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H-designs with the properties of resolvability or (1, 2)-resolvability

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Abstract An H-design is said to be $(1, \alpha)$-resolvable, if its block set can be partitioned into α-parallel classes