n \geq 1$, as was shown in the proof of Lemma 4.4.4, there exists a 2-FG$(3, (\{3\}, \{3\}, \{4\}), 2(3n+1))$ of type 2^{3n+1} with two disjoint blocks, one being of size 4 and the other of size 3. Applying Theorem 4.2.3 with a CSCU-CQS$(8^3 : 2)$ and a CSCU-GDD(8^4), we obtain a CSCU-CQS$(16^{3n+1} : 10)$. Then by applying Theorem 4.2.2 with a CSCU-HSQS$(26 : 10)$ and Theorem 4.2.1 with a BSCU(26), we obtain a BSCU$(48n + 26)$. Here, the ingredient designs come from Theorem 4.2.6, Lemmas 4.3.1, 4.3.3 and 4.3.4. □

Lemma 4.4.11. *There exists a BSCU*$(12n+s)$ *for* $n \in \{1, 3, 12\}$ *and* $s \in \{2, 4\}$.

Proof. For each $v \in \{14, 16, 38\}$, there is a BSCU(v) by Lemmas 4.3.4 and 4.4.2. For each $v \in \{40, 148\}$, there is a BSCU(v) by applying Theorem 4.1.1.(1) with a BSCU(u) for $u \in \{20, 74\}$ in Lemmas 4.3.4 and 4.4.10, respectively.

For $v = 146$, there exists an S$(3, 6, 26)$ in [28]. Deleting two points gives a 2-FG$(3, (\{5\}, \{5\}, \{6\}), 24)$ of type 4^6, which is also a 1-FG$(3, (\{5\}, \{5, 6\}), 24)$ of type 4^6. It can be easily shown that this 2-FG$(3, (\{5\}, \{5\}, \{6\}), 24)$ has two disjoint blocks with one of size 6 and the other of size 5. Applying Theorem 4.2.3 with a CSCU-CQS$(6^5 : 2)$, a CSCU-GDD(6^5) and a CSCU-GDD(6^6), we obtain a CSCU-CQS$(24^6 : 2)$. Then applying Theorem 4.2.2 with a CSCU-HSQS$(26 : 2)$ and Theorem 4.2.1 with a BSCU(26), we obtain a BSCU(146). Here, the ingredient designs come from Theorem 4.1.1, Lemmas 4.3.1 and 4.3.2, where the BSCU(26) in Theorem 4.1.1 is actually a CSCU-HSQS$(26 : 2)$. □

Lemma 4.4.12. *There exists a BSCU(v) for* $v \equiv 28 \pmod{48}$.

Proof. Combining Lemmas 4.4.9 and 4.4.11, we have the fact that there exists a BSCU$(12n + 2)$ for each $n \equiv 1 \pmod 2$. Then apply Theorem 4.1.1.(1). □

Combining Lemmas 4.4.5 and 4.4.9–4.4.12, we have the following conclusion.

Theorem 4.4.13. *The necessary conditions for the existence of a BSCU(v), namely,* $v \equiv 2, 4 \pmod 6$ *and* $v \geq 4$, *are also sufficient with two exceptions* $v = 8, 10$.

Chapter 5

Fault-Tolerant Routings with Levelled Minimum Optical Indices

The design of fault-tolerant routings with levelled minimum optical indices plays an important role in the context of optical networks. However, not much is known for the existence of optimal routings with levelled minimum optical indices besides the results established by Dinitz, Ling and Stinson via the partitionable Steiner quadruple systems approach. In this chapter, we introduce a new concept of a large set of even levelled $\overrightarrow{P_3}$-design of order v and index 2, denoted by $(v, \overrightarrow{P_3}, 2)$-LELD, which is equivalent to an optimal, levelled $(v-2)$-fault tolerant routing with levelled minimum optical indices of the complete network with v nodes. Based on the theory of 3-wise balanced designs and partitionable candelabra systems, several infinite classes of $(v, \overrightarrow{P_3}, 2)$-LELDs are constructed. As a consequence, the existence problem for optimal routings with levelled minimum optical indices is solved nearly one-third.

5.1 Introduction

The design of routings in optical networks has been a topic of considerable recent interest (see, for examples, [2, 5, 6, 7, 53]). In the model of *WDM optical networks*, namely, wavelength division multiplexing optical networks, routing nodes are joined by fiber-optic links, and each link can support some fixed number of wavelengths. Each routing path uses a particular wavelength, and two paths must use different wavelengths if they have common links. Most research concentrates on determining the minimum total number of wavelengths used in the network, which is related to two basic invariants – the *arc-forwarding* and *optical indices*. The *f-tolerant arc-forwarding* and *f-tolerant optical indices* were introduced by Maňuch and Stacho when they considered the fault-tolerant issues in [53]. The parameter f represents the number of faults that can be tolerated in the optical network. That is, we can provide a routing between any two nodes even if some number (up to f) of nodes and/or links fail. In this chapter, we focus on the fault-tolerant routings in the complete optical network.

We first review definitions of several desirable properties that we are going to investigate in the setting of fault-tolerant routings. These terms have previously been defined in papers such as [6, 21, 22].

Let $G = (V(G), A(G))$ be a *symmetric* directed graph, i.e., $(u, v) \in A(G)$ implies $(v, u) \in A(G)$. An *f-fault tolerant routing* is a set of directed paths in G, say $\mathcal{R}_f = \{P_i(u, v) : u \neq v, 0 \leq i \leq f\}$, where the following two properties are satisfied:

1. every $P_i(u, v)$ is a directed path in G from vertex u to vertex v, and
2. for all vertices u and v where $u \neq v$, the $f + 1$ paths $P_i(u, v)$ $(0 \leq i \leq f)$ are internally vertex disjoint.

For $0 \leq i \leq f$, define $\mathcal{L}_i = \{P_i(u, v) : u \neq v\}$, which is called the ith *level* of the routing. For convenience, we write $\mathcal{R}_f$ in the form $\mathcal{R}_f = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_f)$. It is clear that $\mathcal{R}_j = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_j)$ is a j-fault tolerant routing, for $0 \leq j \leq f$. Therefore an f-fault tolerant routing can be regarded as a sequence of j-fault tolerant routings for $0 \leq j \leq f$, namely, $(\mathcal{R}_0, \ldots, \mathcal{R}_f)$.

The *load* $\overrightarrow{\pi}(e)$ on an arc $e \in A(G)$ is defined to be the number of paths in the routing that contain the arc e. Define $\overrightarrow{\pi}(\mathcal{R}_f) = \max\{\overrightarrow{\pi}(e) : e \in A(G)\}$. Further, define $\overrightarrow{\pi}_f(G) = \min_{\mathcal{R}_f}\{\overrightarrow{\pi}(\mathcal{R}_f)\}$ and call $\overrightarrow{\pi}_f(G)$ the *f-fault tolerant arc-forwarding index* of G. The routing $\mathcal{R}_f$ is said to be *optimal* if $\overrightarrow{\pi}_f(G) = \overrightarrow{\pi}(\mathcal{R}_f)$, and to be *optimal, levelled* if $\overrightarrow{\pi}_j(G) = \overrightarrow{\pi}(\mathcal{R}_j)$, for all $0 \leq j \leq f$.

Let $n \geq 2$ be a positive integer and let $\overrightarrow{K_n}$ denote the complete symmetric directed graph on a set of n vertices, say X. By [15, 21, 22], we have

Theorem 5.1.1. *Suppose there is an f-fault tolerant routing of $\overrightarrow{K}_n$, say $\mathcal{R}_f = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_f)$. Then $0 \leq f \leq n - 2$ and $\overrightarrow{\pi}_f(\overrightarrow{K_n}) \geq 2f + 1$ for all f, $0 \leq f \leq n - 2$. Furthermore, equality is attained (i.e., the routing is an optimal, levelled routing) if and only if the following properties are satisfied:*

1. *$\mathcal{L}_0$ consists of all the arcs in $\overrightarrow{K}_n$ (that is, $\mathcal{L}_0$ comprises $n(n-1)$ directed paths, each having length one), and*

2. *for $1 \leq j \leq n-2$, $\mathcal{L}_j$ consists of $n(n-1)$ directed paths, each having length two, such that every arc in $\overrightarrow{K}_n$ is in exactly two directed paths in $\mathcal{L}_j$.*

The following theorem was proved in [15, 21, 22].

Theorem 5.1.2. [15, 21, 22] *For each integer $n \geq 2$, there exists an optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$.*

Let $\mathcal{W}$ be a set of wavelengths. A *wavelength assignment* to the directed paths in $\mathcal{R}_f$ is defined to be a map $\alpha : \mathcal{R}_f \to \mathcal{W}$ such that $\alpha(P) \neq \alpha(Q)$ whenever $P, Q \in \mathcal{R}_f$ are two directed paths that share a common arc. Let $\overrightarrow{w}(\mathcal{R}_f)$ denote the minimum cardinality of a set $\mathcal{W}$ such that an assignment of wavelengths for $\mathcal{R}_f$ exists that satisfies the previous property. Denote $\overrightarrow{w}_f(G) = \min_{\mathcal{R}_f}\{\overrightarrow{w}(\mathcal{R}_f)\}$ and call $\overrightarrow{w}_f(G)$ the *f-fault tolerant optical index* of G. It is obvious that $\overrightarrow{w}_f(G) \geq \overrightarrow{\pi}_f(G)$. An optimal, levelled f-fault tolerant routing $\mathcal{R}_f$ is said to have *minimum optical indices* if $\overrightarrow{w}(\mathcal{R}_i) = \overrightarrow{w}_i(G)$ for all i such that $0 \leq i \leq f$.

For $0 \leq i \leq f$, construct a graph whose vertices are the directed paths in $\mathcal{L}_i$. Two vertices are defined to be *adjacent* if they have a common arc. This graph is called the *path graph* of $\mathcal{L}_i$. In many applications, it could be desirable that wavelength assignments for $\mathcal{R}_{i-1}$ do not change when we determine wavelength assignments for $\mathcal{R}_i$. Under this assumption, it is easy to see that we require at most δ_i "extra" wavelengths when we proceed the assignment from $\mathcal{R}_{i-1}$ to $\mathcal{R}_i$, where δ_i is the chromatic number of the path graph of $\mathcal{L}_i$, for $0 \leq i \leq f$. Define $\overrightarrow{w}_L(\mathcal{R}_i) = \sum_{j=0}^{i} \delta_j$, $0 \leq i \leq f$. It is clear that $\overrightarrow{w}(\mathcal{R}_i) \leq \overrightarrow{w}_L(\mathcal{R}_i)$. An optimal, levelled f-fault tolerant routing $\mathcal{R}_f$ is said to have *levelled minimum optical indices* if $\overrightarrow{w}_L(\mathcal{R}_i) = \overrightarrow{w}_i(G)$ for all i such that $0 \leq i \leq f$. A routing having levelled minimum optical indices has minimum optical indices. The converse is not true. Here is a counterexample given in [15].

Example 5.1.3. [15] *The unique 1-fault tolerant routing of $\overrightarrow{K_3}$ has minimum optical indices, but it does not have levelled minimum optical indices.*

Proof. The unique 1-fault tolerant routing $\mathcal{R}_1$ of $\overrightarrow{K_3}$ on $X = \{0, 1, 2\}$ is as follows:

$$\mathcal{L}_0 : (0,1)^3, (0,2)^2, (1,0)^1, (1,2)^2, (2,0)^1, (2,1)^3$$
$$\mathcal{L}_1 : (0,2,1)^1, (0,1,2)^1, (1,2,0)^3, (1,0,2)^3, (2,1,0)^2, (2,0,1)^2$$

Here, the superscripts denote wavelengths. The wavelength assignment shows that $\overrightarrow{w}(\mathcal{R}_0) = 1$ and $\overrightarrow{w}(\mathcal{R}_1) = 3$, which is minimal.

It is clear that $\overrightarrow{w}_L(\mathcal{R}_0) = \overrightarrow{w}(\mathcal{R}_0) = 1$. However, when we proceed the assignment from $\mathcal{R}_0$ to $\mathcal{R}_1$, three "extra" wavelengths should be used. That is $\overrightarrow{w}_L(\mathcal{R}_1) = 4$. Therefore, the routing $\mathcal{R}_1$ does not have levelled minimum optical indices. □

The path graph of $\mathcal{L}_0$ contains no edges, so $\delta_0 = 1$. For each i, $1 \le i \le f$, the path graph of $\mathcal{L}_i$ is a union of disjoint cycles. It is straightforward that $\overrightarrow{w}_L(\mathcal{R}_i) \ge \overrightarrow{\pi}_i(\overrightarrow{K}_n) \ge 2i + 1$, equality holds when $\delta_i = 2$ for all $1 \le i \le f$, which happens if and only if all the cycles have even length. So we have

Theorem 5.1.4. *An optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$, say $\mathcal{R}_{n-2} = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{n-2})$ has levelled minimum optical indices if and only if the following property is satisfied:*

3. *The path graph of each $\mathcal{L}_i$ $(1 \le i \le n-2)$ has only even cycles.*

Based on the theory of 3-wise balanced designs and partitionable candelabra systems, Ji [39] gave a simple new proof for the existence of large sets of Steiner triple systems. In this chapter, via a similar approach we will concentrate on constructing optimal, levelled $(n-2)$-fault tolerant routings $\mathcal{R}_{n-2}$ with levelled minimum optical indices of the complete directed graph $\overrightarrow{K}_n$. The following results are known.

Theorem 5.1.5. [15] *For each n, $5 \le n \le 8$, $n = 4^k$ or $n = 2(p^k + 1)$ with $p \in \{7, 31\}$, there exists an optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$ that has levelled minimum optical indices.*

The chapter is organized as follows. In Section 5.2, we first define a new class of combinatorial objects, large sets of even levelled $(n, \overrightarrow{P_3}, 2)$-design (LELDs), which are equivalent to the optimal, levelled $(n-2)$-fault tolerant routings with levelled minimum optical indices. Then, we present a recursive construction for LELDs by using the theory of 3-wise balanced designs and partitionable candelabra systems. In Section 5.3, some small ingredient designs are constructed directly. Combining these ingredient designs together with the recursive methods established in Section 5.2, we are able to give several infinite classes of LELDs in Section 5.4, which imply the existence of the corresponding routings having levelled minimum optical indices.

5.2 Definitions and Recursive Constructions

Let $\overrightarrow{P_3} = (a, b, c)$ be a directed path which contains two arcs (a, b) and (b, c). Let $\lambda\overrightarrow{K_n}$ be the directed multigraph on n vertices in which each ordered pair of vertices is joined by λ arcs. A $\overrightarrow{P_3}$-*decomposition* of $\lambda\overrightarrow{K_n}$ is a partition of the arcs of $\lambda\overrightarrow{K_n}$ into paths isomorphic to $\overrightarrow{P_3}$, which is also called a $\overrightarrow{P_3}$-*design* of order n and index λ and denoted by $(n, \overrightarrow{P_3}, \lambda)$-*design*. A similar concept of P_3-decomposition of the undirected graph was given in [46]. If a set $\mathcal{B}$ of $\overrightarrow{P_3}$ paths contains exactly one path from u to v for every two vertices u, v of $\overrightarrow{K_n}$, then we call the set $\mathcal{B}$ a *level*. A level is said to be *even* if its path graph has only even cycles. An $(n, \overrightarrow{P_3}, 2)$-design is said to be *levelled* (*even levelled*) if it is a level (an even level), which is denoted by $(n, \overrightarrow{P_3}, 2)$-LD ($(n, \overrightarrow{P_3}, 2)$-ELD).

A *large set of* $(n, \overrightarrow{P_3}, 2)$-*LD*, denoted by $(n, \overrightarrow{P_3}, 2)$-LLD, is a partition $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ of all $\overrightarrow{P_3}$ paths in $\overrightarrow{K_n}$ such that each $\mathcal{B}_i$ forms an $(n, \overrightarrow{P_3}, 2)$-LD. If each $\mathcal{B}_i$ is even levelled, then we call the partition a *large set* of $(n, \overrightarrow{P_3}, 2)$-ELD, which is denoted by $(n, \overrightarrow{P_3}, 2)$-LELD.

As a consequence of Theorems 5.1.1 and 5.1.4, we have the following theorem.

Theorem 5.2.1. *Suppose that $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ form an $(n, \overrightarrow{P_3}, 2)$-LLD. Let $\mathcal{L}_0$ consist of all arcs in $\overrightarrow{K}_n$, $\mathcal{L}_i$ consist of all paths in $\mathcal{B}_i$, $1 \le i \le n-2$, then $\mathcal{R}_{n-2} = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_{n-2})$ is an optimal, levelled $(n-2)$-fault tolerant routing of $\overrightarrow{K}_n$. The reverse is also true. Furthermore, $\mathcal{R}_{n-2}$ has levelled minimum optical indices if and only if $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_{n-2}$ form an $(n, \overrightarrow{P_3}, 2)$-LELD.*

By Theorems 5.1.2 and 5.1.5, we have the following corollaries.

Corollary 5.2.2. *There exists an* $(n, \overrightarrow{P_3}, 2)$*-LLD for any integer* $n \geq 2$*.*

Corollary 5.2.3. *For each* n*,* $5 \leq n \leq 8$*,* $n = 4^k$ *or* $n = 2(p^k + 1)$ *with* $p \in \{7, 31\}$*, there exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD.*

In the remainder of this section, we will present a recursive construction for LELDs via partitionable candelabra systems having even levelled property. First, we give some notations and terminology. The interested reader may refer to [14] for the undefined terms as well as a general overview of design theory.

A *candelabra* $\overrightarrow{P_3}$*-system* of order n, denoted by $(n, \overrightarrow{P_3})$-CS, is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

1. X is the vertex set of $\overrightarrow{K_n}$;
2. S is a subset of X of size s;
3. $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X \setminus S$, which partition $X \setminus S$;
4. $\mathcal{A}$ consists of all $\overrightarrow{P_3}$ paths of $\overrightarrow{K_n}$ not contained in any subgraph spanned by $S \cup G$ for each $G \in \mathcal{G}$.

A *group divisible* $\overrightarrow{P_3}$*-design* of order n and index λ, denoted by $(n, \overrightarrow{P_3}, \lambda)$-GDD is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

1. X is the vertex set of $\overrightarrow{K_n}$;
2. $\mathcal{G} = \{G_1, G_2, \cdots\}$ is a set of nonempty subsets of X which partition X;
3. $\mathcal{B}$ is a family of $\overrightarrow{P_3}$ paths of $\overrightarrow{K_n}$ such that each path intersects any given group in at most one point;
4. each arc from two different groups is contained in exactly λ paths of $\mathcal{B}$.

By the *group type* of an $(n, \overrightarrow{P_3})$-CS and an $(n, \overrightarrow{P_3}, \lambda)$-GDD, we use the same notations as t-CS and GDD(t, K, v) respectively. An $(n, \overrightarrow{P_3}, 2)$-GDD $(X, \mathcal{G}, \mathcal{B})$ is called a *level*, denoted by $(n, \overrightarrow{P_3}, 2)$-LGDD, if $\mathcal{B}$ contains exactly one path from

u to v, for every two vertices u, v from two different groups. An $(n, \overrightarrow{P_3}, 2)$-GDD is called *even levelled*, denoted by $(n, \overrightarrow{P_3}, 2)$-ELGDD, if it is an even level, that is its path graph has only even cycles.

An $(n, \overrightarrow{P_3})$-CS of type $(g_1^{a_1} g_2^{a_2} \dots g_r^{a_r} : s)$ $(X, S, \mathcal{G}, \mathcal{A})$ with $s \geq 2$ is called *partitionable*, denoted by $(n, \overrightarrow{P_3})$-PCS, if the path set $\mathcal{A}$ can be partitioned into components $\mathcal{A}_x$ $(x \in G, G \in \mathcal{G})$ and $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_{s-2}$ with the following two properties: (i) for any $x \in G$ and $G \in \mathcal{G}$, $\mathcal{A}_x$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-GDD of type $1^{n-s-|G|}(|G|+s)^1$ with $G \cup S$ as the long group; (ii) for $1 \leq i \leq s-2$, $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is an $(n-s, \overrightarrow{P_3}, 2)$-GDD of type $g_1^{a_1} g_2^{a_2} \dots g_r^{a_r}$. If each component of an $(n, \overrightarrow{P_3})$-PCS is levelled (even levelled), then we denote it as $(n, \overrightarrow{P_3})$-LPCS ($(n, \overrightarrow{P_3})$-ELPCS).

In order to use an $(n, \overrightarrow{P_3})$-ELPCS to construct an $(n, \overrightarrow{P_3}, 2)$-LELD, we need a holey large set. Let X be an n-element set and Y be an s-subset of X with $s \geq 2$. Let $\overrightarrow{X}^{(3)}$ and $\overrightarrow{Y}^{(3)}$ denote all $\overrightarrow{P_3}$ paths in the complete symmetric directed graph on X and Y, respectively. A *holey large set of* $(n, \overrightarrow{P_3}, 2)$-*LD* on X with a hole Y, denoted by $(n, s; \overrightarrow{P_3}, 2)$-HLLD, is a partition of $\overrightarrow{X}^{(3)} \setminus \overrightarrow{Y}^{(3)}$ into components $\mathcal{A}_1$, $\mathcal{A}_2$, …, $\mathcal{A}_{n-2}$ with the properties that (1) for $1 \leq i \leq n-s$, each $(X, \mathcal{A}_i)$ is an $(n, \overrightarrow{P_3}, 2)$-LD; (2) for $n-s+1 \leq i \leq n-2$, each $\mathcal{A}_i$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-LGDD of type $1^{n-s}s^1$ with the long group Y. If each component of an $(n, s; \overrightarrow{P_3}, 2)$-HLLD is even, then we denote it by $(n, s; \overrightarrow{P_3}, 2)$-HLELD.

A *generalized* $\overrightarrow{P_3}$*-frame*, denoted by $\mathrm{F}(\overrightarrow{P_3}, v\{m\})$, is a collection of triples $\{(X, \mathcal{G}, \mathcal{B}_r) : r \in X\}$, where X is the vertex set of $\overrightarrow{K_{vm}}$, $\mathcal{G}$ is a partition of X into v sets of m points each, such that $(X \setminus G, \mathcal{G} \setminus \{G\}, \mathcal{B}_r)$ is a $((v-1)m, \overrightarrow{P_3}, 2)$-GDD of type m^{v-1} for each $r \in G$ and $G \in \mathcal{G}$, $\cup_{r \in X} \mathcal{B}_r$ consists of all the $\overrightarrow{P_3}$ paths intersecting every given group in at most one point, and all $\mathcal{B}_r, r \in X$ are pairwise disjoint. If each component of an $\mathrm{F}(\overrightarrow{P_3}, v\{m\})$ is levelled (even levelled), then we denote it by $\mathrm{LF}(\overrightarrow{P_3}, v\{m\})$ ($\mathrm{ELF}(\overrightarrow{P_3}, v\{m\})$).

Now, we give several recursive constructions for LELDs.

Theorem 5.2.4. *Suppose there exists an $H(g^n)$ and an $ELF(\overrightarrow{P_3}, 4\{m\})$, then there exists an $ELF(\overrightarrow{P_3}, n\{gm\})$.*

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be an $\mathrm{H}(g^n)$. Let $X' = X \times Z_m$ and $\mathcal{G}' = \{G' = G \times Z_m :$

$G \in \mathcal{G}\}$. We will construct an ELF$(\overrightarrow{P_3}, n\{gm\})$ on X' with group set $\mathcal{G}'$.

For each $B \in \mathcal{B}$, construct an ELF$(\overrightarrow{P_3}, 4\{m\})$ on $B \times Z_m$ with group set $\{x \times Z_m : x \in B\}$. Denote the path set by $\mathcal{A}_B$ which can be partitioned into $4m$ subsets $\mathcal{A}_B(x,i)$, $(x,i) \in B \times Z_m$, such that each $\mathcal{A}_B(x,i)$ is a $(3m, \overrightarrow{P_3}, 2)$-ELGDD of type m^3 on $(B \setminus \{x\}) \times Z_m$ with group set $\{y \times Z_m : y \in B \setminus \{x\}\}$.

For each $x \in X$ and $i \in Z_m$, let $\mathcal{C}(x,i) = \bigcup_{x \in B \in \mathcal{B}} \mathcal{A}_B(x,i)$. It is easy to check that $\mathcal{C}(x,i)$ is a $(gm(n-1), \overrightarrow{P_3}, 2)$-ELGDD of type $(gm)^{n-1}$ with group set $\mathcal{G}' \setminus \{G' : x \in G\}$. In fact, since every two distinct blocks B and B' from the set $\{B : x \in B \in \mathcal{B}\}$ have at most one common point besides x, every two paths from $\mathcal{A}_B(x,i)$ and $\mathcal{A}_{B'}(x,i)$ respectively have no common arc. Then by the definition of path graph, $\mathcal{C}(x,i)$ is even.

Since all $\mathcal{C}(x,i)$ with $x \in X$ and $i \in Z_m$ are disjoint, they form an ELF$(\overrightarrow{P_3}, n\{gm\})$. □

Theorem 5.2.5. *Suppose there exists an $(n, \overrightarrow{P_3})$-ELPCS of type $(g_0^1 g_1^{a_1} g_2^{a_2} \ldots g_r^{a_r} : s)$ with $n = \sum_{1 \le i \le r} a_i g_i + g_0 + s$. If there is a $(g_i+s, s; \overrightarrow{P_3}, 2)$-HLELD for $1 \le i \le r$, then there is an $(n, g_0+s; \overrightarrow{P_3}, 2)$-HLELD. Furthermore, if a $(g_0+s, \overrightarrow{P_3}, 2)$-LELD exists, then there is an $(n, \overrightarrow{P_3}, 2)$-LELD.*

Proof. Let $(X, S, \mathcal{G}, \mathcal{A})$ be the given $(n, \overrightarrow{P_3})$-ELPCS of type $(g_0^1 g_1^{a_1} g_2^{a_2} \ldots g_r^{a_r} : s)$. By the definition, $\mathcal{A}$ can be partitioned into subsets $\mathcal{A}_y$ ($y \in G$ and $G \in \mathcal{G}$) and $\mathcal{A}_i$ ($1 \le i \le s-2$) with the properties that each $\mathcal{A}_y$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{n-|G|-s}(|G|+s)^1$ with the long group $G \cup S$ and that each $(X \setminus S, \mathcal{G}, \mathcal{A}_i)$ is an $(n-s, \overrightarrow{P_3}, 2)$-ELGDD of type $g_0^1 g_1^{a_1} g_2^{a_2} \cdots g_r^{a_r}$.

Let G_0 be the special group with $|G_0| = g_0$. For each $G \in \mathcal{G}$ with $G \ne G_0$, suppose the given $(|G|+s, s; \overrightarrow{P_3}, 2)$-HLELD consists of $|G|$ $(|G|+s, \overrightarrow{P_3}, 2)$-ELDs with path sets $\mathcal{B}_y$ ($y \in G$) and $s-2$ $(|G|+s, \overrightarrow{P_3}, 2)$-ELGDDs of type $1^{|G|}s^1$ with the long group S and path sets $\mathcal{B}_i^G$ ($1 \le i \le s-2$).

For each $y \in G$, $G \in \mathcal{G}$ with $G \ne G_0$, let $\mathcal{C}_y = \mathcal{A}_y \bigcup \mathcal{B}_y$. Since the path graph of $\mathcal{C}_y$ is the disjoint union of path graphs of $\mathcal{A}_y$ and $\mathcal{B}_y$, each $(X, \mathcal{C}_y)$ is an $(n, \overrightarrow{P_3}, 2)$-ELD. For $1 \le i \le s-2$, let $\mathcal{C}_i = \mathcal{A}_i \bigcup (\bigcup_{G \in \mathcal{G}, G \ne G_0} \mathcal{B}_i^G)$. It is easy to check that the path graph of $\mathcal{C}_i$ is also the disjoint union of path graphs of all its components. Then each $\mathcal{C}_i$ is the path set of an $(n, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{n-g_0-s}(g_0+s)^1$ with the long group $G_0 \cup S$. So $\{\mathcal{C}_y : y \in G \in \mathcal{G}, G \neq G_0\} \bigcup \{\mathcal{A}_y : y \in G_0\} \bigcup \{\mathcal{C}_i : 1 \leq i \leq s-2\}$ forms an $(n, g_0+s; \overrightarrow{P_3}, 2)$-HLELD.

Finally, suppose the given $(g_0+s, \overrightarrow{P_3}, 2)$-LELD on $G_0 \bigcup S$ has g_0+s-2 disjoint $(g_0+s, \overrightarrow{P_3}, 2)$-ELDs with path sets $\mathcal{B}_y$ ($y \in G_0$) and $\mathcal{B}_i$ ($1 \leq i \leq s-2$), respectively. Then the $(X, \mathcal{A}_y \cup \mathcal{B}_y)$ and the $(X, \mathcal{C}_i \bigcup \mathcal{B}_i)$ are all $(n, \overrightarrow{P_3}, 2)$-ELDs, and these $n-2$ ELDs form an $(n, \overrightarrow{P_3}, 2)$-LELD. $\square$

Theorem 5.2.6. *Suppose that there exists an e-FG$(3, (K_1, K_2, \ldots, K_e, K_T), \sum_{i=1}^{r} a_i g_i)$ of type $g_1^{a_1} g_2^{a_2} \ldots g_r^{a_r}$. If there is an $(mk_1+t, \overrightarrow{P_3})$-ELPCS of type $(m^{k_1} : t)$ for each $k_1 \in K_1$, an ELF$(\overrightarrow{P_3}, (k_i+1)\{m\})$ for each $k_i \in K_i$, $2 \leq i \leq e$, and an ELF$(\overrightarrow{P_3}, k\{m\})$ for each $k \in K_T$, then there is an $(m\sum_{i=1}^{r} a_i g_i + t + (e-1)m, \overrightarrow{P_3})$-ELPCS of type $((mg_1)^{a_1}(mg_2)^{a_2} \ldots (mg_r)^{a_r} : t+(e-1)m)$.*

Proof. Let $(X, \mathcal{G}, \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_e, \mathcal{A}_T)$ be an e-FG$(3, (K_1, K_2, \ldots, K_e, K_T), \sum_{i=1}^{r} a_i g_i)$ of type $g_1^{a_1} g_2^{a_2} \ldots g_r^{a_r}$. Let $S = \{\infty\} \times Z_s$, where $s = t + (e-1)m$. We shall construct the desired design on $X' = (X \times Z_m) \cup S$ with the group set $\mathcal{G}' = \{G' = G \times Z_m : G \in \mathcal{G}\}$ and the stem S, where $(X \times Z_m) \cap S = \emptyset$.

Denote $G_x = \{x\} \times Z_m$ for $x \in X$ and $G_A = \{G_x : x \in A\}$ for any subset A of X. Denote $S_1 = \{\infty\} \times Z_t$ and $S_i = \{\infty\} \times \{t+(i-2)m, t+(i-2)m+1, \ldots, t+(i-1)m-1\}$ for $2 \leq i \leq e$.

For each block $A \in \mathcal{A}_1$, construct an $(m|A|+t, \overrightarrow{P_3})$-ELPCS of type $(m^{|A|} : t)$ on $(A \times Z_m) \cup S_1$ having group set G_A and stem S_1. Denote its path set by $\mathcal{D}_A$. By the definition, $\mathcal{D}_A$ can be partitioned into subsets $\mathcal{D}_A(x, i)$ ($(x, i) \in A \times Z_m$) and $\mathcal{D}_A(j)$ ($2 \leq j \leq t-1$) with the properties that each $\mathcal{D}_A(x, i)$ is the path set of an $(m|A|+t, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{m(|A|-1)}(m+t)^1$ with the long group $G_x \cup S_1$ and that each $(A \times Z_m, G_A, \mathcal{D}_A(j))$ is an $(m|A|, \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|}$.

For each block $A \in \mathcal{A}_i$, $2 \leq i \leq e$, construct an ELF$(\overrightarrow{P_3}, (|A|+1)\{m\})$ on $(A \times Z_m) \cup S_i$ having group set $G_A \cup \{S_i\}$. Denote its path set by $\mathcal{C}_A$. By the definition, $\mathcal{C}_A$ can be partitioned into subsets $\mathcal{C}_A(x, i)$ ($(x, i) \in (A \times Z_m) \cup S_i$) with the property that each $\mathcal{C}_A(x, i)$ is the path set of an $(m|A|, \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|}$ with the group set G_A when $x=\infty$ or $(G_A \cup \{S_i\}) \setminus \{G_x\}$ when $x \in A$.

For each block $A \in \mathcal{A}_T$, construct an ELF$(\overrightarrow{P_3}, |A|\{m\})$ on $A \times Z_m$ having group set G_A. Denote its path set by $\mathcal{B}_A$. By the definition, $\mathcal{B}_A$ can be partitioned into subsets $\mathcal{B}_A(x,i)$ ($(x,i) \in A \times Z_m$) with the property that each $\mathcal{B}_A(x,i)$ is the path set of an $(m(|A|-1), \overrightarrow{P_3}, 2)$-ELGDD of type $m^{|A|-1}$ with the group set $G_A \setminus \{G_x\}$.

For any $x \in X$ and $i \in Z_m$, let

$$\mathcal{F}(x,i) = \big(\bigcup_{x\in A\in\mathcal{A}_1} \mathcal{D}_A(x,i)\big)\bigcup\big(\bigcup_{x\in A\in\mathcal{A}_i, 2\le i\le e} \mathcal{C}_A(x,i)\big)\bigcup\big(\bigcup_{x\in A\in\mathcal{A}_T} \mathcal{B}_A(x,i)\big).$$

For any $2 \le i \le t-1$, let

$$\mathcal{F}(\infty,i) = \bigcup_{A\in\mathcal{A}_1} \mathcal{D}_A(\infty,i).$$

For any $t+(j-2)m \le i \le t+(j-1)m-1$, $2 \le j \le e$, let

$$\mathcal{F}(\infty,i) = \bigcup_{A\in\mathcal{A}_j} \mathcal{C}_A(\infty,i).$$

Let

$$\mathcal{F} = \big(\bigcup_{x\in X, i\in Z_m} \mathcal{F}(x,i)\big)\bigcup\big(\bigcup_{2\le i\le s-1} \mathcal{F}(\infty,i)\big).$$

For each $x \in G$ and $i \in Z_m$, $\mathcal{F}(x,i)$ is the path set of an $(m\sum_{i=1}^r a_ig_i + t + (e-1)m, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{m(\sum_{i=1}^r a_ig_i - |G|)}(m|G|+t+(e-1)m)^1$ with the long group $G' \cup S$. Each $(X', \mathcal{G}', \mathcal{F}(\infty,i))$ is an $(m\sum_{i=1}^r a_ig_i, \overrightarrow{P_3}, 2)$-ELGDD of type $(mg_1)^{a_1}(mg_2)^{a_2}\ldots(mg_r)^{a_r}$. So they form an $(m\sum_{i=1}^r a_ig_i + t + (e-1)m, \overrightarrow{P_3})$-ELPCS of type $((mg_1)^{a_1}(mg_2)^{a_2}\ldots(mg_r)^{a_r} : t+(e-1)m)$. $\square$

5.3 Direct Constructions

Lemma 5.3.1. *There does not exist a $(3h, \overrightarrow{P_3}, 2)$-ELGDD of type h^3 for any odd integer $h > 0$.*

Proof. Suppose that there exists a $(3h, \overrightarrow{P_3}, 2)$-ELGDD of type h^3, then we can construct such a design $(X, \mathcal{B})$ on Z_{3h} with group set $\{\{i, i+3, \ldots, i+3(h-1)\} : 0 \le i \le 2\}$, where $|\mathcal{B}| = 6h^2$. From the definition, we know that the three vertices in each path are from distinct groups, i.e., distinct modulo 3. For each $B = (x, y, z) \in \mathcal{B}$, let $\hat{B} \equiv (x, y, z) \pmod 3$ be the path restricted to Z_3. Let $A = \{B \in \mathcal{B} | \hat{B} \in \{(0,1,2), (1,2,0), (2,0,1)\}\}$ and $A' = \{B \in \mathcal{B} | \hat{B} \in \{(0,2,1), (2,1,0), (1,0,2)\}\}$. Then it is easy to check that $\mathcal{B}$ is the disjoint union of A and A'. Since any two paths coming from A and A' respectively have no common arc, they can not be in the same cycle in the path graph of $\mathcal{B}$. But $|A| = |A'| = 3h^2$ is odd, neither A nor A' can be partitioned into even cycles only, which leads to a contradiction. $\square$

By Lemma 5.3.1, we have the following corollaries.

Corollary 5.3.2. *Let $h > 0$ be an odd integer and $s \ge 3$. There does not exist an ELF$(\overrightarrow{P_3}, 4\{h\})$ and a $(3h+s, \overrightarrow{P_3})$-ELPCS of type $(h^3 : s)$.*

By an exhaustive computer search, we have

Lemma 5.3.3. *There does not exist an $(8, \overrightarrow{P_3})$-ELPCS of type $(2^3 : 2)$.*

Lemma 5.3.4. *There exist both a $(9, \overrightarrow{P_3}, 2)$-LELD and a $(10, \overrightarrow{P_3}, 2)$-LELD.*

Proof. We construct the design on Z_n for each $n \in \{9, 10\}$. We list the paths of the initial $(n, \overrightarrow{P_3}, 2)$-ELD, which will be developed under the automorphism group $G = \langle (0\ 1\ 2\ \ldots\ n-3)(n-2)(n-1) \rangle$.

$n = 9:$	$(0,2,1)$	$(1,2,0)$	$(2,3,0)$	$(3,8,0)$	$(4,8,0)$	$(5,4,0)$	$(6,5,0)$	$(7,6,0)$	$(8,7,0)$
	$(0,1,2)$	$(1,4,2)$	$(2,6,1)$	$(3,7,1)$	$(4,5,1)$	$(5,7,1)$	$(6,8,1)$	$(7,5,1)$	$(8,6,1)$
	$(0,1,3)$	$(1,4,3)$	$(2,8,3)$	$(3,8,2)$	$(4,6,2)$	$(5,7,2)$	$(6,5,2)$	$(7,3,2)$	$(8,4,2)$
	$(0,2,4)$	$(1,3,4)$	$(2,7,4)$	$(3,7,4)$	$(4,6,3)$	$(5,2,3)$	$(6,4,3)$	$(7,8,3)$	$(8,6,3)$
	$(0,3,5)$	$(1,0,5)$	$(2,0,5)$	$(3,1,5)$	$(4,1,5)$	$(5,3,4)$	$(6,8,4)$	$(7,6,4)$	$(8,5,4)$
	$(0,3,6)$	$(1,0,6)$	$(2,5,6)$	$(3,0,6)$	$(4,1,6)$	$(5,3,6)$	$(6,7,5)$	$(7,3,5)$	$(8,2,5)$
	$(0,4,7)$	$(1,6,7)$	$(2,4,7)$	$(3,2,7)$	$(4,0,7)$	$(5,8,7)$	$(6,0,7)$	$(7,2,6)$	$(8,5,6)$
	$(0,4,8)$	$(1,7,8)$	$(2,1,8)$	$(3,1,8)$	$(4,5,8)$	$(5,0,8)$	$(6,2,8)$	$(7,0,8)$	$(8,1,7)$

It is readily checked that the path graph consists of a 72-cycle.

$n = 10:$

$(0,2,1)$	$(1,2,0)$	$(2,3,0)$	$(3,7,0)$	$(4,7,0)$	$(5,4,0)$	$(6,4,0)$	$(7,5,0)$	$(8,6,0)$	$(9,8,0)$
$(0,1,2)$	$(1,4,2)$	$(2,4,1)$	$(3,9,1)$	$(4,6,1)$	$(5,6,1)$	$(6,9,1)$	$(7,8,1)$	$(8,5,1)$	$(9,7,1)$
$(0,1,3)$	$(1,4,3)$	$(2,7,3)$	$(3,9,2)$	$(4,6,2)$	$(5,3,2)$	$(6,8,2)$	$(7,5,2)$	$(8,9,2)$	$(9,6,2)$
$(0,2,4)$	$(1,3,4)$	$(2,6,4)$	$(3,8,4)$	$(4,2,3)$	$(5,8,3)$	$(6,7,3)$	$(7,9,3)$	$(8,6,3)$	$(9,5,3)$
$(0,3,5)$	$(1,0,5)$	$(2,8,5)$	$(3,2,5)$	$(4,9,5)$	$(5,8,4)$	$(6,5,4)$	$(7,9,4)$	$(8,3,4)$	$(9,7,4)$
$(0,3,6)$	$(1,0,6)$	$(2,0,6)$	$(3,7,6)$	$(4,9,6)$	$(5,1,6)$	$(6,3,5)$	$(7,4,5)$	$(8,2,5)$	$(9,4,5)$
$(0,4,7)$	$(1,5,7)$	$(2,8,7)$	$(3,0,7)$	$(4,1,7)$	$(5,2,7)$	$(6,5,7)$	$(7,2,6)$	$(8,7,6)$	$(9,3,6)$
$(0,4,8)$	$(1,6,8)$	$(2,9,8)$	$(3,1,8)$	$(4,3,8)$	$(5,0,8)$	$(6,7,8)$	$(7,1,8)$	$(8,1,7)$	$(9,0,7)$
$(0,5,9)$	$(1,5,9)$	$(2,1,9)$	$(3,1,9)$	$(4,8,9)$	$(5,6,9)$	$(6,0,9)$	$(7,2,9)$	$(8,0,9)$	$(9,0,8)$

It is readily checked that the path graph consists of a 28-cycle and a 62-cycle. □

Lemma 5.3.5. *There exists an ELF($\overrightarrow{P_3}, 4\{2\}$).*

Proof. We construct the design on Z_8 with group set $\{\{i, i+4\} : 0 \le i \le 3\}$. We first construct below an initial $(6, \overrightarrow{P_3}, 2)$-ELGDD of type 2^3 on the group set $\{\{i, i+4\} : 1 \le i \le 3\}$ with the path graph consisting of four 6-cycles.

$(2,1,7)$	$(2,3,1)$	$(2,5,3)$	$(2,7,5)$	$(3,1,6)$	$(3,2,1)$	$(3,5,2)$	$(3,6,5)$
$(6,3,5)$	$(6,5,7)$	$(1,2,3)$	$(6,7,1)$	$(7,2,5)$	$(7,5,6)$	$(7,6,1)$	$(1,6,7)$
$(5,2,7)$	$(7,1,2)$	$(5,6,3)$	$(1,7,2)$	$(5,3,2)$	$(5,7,6)$	$(6,1,3)$	$(1,3,6)$

Developing the above paths under the automorphism group $G = \langle \pi : i \to i+1 \rangle$, we get eight $(6, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form an ELF($\overrightarrow{P_3}, 4\{2\}$). □

Lemma 5.3.6. *There exists an ELF($\overrightarrow{P_3}, 5\{4\}$).*

Proof. We construct the design on Z_{20} with group set $\{\{i, i+5, i+10, i+15\} : 0 \le i \le 4\}$. We list the path set of an initial $(16, \overrightarrow{P_3}, 2)$-ELGDD of type 4^4 on the group set $\{\{i, i+5, i+10, i+15\} : 1 \le i \le 4\}$ with a multiplicative automorphism group $G' = \langle (0)(1\ 3\ 9\ 7)(2\ 6\ 18\ 14)(4\ 12\ 16\ 8)(5\ 15)(10)(11\ 13\ 19\ 17) \rangle$.

$(19,12,8)$	$(13,2,6)$	$(9,16,17)$	$(9,8,7)$	$(6,19,8)$	$(6,13,7)$	$(1,14,2)$
$(13,7,14)$	$(16,18,4)$	$(16,7,9)$	$(1,19,17)$	$(4,12,11)$	$(16,3,7)$	$(14,17,1)$
$(13,6,4)$	$(6,4,17)$	$(19,8,1)$	$(13,6,17)$	$(17,6,9)$	$(6,3,12)$	$(11,19,13)$
$(6,8,2)$	$(6,19,13)$	$(8,17,11)$	$(1,17,8)$	$(18,2,9)$	$(17,16,8)$	$(9,17,8)$
$(8,6,14)$	$(7,1,13)$	$(4,18,7)$	$(14,8,16)$	$(18,1,14)$	$(8,2,19)$	$(1,8,14)$
$(4,11,18)$	$(1,7,4)$	$(19,3,6)$	$(8,16,4)$	$(11,3,17)$	$(7,3,9)$	$(14,8,11)$
$(12,3,4)$	$(7,19,6)$	$(2,14,18)$	$(1,18,9)$	$(4,7,6)$	$(19,11,7)$	

The path graph of this ELGDD consists of one 188-cycle and one 4-cycle. Developing the initial ELGDD under the automorphism group $G = \langle \pi : i \to i+1 \rangle$, we get twenty $(16, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form an ELF($\overrightarrow{P_3}, 5\{4\}$). □

Lemma 5.3.7. *There exists an ELF*$(\overrightarrow{P_3}, 6\{4\})$.

Proof. We construct the design on Z_{24} with group set $\{\{i, i+6, i+12, i+18\} : 0 \le i \le 5\}$. We list the path set of an initial $(20, \overrightarrow{P_3}, 2)$-ELGDD of type 4^5 on the group set $\{\{i, i+6, i+12, i+18\} : 1 \le i \le 5\}$ with a multiplicative automorphism group $G' = \langle \eta : i \to 17i \rangle$.

$(15,5,20)$	$(19,3,22)$	$(11,13,21)$	$(11,7,20)$	$(14,3,7)$	$(9,14,22)$	$(10,17,19)$
$(9,14,7)$	$(5,19,14)$	$(15,4,19)$	$(8,16,21)$	$(15,11,16)$	$(9,23,19)$	$(7,21,22)$
$(11,19,22)$	$(2,3,16)$	$(3,16,23)$	$(9,4,13)$	$(21,5,22)$	$(21,2,13)$	$(21,19,20)$
$(14,17,22)$	$(19,17,15)$	$(13,15,16)$	$(3,19,14)$	$(20,3,22)$	$(20,10,15)$	$(15,20,22)$
$(16,15,23)$	$(11,20,1)$	$(11,21,10)$	$(20,23,16)$	$(14,5,16)$	$(10,14,21)$	$(15,14,17)$
$(8,19,23)$	$(7,3,17)$	$(10,13,23)$	$(13,21,20)$	$(7,16,8)$	$(23,22,20)$	$(3,19,16)$
$(8,1,4)$	$(9,11,8)$	$(1,16,20)$	$(9,13,17)$	$(16,14,19)$	$(7,23,21)$	$(10,19,9)$
$(1,5,8)$	$(14,9,10)$	$(17,20,7)$	$(17,2,13)$	$(20,19,11)$	$(20,9,7)$	$(15,16,2)$
$(11,13,4)$	$(21,8,23)$	$(3,13,20)$	$(20,11,3)$	$(10,23,7)$	$(16,17,13)$	$(22,17,3)$
$(21,7,10)$	$(14,16,1)$	$(1,20,15)$	$(8,19,15)$	$(8,22,19)$	$(21,1,16)$	$(19,4,11)$
$(21,23,1)$	$(4,1,21)$	$(13,22,2)$	$(13,14,23)$	$(23,15,19)$	$(16,2,3)$	$(11,9,16)$
$(11,16,13)$	$(15,5,13)$	$(14,3,23)$	$(16,19,2)$	$(7,22,20)$	$(17,21,22)$	$(10,23,13)$
$(2,21,4)$	$(23,2,10)$	$(9,1,10)$	$(11,21,8)$	$(4,7,11)$	$(1,9,16)$	$(16,3,17)$
$(13,11,14)$	$(17,14,19)$	$(1,8,4)$	$(10,9,5)$	$(1,23,22)$	$(3,2,11)$	$(8,9,17)$
$(13,10,8)$	$(7,22,15)$	$(15,17,7)$	$(3,10,1)$	$(16,7,9)$	$(11,4,3)$	$(21,17,19)$
$(3,4,5)$	$(22,23,20)$	$(10,7,17)$	$(7,5,14)$	$(1,8,21)$	$(4,1,2)$	$(4,14,13)$
$(14,16,5)$	$(22,7,9)$	$(1,17,3)$	$(22,2,21)$	$(14,9,11)$	$(13,20,21)$	$(22,14,5)$
$(19,10,9)$	$(4,20,1)$	$(19,2,23)$	$(1,20,9)$	$(19,15,10)$	$(3,4,2)$	$(1,15,2)$
$(2,4,15)$	$(22,21,11)$	$(10,11,2)$	$(4,8,17)$	$(8,13,16)$	$(23,10,8)$	$(13,3,5)$
$(10,13,3)$	$(22,15,1)$	$(8,15,22)$	$(10,14,11)$	$(9,20,4)$	$(14,13,15)$	$(5,10,20)$
$(7,16,9)$	$(8,5,13)$	$(20,21,13)$	$(5,7,3)$	$(23,10,2)$	$(13,2,15)$	$(7,20,5)$
$(17,14,1)$	$(10,11,1)$	$(5,16,19)$	$(2,17,22)$	$(5,2,1)$	$(7,8,3)$	$(4,23,20)$
$(23,15,7)$	$(4,5,9)$	$(1,17,10)$	$(5,1,9)$	$(5,9,2)$	$(20,16,23)$	

The path graph of this ELGDD consists of two 152-cycles, two 6-cycles and one 4-cycle. Developing the initial ELGDD under the automorphism group $G = \langle \pi : i \to i+1 \rangle$, we get twenty four $(20, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form an ELF$(\overrightarrow{P_3}, 6\{4\})$. □

Lemma 5.3.8. *There exists an* $(11, \overrightarrow{P_3})$*-ELPCS of type* $(3^3 : 2)$.

Proof. We construct the design on Z_{11} with the group set $\{\{i, i+3, i+6\} : 0 \le i \le 2\}$ and the stem $\{9, 10\}$. We first construct an initial $(11, \overrightarrow{P_3}, 2)$-ELGDD of type $1^6 5^1$ with the long group $\{0, 3, 6, 9, 10\}$ and the following path set.

$(9,2,4)$	$(4,3,2)$	$(0,2,8)$	$(5,9,4)$	$(0,5,2)$	$(0,8,5)$	$(1,2,0)$	$(1,4,3)$	$(3,1,4)$
$(7,9,2)$	$(2,8,3)$	$(10,7,2)$	$(7,8,10)$	$(7,0,1)$	$(7,0,4)$	$(2,1,6)$	$(5,6,7)$	$(7,4,6)$
$(2,6,5)$	$(10,2,7)$	$(4,9,5)$	$(1,3,5)$	$(8,4,6)$	$(1,0,7)$	$(8,4,9)$	$(6,7,4)$	$(5,1,10)$
$(0,5,4)$	$(7,10,5)$	$(5,7,9)$	$(6,4,1)$	$(6,2,5)$	$(4,1,6)$	$(10,4,5)$	$(6,8,2)$	$(4,0,7)$
$(2,5,0)$	$(9,8,7)$	$(2,9,7)$	$(2,3,8)$	$(8,7,10)$	$(8,10,7)$	$(6,5,8)$	$(7,3,8)$	$(4,7,3)$
$(5,8,0)$	$(3,7,5)$	$(5,3,2)$	$(8,5,3)$	$(0,2,7)$	$(4,0,1)$	$(9,5,1)$	$(2,4,10)$	$(3,4,8)$
$(3,1,2)$	$(1,0,4)$	$(1,8,9)$	$(1,5,10)$	$(3,7,1)$	$(8,2,0)$	$(5,10,1)$	$(10,1,8)$	$(7,5,0)$
$(1,9,8)$	$(1,10,2)$	$(7,1,3)$	$(4,10,8)$	$(8,9,1)$	$(3,4,7)$	$(8,3,5)$	$(2,1,9)$	$(10,8,1)$
$(8,6,4)$	$(6,1,7)$	$(2,6,1)$	$(0,8,1)$	$(4,5,9)$	$(2,10,4)$	$(5,2,3)$	$(5,7,6)$	$(8,6,2)$
$(7,2,9)$	$(9,7,8)$	$(1,7,6)$	$(4,8,0)$	$(5,6,8)$	$(9,1,5)$	$(9,4,2)$	$(10,5,4)$	$(4,2,10)$

It is readily checked that the path graph consists of four 6-cycles and one 66-cycle. Developing the paths under the automorphism group $G = \langle(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8)(9)(10)\rangle$, we get nine $(11, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form an $(11, \overrightarrow{P_3})$-ELPCS of type $(3^3 : 2)$. □

Lemma 5.3.9. *There exists a* $(14, \overrightarrow{P_3})$*-ELPCS of type* $(4^3 : 2)$*.*

Proof. We construct the design on Z_{14} with the group set $\{\{i, i+3, i+6, i+9\} : 0 \le i \le 2\}$ and the stem $\{12, 13\}$. We list below the path sets of two initial $(14, \overrightarrow{P_3}, 2)$-ELGDDs of type $1^8 6^1$ with the long group $\{0, 3, 6, 9, 12, 13\}$, both of which have an automorphism group $G' = \langle(0)(1\ 5)(2\ 10)(3)(4\ 8)(6)(7\ 11)(9)(12)(13)\rangle$.
The first initial ELGDD with the path graph consisting of two 74-cycles and one 4-cycle:

$(1,13,8)$	$(0,2,10)$	$(8,0,4)$	$(8,11,9)$	$(5,12,7)$	$(6,1,7)$	$(10,9,8)$
$(2,4,10)$	$(7,1,0)$	$(11,10,13)$	$(13,5,1)$	$(9,2,1)$	$(3,11,8)$	$(12,2,7)$
$(3,4,10)$	$(9,4,11)$	$(11,13,7)$	$(7,12,2)$	$(1,11,13)$	$(2,0,5)$	$(4,2,3)$
$(0,4,1)$	$(5,3,8)$	$(7,1,9)$	$(1,6,7)$	$(11,12,4)$	$(13,10,2)$	$(5,7,6)$
$(11,1,8)$	$(7,5,3)$	$(1,10,3)$	$(12,4,8)$	$(8,11,0)$	$(4,13,2)$	$(2,13,1)$
$(7,3,10)$	$(5,10,0)$	$(4,8,6)$	$(5,4,12)$	$(8,1,13)$	$(13,4,11)$	$(8,0,11)$
$(0,5,11)$	$(5,8,1)$	$(5,2,9)$	$(2,7,11)$	$(2,8,6)$	$(5,0,2)$	$(3,10,1)$
$(9,11,2)$	$(7,8,12)$	$(2,6,8)$	$(11,10,6)$	$(4,9,11)$	$(10,8,13)$	$(2,12,7)$
$(6,10,4)$	$(1,6,2)$	$(11,9,1)$	$(2,5,9)$	$(3,11,7)$	$(12,1,5)$	$(8,3,1)$
$(8,9,2)$	$(6,7,10)$	$(2,11,0)$	$(4,2,12)$	$(8,3,5)$	$(13,11,4)$	$(12,1,2)$
$(0,11,8)$	$(7,6,1)$	$(6,8,5)$	$(10,5,12)$	$(9,5,8)$	$(2,11,3)$	

The second initial ELGDD with the path graph consisting of two 76-cycles:

$(7,11,13)$	$(9,4,1)$	$(10,2,12)$	$(4,8,1)$	$(10,12,8)$	$(1,8,9)$	$(8,12,7)$

(4, 7, 6)	(8, 3, 2)	(6, 7, 2)	(11, 5, 3)	(1, 9, 7)	(10, 13, 2)	(4, 9, 2)
(11, 2, 9)	(11, 7, 12)	(9, 1, 2)	(0, 5, 2)	(7, 13, 8)	(5, 12, 1)	(11, 2, 6)
(6, 5, 8)	(8, 4, 13)	(11, 9, 5)	(1, 6, 10)	(5, 0, 10)	(7, 4, 0)	(9, 2, 11)
(11, 1, 7)	(3, 10, 4)	(4, 3, 7)	(8, 1, 12)	(12, 11, 10)	(10, 4, 0)	(2, 3, 5)
(8, 13, 1)	(7, 3, 4)	(6, 1, 5)	(11, 0, 1)	(2, 10, 1)	(3, 5, 2)	(0, 4, 11)
(0, 7, 8)	(1, 4, 6)	(11, 6, 2)	(12, 10, 5)	(4, 10, 3)	(6, 4, 11)	(10, 6, 11)
(8, 5, 0)	(3, 4, 7)	(12, 2, 4)	(13, 7, 5)	(5, 10, 13)	(10, 5, 13)	(5, 7, 12)
(9, 11, 4)	(4, 12, 8)	(5, 11, 0)	(13, 4, 2)	(12, 5, 7)	(3, 11, 5)	(13, 5, 4)
(5, 1, 3)	(7, 4, 2)	(1, 13, 11)	(1, 9, 8)	(0, 2, 5)	(5, 6, 8)	(10, 0, 4)
(8, 2, 9)	(10, 0, 7)	(10, 7, 9)	(2, 4, 6)	(2, 7, 3)	(13, 2, 7)	

Let $G = \langle (0\ 2\ 4\ 6\ 8\ 10)(1\ 3\ 5\ 7\ 9\ 11)(12)(13) \rangle$. Developing the above two initial designs under the automorphism group G, we get twelve $(14, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form a $(14, \overrightarrow{P_3})$-ELPCS of type $(4^3 : 2)$. □

Lemma 5.3.10. *There exists an* $(18, \overrightarrow{P_3})$*-ELPCS of type* $(4^4 : 2)$.

Proof. We construct the design on Z_{18} with the group set $\{\{i, i+4, i+8, i+12\} : 0 \le i \le 3\}$ and the stem $\{16, 17\}$. Let

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15)(16)(17) \rangle, \text{ and } G' = \langle \eta : i \to 7i \rangle.$$

We list below the path set of an initial $(18, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}6^1$ on Z_{18} with the long group $\{0, 4, 8, 12, 16, 17\}$ and the automorphism group G', where the path graph consists of one 264-cycle and three 4-cycles.

(6, 12, 1)	(1, 5, 3)	(6, 4, 14)	(1, 13, 14)	(5, 14, 17)	(8, 14, 6)	(0, 10, 6)
(6, 15, 2)	(8, 5, 13)	(15, 6, 3)	(3, 2, 8)	(4, 15, 2)	(13, 11, 12)	(9, 16, 10)
(15, 10, 1)	(2, 11, 6)	(13, 4, 9)	(15, 0, 13)	(10, 2, 12)	(15, 9, 8)	(4, 2, 15)
(8, 15, 3)	(0, 1, 3)	(11, 17, 1)	(4, 2, 9)	(7, 3, 14)	(13, 3, 10)	(4, 9, 3)
(5, 17, 6)	(3, 0, 14)	(16, 2, 15)	(2, 17, 13)	(7, 10, 8)	(4, 11, 13)	(17, 5, 7)
(17, 2, 3)	(0, 3, 13)	(13, 7, 16)	(5, 13, 14)	(3, 4, 11)	(6, 13, 7)	(10, 17, 15)
(7, 13, 12)	(10, 3, 9)	(14, 0, 3)	(8, 6, 1)	(6, 0, 13)	(12, 9, 13)	(3, 5, 1)
(2, 8, 9)	(1, 14, 16)	(3, 16, 13)	(17, 3, 13)	(2, 1, 10)	(10, 1, 8)	(11, 4, 5)
(13, 8, 1)	(9, 2, 16)	(4, 3, 6)	(3, 1, 16)	(9, 17, 2)	(1, 17, 15)	(9, 16, 14)
(13, 0, 2)	(16, 6, 11)	(5, 8, 3)	(4, 13, 1)	(8, 6, 14)	(16, 1, 2)	(1, 12, 6)
(11, 3, 0)	(9, 11, 15)	(9, 5, 12)	(5, 1, 0)	(3, 9, 15)	(6, 2, 0)	(0, 15, 14)
(6, 3, 17)	(16, 7, 6)	(9, 0, 6)	(13, 2, 6)	(4, 10, 7)	(0, 15, 1)	(14, 11, 7)
(4, 6, 5)	(17, 13, 6)	(6, 16, 5)	(14, 2, 13)	(1, 3, 12)	(11, 10, 8)	(1, 8, 11)
(5, 6, 10)	(14, 10, 0)	(6, 11, 16)	(14, 1, 17)	(2, 9, 8)	(13, 2, 5)	(2, 4, 14)
(9, 4, 3)	(14, 3, 12)	(7, 11, 0)	(15, 6, 17)	(10, 4, 6)	(6, 2, 12)	(17, 6, 9)

$(14,5,16)$	$(11,9,12)$	$(13,15,17)$	$(2,10,12)$	$(14,13,5)$	$(7,1,6)$	$(16,9,3)$
$(15,10,11)$	$(5,2,4)$	$(1,11,17)$	$(2,14,7)$	$(9,1,0)$	$(3,15,4)$	$(5,8,1)$
$(12,11,6)$	$(1,12,7)$	$(10,16,5)$	$(13,8,14)$	$(6,15,11)$	$(17,1,14)$	$(8,13,9)$
$(0,7,15)$	$(1,7,9)$	$(4,7,14)$	$(1,4,5)$	$(9,1,4)$	$(3,6,9)$	$(9,4,1)$
$(14,1,9)$	$(11,16,13)$	$(11,5,9)$	$(16,9,7)$	$(1,15,13)$		

Developing the above initial design under the automorphism group G, we get sixteen $(18,\overrightarrow{P_3},2)$-ELGDDs all together, which form an $(18,\overrightarrow{P_3})$-ELPCS of type $(4^4:2)$. $\square$

Lemma 5.3.11. *There exists a* $(22,\overrightarrow{P_3})$*-ELPCS of type* $(4^5:2)$.

Proof. We construct the design on Z_{22} with the group set $\{\{i,i+5,i+10,i+15\}:0\le i\le 4\}$ and the stem $\{20,21\}$. Let

$$G=\langle(0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19)(20)(21)\rangle, \text{and}$$
$$G'=\langle(0)(1\ 3\ 9\ 7)(2\ 6\ 18\ 14)(4\ 12\ 16\ 8)(5\ 15)(10)(11\ 13\ 19\ 17)(20)(21)\rangle.$$

We list below the path set of an initial $(22,\overrightarrow{P_3},2)$-ELGDD of type $1^{16}6^1$ on Z_{22} with the long group $\{0,5,10,15,20,21\}$ and the automorphism group G', where the path graph consists of one 180-cycle, one 88-cycle, two 60-cycles, one 40-cycle and one 4-cycle.

$(5,11,12)$	$(6,10,13)$	$(1,14,16)$	$(0,3,13)$	$(1,16,12)$	$(7,0,19)$	$(7,8,17)$
$(3,18,17)$	$(18,8,0)$	$(6,0,4)$	$(18,17,4)$	$(19,12,5)$	$(0,16,7)$	$(19,20,12)$
$(7,21,14)$	$(10,9,4)$	$(2,1,5)$	$(8,7,17)$	$(5,1,3)$	$(12,7,14)$	$(7,10,2)$
$(7,6,0)$	$(15,11,6)$	$(7,3,6)$	$(19,21,1)$	$(18,5,16)$	$(21,2,3)$	$(10,12,9)$
$(21,13,17)$	$(9,3,1)$	$(6,7,2)$	$(4,10,16)$	$(10,3,19)$	$(12,8,18)$	$(1,19,20)$
$(11,0,13)$	$(8,20,14)$	$(14,17,19)$	$(4,21,12)$	$(5,6,11)$	$(0,6,4)$	$(16,6,5)$
$(6,9,17)$	$(9,6,21)$	$(13,9,21)$	$(3,5,4)$	$(18,21,9)$	$(12,7,4)$	$(17,2,4)$
$(17,9,2)$	$(16,14,9)$	$(20,8,11)$	$(11,8,4)$	$(19,16,2)$	$(1,8,4)$	$(5,3,4)$
$(11,16,0)$	$(4,10,19)$	$(15,4,2)$	$(16,18,3)$	$(19,11,10)$	$(20,13,7)$	$(8,6,3)$
$(17,7,5)$	$(4,8,13)$	$(4,15,14)$	$(17,12,1)$	$(11,15,19)$	$(7,8,15)$	$(9,13,15)$
$(8,13,19)$	$(16,3,0)$	$(1,5,14)$	$(8,7,10)$	$(10,2,6)$	$(7,3,9)$	$(8,12,20)$
$(21,13,12)$	$(17,0,9)$	$(15,6,11)$	$(2,20,9)$	$(18,1,20)$	$(18,8,15)$	$(11,13,14)$
$(19,15,4)$	$(7,20,11)$	$(11,17,1)$	$(18,11,2)$	$(14,18,10)$	$(4,13,15)$	$(7,19,10)$
$(6,17,18)$	$(2,14,12)$	$(20,1,12)$	$(12,11,9)$	$(14,15,1)$	$(0,14,2)$	$(15,13,3)$
$(7,13,1)$	$(11,18,2)$	$(18,15,3)$	$(20,6,14)$	$(21,4,6)$	$(17,14,20)$	$(18,16,21)$
$(2,16,13)$	$(4,11,21)$	$(17,18,19)$				

Developing the above initial design under the automorphism group G, we get twenty $(22, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form a $(22, \overrightarrow{P_3})$-ELPCS of type $(4^5 : 2)$. $\square$

Lemma 5.3.12. *There exists a* $(20, \overrightarrow{P_3})$*-ELPCS of type* $(6^3 : 2)$.

Proof. We construct the design on Z_{20} with the group set $\{\{i, i+3, \ldots, i+15\} : 0 \leq i \leq 2\}$ and the stem $\{18, 19\}$. Let

$$\begin{aligned} G = \langle &(0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19), \\ &(0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)\rangle, \text{and} \\ G' = \langle &(0)(1\ 7\ 13)(2\ 14\ 8)(3)(4\ 10\ 16)(5\ 17\ 11)(6)(9)(12)(15)(18)(19)\rangle. \end{aligned}$$

We list below the path set of a $(20, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}8^1$ on Z_{20} with the long group $\{0, 3, 6, 9, 12, 15, 18, 19\}$ and the automorphism group G'. The path graph consists of one 54-cycle, one 264-cycle and one 6-cycle.

$(7,2,19)$	$(10,3,16)$	$(2,6,14)$	$(4,16,15)$	$(3,13,1)$	$(17,13,19)$	$(7,17,11)$
$(8,0,5)$	$(5,0,17)$	$(15,11,16)$	$(11,12,14)$	$(3,4,8)$	$(8,4,17)$	$(3,8,5)$
$(4,0,16)$	$(0,8,10)$	$(17,18,4)$	$(4,12,17)$	$(4,10,0)$	$(0,2,5)$	$(4,19,11)$
$(1,7,0)$	$(1,5,2)$	$(15,16,13)$	$(5,14,10)$	$(13,5,18)$	$(16,10,12)$	$(7,15,13)$
$(4,1,3)$	$(6,17,7)$	$(6,8,2)$	$(13,7,3)$	$(4,18,8)$	$(5,16,18)$	$(2,6,4)$
$(8,11,0)$	$(12,4,7)$	$(14,12,7)$	$(7,12,8)$	$(11,9,17)$	$(9,14,2)$	$(0,1,14)$
$(1,16,11)$	$(17,5,6)$	$(13,4,9)$	$(8,1,19)$	$(13,18,11)$	$(11,19,7)$	$(2,7,18)$
$(17,6,7)$	$(1,2,8)$	$(3,7,10)$	$(19,17,16)$	$(17,11,3)$	$(9,7,5)$	$(2,11,15)$
$(12,13,4)$	$(16,19,2)$	$(7,6,10)$	$(2,11,12)$	$(14,1,13)$	$(18,1,2)$	$(17,2,0)$
$(14,15,16)$	$(16,8,19)$	$(2,10,16)$	$(0,16,7)$	$(10,5,9)$	$(17,19,10)$	$(14,11,3)$
$(5,1,12)$	$(18,17,1)$	$(7,9,16)$	$(15,13,5)$	$(9,2,13)$	$(13,14,12)$	$(6,1,16)$
$(2,3,8)$	$(13,0,7)$	$(12,17,14)$	$(10,2,18)$	$(19,1,11)$	$(10,3,17)$	$(1,14,15)$
$(9,11,16)$	$(5,13,15)$	$(4,9,13)$	$(10,8,13)$	$(10,15,14)$	$(6,11,17)$	$(16,17,13)$
$(2,18,13)$	$(19,4,14)$	$(5,15,2)$	$(11,14,9)$	$(11,10,1)$	$(13,16,6)$	$(14,3,17)$
$(18,14,4)$	$(11,4,2)$	$(4,7,6)$	$(14,8,9)$	$(8,16,6)$	$(18,4,17)$	$(1,9,16)$
$(12,4,5)$	$(19,2,1)$	$(15,5,14)$				

Developing the above initial design under the automorphism group G, we get eighteen $(20, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form a $(20, \overrightarrow{P_3})$-ELPCS of type $(6^3 : 2)$. $\square$

Lemma 5.3.13. *There exists a* $(32, \overrightarrow{P_3})$*-ELPCS of type* $(6^5 : 2)$.

Proof. We construct the design on Z_{32} with the group set $\{\{i, i+5, \ldots, i+25\} : 0 \le i \le 4\}$ and the stem $\{30, 31\}$. Let

$$G = \langle (0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 22\ 23\ 24\ 25\ 26\ 27\ 28\ 29)(30)(31)\rangle, \text{and}$$

$$G' = \langle (0)(1\ 7\ 19\ 13)(2\ 14\ 8\ 26)(3\ 21\ 27\ 9)(4\ 28\ 16\ 22)(5)(6\ 12\ 24\ 18)(10)(11\ 17\ 29\ 23)(15)(20)(25)(30)(31)\rangle.$$

We list below the path set of a $(32, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{24}8^1$ on Z_{32} with the long group $\{0, 5, 10, 15, 20, 25, 30, 31\}$ and the automorphism group G'. The path graph consists of one 752-cycle, four 36-cycles, two 18-cycles and one 4-cycle.

$(24,30,28)$	$(28,17,14)$	$(19,24,25)$	$(12,7,13)$	$(24,8,10)$	$(15,13,23)$	$(29,17,12)$
$(26,31,29)$	$(27,8,2)$	$(2,13,27)$	$(6,13,25)$	$(16,5,7)$	$(19,4,6)$	$(11,23,26)$
$(28,30,11)$	$(16,9,27)$	$(22,17,13)$	$(8,22,20)$	$(31,21,13)$	$(17,29,11)$	$(21,23,30)$
$(9,29,10)$	$(12,2,18)$	$(18,7,23)$	$(16,26,12)$	$(10,22,26)$	$(27,18,4)$	$(23,28,16)$
$(28,30,16)$	$(5,13,23)$	$(25,3,7)$	$(23,22,25)$	$(0,22,12)$	$(28,29,4)$	$(23,13,7)$
$(2,24,15)$	$(18,11,28)$	$(18,14,9)$	$(5,28,18)$	$(17,12,19)$	$(7,10,19)$	$(15,17,8)$
$(6,4,27)$	$(16,24,21)$	$(6,25,29)$	$(21,30,8)$	$(20,7,2)$	$(16,15,24)$	$(2,4,1)$
$(5,8,9)$	$(12,22,1)$	$(3,16,30)$	$(21,16,25)$	$(15,16,9)$	$(24,17,2)$	$(26,9,19)$
$(12,21,15)$	$(26,25,11)$	$(18,1,24)$	$(19,8,26)$	$(24,5,21)$	$(8,3,26)$	$(22,19,2)$
$(20,8,11)$	$(17,20,16)$	$(14,26,5)$	$(10,13,19)$	$(15,23,13)$	$(0,14,7)$	$(9,13,24)$
$(19,23,1)$	$(3,19,4)$	$(21,14,24)$	$(20,2,12)$	$(13,30,22)$	$(28,26,20)$	$(31,22,9)$
$(14,29,0)$	$(3,16,19)$	$(12,20,27)$	$(31,17,18)$	$(15,24,16)$	$(22,28,11)$	$(22,9,10)$
$(29,7,31)$	$(20,12,1)$	$(0,6,11)$	$(1,25,8)$	$(4,8,11)$	$(3,5,29)$	$(19,0,28)$
$(16,7,1)$	$(30,6,23)$	$(17,30,1)$	$(7,14,22)$	$(29,22,16)$	$(25,28,14)$	$(15,21,6)$
$(2,20,18)$	$(17,3,15)$	$(2,5,28)$	$(7,5,17)$	$(11,15,2)$	$(26,31,12)$	$(4,30,27)$
$(3,11,26)$	$(13,31,27)$	$(11,12,6)$	$(28,13,22)$	$(23,24,14)$	$(11,21,10)$	$(29,23,9)$
$(18,27,2)$	$(12,22,16)$	$(17,19,20)$	$(11,3,30)$	$(13,2,17)$	$(5,12,14)$	$(28,15,2)$
$(1,22,0)$	$(2,19,22)$	$(21,4,20)$	$(8,15,1)$	$(4,7,15)$	$(11,22,0)$	$(5,8,28)$
$(3,17,13)$	$(2,14,25)$	$(17,3,8)$	$(0,2,21)$	$(8,0,27)$	$(30,18,12)$	$(27,25,3)$
$(22,24,3)$	$(18,10,8)$	$(24,28,23)$	$(10,18,23)$	$(29,26,19)$	$(19,31,8)$	$(21,6,0)$
$(5,6,1)$	$(7,10,29)$	$(19,18,30)$	$(4,26,30)$	$(4,9,8)$	$(1,23,15)$	$(25,12,23)$
$(16,23,0)$	$(0,21,16)$	$(17,23,24)$	$(12,24,30)$	$(27,23,19)$	$(8,1,14)$	$(4,27,31)$
$(19,0,7)$	$(23,25,12)$	$(23,5,21)$	$(26,29,10)$	$(20,21,28)$	$(3,29,5)$	$(27,26,23)$
$(31,1,17)$	$(23,22,27)$	$(12,4,5)$	$(1,26,18)$	$(29,3,27)$	$(8,15,16)$	$(3,27,24)$
$(3,12,15)$	$(8,17,31)$	$(9,12,27)$	$(30,7,4)$	$(11,8,16)$	$(30,12,8)$	$(27,13,22)$
$(31,13,1)$	$(24,3,13)$	$(11,24,5)$	$(19,3,22)$	$(14,2,26)$	$(18,9,13)$	$(19,27,9)$
$(31,11,8)$	$(1,21,12)$	$(26,9,3)$	$(25,16,4)$	$(0,6,14)$	$(4,19,5)$	$(21,25,14)$
$(24,2,0)$	$(25,1,12)$	$(14,3,24)$	$(9,29,18)$	$(7,18,10)$	$(1,9,5)$	$(13,1,18)$
$(23,3,11)$	$(30,8,21)$	$(1,8,31)$	$(3,20,22)$	$(25,13,3)$	$(10,4,18)$	$(7,12,20)$

$(18,24,6)$	$(10,21,3)$	$(2,10,9)$	$(24,23,16)$	$(30,26,28)$	$(8,12,13)$	$(14,19,17)$
$(13,15,21)$	$(24,22,8)$	$(10,2,28)$	$(2,22,6)$	$(7,20,11)$	$(27,16,17)$	$(19,11,27)$
$(27,30,9)$	$(3,0,11)$	$(8,24,11)$	$(4,10,29)$	$(18,0,29)$	$(12,24,31)$	$(22,2,1)$
$(20,1,3)$	$(22,23,25)$	$(8,29,30)$	$(7,9,2)$	$(9,0,1)$	$(11,20,29)$	$(26,1,28)$
$(16,10,18)$	$(4,28,24)$	$(18,3,20)$				

Developing the above initial design under the automorphism group G, we get thirty $(32, \overrightarrow{P_3}, 2)$-ELGDDs all together, which form a $(32, \overrightarrow{P_3})$-ELPCS of type $(6^5 : 2)$. □

Lemma 5.3.14. *There exists a* $(23, \overrightarrow{P_3})$*-ELPCS of type* $(6^3 : 5)$.

Proof. We construct the design on Z_{23} with group set $\{\{i, i+3, \ldots, i+15\} : 0 \leq i \leq 2\}$ and stem $\{18, 19, 20, 21, 22\}$. Let

$$
\begin{aligned}
G_1 &= \langle (0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19)(20)(21)(22)\rangle,\\
G_2 &= \langle (0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16)(1\ 3\ 5\ 7\ 9\ 11\ 13\ 15\ 17)(18)(19)(20)(21)(22),\\
&\qquad (0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)(20)(21)(22)\rangle,\\
G' &= \langle (0)(1\ 7\ 13)(2\ 14\ 8)(3)(4\ 10\ 16)(5\ 17\ 11)(6)(9)(12)(15)(18)(19)(20)(21)\\
&\qquad (22)\rangle, \text{and}\\
G'' &= \langle (0)(1\ 5\ 7\ 17\ 13\ 11)(2\ 10\ 14\ 16\ 8\ 4)(3\ 15)(6\ 12)(9)(18)(19)(20)(21)(22),\\
&\qquad (0\ 6\ 12)(1\ 7\ 13)(2\ 8\ 14)(3\ 9\ 15)(4\ 10\ 16)(5\ 11\ 17)(18)(19)(20)(21)(22)\rangle.
\end{aligned}
$$

We list below the path set of an $(18, \overrightarrow{P_3}, 2)$-ELGDD of type 6^3 on Z_{18} with group set $\{\{i, i+3, \ldots, i+15\} : 0 \leq i \leq 2\}$ and the automorphism group G''. The path graph consists of six 36-cycles.

$(4,9,5)$	$(17,16,3)$	$(10,8,6)$	$(9,10,17)$	$(9,8,16)$	$(0,5,16)$	$(11,3,13)$
$(6,1,17)$	$(13,17,0)$	$(14,0,16)$	$(5,6,10)$	$(14,7,3)$		

Then, we list below the path set of a $(23, \overrightarrow{P_3}, 2)$-ELGDD of type $1^{12}11^1$ on Z_{23} with the long group $\{0, 3, 6, 9, 12, 15, 18, 19, 20, 21, 22\}$ and the automorphism group G'. The path graph consists of one 138-cycle, three 84-cycles and one 6-cycle.

$(21,7,8)$	$(6,7,1)$	$(16,22,11)$	$(19,8,10)$	$(1,7,12)$	$(10,16,6)$	$(13,5,22)$
$(16,3,7)$	$(7,4,9)$	$(19,1,11)$	$(11,10,22)$	$(20,16,17)$	$(19,7,2)$	$(2,6,11)$
$(7,12,17)$	$(18,4,2)$	$(14,7,22)$	$(6,17,8)$	$(10,21,5)$	$(18,10,17)$	$(14,22,10)$
$(2,0,5)$	$(14,9,2)$	$(4,14,22)$	$(15,11,17)$	$(20,5,4)$	$(4,12,16)$	$(5,16,12)$
$(8,15,10)$	$(1,10,0)$	$(16,21,2)$	$(10,20,2)$	$(17,20,1)$	$(22,2,1)$	$(8,1,18)$
$(16,19,5)$	$(16,17,20)$	$(2,12,13)$	$(4,16,3)$	$(4,17,12)$	$(17,10,18)$	$(22,2,4)$
$(5,14,15)$	$(0,1,7)$	$(6,1,4)$	$(14,2,12)$	$(2,4,20)$	$(17,0,11)$	$(11,18,1)$
$(22,1,17)$	$(20,5,13)$	$(16,15,13)$	$(1,0,4)$	$(21,16,5)$	$(14,1,21)$	$(0,2,14)$
$(5,11,6)$	$(14,8,9)$	$(2,10,15)$	$(2,7,19)$	$(18,17,16)$	$(2,6,14)$	$(2,18,1)$
$(9,5,2)$	$(16,9,4)$	$(14,5,0)$	$(11,3,8)$	$(1,15,8)$	$(1,19,17)$	$(9,4,1)$
$(5,15,8)$	$(13,7,3)$	$(21,11,7)$	$(15,10,4)$	$(2,19,10)$	$(19,8,7)$	$(5,2,3)$
$(7,14,20)$	$(2,20,7)$	$(1,6,10)$	$(5,1,20)$	$(4,0,7)$	$(5,9,14)$	$(17,13,21)$
$(15,13,2)$	$(21,14,16)$	$(1,18,14)$	$(11,5,9)$	$(6,8,5)$	$(12,10,1)$	$(2,17,6)$
$(11,14,0)$	$(12,2,8)$	$(9,1,10)$	$(9,5,17)$	$(11,19,16)$	$(3,17,2)$	$(5,10,19)$
$(12,8,11)$	$(3,16,4)$	$(5,22,10)$	$(13,3,1)$	$(4,1,0)$	$(3,4,7)$	$(7,17,18)$
$(14,17,3)$	$(7,20,14)$	$(8,3,5)$	$(0,14,4)$	$(4,18,2)$	$(22,1,8)$	$(13,6,10)$
$(17,21,7)$	$(7,9,1)$	$(1,22,11)$	$(15,17,13)$	$(10,1,9)$	$(16,8,18)$	$(12,7,4)$
$(16,1,15)$	$(7,14,21)$	$(0,4,5)$	$(3,14,5)$	$(16,2,21)$	$(5,21,16)$	$(1,5,19)$
$(1,11,15)$	$(11,12,17)$	$(4,8,19)$	$(1,4,6)$	$(20,4,8)$	$(18,17,7)$	

Develop the initial $(18,\overrightarrow{P_3},2)$-ELGDD of type 6^3 under the automorphism group G_1 to get three $(18,\overrightarrow{P_3},2)$-ELGDDs of type 6^3, and develop the initial $(23,\overrightarrow{P_3},2)$-ELGDD of type $1^{12}11^1$ under the automorphism group G_2 to get eighteen $(23,\overrightarrow{P_3},2)$-ELGDDs of type $1^{12}11^1$, all of which form a $(23,\overrightarrow{P_3})$-ELPCS of type $(6^3:5)$. $\square$

5.4 Infinite Families of LELDs

Now, we are in a position to establish several infinite classes for the existence of LELDs by recursion.

Lemma 5.4.1. *There exists an $(n,\overrightarrow{P_3},2)$-LELD for any integer $n\geq 6$ with $n\equiv 2,6,14 \pmod{16}$ or $n\equiv 8 \pmod{12}$ and $n\neq 34,50$.*

Proof. For $n=6,8$, there is an $(n,\overrightarrow{P_3},2)$-LELD by Corollary 5.2.3.

For each $n=16m+2$, $n=16m+6$ or $n=16m+14$, $n\geq 14$ and $n\neq 34,50$, there is a 1-FG$(3,(\{3,4,5\},\{4,5,6\}),(n-2)/4)$ of type $1^{(n-2)/4}$, which is obtained by deleting one point from an S$(3,\{4,5,6\},(n+2)/4)$ (see

[38]). Applying Theorem 5.2.6 with a $(4k-2,\overrightarrow{P_3})$-ELPCS of type $(4^{k-1}:2)$ and an $\mathrm{ELF}(\overrightarrow{P_3},k\{4\})$ with $k\in\{4,5,6\}$, we get an $(n,\overrightarrow{P_3})$-ELPCS of type $(4^{(n-2)/4}:2)$. Then, applying Theorem 5.2.5 with a $(6,\overrightarrow{P_3},2)$-LELD, we obtain an $(n,\overrightarrow{P_3},2)$-LELD. Here, the input $(4k-2,\overrightarrow{P_3})$-ELPCSs of types $(4^{k-1}:2)$ with $k\in\{4,5,6\}$ exist by Lemmas 5.3.9–5.3.11. The input $\mathrm{ELF}(\overrightarrow{P_3},5\{4\})$ and $\mathrm{ELF}(\overrightarrow{P_3},6\{4\})$ exist by Lemmas 5.3.6 and 5.3.7. The input $\mathrm{ELF}(\overrightarrow{P_3},4\{4\})$ is obtained by applying Theorem 5.2.4 with an $\mathrm{H}(2^4)$ and an $\mathrm{ELF}(\overrightarrow{P_3},4\{2\})$, which exist by Theorem 1.1.1 and Lemma 5.3.5, respectively.

For each $n=12m-4$ and $m>1$, there is a 1-FG$(3,(\{3,5\},\{4,6\}),2m-1)$ of type 1^{2m-1}, which is obtained by deleting one point from an $\mathrm{S}(3,\{4,6\},2m)$ (see [25]). Applying Theorem 5.2.6 with a $(6k-4,\overrightarrow{P_3})$-ELPCS of type $(6^{k-1}:2)$ and an $\mathrm{ELF}(\overrightarrow{P_3},k\{6\})$ with $k\in\{4,6\}$, we get a $(12m-4,\overrightarrow{P_3})$-ELPCS of type $(6^{2m-1}:2)$. Then, applying Theorem 5.2.5 with an $(8,\overrightarrow{P_3},2)$-LELD, we obtain an $(n,\overrightarrow{P_3},2)$-LELD. Here, the input $(6k-4,\overrightarrow{P_3})$-ELPCSs of types $(6^{k-1}:2)$ with $k\in\{4,6\}$ exist by Lemmas 5.3.12 and 5.3.13. The $\mathrm{ELF}(\overrightarrow{P_3},k\{6\})$ with $k\in\{4,6\}$ is obtained by applying Theorem 5.2.4 with an $\mathrm{H}(3^k)$ and an $\mathrm{ELF}(\overrightarrow{P_3},4\{2\})$. □

Lemma 5.4.2. *There exists an $(n,\overrightarrow{P_3},2)$-LELD for each positive integer $n\equiv 11,23 \pmod{36}$.*

Proof. For $n=11$, we obtain the design by applying Theorem 5.2.5 with a $(5,\overrightarrow{P_3},2)$-LELD and an $(11,\overrightarrow{P_3})$-ELPCS of type $(3^3:2)$. Simultaneously, we get an $(11,5;\overrightarrow{P_3},2)$-HLELD.

For each $n=36m+11$ or $n=36m+23$ and $n\geq 23$, there is a 1-FG$(3,(3,4),(n-5)/6)$ of type $1^{(n-5)/6}$, which is obtained by deleting one point from an $\mathrm{SQS}((n+1)/6)$ (see [23]). Applying Theorem 5.2.6 with a $(23,\overrightarrow{P_3})$-ELPCS of type $(6^3:5)$ from Lemma 5.3.14 and an $\mathrm{ELF}(\overrightarrow{P_3},4\{6\})$, we get an $(n,\overrightarrow{P_3})$-ELPCS of type $(6^{(n-5)/6}:5)$. Since there exists an $(11,5;\overrightarrow{P_3},2)$-HLELD and an $(11,\overrightarrow{P_3},2)$-LELD, we obtain the desired $(n,\overrightarrow{P_3},2)$-LELD by Theorem 5.2.5. □

Combining Corollary 5.2.3, Lemmas 5.3.4, 5.4.1 and 5.4.2, we have the following theorem.

Theorem 5.4.3. *For each positive integer n, $4\leq n\leq 11$ or $n\geq 14$, $n\equiv k \pmod{144}$ with $k\in\{2,6,8,11,14,18,20,22,23,30,32,34,38,44,46,47,50,54,56,59,62,$*

$66, 68, 70, 78, 80, 82, 83, 86, 92, 94, 95, 98, 102, 104, 110, 114, 116, 118, 119, 126, 128, 130, 131, 134, 140, 142\}$ *and* $n \neq 34, 50$, *there exists an* $(n, \overrightarrow{P_3}, 2)$*-LELD and an optimal, levelled* $(n-2)$*-fault tolerant routing of* $\overrightarrow{K}_n$ *that has levelled minimum optical indices.*

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Small Resolvable H-designs

Lemma .0.4. *There exists an RH*(4^{19}).

Proof. In [10, Lemma 5.4], Cao, Ji and Zhu constructed an H(2^{19}) on Z_{38} with group set $\{\{j, j+19\}, j = 0, 1, \cdots, 18\}$ and the following shortened list of base blocks.

{0,2,16,25}	{0,2,15,24}	{0,4,10,11}	{0,4,31,32}	{0,8,20,22}	{0,8,24,26}
{1,3,17,26}	{1,3,16,25}	{1,5,11,12}	{1,5,32,33 }	{1,9,21,23}	{1,9,25,27}
{0,1,2,3}	{1,2,4,6}	{0,1,4,5}	{1,2,5,9}	{0,1,6,12}	{1,2,7,23}
{0,1,8,16}	{0,1,9,30}	{1,2,10,13}	{0,1,10,22}	{1,2,11,31}	{0,1,13,23}
{0,1,14,15}	{1,2,15,18}	{1,2,16,37}	{1,2,17,27}	{0,1,17,21}	{1,2,19,25}
{1,2,22,35}	{1,2,24,34}	{1,2,26,30}	{0,1,26,29}	{1,3,7,31}	{0,2,8,23}
{0,2,10,12}	{1,3,21,24}	{0,3,6,15}	{0,3,8,18}	{0,3,9,21}	{1,4,14,22}

Here, the blocks of the last five rows are developed by a multiplier 7 of order 3. These 90 blocks and the blocks in the first two rows form the set $\mathcal{B}'$ of all base blocks, which are developed under the automorphism group $\langle(0\ 2 \ldots 34\ 36)(1\ 3\ \ldots\ 35\ 37)\rangle$.

For each block $B = \{a, b, c, d\} \in \mathcal{B}'$, construct an H$(2^4)$ with group set $\{\{x, x+38\} : x \in B\}$ and block set $\mathcal{A}_B = \{\{a+38i, b+38(i+k), c+38j, d+38(j+k)\} : i, j, k \in Z_2\}$. Let $\mathcal{B} = \cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an H(4^{19}) on $I_{76} = \{0, 1, 2, \ldots, 75\}$ with group set $\{\{j, j+19, j+38, j+57\}, j = 0, 1, \cdots, 18\}$ and an automorphism group $\langle\alpha\rangle$, where $\alpha = (0\ 2 \ldots 34\ 36)(1\ 3 \ldots 35\ 37)(38\ 40 \ldots 72\ 74)(39\ 41 \ldots 73\ 75)$. Now, we need to show the resolution.

Note that there are several blocks in $\mathcal{B}$, each of which contains exactly one element from each cycle of α. We first list below some of these blocks and denote them by Δ, each block of which gives a parallel class when developed under the automorphism group $\langle\alpha\rangle$.

$\{1, 2, 54, 75\}$	$\{1, 2, 60, 73\}$	$\{0, 1, 64, 67\}$	$\{0, 3, 44, 53\}$	$\{0, 7, 52, 59\}$
$\{0, 7, 63, 58\}$	$\{0, 7, 60, 67\}$	$\{7, 14, 74, 69\}$	$\{7, 14, 40, 55\}$	$\{0, 11, 44, 55\}$

$\{0, 11, 61, 64\}$	$\{11, 22, 51, 46\}$	$\{11, 22, 62, 65\}$	$\{0, 33, 66, 51\}$	$\{38, 39, 2, 3\}$
$\{38, 39, 9, 30\}$	$\{39, 40, 15, 18\}$	$\{39, 40, 16, 37\}$	$\{38, 39, 26, 29\}$	$\{38, 45, 14, 21\}$
$\{38, 45, 25, 20\}$	$\{45, 52, 32, 15\}$	$\{38, 45, 22, 29\}$	$\{45, 52, 29, 12\}$	$\{45, 52, 36, 31\}$
$\{45, 52, 2, 17\}$	$\{38, 45, 30, 13\}$	$\{38, 49, 22, 33\}$	$\{38, 49, 6, 17\}$	$\{38, 49, 23, 26\}$
$\{49, 60, 34, 29\}$	$\{49, 60, 14, 5\}$	$\{38, 49, 20, 15\}$	$\{38, 71, 28, 13\}$	$\{38, 1, 2, 41\}$
$\{38, 1, 4, 43\}$	$\{38, 1, 14, 53\}$	$\{38, 1, 26, 67\}$	$\{38, 3, 6, 53\}$	$\{38, 7, 14, 59\}$
$\{38, 7, 28, 73\}$	$\{38, 7, 22, 67\}$	$\{45, 14, 29, 50\}$	$\{38, 7, 30, 51\}$	$\{38, 21, 4, 67\}$
$\{38, 11, 6, 55\}$	$\{38, 11, 2, 51\}$	$\{38, 33, 28, 51\}$	$\{0, 39, 40, 3\}$	$\{0, 39, 42, 5\}$
$\{0, 39, 52, 15\}$	$\{1, 40, 53, 18\}$	$\{0, 39, 64, 29\}$	$\{0, 45, 66, 35\}$	$\{0, 45, 68, 13\}$
$\{0, 59, 42, 29\}$	$\{0, 49, 60, 33\}$	$\{0, 71, 66, 13\}$	$\{38, 1, 47, 30\}$	$\{38, 7, 63, 20\}$
$\{45, 14, 70, 15\}$	$\{45, 14, 74, 31\}$	$\{38, 11, 61, 26\}$	$\{49, 22, 72, 29\}$	$\{49, 22, 62, 27\}$
$\{0, 39, 9, 68\}$	$\{1, 40, 16, 75\}$	$\{1, 40, 22, 73\}$	$\{0, 45, 25, 58\}$	$\{7, 52, 32, 53\}$
$\{7, 52, 36, 69\}$	$\{0, 49, 23, 64\}$	$\{11, 60, 34, 67\}$	$\{11, 60, 24, 65\}$	$\{11, 60, 14, 43\}$

Then we shift each of the remaining base blocks in $\mathcal{B}$ by a suitable automorphism α^i for some integer i. The result is listed below, where the blocks in each of the four consecutive rows, namely the ith, $(i+1)$th, $(i+2)$th and $(i+3)$th rows for $i \in \{4k+1 : k = 0, 1, \ldots, 38\}$, form a parallel class.

$\{0, 1, 2, 3\}$	$\{5, 6, 8, 10\}$	$\{12, 13, 16, 17\}$	$\{19, 20, 23, 27\}$	$\{24, 25, 30, 36\}$
$\{31, 32, 37, 15\}$	$\{33, 34, 4, 7\}$	$\{26, 29, 35, 9\}$	$\{14, 21, 18, 22\}$	$\{38, 39, 40, 41\}$
$\{43, 44, 46, 48\}$	$\{50, 51, 54, 55\}$	$\{57, 58, 61, 65\}$	$\{62, 63, 68, 74\}$	$\{59, 60, 75, 47\}$
$\{66, 73, 42, 49\}$	$\{56, 67, 52, 70\}$	$\{45, 53, 69, 71\}$	$\{64, 28, 72, 11\}$	
$\{0, 1, 8, 16\}$	$\{2, 3, 11, 32\}$	$\{4, 5, 14, 26\}$	$\{17, 18, 27, 9\}$	$\{6, 7, 19, 29\}$
$\{20, 21, 34, 35\}$	$\{22, 23, 10, 13\}$	$\{12, 33, 30, 24\}$	$\{25, 36, 31, 37\}$	$\{39, 40, 45, 61\}$
$\{42, 43, 50, 58\}$	$\{46, 47, 55, 38\}$	$\{51, 52, 60, 63\}$	$\{65, 66, 75, 57\}$	$\{72, 73, 48, 49\}$
$\{53, 54, 67, 70\}$	$\{64, 71, 69, 59\}$	$\{41, 62, 56, 74\}$	$\{28, 68, 44, 15\}$	
$\{1, 2, 15, 18\}$	$\{5, 6, 20, 3\}$	$\{7, 8, 23, 33\}$	$\{10, 11, 27, 31\}$	$\{13, 14, 34, 9\}$
$\{37, 0, 22, 32\}$	$\{25, 26, 12, 16\}$	$\{17, 24, 21, 19\}$	$\{38, 39, 48, 60\}$	$\{40, 41, 53, 63\}$
$\{49, 50, 64, 47\}$	$\{44, 45, 61, 65\}$	$\{51, 52, 69, 75\}$	$\{73, 74, 58, 68\}$	$\{59, 66, 42, 56\}$
$\{55, 62, 67, 71\}$	$\{70, 43, 29, 35\}$	$\{46, 54, 28, 30\}$	$\{72, 36, 4, 57\}$	
$\{1, 2, 19, 25\}$	$\{3, 5, 9, 33\}$	$\{4, 6, 12, 27\}$	$\{8, 10, 18, 20\}$	$\{11, 13, 31, 34\}$
$\{14, 17, 22, 32\}$	$\{16, 23, 30, 37\}$	$\{0, 7, 28, 35\}$	$\{39, 40, 60, 73\}$	$\{41, 42, 66, 70\}$
$\{46, 47, 72, 75\}$	$\{51, 53, 57, 43\}$	$\{48, 50, 56, 71\}$	$\{61, 64, 74, 44\}$	$\{38, 49, 67, 63\}$
$\{58, 62, 68, 69\}$	$\{29, 36, 65, 59\}$	$\{52, 54, 24, 26\}$	$\{15, 55, 21, 45\}$	
$\{0, 3, 6, 15\}$	$\{1, 4, 14, 22\}$	$\{11, 18, 32, 8\}$	$\{9, 16, 37, 27\}$	$\{12, 19, 30, 10\}$
$\{26, 33, 20, 28\}$	$\{36, 5, 13, 7\}$	$\{34, 17, 21, 29\}$	$\{31, 35, 24, 25\}$	$\{38, 40, 48, 50\}$
$\{39, 41, 59, 62\}$	$\{46, 49, 52, 61\}$	$\{64, 67, 72, 44\}$	$\{54, 57, 63, 75\}$	$\{66, 73, 60, 68\}$
$\{74, 47, 42, 53\}$	$\{70, 43, 55, 58\}$	$\{45, 56, 69, 65\}$	$\{71, 2, 23, 51\}$	
$\{0, 7, 25, 20\}$	$\{9, 16, 34, 17\}$	$\{11, 18, 5, 31\}$	$\{6, 13, 28, 35\}$	$\{19, 26, 3, 24\}$
$\{23, 30, 21, 15\}$	$\{32, 1, 37, 27\}$	$\{33, 2, 4, 36\}$	$\{38, 45, 66, 73\}$	$\{47, 54, 75, 65\}$

$\{42, 49, 46, 50\}$	$\{51, 58, 55, 53\}$	$\{64, 71, 44, 62\}$	$\{60, 74, 40, 69\}$	$\{61, 63, 39, 48\}$
$\{59, 67, 41, 43\}$	$\{57, 52, 10, 22\}$	$\{56, 29, 14, 70\}$	$\{12, 72, 8, 68\}$	
$\{7, 14, 36, 31\}$	$\{9, 16, 21, 25\}$	$\{13, 20, 8, 23\}$	$\{11, 18, 34, 24\}$	$\{30, 37, 22, 5\}$
$\{15, 29, 19, 35\}$	$\{17, 0, 32, 12\}$	$\{33, 6, 4, 10\}$	$\{38, 45, 63, 58\}$	$\{47, 54, 72, 55\}$
$\{49, 56, 43, 69\}$	$\{44, 51, 59, 53\}$	$\{57, 64, 41, 62\}$	$\{73, 42, 71, 65\}$	$\{52, 66, 46, 60\}$
$\{67, 40, 70, 61\}$	$\{27, 68, 2, 48\}$	$\{26, 75, 28, 39\}$	$\{1, 50, 3, 74\}$	
$\{0, 14, 18, 9\}$	$\{2, 16, 34, 10\}$	$\{11, 25, 37, 20\}$	$\{22, 5, 26, 13\}$	$\{6, 17, 28, 1\}$
$\{24, 35, 30, 3\}$	$\{31, 4, 21, 7\}$	$\{12, 23, 32, 27\}$	$\{38, 45, 60, 67\}$	$\{49, 56, 40, 73\}$
$\{47, 54, 42, 57\}$	$\{55, 62, 64, 58\}$	$\{39, 53, 65, 48\}$	$\{61, 72, 69, 59\}$	$\{63, 74, 50, 46\}$
$\{71, 75, 43, 44\}$	$\{8, 15, 51, 41\}$	$\{19, 68, 52, 36\}$	$\{66, 29, 70, 33\}$	
$\{11, 22, 6, 28\}$	$\{2, 13, 30, 20\}$	$\{12, 23, 24, 36\}$	$\{8, 19, 31, 34\}$	$\{15, 26, 0, 33\}$
$\{18, 29, 14, 32\}$	$\{10, 21, 1, 35\}$	$\{4, 37, 27, 7\}$	$\{17, 25, 3, 5\}$	$\{45, 52, 68, 58\}$
$\{40, 47, 70, 53\}$	$\{51, 65, 55, 71\}$	$\{42, 63, 67, 75\}$	$\{43, 54, 66, 61\}$	$\{38, 60, 72, 56\}$
$\{57, 41, 49, 44\}$	$\{74, 69, 59, 39\}$	$\{48, 50, 64, 73\}$	$\{46, 9, 16, 62\}$	
$\{11, 22, 7, 37\}$	$\{2, 13, 4, 15\}$	$\{17, 28, 19, 14\}$	$\{23, 34, 36, 1\}$	$\{5, 16, 29, 25\}$
$\{24, 35, 21, 27\}$	$\{33, 6, 3, 31\}$	$\{10, 18, 30, 32\}$	$\{38, 59, 42, 67\}$	$\{40, 61, 58, 52\}$
$\{44, 55, 66, 39\}$	$\{51, 62, 46, 68\}$	$\{53, 64, 49, 41\}$	$\{60, 71, 57, 63\}$	$\{54, 65, 74, 69\}$
$\{70, 72, 47, 56\}$	$\{9, 20, 43, 73\}$	$\{45, 8, 48, 12\}$	$\{26, 75, 0, 50\}$	
$\{11, 22, 14, 5\}$	$\{13, 24, 0, 34\}$	$\{19, 3, 9, 7\}$	$\{4, 26, 16, 29\}$	$\{10, 32, 6, 28\}$
$\{31, 15, 23, 18\}$	$\{17, 12, 8, 20\}$	$\{49, 60, 55, 61\}$	$\{40, 51, 68, 58\}$	$\{53, 64, 43, 67\}$
$\{62, 73, 74, 48\}$	$\{52, 63, 54, 65\}$	$\{59, 70, 72, 75\}$	$\{57, 41, 47, 45\}$	$\{56, 39, 36, 30\}$
$\{38, 21, 25, 71\}$	$\{27, 66, 69, 35\}$	$\{37, 44, 46, 2\}$	$\{1, 50, 42, 33\}$	
$\{0, 33, 28, 13\}$	$\{2, 35, 14, 10\}$	$\{4, 6, 20, 29\}$	$\{22, 24, 37, 8\}$	$\{32, 36, 25, 26\}$
$\{5, 7, 21, 30\}$	$\{19, 27, 1, 3\}$	$\{49, 60, 51, 46\}$	$\{53, 64, 62, 68\}$	$\{44, 66, 56, 69\}$
$\{42, 75, 70, 55\}$	$\{40, 73, 52, 48\}$	$\{50, 58, 74, 38\}$	$\{39, 41, 54, 63\}$	$\{17, 18, 65, 47\}$
$\{12, 15, 59, 71\}$	$\{61, 34, 9, 43\}$	$\{23, 72, 57, 11\}$	$\{67, 31, 45, 16\}$	
$\{0, 4, 10, 11\}$	$\{6, 14, 30, 32\}$	$\{1, 3, 16, 25\}$	$\{9, 13, 19, 20\}$	$\{49, 44, 40, 52\}$
$\{38, 42, 69, 70\}$	$\{46, 54, 66, 68\}$	$\{41, 45, 72, 73\}$	$\{7, 8, 48, 50\}$	$\{17, 18, 59, 63\}$
$\{21, 22, 65, 43\}$	$\{34, 35, 47, 57\}$	$\{5, 12, 71, 61\}$	$\{33, 28, 62, 74\}$	$\{29, 37, 53, 55\}$
$\{51, 58, 36, 26\}$	$\{39, 23, 31, 64\}$	$\{67, 2, 75, 27\}$	$\{15, 60, 24, 56\}$	
$\{0, 1, 44, 50\}$	$\{2, 3, 48, 56\}$	$\{4, 5, 52, 64\}$	$\{7, 8, 61, 71\}$	$\{10, 11, 65, 69\}$
$\{17, 18, 73, 41\}$	$\{13, 14, 74, 46\}$	$\{29, 30, 54, 58\}$	$\{19, 21, 63, 49\}$	$\{20, 22, 66, 43\}$
$\{24, 27, 70, 42\}$	$\{9, 12, 60, 68\}$	$\{25, 32, 57, 45\}$	$\{53, 16, 33, 39\}$	$\{37, 38, 55, 23\}$
$\{6, 51, 59, 15\}$	$\{62, 26, 72, 36\}$	$\{47, 31, 75, 35\}$	$\{28, 67, 34, 40\}$	
$\{0, 2, 48, 50\}$	$\{1, 3, 59, 62\}$	$\{7, 14, 66, 42\}$	$\{10, 17, 52, 56\}$	$\{9, 16, 51, 49\}$
$\{4, 11, 60, 40\}$	$\{6, 13, 38, 46\}$	$\{8, 15, 61, 55\}$	$\{19, 26, 69, 73\}$	$\{23, 30, 70, 64\}$
$\{21, 28, 44, 72\}$	$\{18, 32, 74, 65\}$	$\{12, 33, 75, 45\}$	$\{57, 58, 37, 5\}$	$\{29, 68, 54, 20\}$
$\{27, 41, 53, 36\}$	$\{71, 35, 39, 25\}$	$\{22, 63, 31, 43\}$	$\{34, 67, 24, 47\}$	
$\{7, 21, 49, 65\}$	$\{0, 14, 70, 46\}$	$\{9, 23, 73, 56\}$	$\{4, 25, 60, 54\}$	$\{11, 32, 64, 44\}$
$\{17, 28, 50, 72\}$	$\{13, 24, 57, 63\}$	$\{20, 31, 48, 38\}$	$\{15, 26, 43, 67\}$	$\{16, 27, 66, 40\}$
$\{8, 19, 75, 71\}$	$\{18, 29, 53, 59\}$	$\{30, 34, 61, 62\}$	$\{45, 52, 5, 37\}$	$\{74, 47, 10, 22\}$
$\{68, 33, 1, 51\}$	$\{69, 35, 3, 42\}$	$\{39, 12, 41, 36\}$	$\{6, 55, 2, 58\}$	

$\{0,11,72,52\}$	$\{13,24,75,71\}$	$\{15,26,61,51\}$	$\{17,28,42,38\}$	$\{21,32,68,74\}$
$\{19,3,47,45\}$	$\{14,36,64,39\}$	$\{22,6,56,40\}$	$\{23,7,53,48\}$	$\{4,37,54,50\}$
$\{2,35,63,43\}$	$\{16,18,70,41\}$	$\{59,60,25,29\}$	$\{65,67,9,12\}$	$\{62,69,1,33\}$
$\{58,31,5,46\}$	$\{57,30,27,55\}$	$\{20,66,44,8\}$	$\{34,73,10,49\}$	
$\{0,2,53,62\}$	$\{6,10,54,55\}$	$\{8,16,66,68\}$	$\{12,20,74,38\}$	$\{7,9,61,70\}$
$\{3,5,56,65\}$	$\{11,15,59,60\}$	$\{13,17,44,45\}$	$\{19,27,39,41\}$	$\{57,58,22,24\}$
$\{48,49,18,26\}$	$\{46,47,21,31\}$	$\{51,52,29,1\}$	$\{67,32,4,50\}$	$\{64,33,30,72\}$
$\{40,35,25,43\}$	$\{34,75,42,14\}$	$\{28,73,71,23\}$	$\{63,36,69,37\}$	
$\{38,39,6,12\}$	$\{41,42,9,25\}$	$\{44,45,16,28\}$	$\{47,48,19,1\}$	$\{50,51,29,33\}$
$\{53,54,0,10\}$	$\{55,56,4,8\}$	$\{59,61,27,13\}$	$\{60,62,30,7\}$	$\{40,43,11,23\}$
$\{49,52,24,32\}$	$\{63,70,15,5\}$	$\{64,71,31,21\}$	$\{73,37,3,65\}$	$\{58,34,14,66\}$
$\{69,26,22,72\}$	$\{17,57,75,2\}$	$\{18,67,46,36\}$	$\{35,74,20,68\}$	
$\{38,41,8,18\}$	$\{45,52,28,4\}$	$\{40,47,6,10\}$	$\{49,56,15,13\}$	$\{54,61,34,14\}$
$\{60,67,16,24\}$	$\{51,58,7,33\}$	$\{57,64,31,35\}$	$\{55,62,26,20\}$	$\{59,73,25,3\}$
$\{66,42,22,36\}$	$\{70,53,19,27\}$	$\{39,50,9,37\}$	$\{29,43,71,11\}$	$\{12,72,46,30\}$
$\{32,65,44,2\}$	$\{74,5,68,0\}$	$\{17,69,21,75\}$	$\{23,63,1,48\}$	
$\{38,52,18,9\}$	$\{45,59,33,16\}$	$\{47,68,24,4\}$	$\{49,60,6,28\}$	$\{51,62,19,25\}$
$\{42,53,32,22\}$	$\{55,66,7,31\}$	$\{50,61,8,26\}$	$\{63,74,21,13\}$	$\{67,40,15,11\}$
$\{43,54,14,20\}$	$\{56,58,34,5\}$	$\{65,69,37,0\}$	$\{41,10,1,71\}$	$\{64,30,36,75\}$
$\{17,57,70,3\}$	$\{44,27,48,35\}$	$\{46,12,39,2\}$	$\{23,72,29,73\}$	
$\{38,49,29,25\}$	$\{51,62,0,34\}$	$\{53,75,5,3\}$	$\{44,66,18,31\}$	$\{46,68,4,26\}$
$\{57,41,11,6\}$	$\{40,73,14,10\}$	$\{48,43,33,13\}$	$\{50,52,27,36\}$	$\{60,64,15,16\}$
$\{72,42,20,22\}$	$\{45,47,23,32\}$	$\{69,71,8,17\}$	$\{67,30,1,59\}$	$\{56,19,35,39\}$
$\{24,63,70,2\}$	$\{74,12,54,7\}$	$\{55,21,65,28\}$	$\{9,58,37,61\}$	
$\{38,42,10,11\}$	$\{39,43,32,33\}$	$\{41,49,23,25\}$	$\{51,59,37,1\}$	$\{53,16,18,58\}$
$\{61,24,27,69\}$	$\{40,3,8,52\}$	$\{63,26,31,47\}$	$\{44,7,15,74\}$	$\{71,34,4,45\}$
$\{50,13,22,72\}$	$\{75,0,14,73\}$	$\{48,17,35,68\}$	$\{20,65,62,28\}$	$\{21,66,46,29\}$
$\{6,55,56,30\}$	$\{67,36,57,9\}$	$\{70,5,60,12\}$	$\{19,64,2,54\}$	
$\{38,1,13,61\}$	$\{39,2,17,65\}$	$\{41,4,24,75\}$	$\{45,8,30,40\}$	$\{51,14,0,42\}$
$\{46,10,18,58\}$	$\{43,7,25,66\}$	$\{50,15,20,68\}$	$\{67,36,12,64\}$	$\{53,22,19,55\}$
$\{54,23,34,52\}$	$\{63,32,16,49\}$	$\{57,26,31,73\}$	$\{3,74,70,6\}$	$\{62,28,72,35\}$
$\{21,60,37,47\}$	$\{9,48,27,71\}$	$\{11,56,5,69\}$	$\{33,44,29,59\}$	
$\{45,14,32,53\}$	$\{40,9,34,42\}$	$\{47,16,3,67\}$	$\{46,15,23,55\}$	$\{38,7,5,71\}$
$\{49,18,6,59\}$	$\{51,20,22,54\}$	$\{57,26,4,70\}$	$\{61,37,27,43\}$	$\{60,36,2,69\}$
$\{63,1,13,72\}$	$\{65,0,33,39\}$	$\{29,68,50,25\}$	$\{11,56,44,21\}$	$\{10,62,66,19\}$
$\{28,74,48,12\}$	$\{35,75,17,58\}$	$\{31,52,8,64\}$	$\{24,73,30,41\}$	
$\{38,21,18,50\}$	$\{45,28,22,40\}$	$\{51,24,8,68\}$	$\{42,15,32,60\}$	$\{53,26,5,67\}$
$\{46,19,20,70\}$	$\{57,30,4,75\}$	$\{71,6,29,59\}$	$\{44,17,35,69\}$	$\{61,34,36,39\}$
$\{52,25,11,55\}$	$\{43,16,14,58\}$	$\{13,73,41,1\}$	$\{49,12,65,37\}$	$\{47,10,72,0\}$
$\{54,27,74,31\}$	$\{9,48,23,64\}$	$\{7,56,2,62\}$	$\{3,63,33,66\}$	
$\{49,22,14,43\}$	$\{51,24,0,72\}$	$\{53,37,5,41\}$	$\{42,26,16,67\}$	$\{44,28,2,62\}$
$\{38,33,12,46\}$	$\{40,4,18,65\}$	$\{68,32,7,54\}$	$\{64,30,19,58\}$	$\{66,36,10,50\}$

$\{45, 9, 23, 70\}$	$\{21, 60, 69, 13\}$	$\{20, 59, 75, 3\}$	$\{29, 74, 71, 31\}$	$\{34, 55, 52, 8\}$
$\{1, 47, 63, 27\}$	$\{73, 11, 39, 17\}$	$\{35, 57, 25, 61\}$	$\{15, 48, 6, 56\}$	
$\{38, 8, 24, 64\}$	$\{39, 3, 16, 63\}$	$\{41, 7, 34, 73\}$	$\{43, 13, 25, 65\}$	$\{45, 15, 31, 71\}$
$\{1, 40, 42, 6\}$	$\{10, 49, 54, 22\}$	$\{11, 50, 55, 33\}$	$\{12, 51, 59, 4\}$	$\{5, 44, 52, 17\}$
$\{18, 57, 66, 2\}$	$\{21, 60, 74, 19\}$	$\{37, 48, 46, 14\}$	$\{58, 27, 62, 28\}$	$\{29, 68, 32, 72\}$
$\{30, 70, 0, 53\}$	$\{9, 61, 35, 56\}$	$\{36, 47, 20, 69\}$	$\{26, 75, 23, 67\}$	
$\{0, 39, 51, 23\}$	$\{1, 40, 55, 27\}$	$\{3, 42, 64, 36\}$	$\{5, 45, 49, 35\}$	$\{6, 46, 52, 29\}$
$\{8, 48, 56, 20\}$	$\{12, 53, 59, 33\}$	$\{9, 50, 60, 30\}$	$\{13, 58, 72, 10\}$	$\{18, 63, 74, 16\}$
$\{32, 43, 61, 19\}$	$\{66, 31, 75, 11\}$	$\{68, 37, 73, 25\}$	$\{26, 65, 28, 67\}$	$\{15, 54, 2, 44\}$
$\{17, 62, 21, 57\}$	$\{34, 41, 14, 70\}$	$\{24, 38, 4, 71\}$	$\{7, 47, 22, 69\}$	
$\{7, 52, 73, 25\}$	$\{0, 45, 63, 20\}$	$\{2, 47, 72, 4\}$	$\{9, 54, 41, 29\}$	$\{11, 56, 40, 35\}$
$\{13, 58, 49, 5\}$	$\{15, 60, 65, 31\}$	$\{19, 64, 42, 32\}$	$\{14, 66, 46, 22\}$	$\{17, 38, 70, 12\}$
$\{53, 16, 57, 23\}$	$\{74, 37, 62, 27\}$	$\{39, 8, 43, 3\}$	$\{50, 33, 68, 24\}$	$\{61, 34, 48, 6\}$
$\{51, 21, 75, 1\}$	$\{28, 67, 36, 44\}$	$\{30, 69, 18, 59\}$	$\{26, 71, 10, 55\}$	
$\{0, 59, 63, 33\}$	$\{11, 60, 55, 23\}$	$\{13, 62, 41, 27\}$	$\{4, 53, 65, 30\}$	$\{17, 66, 40, 35\}$
$\{2, 51, 74, 16\}$	$\{21, 70, 72, 37\}$	$\{19, 68, 43, 1\}$	$\{12, 61, 47, 15\}$	$\{31, 42, 56, 14\}$
$\{34, 38, 44, 7\}$	$\{46, 9, 52, 20\}$	$\{67, 36, 50, 26\}$	$\{39, 3, 54, 25\}$	$\{6, 45, 10, 49\}$
$\{8, 48, 18, 58\}$	$\{28, 73, 5, 75\}$	$\{24, 69, 29, 57\}$	$\{22, 64, 32, 71\}$	
$\{11, 60, 57, 9\}$	$\{2, 62, 52, 27\}$	$\{13, 73, 43, 0\}$	$\{4, 75, 65, 7\}$	$\{6, 46, 59, 30\}$
$\{14, 56, 45, 8\}$	$\{1, 41, 55, 26\}$	$\{5, 47, 53, 16\}$	$\{3, 49, 61, 25\}$	$\{66, 29, 68, 31\}$
$\{71, 34, 51, 19\}$	$\{42, 15, 54, 28\}$	$\{74, 20, 48, 23\}$	$\{18, 63, 22, 64\}$	$\{33, 40, 17, 38\}$
$\{37, 44, 35, 67\}$	$\{36, 69, 21, 39\}$	$\{12, 58, 32, 72\}$	$\{24, 70, 10, 50\}$	
$\{1, 43, 70, 33\}$	$\{39, 2, 45, 23\}$	$\{40, 3, 48, 18\}$	$\{42, 5, 52, 26\}$	$\{53, 16, 63, 7\}$
$\{46, 9, 59, 31\}$	$\{50, 13, 64, 27\}$	$\{71, 34, 47, 12\}$	$\{51, 14, 74, 8\}$	$\{73, 4, 58, 10\}$
$\{38, 11, 44, 17\}$	$\{55, 28, 41, 37\}$	$\{75, 21, 67, 24\}$	$\{65, 22, 56, 30\}$	$\{72, 36, 49, 20\}$
$\{69, 35, 62, 25\}$	$\{15, 60, 0, 66\}$	$\{32, 54, 6, 57\}$	$\{19, 61, 29, 68\}$	
$\{38, 1, 55, 21\}$	$\{39, 3, 59, 24\}$	$\{40, 5, 46, 17\}$	$\{42, 7, 50, 22\}$	$\{43, 8, 56, 26\}$
$\{44, 13, 58, 27\}$	$\{64, 33, 54, 23\}$	$\{68, 37, 48, 28\}$	$\{47, 16, 41, 29\}$	$\{45, 14, 67, 12\}$
$\{71, 2, 69, 25\}$	$\{57, 30, 52, 36\}$	$\{62, 35, 53, 11\}$	$\{61, 34, 70, 0\}$	$\{6, 65, 10, 73\}$
$\{31, 4, 72, 63\}$	$\{66, 49, 32, 19\}$	$\{74, 9, 18, 51\}$	$\{15, 60, 75, 20\}$	
$\{38, 7, 53, 9\}$	$\{42, 11, 64, 33\}$	$\{45, 14, 57, 23\}$	$\{47, 16, 56, 12\}$	$\{44, 13, 74, 19\}$
$\{52, 28, 46, 22\}$	$\{49, 25, 75, 20\}$	$\{48, 31, 73, 5\}$	$\{51, 34, 66, 8\}$	$\{65, 0, 55, 3\}$
$\{54, 27, 50, 30\}$	$\{2, 41, 15, 63\}$	$\{32, 39, 26, 72\}$	$\{10, 21, 70, 43\}$	$\{18, 29, 58, 69\}$
$\{59, 60, 4, 17\}$	$\{61, 24, 37, 40\}$	$\{71, 6, 35, 68\}$	$\{67, 36, 62, 1\}$	
$\{38, 11, 60, 33\}$	$\{49, 22, 45, 37\}$	$\{40, 13, 42, 15\}$	$\{44, 17, 41, 9\}$	$\{50, 34, 46, 30\}$
$\{64, 21, 54, 1\}$	$\{62, 19, 74, 32\}$	$\{66, 23, 51, 31\}$	$\{56, 20, 72, 5\}$	$\{8, 47, 18, 68\}$
$\{12, 57, 26, 71\}$	$\{10, 55, 2, 61\}$	$\{4, 63, 29, 75\}$	$\{16, 65, 36, 69\}$	$\{35, 39, 28, 67\}$
$\{7, 14, 70, 53\}$	$\{6, 27, 48, 73\}$	$\{58, 59, 24, 25\}$	$\{3, 52, 43, 0\}$	
$\{38, 8, 58, 22\}$	$\{40, 10, 64, 28\}$	$\{39, 9, 59, 23\}$	$\{3, 42, 7, 49\}$	$\{5, 44, 11, 65\}$
$\{15, 54, 25, 45\}$	$\{2, 41, 19, 61\}$	$\{14, 55, 20, 67\}$	$\{16, 57, 24, 72\}$	$\{17, 62, 29, 71\}$
$\{0, 52, 32, 46\}$	$\{26, 27, 68, 69\}$	$\{33, 34, 47, 50\}$	$\{30, 37, 60, 43\}$	$\{74, 75, 12, 13\}$
$\{18, 63, 70, 1\}$	$\{6, 51, 66, 35\}$	$\{4, 53, 48, 21\}$	$\{73, 36, 56, 31\}$	

$\{0,45,28,73\}$	$\{7,52,35,63\}$	$\{6,65,24,56\}$	$\{2,51,30,58\}$	$\{4,53,33,67\}$
$\{13,62,37,71\}$	$\{11,60,19,47\}$	$\{21,70,8,42\}$	$\{23,72,32,38\}$	$\{26,27,40,41\}$
$\{5,16,66,61\}$	$\{43,44,14,17\}$	$\{54,57,22,31\}$	$\{39,50,3,36\}$	$\{75,48,12,15\}$
$\{74,9,20,69\}$	$\{10,59,68,25\}$	$\{55,18,64,29\}$	$\{1,46,34,49\}$	
$\{0,71,12,46\}$	$\{2,42,18,65\}$	$\{4,44,19,66\}$	$\{6,48,37,38\}$	$\{1,47,21,61\}$
$\{3,49,27,67\}$	$\{10,11,50,51\}$	$\{22,23,69,52\}$	$\{7,8,54,57\}$	$\{28,35,56,63\}$
$\{17,24,39,60\}$	$\{20,31,40,73\}$	$\{74,43,26,33\}$	$\{72,45,36,9\}$	$\{14,55,58,29\}$
$\{30,41,70,5\}$	$\{53,16,68,13\}$	$\{59,32,62,15\}$	$\{25,64,34,75\}$	

□

Lemma .0.5. *There exists an RH*(4^{41}).

Proof. In [10, Lemma 5.2], Cao, Ji and Zhu constructed an $\mathrm{H}(2^{41})$ on Z_{82} with group set $\{\{j, j+41\}, j = 0, 1, \cdots, 40\}$ and the following shortened list of base blocks, which are developed by the automorphism group $\langle\alpha', \beta'\rangle$, where $\alpha' = (0\ 1 \ldots 80\ 81)$ and β' is a multiplier 37 of order 5 in Z_{82}.

$\mathcal{B}'$:	{0,1,2,4}	{0,1,5,6}	{0,1,7,8}	{0,1,9,10}	{0,1,11,12}	{0,1,13,14}
	{0,1,15,16}	{0,1,17,18}	{0,1,19,20}	{0,1,21,22}	{0,1,23,24}	{0,1,27,28}
	{0,1,29,30}	{0,1,31,32}	{0,1,33,34}	{0,1,35,36}	{0,1,39,43}	{0,1,40,80}
	{0,1,44,79}	{0,2,5,7}	{0,2,6,8}	{0,2,9,12}	{0,2,10,13}	{0,2,11,15}
	{0,2,14,16}	{0,2,17,19}	{0,2,18,24}	{0,2,20,48}	{0,2,21,26}	{0,2,28,46}
	{0,2,29,50}	{0,2,30,72}	{0,2,32,58}	{0,2,35,49}	{0,2,55,66}	{0,2,56,69}
	{0,2,63,75}	{0,3,7,22}	{0,3,9,56}	{0,3,12,64}	{0,3,15,70}	{0,3,17,61}
	{0,3,19,29}	{0,3,24,52}	{0,3,27,76}	{0,3,33,55}	{0,3,43,58}	{0,4,9,72}
	{0,4,10,58}	{0,4,14,38}	{0,4,39,77}	{0,4,48,65}		

For each block $B = \{a, b, c, d\} \in \mathcal{B}'$, construct an $\mathrm{H}(2^4)$ with group set $\{\{x, x+82\} : x \in B\}$ and block set $\mathcal{A}_B = \{\{a+82i, b+82(i+k), c+82j, d+82(j+k)\} : i, j, k \in Z_2\}$. Let $\mathcal{B} = \cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an $\mathrm{H}(4^{41})$ on $I_{164} = \{0, 1, 2, \ldots, 163\}$ with group set $\mathcal{G} = \{\{j, j+41, j+82, j+123\}, j = 0, 1, \cdots, 40\}$ and an automorphism group $\langle\alpha, \beta\rangle$, where

$$\alpha = (0\ 1 \ldots 80\ 81)(82\ 83 \ldots 162\ 163) \text{ and}$$

$$\beta = \begin{cases} \beta'(x), & \text{if } x < 82, \\ \beta'(x-82) + 82, & \text{if } x \geq 82. \end{cases}$$

Now, we need to show the resolution.

Note that there are several blocks in $\mathcal{B}$, each of which contains exactly one even and one odd elements from each cycle of α. We first list below some of these blocks and denote them by Δ, each block of which gives a parallel class when developed under the automorphism group $\langle \alpha^2, \beta \rangle$.

$\{0,1,87,88\}$	$\{0,1,93,94\}$	$\{0,1,101,102\}$	$\{0,1,103,104\}$	$\{0,1,105,106\}$
$\{0,1,109,110\}$	$\{0,1,111,112\}$	$\{0,1,113,114\}$	$\{0,1,115,116\}$	$\{0,1,117,118\}$
$\{0,1,126,161\}$	$\{0,3,91,138\}$	$\{0,3,97,152\}$	$\{0,3,125,140\}$	$\{82,83,5,6\}$
$\{82,83,7,8\}$	$\{82,83,9,10\}$	$\{82,83,11,12\}$	$\{82,83,13,14\}$	$\{82,83,15,16\}$
$\{82,83,17,18\}$	$\{82,83,19,20\}$	$\{82,83,21,22\}$	$\{82,83,23,24\}$	$\{82,83,27,28\}$
$\{82,83,29,30\}$	$\{82,83,31,32\}$	$\{82,83,33,34\}$	$\{82,83,35,36\}$	$\{82,85,9,56\}$
$\{82,85,15,70\}$	$\{82,85,43,58\}$	$\{82,1,44,161\}$	$\{82,2,5,89\}$	$\{82,2,17,101\}$
$\{82,2,35,131\}$	$\{82,4,39,159\}$	$\{0,83,126,79\}$	$\{0,84,99,19\}$	$\{0,84,117,49\}$
$\{0,84,145,75\}$	$\{0,85,9,138\}$	$\{0,85,27,158\}$	$\{0,85,43,140\}$	$\{82,1,87,6\}$
$\{82,1,89,8\}$	$\{82,1,91,10\}$	$\{82,1,93,12\}$	$\{82,1,97,16\}$	$\{82,1,101,20\}$
$\{82,1,105,24\}$	$\{82,1,109,28\}$	$\{82,1,111,30\}$	$\{82,1,113,32\}$	$\{82,1,115,34\}$
$\{82,1,117,36\}$	$\{82,2,87,7\}$	$\{82,2,93,15\}$	$\{82,2,99,19\}$	$\{82,2,117,49\}$
$\{82,2,145,75\}$	$\{82,3,89,22\}$	$\{82,3,91,56\}$	$\{82,3,97,70\}$	$\{82,3,125,58\}$
$\{82,4,121,77\}$	$\{0,83,5,88\}$	$\{0,83,7,90\}$	$\{0,83,9,92\}$	$\{0,83,11,94\}$
$\{0,83,13,96\}$	$\{0,83,15,98\}$	$\{0,83,17,100\}$	$\{0,83,19,102\}$	$\{0,83,23,106\}$
$\{0,83,27,110\}$	$\{0,83,29,112\}$	$\{0,83,31,114\}$	$\{0,83,33,116\}$	$\{0,83,35,118\}$
$\{0,84,5,89\}$	$\{0,84,11,97\}$	$\{0,84,17,101\}$	$\{0,84,63,157\}$	$\{0,85,7,104\}$
$\{0,1,89,90\}$	$\{0,1,95,96\}$	$\{0,1,97,98\}$		

Then we shift each of the remaining base blocks in $\mathcal{B}$ by a suitable automorphism $\alpha^i\beta^j$ for some integers i and j. The result is listed below, where the blocks in each of the eleven consecutive rows, namely the ith, $(i+1)$th, ..., and $(i+10)$th rows for $i \in \{11k+1 : k = 0,1,\ldots,7\}$, form a parallel class.

$\{0,1,2,4\}$	$\{5,6,10,11\}$	$\{7,8,14,15\}$	$\{12,13,21,22\}$
$\{16,17,27,28\}$	$\{18,19,31,32\}$	$\{23,24,38,39\}$	$\{25,26,42,43\}$
$\{29,30,48,49\}$	$\{33,34,54,55\}$	$\{35,36,58,59\}$	$\{40,41,67,68\}$
$\{44,45,73,74\}$	$\{46,47,77,78\}$	$\{81,56,76,51\}$	$\{60,37,75,52\}$
$\{20,57,69,53\}$	$\{64,66,70,72\}$	$\{61,63,79,3\}$	$\{82,83,84,86\}$
$\{87,88,92,93\}$	$\{89,90,96,97\}$	$\{94,95,103,104\}$	$\{98,99,109,110\}$
$\{100,101,113,114\}$	$\{105,106,120,121\}$	$\{107,108,124,125\}$	$\{111,112,130,131\}$
$\{115,116,136,137\}$	$\{117,118,140,141\}$	$\{122,123,149,150\}$	$\{126,127,155,156\}$
$\{128,129,159,160\}$	$\{162,139,85,157\}$	$\{153,145,138,135\}$	$\{144,147,163,91\}$
$\{158,161,119,134\}$	$\{142,146,151,132\}$	$\{65,148,152,71\}$	$\{50,133,143,62\}$
$\{80,154,102,9\}$			
$\{0,1,40,80\}$	$\{2,3,46,81\}$	$\{4,6,9,11\}$	$\{5,7,14,17\}$
$\{8,10,18,21\}$	$\{13,15,24,28\}$	$\{20,22,34,36\}$	$\{25,27,42,44\}$
$\{29,31,49,77\}$	$\{30,32,51,56\}$	$\{33,35,61,79\}$	$\{69,71,16,37\}$
$\{48,50,78,38\}$	$\{43,45,75,19\}$	$\{26,62,41,47\}$	$\{72,64,57,54\}$

$\{65,67,39,52\}$ $\{53,60,70,66\}$ $\{82,83,115,116\}$ $\{84,85,119,120\}$
$\{88,89,128,86\}$ $\{90,91,134,87\}$ $\{92,94,97,99\}$ $\{96,98,102,104\}$
$\{101,103,110,113\}$ $\{112,114,122,125\}$ $\{106,108,117,121\}$ $\{93,95,107,109\}$
$\{124,126,141,143\}$ $\{127,129,145,151\}$ $\{146,138,148,118\}$ $\{131,133,152,157\}$
$\{162,154,132,142\}$ $\{147,139,130,156\}$ $\{161,111,155,158\}$ $\{137,140,144,159\}$
$\{73,23,100,160\}$ $\{153,63,76,105\}$ $\{136,74,59,150\}$ $\{12,123,149,55\}$
$\{163,58,135,68\}$
$\{0,2,63,75\}$ $\{1,4,8,23\}$ $\{3,6,12,59\}$ $\{7,10,19,71\}$
$\{11,14,26,81\}$ $\{13,16,30,74\}$ $\{15,18,39,67\}$ $\{28,31,55,22\}$
$\{17,20,50,72\}$ $\{21,24,64,79\}$ $\{37,41,46,27\}$ $\{56,60,66,32\}$
$\{38,42,52,76\}$ $\{69,53,36,48\}$ $\{43,25,73,58\}$ $\{82,84,111,132\}$
$\{83,85,113,155\}$ $\{86,88,118,144\}$ $\{87,89,150,162\}$ $\{90,93,99,146\}$
$\{92,95,104,156\}$ $\{91,94,106,161\}$ $\{97,100,114,158\}$ $\{102,105,126,154\}$
$\{107,110,134,101\}$ $\{130,133,163,103\}$ $\{139,143,149,115\}$ $\{121,125,135,159\}$
$\{141,145,98,136\}$ $\{124,108,96,151\}$ $\{33,34,117,119\}$ $\{77,78,116,120\}$
$\{57,49,160,152\}$ $\{44,80,122,148\}$ $\{131,123,70,62\}$ $\{138,140,65,68\}$
$\{153,9,5,147\}$ $\{29,112,128,47\}$ $\{40,142,127,54\}$ $\{45,129,157,35\}$
$\{51,137,61,109\}$
$\{0,1,122,162\}$ $\{2,4,90,92\}$ $\{3,5,94,97\}$ $\{6,8,98,101\}$
$\{7,9,100,104\}$ $\{10,12,106,108\}$ $\{11,13,110,112\}$ $\{14,16,116,144\}$
$\{15,17,118,123\}$ $\{18,20,128,146\}$ $\{19,21,130,151\}$ $\{23,25,135,95\}$
$\{24,26,138,82\}$ $\{30,32,85,96\}$ $\{33,35,89,102\}$ $\{36,38,99,111\}$
$\{39,42,133,103\}$ $\{44,47,143,105\}$ $\{28,31,129,139\}$ $\{34,37,140,86\}$
$\{54,57,87,109\}$ $\{41,45,132,113\}$ $\{49,53,141,107\}$ $\{46,50,142,84\}$
$\{75,79,114,152\}$ $\{69,73,117,134\}$ $\{136,137,56,58\}$ $\{119,120,76,80\}$
$\{126,163,48,52\}$ $\{148,150,72,74\}$ $\{121,157,55,68\}$ $\{88,124,81,71\}$
$\{149,153,77,43\}$ $\{147,66,78,161\}$ $\{91,29,63,155\}$ $\{127,65,62,131\}$
$\{115,22,60,159\}$ $\{154,61,51,160\}$ $\{27,145,156,67\}$ $\{70,83,125,59\}$
$\{40,158,64,93\}$
$\{82,84,14,16\}$ $\{83,85,18,20\}$ $\{86,88,22,28\}$ $\{87,89,25,53\}$
$\{90,92,29,34\}$ $\{91,93,37,55\}$ $\{94,96,41,62\}$ $\{95,97,43,3\}$
$\{98,100,48,74\}$ $\{99,101,52,66\}$ $\{102,104,75,4\}$ $\{103,105,77,8\}$
$\{106,108,5,17\}$ $\{109,112,39,9\}$ $\{110,113,45,7\}$ $\{114,117,51,61\}$
$\{118,121,60,6\}$ $\{119,122,70,10\}$ $\{111,115,38,19\}$ $\{124,128,56,80\}$
$\{127,131,2,40\}$ $\{129,133,13,30\}$ $\{116,35,36,120\}$ $\{135,54,58,141\}$
$\{138,57,63,146\}$ $\{140,59,67,150\}$ $\{149,68,78,161\}$ $\{153,46,24,163\}$
$\{107,26,42,125\}$ $\{134,27,69,126\}$ $\{148,21,23,142\}$ $\{156,49,73,130\}$
$\{154,47,81,145\}$ $\{155,65,33,144\}$ $\{159,15,31,147\}$ $\{132,32,71,136\}$
$\{72,143,137,12\}$ $\{50,139,123,76\}$ $\{160,79,162,0\}$ $\{1,157,11,151\}$
$\{44,158,64,152\}$
$\{82,1,27,110\}$ $\{83,2,30,113\}$ $\{84,3,33,116\}$ $\{85,4,36,119\}$
$\{86,5,39,122\}$ $\{89,8,47,87\}$ $\{90,10,14,98\}$ $\{91,11,23,107\}$

$\{92, 12, 38, 138\}$	$\{93, 13, 40, 143\}$	$\{95, 15, 45, 153\}$	$\{101, 21, 75, 88\}$
$\{99, 20, 24, 121\}$	$\{104, 25, 31, 160\}$	$\{114, 35, 44, 96\}$	$\{97, 18, 32, 158\}$
$\{105, 26, 42, 134\}$	$\{130, 51, 72, 100\}$	$\{108, 29, 53, 102\}$	$\{147, 68, 16, 120\}$
$\{127, 49, 55, 103\}$	$\{124, 46, 56, 162\}$	$\{135, 57, 19, 118\}$	$\{48, 131, 132, 52\}$
$\{50, 133, 139, 58\}$	$\{54, 137, 145, 64\}$	$\{59, 142, 154, 73\}$	$\{66, 149, 163, 0\}$
$\{43, 126, 144, 63\}$	$\{17, 156, 148, 41\}$	$\{60, 117, 141, 34\}$	$\{9, 128, 106, 61\}$
$\{74, 157, 109, 28\}$	$\{7, 140, 161, 69\}$	$\{150, 70, 159, 80\}$	$\{151, 76, 125, 81\}$
$\{115, 37, 129, 71\}$	$\{79, 136, 6, 152\}$	$\{111, 65, 22, 94\}$	$\{62, 146, 155, 77\}$
$\{123, 78, 112, 67\}$			
$\{0, 83, 111, 30\}$	$\{1, 84, 114, 33\}$	$\{2, 85, 117, 36\}$	$\{3, 86, 124, 46\}$
$\{4, 88, 92, 12\}$	$\{5, 89, 96, 17\}$	$\{6, 90, 98, 19\}$	$\{7, 91, 103, 23\}$
$\{10, 94, 110, 34\}$	$\{11, 95, 113, 59\}$	$\{9, 93, 119, 55\}$	$\{15, 99, 126, 65\}$
$\{13, 97, 127, 71\}$	$\{16, 101, 105, 38\}$	$\{21, 106, 112, 77\}$	$\{40, 125, 134, 22\}$
$\{43, 128, 140, 31\}$	$\{24, 109, 130, 76\}$	$\{35, 120, 150, 8\}$	$\{47, 133, 138, 37\}$
$\{49, 135, 141, 25\}$	$\{58, 144, 154, 14\}$	$\{56, 142, 104, 39\}$	$\{107, 26, 146, 68\}$
$\{129, 48, 87, 45\}$	$\{131, 51, 137, 57\}$	$\{122, 42, 132, 53\}$	$\{147, 67, 161, 81\}$
$\{121, 41, 139, 63\}$	$\{160, 70, 162, 50\}$	$\{158, 78, 108, 52\}$	$\{82, 74, 149, 64\}$
$\{153, 60, 118, 66\}$	$\{73, 157, 20, 123\}$	$\{32, 143, 18, 152\}$	$\{80, 69, 145, 102\}$
$\{155, 75, 54, 148\}$	$\{72, 156, 159, 79\}$	$\{28, 100, 115, 61\}$	$\{151, 44, 136, 29\}$
$\{27, 116, 62, 163\}$			
$\{1, 87, 49, 148\}$	$\{82, 2, 103, 26\}$	$\{83, 3, 111, 47\}$	$\{84, 4, 113, 52\}$
$\{85, 5, 115, 75\}$	$\{89, 9, 145, 76\}$	$\{90, 11, 102, 72\}$	$\{91, 12, 110, 38\}$
$\{92, 13, 125, 65\}$	$\{88, 10, 97, 78\}$	$\{94, 16, 104, 70\}$	$\{96, 18, 144, 79\}$
$\{15, 98, 17, 101\}$	$\{29, 112, 69, 109\}$	$\{44, 127, 6, 123\}$	$\{24, 108, 30, 114\}$
$\{22, 106, 31, 116\}$	$\{23, 107, 33, 118\}$	$\{21, 105, 35, 119\}$	$\{51, 135, 71, 99\}$
$\{42, 126, 63, 150\}$	$\{7, 163, 59, 151\}$	$\{50, 134, 80, 122\}$	$\{55, 139, 28, 121\}$
$\{8, 93, 20, 154\}$	$\{39, 124, 56, 100\}$	$\{45, 130, 64, 156\}$	$\{74, 159, 25, 129\}$
$\{77, 149, 34, 143\}$	$\{0, 86, 14, 120\}$	$\{81, 147, 48, 142\}$	$\{66, 67, 157, 158\}$
$\{53, 54, 152, 153\}$	$\{57, 60, 146, 161\}$	$\{162, 137, 46, 73\}$	$\{131, 160, 62, 43\}$
$\{133, 140, 32, 37\}$	$\{132, 27, 141, 36\}$	$\{136, 61, 117, 40\}$	$\{19, 138, 58, 95\}$
$\{41, 155, 68, 128\}$			

□

Lemma .0.6. *There exists an $RH(12^9)$.*

Proof. In [57], Mills constructed an $H(6^9)$ on $Z_{27}\times Z_2$ with group set $\{\{(m,0),(m+9,0),(m+18,0),(m,1),(m+9,1),(m+18,1)\} : m \in Z_9\}\}$ and the following 42 forms of blocks, where $m \in Z_{27}$, $a, b \in Z_2$.

$\{(m,a),(m+2,a),(m+5,b),(m+7,b)\}$,$\{(m,a),(m+5,b),(m+12,a+b),(m+20,a+b+1)\}$
$\{(m,a),(m+1,a),(m+5,b),(m+6,b)\}$,$\{(m,a),(m+5,b),(m+10,a+1),(m+21,a+b+1)\}$
$\{(m,a),(m+1,a),(m+7,b),(m+8,b)\}$,$\{(m,a),(m+5,b),(m+12,a+b+1),(m+17,a+1)\}$
$\{(m,a),(m+2,a),(m+8,b),(m+10,b)\}$,$\{(m,a),(m+6,b),(m+13,a+b+1),(m+19,a+1)\}$

$\{(m,a),(m+2,b),(m+12,b),(m+17,a)\}$,$\{(m,a),(m+1,b),(m+15,a+b+1),(m+17,a+1)\}$
$\{(m,a),(m+1,b),(m+2,a+b),(m+4,b)\}$,$\{(m,a),(m+1,b),(m+12,a+b+1),(m+14,b+1)\}$
$\{(m,a),(m+2,a+1),(m+5,b),(m+8,b)\}$,$\{(m,a),(m+2,a+1),(m+7,b),(m+23,a+b+1)\}$
$\{(m,a),(m+3,a),(m+7,b),(m+20,a+b)\}$,$\{(m,a),(m+2,a+1),(m+6,b),(m+21,a+b+1)\}$
$\{(m,a),(m+4,b),(m+8,a),(m+15,a+b)\}$,$\{(m,a),(m+4,b),(m+8,a+1),(m+16,a+b+1)\}$
$\{(m,a),(m+1,a+1),(m+5,b),(m+6,b+1)\}$,$\{(m,a),(m+3,a),(m+16,b),(m+23,a+b+1)\}$
$\{(m,a),(m+3,a),(m+11,b),(m+14,b+1)\}$,$\{(m,a),(m+4,b),(m+14,b+1),(m+21,a+b)\}$
$\{(m,a),(m+1,a+1),(m+7,b),(m+8,b+1)\}$,$\{(m,a),(m+5,b),(m+10,a),(m+16,a+b+1)\}$
$\{(m,a),(m+6,b),(m+12,a+b),(m+19,a)\}$,$\{(m,a),(m+2,a+1),(m+10,b),(m+24,a+b)\}$
$\{(m,a),(m+2,a+1),(m+19,b),(m+22,b)\}$,$\{(m,a),(m+4,b),(m+10,a+b),(m+20,a+b)\}$
$\{(m,a),(m+1,b),(m+11,b),(m+12,a+b)\}$,$\{(m,a),(m+3,a),(m+15,b),(m+19,a+b+1)\}$
$\{(m,a),(m+1,b),(m+11,b+1),(m+13,a)\}$,$\{(m,a),(m+1,b),(m+14,b),(m+16,a+b+1)\}$
$\{(m,a),(m+1,b),(m+3,a+1),(m+4,b+1)\}$,$\{(m,a),(m+3,a+1),(m+11,b),(m+15,a+b)\}$
$\{(m,a),(m+3,a+1),(m+8,b),(m+22,a+b)\}$,$\{(m,a),(m+1,b),(m+13,a+1),(m+15,a+b)\}$
$\{(m,a),(m+3,b),(m+6,b+1),(m+17,a+1)\}$,$\{(m,a),(m+5,b),(m+11,a+b),(m+19,a+b)\}$
$\{(m,a),(m+3,a+1),(m+7,b),(m+10,b+1)\}$,$\{(m,a),(m+3,b),(m+13,a+b+1),(m+17,a)\}$
$\{(m,a),(m+2,a),(m+6,b),(m+23,a+b+1)\}$,$\{(m,a),(m+1,b),(m+2,a+b+1),(m+25,b)\}$

For each form of blocks, taking $m=0$ and $a,b\in Z_2$, we get four blocks. Thus we get 168 base blocks of the H(6^9) all together, denote the set by $\mathcal{B}''$, which are developed by $(+1\ mod\ 27,-)$. Define a map $\phi:(x,y)\rightarrow 27y+x$, for each element $(x,y)\in Z_{27}\times Z_2$. Then $\mathcal{B}'=\phi(\mathcal{B}'')$ is the base block set of an H(6^9) on $I_{54}=\{0,1,\ldots,53\}$ with group set $\{\{m,m+9,\ldots,m+45\}:m\in Z_9\}$ and an automorphism group $\langle(0\ 1\ldots 26)(27\ 28\ldots 53)\rangle$.

For each block $B=\{a,b,c,d\}\in\mathcal{B}'$, construct an H($2^4$) with group set $\{\{x,x+54\}:x\in B\}$ and block set $\mathcal{A}_B=\{\{a+54i,b+54(i+k),c+54j,d+54(j+k)\}:i,j,k\in Z_2\}$. Let $\mathcal{B}=\cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an H(12^9) on $I_{108}=\{0,1,\ldots,107\}$ with group set $\{\{m,m+9,\ldots,m+99\}:m\in Z_9\}$ and an automorphism group $\langle\alpha\rangle$ with $\alpha=(0\ 1\ldots 26)(27\ 28\ldots 53)(54\ 55\ldots 80)(81\ 82\ldots 107)$. Now we need to show the resolution.

Note that there are several blocks in $\mathcal{B}$, each of which contains exactly one element in each cycle of α. We first list below some of these blocks and denote them by Δ, each block of which gives a parallel class when developed under the automorphism group $\langle\alpha\rangle$.

$\{0,28,59,87\}$	$\{0,28,61,89\}$	$\{0,29,60,102\}$	$\{0,30,61,91\}$	$\{0,28,86,60\}$
$\{0,28,88,62\}$	$\{0,28,56,106\}$	$\{0,28,84,58\}$	$\{0,28,95,70\}$	$\{0,29,93,71\}$

$\{0,32,66,98\}$	$\{0,32,93,74\}$	$\{0,29,88,77\}$	$\{0,30,60,98\}$	$\{0,32,91,75\}$
$\{0,31,89,70\}$	$\{0,30,88,64\}$	$\{27,1,59,87\}$	$\{27,1,61,89\}$	$\{27,1,83,58\}$
$\{27,1,57,85\}$	$\{27,1,65,93\}$	$\{27,1,66,95\}$	$\{27,1,67,96\}$	$\{27,5,93,74\}$
$\{27,3,67,98\}$	$\{27,3,87,71\}$	$\{27,3,65,96\}$	$\{27,3,61,91\}$	$\{27,1,88,62\}$
$\{27,2,91,78\}$	$\{27,3,89,76\}$	$\{27,3,88,64\}$	$\{54,82,5,33\}$	$\{54,82,7,35\}$
$\{54,83,7,50\}$	$\{54,82,32,6\}$	$\{54,82,34,8\}$	$\{54,82,2,52\}$	$\{54,82,30,4\}$
$\{54,82,41,16\}$	$\{54,82,15,44\}$	$\{54,83,39,17\}$	$\{54,86,12,44\}$	$\{54,86,39,20\}$
$\{54,85,8,42\}$	$\{54,83,34,23\}$	$\{54,83,33,21\}$	$\{54,84,6,44\}$	$\{54,86,37,21\}$
$\{81,55,5,33\}$	$\{81,55,7,35\}$	$\{81,55,29,4\}$	$\{81,55,11,39\}$	$\{81,55,12,41\}$
$\{81,55,13,42\}$	$\{81,56,12,44\}$	$\{81,59,39,20\}$	$\{81,56,10,51\}$	$\{81,57,33,17\}$
$\{81,59,37,16\}$	$\{81,57,11,42\}$	$\{81,57,7,37\}$	$\{81,55,32,6\}$	$\{81,55,34,8\}$
$\{81,56,37,24\}$	$\{81,57,35,22\}$	$\{81,57,38,15\}$	$\{81,57,34,10\}$	$\{0,55,30,85\}$
$\{0,59,39,98\}$	$\{0,57,33,98\}$	$\{0,59,37,102\}$	$\{0,58,35,97\}$	$\{0,55,32,87\}$
$\{0,82,41,70\}$	$\{0,86,39,74\}$	$\{0,83,34,77\}$	$\{0,56,32,88\}$	$\{0,83,33,75\}$
$\{0,56,35,91\}$	$\{0,86,37,75\}$	$\{0,84,34,64\}$	$\{0,57,34,101\}$	$\{0,61,40,98\}$
$\{0,62,41,101\}$	$\{27,82,5,60\}$	$\{27,55,5,87\}$	$\{27,82,7,62\}$	$\{27,55,7,89\}$
$\{27,55,3,85\}$	$\{27,55,11,93\}$	$\{27,55,12,95\}$	$\{27,56,12,98\}$	$\{27,83,6,77\}$
$\{27,83,8,64\}$	$\{27,57,8,103\}$	$\{27,57,13,98\}$	$\{27,57,11,96\}$	$\{27,84,15,73\}$
$\{0,67,34,98\}$	$\{27,82,13,69\}$	$\{27,84,6,71\}$	$\{27,85,10,74\}$	$\{27,86,11,73\}$
$\{54,1,84,31\}$	$\{54,1,93,41\}$	$\{54,3,87,44\}$	$\{54,5,91,48\}$	$\{54,4,89,43\}$
$\{54,28,86,6\}$	$\{54,1,88,35\}$	$\{54,28,88,8\}$	$\{54,28,84,4\}$	$\{54,28,95,16\}$
$\{54,29,93,17\}$	$\{54,32,93,20\}$	$\{54,2,86,34\}$	$\{54,29,87,21\}$	$\{54,2,89,37\}$
$\{54,32,91,21\}$	$\{54,30,88,10\}$	$\{54,3,88,47\}$	$\{54,6,94,46\}$	$\{54,7,94,44\}$
$\{81,1,59,33\}$	$\{81,28,61,8\}$	$\{81,1,61,35\}$	$\{81,1,65,39\}$	$\{81,1,66,41\}$
$\{81,1,67,42\}$	$\{81,2,66,44\}$	$\{81,29,62,10\}$	$\{81,2,64,51\}$	$\{81,3,67,44\}$
$\{81,3,65,42\}$	$\{81,30,69,19\}$	$\{81,30,70,23\}$	$\{81,3,61,37\}$	$\{54,13,87,46\}$
$\{81,28,57,4\}$	$\{81,28,67,15\}$	$\{81,30,60,17\}$	$\{81,31,64,20\}$	$\{81,32,65,19\}$
$\{0,82,59,33\}$	$\{0,82,61,35\}$	$\{0,55,93,41\}$	$\{0,55,96,44\}$	$\{0,59,93,44\}$
$\{0,57,87,44\}$	$\{0,59,91,48\}$	$\{0,84,61,37\}$	$\{0,55,86,33\}$	$\{0,55,88,35\}$
$\{0,82,56,52\}$	$\{0,82,69,44\}$	$\{0,85,62,42\}$	$\{0,56,86,34\}$	$\{0,56,89,37\}$
$\{0,84,60,44\}$	$\{0,60,94,46\}$	$\{0,61,94,44\}$	$\{0,62,95,47\}$	$\{27,82,61,8\}$
$\{27,55,83,4\}$	$\{27,83,59,7\}$	$\{27,83,60,23\}$	$\{27,83,62,10\}$	$\{27,57,87,17\}$
$\{27,59,91,16\}$	$\{27,84,69,19\}$	$\{27,84,70,23\}$	$\{27,55,86,6\}$	$\{0,67,87,46\}$
$\{27,57,89,22\}$	$\{27,86,65,19\}$	$\{27,57,92,15\}$	$\{27,57,88,10\}$	$\{54,28,5,87\}$
$\{54,28,7,89\}$	$\{54,1,30,85\}$	$\{54,1,42,98\}$	$\{54,5,39,98\}$	$\{54,29,7,104\}$
$\{54,29,6,102\}$	$\{54,3,33,98\}$	$\{54,5,37,102\}$	$\{54,4,35,97\}$	$\{54,30,7,91\}$
$\{54,1,32,87\}$	$\{54,1,34,89\}$	$\{54,28,2,106\}$	$\{54,28,15,98\}$	$\{54,32,12,98\}$
$\{54,31,8,96\}$	$\{54,2,32,88\}$	$\{54,2,35,91\}$	$\{54,30,6,98\}$	$\{54,3,34,101\}$
$\{81,28,7,62\}$	$\{81,1,29,58\}$	$\{81,5,39,74\}$	$\{81,29,5,61\}$	$\{81,29,6,77\}$
$\{81,29,8,64\}$	$\{81,3,33,71\}$	$\{81,30,15,73\}$	$\{81,30,16,77\}$	$\{81,1,32,60\}$
$\{54,13,33,100\}$	$\{54,13,34,98\}$	$\{81,1,34,62\}$	$\{81,28,3,58\}$	$\{81,28,13,69\}$
$\{81,2,37,78\}$	$\{81,3,35,76\}$	$\{81,30,6,71\}$	$\{81,31,10,74\}$	$\{81,3,38,69\}$

$\{81,3,34,64\}$	$\{27,84,16,77\}$	$\{0,67,33,100\}$	$\{27,2,64,105\}$	$\{27,3,62,103\}$
$\{54,85,35,16\}$	$\{54,84,34,10\}$	$\{0,82,34,62\}$	$\{0,82,30,58\}$	$\{54,8,95,47\}$
$\{81,28,59,6\}$	$\{0,83,61,50\}$	$\{0,83,60,48\}$	$\{0,67,88,44\}$	$\{27,82,57,4\}$
$\{54,8,41,101\}$	$\{81,28,5,60\}$			

Then we shift each of the remaining base blocks in $\mathcal{B}$ by a suitable automorphism α^i for some integer i. The result is listed below, where the blocks in each of the six consecutive rows, namely the ith, $(i+1)$th, ..., and $(i+5)$th rows for $i \in \{6k+1 : k = 0, 1, \ldots, 40\}$, form a parallel class.

$\{0,1,5,6\}$	$\{2,30,7,35\}$	$\{3,4,10,11\}$	$\{8,36,15,43\}$	$\{12,13,14,16\}$
$\{19,20,48,17\}$	$\{21,22,51,52\}$	$\{23,24,34,9\}$	$\{25,26,37,39\}$	$\{18,46,47,49\}$
$\{27,28,32,33\}$	$\{40,41,42,38\}$	$\{54,55,59,60\}$	$\{56,84,61,89\}$	$\{57,58,64,65\}$
$\{62,90,69,97\}$	$\{66,67,68,70\}$	$\{73,74,102,71\}$	$\{75,76,105,106\}$	$\{77,78,88,63\}$
$\{80,81,91,92\}$	$\{72,107,86,93\}$	$\{83,85,95,100\}$	$\{94,96,99,101\}$	$\{44,45,103,104\}$
$\{53,82,31,87\}$	$\{50,79,29,98\}$			
$\{0,1,11,12\}$	$\{2,3,42,17\}$	$\{4,5,18,47\}$	$\{6,7,48,50\}$	$\{8,10,20,25\}$
$\{14,19,53,31\}$	$\{9,38,16,32\}$	$\{15,44,21,36\}$	$\{23,26,29,40\}$	$\{13,41,51,52\}$
$\{24,27,35,39\}$	$\{43,45,28,33\}$	$\{54,55,65,66\}$	$\{56,57,95,97\}$	$\{58,59,98,73\}$
$\{60,61,74,103\}$	$\{62,63,104,106\}$	$\{70,75,82,87\}$	$\{64,69,76,84\}$	$\{71,100,78,94\}$
$\{72,101,77,80\}$	$\{83,88,93,99\}$	$\{86,90,67,102\}$	$\{89,92,105,85\}$	$\{22,79,37,68\}$
$\{34,91,49,107\}$	$\{81,30,96,46\}$			
$\{0,5,12,47\}$	$\{2,6,10,17\}$	$\{9,11,14,16\}$	$\{13,42,18,21\}$	$\{1,3,7,51\}$
$\{25,27,8,22\}$	$\{4,33,23,26\}$	$\{15,20,52,36\}$	$\{24,53,29,32\}$	$\{19,48,38,41\}$
$\{30,31,44,46\}$	$\{43,45,49,39\}$	$\{54,56,66,71\}$	$\{55,59,63,70\}$	$\{60,62,65,67\}$
$\{72,74,78,95\}$	$\{58,87,64,106\}$	$\{57,86,76,79\}$	$\{80,83,61,75\}$	$\{68,73,105,89\}$
$\{69,98,88,91\}$	$\{102,103,104,100\}$	$\{92,97,77,85\}$	$\{99,101,107,82\}$	$\{93,96,28,35\}$
$\{34,90,40,84\}$	$\{37,94,50,81\}$			
$\{0,2,8,10\}$	$\{1,31,9,23\}$	$\{3,6,43,20\}$	$\{11,15,21,4\}$	$\{7,12,17,50\}$
$\{13,18,24,5\}$	$\{22,26,30,38\}$	$\{19,47,51,25\}$	$\{14,42,16,39\}$	$\{27,28,34,35\}$
$\{46,49,32,36\}$	$\{37,40,52,29\}$	$\{54,56,62,64\}$	$\{55,84,65,79\}$	$\{57,60,97,74\}$
$\{58,61,91,102\}$	$\{66,70,76,59\}$	$\{63,68,73,106\}$	$\{69,72,80,83\}$	$\{71,101,78,81\}$
$\{77,107,85,99\}$	$\{95,98,75,88\}$	$\{86,87,93,94\}$	$\{89,90,103,105\}$	$\{44,45,100,96\}$
$\{48,104,33,92\}$	$\{67,41,53,82\}$			
$\{0,30,11,15\}$	$\{1,4,16,47\}$	$\{2,5,13,43\}$	$\{3,6,19,53\}$	$\{7,37,14,44\}$
$\{17,20,24,10\}$	$\{8,9,40,41\}$	$\{21,22,28,29\}$	$\{18,46,52,26\}$	$\{23,51,36,38\}$
$\{25,27,35,49\}$	$\{54,59,65,73\}$	$\{56,60,91,99\}$	$\{55,85,66,70\}$	$\{61,64,76,107\}$
$\{69,72,58,92\}$	$\{62,63,94,95\}$	$\{68,96,100,74\}$	$\{79,80,86,87\}$	$\{71,77,84,90\}$
$\{93,67,104,106\}$	$\{82,57,89,105\}$	$\{31,33,97,102\}$	$\{75,81,34,42\}$	$\{83,88,12,50\}$
$\{101,48,103,45\}$	$\{39,98,78,32\}$			
$\{0,28,30,4\}$	$\{1,29,12,14\}$	$\{3,31,15,17\}$	$\{5,33,46,21\}$	$\{7,35,22,51\}$

$\{8,37,47,25\}$	$\{6,38,18,50\}$	$\{9,41,48,2\}$	$\{11,42,19,53\}$	$\{20,49,27,16\}$
$\{24,26,32,34\}$	$\{39,40,23,52\}$	$\{54,57,61,74\}$	$\{55,83,89,63\}$	$\{56,84,85,87\}$
$\{65,93,67,90\}$	$\{58,86,88,62\}$	$\{64,92,75,77\}$	$\{66,94,78,80\}$	$\{72,101,106,68\}$
$\{71,100,81,95\}$	$\{59,91,97,105\}$	$\{69,99,103,79\}$	$\{96,70,98,73\}$	$\{10,13,107,60\}$
$\{43,45,102,104\}$	$\{44,76,82,36\}$			
$\{0,2,32,34\}$	$\{3,5,36,26\}$	$\{4,33,37,25\}$	$\{8,38,43,30\}$	$\{1,31,14,18\}$
$\{9,39,15,53\}$	$\{12,44,49,6\}$	$\{24,29,7,13\}$	$\{16,48,27,35\}$	$\{11,41,45,21\}$
$\{46,20,22,50\}$	$\{54,82,94,96\}$	$\{56,84,97,72\}$	$\{55,83,70,99\}$	$\{59,88,98,76\}$
$\{57,89,69,101\}$	$\{68,100,107,61\}$	$\{71,102,79,86\}$	$\{58,60,90,92\}$	$\{74,104,80,91\}$
$\{65,73,106,85\}$	$\{103,77,64,66\}$	$\{42,17,75,63\}$	$\{67,95,51,52\}$	$\{23,78,10,93\}$
$\{47,105,28,62\}$	$\{19,81,87,40\}$			
$\{0,31,37,47\}$	$\{1,32,36,17\}$	$\{2,5,44,21\}$	$\{4,7,42,18\}$	$\{3,6,46,26\}$
$\{9,12,43,29\}$	$\{16,20,10,30\}$	$\{8,14,48,27\}$	$\{45,19,23,51\}$	$\{41,15,25,53\}$
$\{39,13,50,52\}$	$\{34,35,49,24\}$	$\{54,83,86,89\}$	$\{55,57,88,78\}$	$\{58,87,91,79\}$
$\{59,61,94,96\}$	$\{60,90,73,77\}$	$\{62,93,99,82\}$	$\{69,101,106,63\}$	$\{67,98,102,56\}$
$\{100,74,75,71\}$	$\{92,68,76,107\}$	$\{105,81,66,70\}$	$\{64,97,104,85\}$	$\{72,103,28,38\}$
$\{11,95,22,80\}$	$\{65,40,84,33\}$			
$\{0,6,12,19\}$	$\{1,7,21,42\}$	$\{3,10,43,47\}$	$\{5,13,46,52\}$	$\{30,31,8,9\}$
$\{34,35,14,15\}$	$\{37,11,17,45\}$	$\{48,22,50,25\}$	$\{49,23,24,20\}$	$\{28,2,16,18\}$
$\{51,27,4,44\}$	$\{33,36,40,26\}$	$\{54,86,64,70\}$	$\{55,85,93,97\}$	$\{56,59,98,75\}$
$\{57,60,95,71\}$	$\{62,65,105,58\}$	$\{69,72,103,89\}$	$\{73,77,67,87\}$	$\{68,74,61,82\}$
$\{92,66,76,104\}$	$\{96,99,80,83\}$	$\{100,101,88,63\}$	$\{91,94,102,78\}$	$\{38,41,107,84\}$
$\{79,53,81,29\}$	$\{90,39,106,32\}$			
$\{27,1,12,41\}$	$\{28,2,14,43\}$	$\{29,3,16,18\}$	$\{30,5,15,47\}$	$\{32,10,17,22\}$
$\{33,11,45,26\}$	$\{31,8,39,46\}$	$\{44,19,24,13\}$	$\{40,42,21,23\}$	$\{53,4,9,20\}$
$\{49,25,6,37\}$	$\{54,60,66,73\}$	$\{55,62,95,99\}$	$\{85,86,63,64\}$	$\{83,57,61,89\}$
$\{87,88,67,68\}$	$\{96,70,76,104\}$	$\{98,72,74,102\}$	$\{91,65,77,106\}$	$\{105,56,90,71\}$
$\{101,78,84,94\}$	$\{97,75,107,59\}$	$\{35,38,69,82\}$	$\{52,80,36,92\}$	$\{79,0,58,48\}$
$\{100,50,81,7\}$	$\{103,51,34,93\}$			
$\{27,29,5,7\}$	$\{28,3,6,9\}$	$\{31,33,10,0\}$	$\{37,12,16,4\}$	$\{38,13,21,35\}$
$\{43,18,8,11\}$	$\{39,15,20,34\}$	$\{47,24,30,40\}$	$\{36,14,46,25\}$	$\{45,48,2,32\}$
$\{41,42,52,26\}$	$\{81,55,66,95\}$	$\{82,56,69,71\}$	$\{83,58,68,100\}$	$\{85,63,70,75\}$
$\{84,61,92,99\}$	$\{98,73,78,67\}$	$\{94,96,72,74\}$	$\{106,54,57,60\}$	$\{59,93,101,107\}$
$\{80,90,97,103\}$	$\{104,105,88,62\}$	$\{50,51,89,64\}$	$\{102,76,23,19\}$	$\{86,91,44,22\}$
$\{1,87,65,17\}$	$\{49,77,79,53\}$			
$\{27,3,13,44\}$	$\{28,4,34,18\}$	$\{29,7,40,48\}$	$\{31,8,12,20\}$	$\{33,36,21,25\}$
$\{47,50,9,16\}$	$\{35,11,15,45\}$	$\{6,19,39,52\}$	$\{17,51,32,38\}$	$\{41,42,53,1\}$
$\{81,83,60,77\}$	$\{82,57,61,76\}$	$\{84,86,65,67\}$	$\{88,63,71,85\}$	$\{87,62,79,55\}$
$\{94,70,75,89\}$	$\{105,54,64,95\}$	$\{90,66,96,80\}$	$\{91,69,74,58\}$	$\{92,93,97,98\}$
$\{10,14,99,107\}$	$\{2,5,72,106\}$	$\{68,73,24,30\}$	$\{102,103,26,0\}$	$\{100,101,22,23\}$
$\{49,78,59,46\}$	$\{56,37,43,104\}$			
$\{27,1,32,6\}$	$\{0,13,34,44\}$	$\{2,35,42,50\}$	$\{4,39,45,52\}$	$\{3,40,47,53\}$

$\{31,5,38,12\}$	$\{48,49,24,25\}$	$\{28,29,14,16\}$	$\{33,37,41,21\}$	$\{51,26,43,46\}$
$\{81,59,92,100\}$	$\{83,60,64,72\}$	$\{82,85,71,78\}$	$\{86,62,66,96\}$	$\{84,58,89,63\}$
$\{55,68,88,101\}$	$\{61,74,95,105\}$	$\{99,73,106,80\}$	$\{90,91,65,94\}$	$\{104,107,56,67\}$
$\{8,9,102,77\}$	$\{54,57,7,20\}$	$\{69,103,30,36\}$	$\{15,70,76,23\}$	$\{11,93,97,17\}$
$\{18,75,87,10\}$	$\{98,19,22,79\}$			
$\{60,9,76,29\}$	$\{96,43,84,5\}$	$\{27,86,37,97\}$	$\{32,7,91,94\}$	$\{65,12,13,69\}$
$\{1,61,67,20\}$	$\{103,107,30,10\}$	$\{3,59,8,64\}$	$\{31,63,41,74\}$	$\{106,26,90,38\}$
$\{21,77,2,58\}$	$\{70,44,50,78\}$	$\{6,88,101,22\}$	$\{25,33,93,100\}$	$\{66,72,24,4\}$
$\{95,98,81,85\}$	$\{47,105,55,36\}$	$\{52,57,62,19\}$	$\{92,15,73,0\}$	$\{39,42,28,35\}$
$\{89,40,18,83\}$	$\{102,53,87,11\}$	$\{17,46,54,68\}$	$\{48,23,80,56\}$	$\{45,104,82,34\}$
$\{75,49,51,79\}$	$\{16,71,99,14\}$			
$\{77,106,28,31\}$	$\{19,20,85,87\}$	$\{48,76,86,34\}$	$\{94,97,53,30\}$	$\{16,44,81,82\}$
$\{13,72,24,59\}$	$\{33,61,35,64\}$	$\{70,73,27,3\}$	$\{80,2,95,18\}$	$\{50,105,91,39\}$
$\{17,23,56,63\}$	$\{9,65,69,32\}$	$\{6,43,104,83\}$	$\{107,0,67,15\}$	$\{92,41,45,58\}$
$\{55,57,7,51\}$	$\{62,36,74,22\}$	$\{98,99,46,42\}$	$\{75,79,4,14\}$	$\{21,78,88,11\}$
$\{68,96,26,1\}$	$\{84,5,103,52\}$	$\{38,40,100,102\}$	$\{8,90,37,93\}$	$\{60,10,54,47\}$
$\{66,49,29,89\}$	$\{71,101,25,12\}$			
$\{99,100,58,60\}$	$\{98,75,25,6\}$	$\{103,106,83,69\}$	$\{81,5,66,17\}$	$\{59,64,16,24\}$
$\{87,91,70,80\}$	$\{93,14,22,90\}$	$\{72,19,56,3\}$	$\{97,102,54,62\}$	$\{76,78,28,18\}$
$\{46,51,2,40\}$	$\{30,35,42,20\}$	$\{39,67,71,45\}$	$\{38,13,44,32\}$	$\{49,77,27,55\}$
$\{86,61,11,26\}$	$\{9,92,15,84\}$	$\{89,37,101,52\}$	$\{31,36,41,47\}$	$\{65,68,0,34\}$
$\{48,105,1,95\}$	$\{21,104,94,43\}$	$\{23,79,29,73\}$	$\{12,96,74,7\}$	$\{53,82,85,33\}$
$\{50,107,88,10\}$	$\{57,4,8,63\}$			
$\{32,61,24,54\}$	$\{5,34,91,94\}$	$\{86,6,10,92\}$	$\{88,37,47,105\}$	$\{60,9,46,77\}$
$\{100,21,80,15\}$	$\{79,28,11,69\}$	$\{102,49,87,8\}$	$\{26,81,93,41\}$	$\{82,83,43,18\}$
$\{70,107,33,39\}$	$\{30,85,35,90\}$	$\{55,7,67,20\}$	$\{59,62,48,1\}$	$\{104,0,4,66\}$
$\{14,98,22,63\}$	$\{57,65,44,50\}$	$\{13,97,101,23\}$	$\{25,3,64,99\}$	$\{42,71,103,38\}$
$\{72,73,52,53\}$	$\{76,78,27,29\}$	$\{96,17,106,12\}$	$\{2,58,68,19\}$	$\{89,36,40,95\}$
$\{84,31,45,74\}$	$\{75,51,56,16\}$			
$\{69,21,62,29\}$	$\{76,26,16,90\}$	$\{41,42,73,74\}$	$\{27,31,10,20\}$	$\{38,39,107,55\}$
$\{65,14,80,30\}$	$\{33,34,8,37\}$	$\{32,9,67,75\}$	$\{97,45,102,50\}$	$\{72,47,52,68\}$
$\{83,87,12,22\}$	$\{106,82,40,44\}$	$\{94,43,28,86\}$	$\{11,95,103,53\}$	$\{46,24,58,63\}$
$\{1,2,60,61\}$	$\{48,78,59,36\}$	$\{35,92,51,85\}$	$\{91,66,17,6\}$	$\{96,99,84,88\}$
$\{79,0,64,15\}$	$\{101,77,81,57\}$	$\{25,3,89,100\}$	$\{105,56,7,18\}$	$\{93,98,23,4\}$
$\{71,19,104,13\}$	$\{49,54,5,70\}$			
$\{77,2,8,69\}$	$\{26,82,92,16\}$	$\{67,18,23,83\}$	$\{21,107,4,64\}$	$\{52,55,65,42\}$
$\{89,10,95,29\}$	$\{80,57,7,14\}$	$\{78,27,62,12\}$	$\{59,60,47,49\}$	$\{5,88,15,56\}$
$\{19,102,11,68\}$	$\{38,39,99,100\}$	$\{50,53,84,70\}$	$\{96,20,25,63\}$	$\{87,61,44,46\}$
$\{98,73,51,40\}$	$\{101,48,22,105\}$	$\{3,9,97,103\}$	$\{45,104,1,93\}$	$\{66,13,106,0\}$
$\{35,90,94,41\}$	$\{76,79,6,36\}$	$\{31,34,74,54\}$	$\{17,72,58,33\}$	$\{30,85,86,28\}$
$\{75,24,37,71\}$	$\{81,32,91,43\}$			
$\{9,37,74,76\}$	$\{18,101,25,95\}$	$\{26,2,60,73\}$	$\{19,79,66,33\}$	$\{100,50,54,35\}$

$\{63,64,11,13\}$	$\{27,55,69,17\}$	$\{40,68,80,28\}$	$\{61,89,36,38\}$	$\{47,22,93,96\}$
$\{7,92,15,103\}$	$\{94,41,78,53\}$	$\{46,51,57,65\}$	$\{52,0,86,102\}$	$\{21,49,88,90\}$
$\{56,5,44,75\}$	$\{62,12,70,23\}$	$\{30,85,99,20\}$	$\{3,6,91,104\}$	$\{24,82,34,98\}$
$\{84,4,59,1\}$	$\{29,58,48,105\}$	$\{67,42,77,10\}$	$\{16,107,87,39\}$	$\{81,31,8,97\}$
$\{83,32,45,106\}$	$\{43,71,72,14\}$			
$\{90,92,14,16\}$	$\{69,71,0,5\}$	$\{22,104,106,26\}$	$\{20,103,30,98\}$	$\{31,6,38,27\}$
$\{82,2,4,86\}$	$\{102,76,8,10\}$	$\{66,42,101,34\}$	$\{75,50,85,45\}$	$\{99,21,107,13\}$
$\{61,37,15,56\}$	$\{3,89,96,23\}$	$\{33,35,41,43\}$	$\{52,80,64,39\}$	$\{100,47,53,81\}$
$\{1,84,65,25\}$	$\{83,88,95,73\}$	$\{62,40,72,24\}$	$\{91,93,97,87\}$	$\{48,77,55,17\}$
$\{12,67,19,74\}$	$\{18,46,57,59\}$	$\{32,7,78,54\}$	$\{58,60,9,11\}$	$\{68,70,49,51\}$
$\{105,29,63,44\}$	$\{79,28,36,94\}$			
$\{24,55,18,92\}$	$\{20,75,85,6\}$	$\{5,62,43,73\}$	$\{52,0,35,22\}$	$\{82,2,14,97\}$
$\{81,29,87,50\}$	$\{7,89,45,100\}$	$\{1,56,39,68\}$	$\{102,53,31,91\}$	$\{4,86,11,93\}$
$\{71,46,9,66\}$	$\{94,15,47,90\}$	$\{59,34,64,13\}$	$\{21,106,83,10\}$	$\{69,17,23,79\}$
$\{37,42,101,107\}$	$\{49,78,95,44\}$	$\{76,27,32,70\}$	$\{63,12,103,26\}$	$\{74,25,57,36\}$
$\{61,96,48,28\}$	$\{30,60,41,72\}$	$\{38,40,98,88\}$	$\{105,80,84,99\}$	$\{19,77,67,33\}$
$\{54,3,16,104\}$	$\{8,65,51,58\}$			
$\{41,43,76,78\}$	$\{48,77,53,83\}$	$\{12,13,95,64\}$	$\{27,55,2,79\}$	$\{59,33,99,47\}$
$\{80,0,82,24\}$	$\{100,51,31,63\}$	$\{88,93,44,50\}$	$\{26,57,9,73\}$	$\{34,11,98,81\}$
$\{7,10,72,102\}$	$\{8,92,97,30\}$	$\{67,14,20,75\}$	$\{23,106,84,19\}$	$\{91,68,45,52\}$
$\{29,4,89,104\}$	$\{17,74,6,94\}$	$\{25,28,87,101\}$	$\{58,5,42,71\}$	$\{1,61,21,96\}$
$\{32,62,39,69\}$	$\{15,16,56,85\}$	$\{18,22,66,86\}$	$\{37,40,70,54\}$	$\{103,107,3,38\}$
$\{36,65,46,60\}$	$\{35,90,49,105\}$			
$\{57,58,14,15\}$	$\{1,59,9,70\}$	$\{24,56,62,16\}$	$\{72,76,12,32\}$	$\{27,86,39,71\}$
$\{3,60,64,23\}$	$\{96,43,26,82\}$	$\{90,65,41,44\}$	$\{34,89,95,42\}$	$\{31,8,93,100\}$
$\{50,106,83,46\}$	$\{101,48,88,36\}$	$\{28,85,97,47\}$	$\{40,68,78,52\}$	$\{105,107,5,7\}$
$\{20,49,66,69\}$	$\{17,19,77,94\}$	$\{61,37,67,51\}$	$\{73,21,54,2\}$	$\{4,87,91,25\}$
$\{29,33,10,45\}$	$\{98,99,74,75\}$	$\{79,80,38,13\}$	$\{6,92,18,104\}$	$\{0,84,35,103\}$
$\{53,81,55,30\}$	$\{102,22,63,11\}$			
$\{67,96,50,37\}$	$\{94,17,104,33\}$	$\{81,5,92,46\}$	$\{89,10,40,97\}$	$\{16,71,18,74\}$
$\{45,49,55,65\}$	$\{44,72,58,6\}$	$\{59,39,47,107\}$	$\{95,15,100,20\}$	$\{98,19,9,66\}$
$\{75,27,88,42\}$	$\{87,90,21,25\}$	$\{14,105,31,91\}$	$\{43,101,24,86\}$	$\{85,61,93,80\}$
$\{79,28,32,62\}$	$\{26,2,68,99\}$	$\{54,82,11,13\}$	$\{51,0,35,12\}$	$\{1,56,60,7\}$
$\{77,78,3,4\}$	$\{106,53,36,64\}$	$\{8,38,70,57\}$	$\{22,23,83,84\}$	$\{73,103,30,34\}$
$\{102,52,29,63\}$	$\{41,69,48,76\}$			
$\{0,2,87,77\}$	$\{7,64,14,54\}$	$\{81,3,92,15\}$	$\{6,34,89,91\}$	$\{61,35,76,51\}$
$\{63,10,65,13\}$	$\{16,74,78,4\}$	$\{1,57,88,24\}$	$\{100,20,30,86\}$	$\{32,37,69,107\}$
$\{98,99,83,58\}$	$\{9,95,47,82\}$	$\{39,41,71,73\}$	$\{101,48,85,5\}$	$\{96,45,53,56\}$
$\{50,26,31,18\}$	$\{72,75,52,38\}$	$\{59,33,97,44\}$	$\{11,94,70,19\}$	$\{93,40,68,43\}$
$\{22,106,62,12\}$	$\{105,25,8,90\}$	$\{36,67,17,79\}$	$\{21,23,80,55\}$	$\{102,103,28,29\}$
$\{66,42,104,27\}$	$\{46,49,84,60\}$			
$\{29,3,71,73\}$	$\{26,56,68,45\}$	$\{105,106,36,11\}$	$\{5,6,89,90\}$	$\{102,23,80,2\}$

$\{25,107,64,12\}$	$\{93,41,44,100\}$	$\{103,27,7,96\}$	$\{54,57,42,19\}$	$\{1,61,13,74\}$
$\{40,99,77,34\}$	$\{67,46,53,88\}$	$\{8,38,75,79\}$	$\{51,52,92,94\}$	$\{83,84,9,10\}$
$\{101,49,82,30\}$	$\{14,69,70,18\}$	$\{58,32,65,39\}$	$\{21,22,59,60\}$	$\{95,43,47,91\}$
$\{97,17,104,24\}$	$\{63,37,20,76\}$	$\{16,98,81,28\}$	$\{33,62,72,50\}$	$\{48,78,55,31\}$
$\{66,15,0,85\}$	$\{4,86,87,35\}$			
$\{39,13,80,55\}$	$\{99,101,50,52\}$	$\{74,23,90,16\}$	$\{70,48,27,89\}$	$\{25,86,94,46\}$
$\{47,104,31,61\}$	$\{0,82,88,8\}$	$\{81,5,38,100\}$	$\{35,64,14,56\}$	$\{37,12,71,60\}$
$\{10,69,22,84\}$	$\{57,32,36,78\}$	$\{106,2,87,40\}$	$\{9,96,103,30\}$	$\{43,98,54,29\}$
$\{97,18,49,91\}$	$\{72,19,83,4\}$	$\{20,75,33,62\}$	$\{44,73,24,67\}$	$\{65,68,51,1\}$
$\{85,7,11,95\}$	$\{34,66,45,107\}$	$\{79,3,63,17\}$	$\{102,26,58,15\}$	$\{92,41,105,28\}$
$\{21,76,59,6\}$	$\{93,42,77,53\}$			
$\{32,36,67,102\}$	$\{72,19,2,57\}$	$\{38,39,103,77\}$	$\{85,61,96,73\}$	$\{83,58,48,51\}$
$\{54,28,59,33\}$	$\{47,76,107,41\}$	$\{68,70,22,24\}$	$\{46,49,30,6\}$	$\{26,81,10,66\}$
$\{62,91,13,16\}$	$\{35,90,42,97\}$	$\{56,31,63,52\}$	$\{29,4,34,37\}$	$\{94,17,50,87\}$
$\{8,64,20,79\}$	$\{60,7,65,12\}$	$\{40,95,15,98\}$	$\{21,78,5,89\}$	$\{18,100,23,105\}$
$\{99,74,82,69\}$	$\{25,0,84,86\}$	$\{1,14,88,101\}$	$\{92,93,43,44\}$	$\{71,45,55,3\}$
$\{106,53,27,104\}$	$\{9,11,75,80\}$			
$\{66,73,52,29\}$	$\{69,44,34,91\}$	$\{86,35,12,106\}$	$\{24,54,94,20\}$	$\{82,84,36,38\}$
$\{13,95,15,92\}$	$\{62,14,1,103\}$	$\{42,71,101,50\}$	$\{80,3,63,19\}$	$\{56,5,9,76\}$
$\{30,33,68,98\}$	$\{57,7,11,72\}$	$\{75,61,27,40\}$	$\{28,60,67,18\}$	$\{99,21,4,89\}$
$\{107,55,32,47\}$	$\{78,64,31,41\}$	$\{17,102,81,37\}$	$\{16,100,22,87\}$	$\{96,45,26,83\}$
$\{90,39,97,2\}$	$\{23,79,58,6\}$	$\{10,65,104,25\}$	$\{48,49,59,88\}$	$\{70,46,77,53\}$
$\{105,0,8,93\}$	$\{43,74,51,85\}$			
$\{81,82,41,43\}$	$\{76,53,57,37\}$	$\{50,51,63,65\}$	$\{83,84,15,17\}$	$\{6,92,97,0\}$
$\{16,75,26,86\}$	$\{67,45,23,56\}$	$\{4,7,91,102\}$	$\{55,3,34,78\}$	$\{62,68,48,27\}$
$\{90,13,44,105\}$	$\{88,89,18,47\}$	$\{85,9,95,20\}$	$\{12,94,77,25\}$	$\{71,46,103,52\}$
$\{14,69,19,74\}$	$\{80,28,38,70\}$	$\{64,93,2,5\}$	$\{104,24,10,66\}$	$\{35,40,101,79\}$
$\{61,8,36,59\}$	$\{39,42,54,58\}$	$\{29,32,99,106\}$	$\{72,100,31,33\}$	$\{96,98,21,11\}$
$\{107,1,87,49\}$	$\{73,22,30,60\}$			
$\{74,80,13,34\}$	$\{43,100,83,33\}$	$\{5,62,45,76\}$	$\{102,107,6,41\}$	$\{98,19,63,12\}$
$\{20,21,54,55\}$	$\{84,87,10,50\}$	$\{9,68,73,52\}$	$\{86,37,69,53\}$	$\{2,85,48,105\}$
$\{46,101,31,60\}$	$\{88,91,23,3\}$	$\{90,11,15,57\}$	$\{51,81,8,92\}$	$\{64,38,39,95\}$
$\{56,7,14,103\}$	$\{71,49,82,36\}$	$\{42,97,44,94\}$	$\{40,96,106,30\}$	$\{72,22,1,65\}$
$\{26,28,58,61\}$	$\{75,104,4,18\}$	$\{70,17,77,24\}$	$\{67,16,78,27\}$	$\{32,89,93,25\}$
$\{79,29,35,99\}$	$\{0,59,66,47\}$			
$\{73,50,83,39\}$	$\{99,74,10,13\}$	$\{49,79,57,44\}$	$\{26,6,93,97\}$	$\{17,100,63,12\}$
$\{15,16,107,55\}$	$\{62,94,46,27\}$	$\{58,59,34,35\}$	$\{72,48,2,60\}$	$\{36,38,41,43\}$
$\{9,91,21,77\}$	$\{75,80,31,42\}$	$\{101,76,25,1\}$	$\{37,96,22,84\}$	$\{102,105,28,14\}$
$\{40,95,106,0\}$	$\{54,29,32,89\}$	$\{33,90,71,47\}$	$\{20,103,52,82\}$	$\{18,23,56,64\}$
$\{5,7,67,69\}$	$\{30,61,65,19\}$	$\{87,92,70,81\}$	$\{88,66,45,53\}$	$\{11,98,51,86\}$
$\{104,24,3,85\}$	$\{4,8,68,78\}$			
$\{103,54,7,12\}$	$\{24,25,83,84\}$	$\{62,67,20,28\}$	$\{102,104,33,38\}$	$\{75,23,0,98\}$

{26, 4, 63, 96}	{50, 78, 57, 31}	{19, 48, 92, 95}	{21, 53, 86, 94}	{68, 44, 49, 90}
{69, 18, 107, 2}	{73, 74, 51, 52}	{105, 81, 3, 14}	{16, 17, 85, 87}	{13, 70, 56, 36}
{55, 58, 34, 45}	{106, 1, 6, 101}	{8, 65, 100, 22}	{42, 97, 30, 59}	{71, 27, 88, 40}
{47, 79, 5, 64}	{11, 41, 76, 80}	{37, 66, 15, 72}	{46, 77, 29, 93}	{32, 61, 39, 82}
{89, 9, 10, 60}	{35, 91, 43, 99}			
{27, 32, 12, 47}	{28, 31, 7, 18}	{29, 34, 13, 21}	{33, 9, 40, 16}	{81, 82, 65, 94}
{83, 87, 91, 71}	{84, 59, 89, 92}	{85, 60, 104, 107}	{0, 1, 56, 58}	{3, 8, 96, 101}
{6, 10, 68, 75}	{11, 14, 105, 55}	{24, 26, 86, 88}	{15, 44, 106, 93}	{48, 22, 77, 73}
{61, 62, 36, 5}	{63, 64, 23, 52}	{25, 54, 4, 102}	{78, 53, 70, 19}	{72, 46, 74, 43}
{67, 45, 79, 30}	{100, 49, 80, 39}	{90, 37, 95, 42}	{57, 38, 98, 51}	{35, 66, 97, 50}
{41, 99, 103, 2}	{69, 17, 20, 76}			
{0, 31, 91, 101}	{1, 33, 65, 71}	{2, 32, 94, 98}	{3, 6, 99, 76}	{4, 7, 96, 72}
{5, 11, 79, 100}	{8, 16, 103, 82}	{27, 28, 61, 62}	{39, 13, 104, 106}	{34, 36, 67, 57}
{37, 15, 74, 58}	{40, 18, 105, 86}	{9, 22, 97, 107}	{14, 47, 81, 89}	{44, 45, 73, 102}
{29, 30, 59, 60}	{38, 43, 77, 85}	{48, 52, 83, 63}	{68, 69, 19, 20}	{50, 78, 10, 66}
{53, 54, 12, 95}	{41, 70, 24, 92}	{25, 84, 64, 42}	{26, 56, 87, 46}	{35, 90, 75, 23}
{51, 80, 88, 21}	{93, 17, 49, 55}			
{0, 34, 96, 102}	{27, 30, 94, 98}	{54, 55, 38, 13}	{56, 57, 41, 43}	{59, 64, 44, 49}
{58, 62, 39, 47}	{60, 90, 17, 21}	{63, 66, 24, 28}	{68, 97, 33, 36}	{61, 91, 42, 29}
{69, 99, 1, 5}	{73, 105, 2, 8}	{100, 74, 7, 9}	{85, 88, 15, 45}	{82, 83, 3, 32}
{107, 81, 37, 11}	{89, 92, 46, 22}	{12, 67, 23, 78}	{75, 50, 4, 72}	{106, 26, 10, 93}
{65, 95, 18, 48}	{103, 77, 25, 53}	{16, 71, 31, 87}	{19, 101, 51, 79}	{40, 70, 20, 104}
{52, 80, 86, 6}	{84, 35, 14, 76}			
{81, 58, 37, 47}	{82, 84, 34, 51}	{0, 55, 29, 79}	{2, 85, 7, 64}	{3, 60, 18, 103}
{4, 88, 11, 95}	{1, 83, 41, 97}	{8, 90, 23, 106}	{9, 93, 22, 80}	{5, 89, 43, 101}
{28, 56, 16, 72}	{27, 59, 39, 74}	{32, 62, 38, 76}	{25, 86, 40, 100}	{49, 104, 33, 61}
{45, 75, 53, 67}	{66, 14, 71, 19}	{17, 46, 78, 94}	{31, 6, 70, 102}	{69, 98, 21, 36}
{10, 65, 44, 99}	{13, 96, 52, 57}	{50, 105, 26, 54}	{73, 48, 107, 15}	{92, 12, 68, 42}
{30, 87, 63, 20}	{77, 24, 35, 91}			
{0, 89, 41, 102}	{27, 82, 11, 94}	{28, 85, 35, 75}	{54, 1, 68, 43}	{55, 6, 67, 48}
{56, 31, 62, 50}	{58, 7, 65, 24}	{57, 33, 70, 20}	{90, 10, 92, 13}	{84, 8, 96, 23}
{81, 2, 60, 21}	{95, 17, 101, 4}	{91, 38, 98, 45}	{72, 52, 87, 39}	{88, 9, 93, 42}
{97, 19, 104, 26}	{5, 61, 64, 12}	{78, 29, 36, 71}	{76, 51, 32, 100}	{80, 30, 34, 69}
{18, 47, 105, 66}	{40, 14, 99, 73}	{103, 79, 3, 44}	{83, 59, 15, 46}	{74, 25, 86, 37}
{22, 77, 106, 53}	{63, 16, 49, 107}			
{81, 30, 92, 14}	{0, 58, 64, 20}	{1, 85, 66, 16}	{2, 59, 67, 43}	{3, 86, 89, 38}
{4, 87, 95, 28}	{5, 91, 97, 51}	{31, 60, 63, 12}	{32, 61, 65, 26}	{27, 56, 73, 22}
{40, 71, 104, 33}	{36, 93, 70, 29}	{50, 105, 88, 8}	{35, 94, 101, 25}	{34, 90, 96, 44}
{72, 47, 23, 80}	{62, 13, 19, 54}	{6, 37, 68, 102}	{39, 17, 103, 55}	{42, 18, 107, 57}
{9, 69, 49, 82}	{74, 21, 106, 53}	{75, 52, 83, 10}	{98, 46, 76, 24}	{100, 48, 79, 15}
{45, 77, 84, 11}	{41, 99, 78, 7}			
{27, 84, 97, 50}	{54, 1, 40, 69}	{55, 2, 15, 98}	{56, 4, 14, 73}	{59, 8, 16, 100}

$\{57,31,35,63\}$	$\{60,34,44,99\}$	$\{62,36,20,76\}$	$\{58,32,45,74\}$	$\{85,5,19,75\}$
$\{86,7,17,103\}$	$\{82,6,13,72\}$	$\{101,22,0,70\}$	$\{81,28,39,68\}$	$\{91,38,51,107\}$
$\{94,42,48,104\}$	$\{88,9,53,83\}$	$\{24,52,66,95\}$	$\{10,65,49,105\}$	$\{21,106,29,64\}$
$\{33,89,11,67\}$	$\{78,25,93,41\}$	$\{96,18,77,37\}$	$\{80,12,87,43\}$	$\{3,61,92,46\}$
$\{47,102,79,26\}$	$\{71,23,30,90\}$			

$\square$

Papers Completed

[1] Xiande Zhang and Gennian Ge, *Super-simple resolvable balanced incomplete block designs with block size* 4 *and index* 2, J. Combin. Des. **15** (2007), 341–356. (SCI Indexed)

[2] Xiande Zhang and Gennian Ge, *On the existence of partitionable skew room frames*, Discrete Math. **307** (2007), 2786–2807. (SCI Indexed)

[3] Xiande Zhang and Gennian Ge, *Existence of Z-cyclic* $3PDTWh(p)$ *for prime* $p \equiv 1$ (mod 4), Des. Codes Cryptogr. **45** (2007), 139–155. (SCI Indexed)

[4] Gennian Ge, Ying Miao and Xiande Zhang, *On block sequences of Steiner quadruple systems with error correcting consecutive unions*, SIAM J. Discrete Math. **23** (2009), 940–958. (SCI Indexed)

[5] Xiande Zhang and Gennian Ge, *Maximal resolvable packings and minimal resolvable coverings of triples by quadruples*, J. Combin. Des., to appear. (SCI Indexed)

[6] Xiande Zhang and Gennian Ge, *Combinatorial constructions of fault-tolerant routings with levelled minimum optical indices*, preprint.

[7] Xiande Zhang and Gennian Ge, *A new existence proof for Steiner quadruple systems via H-designs of type* 2^n, preprint.

[8] Xiande Zhang and Gennian Ge, *A new existence proof for resolvable Steiner quadruple systems via resolvable H-designs*, preprint.

[9] Xiande Zhang and Gennian Ge, *H-designs with the properties of resolvability or* $(1,2)$*-resolvability*, preprint.

Resume

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• Combinatorial structures with applications in bioinformatics, network, cryptography and coding theory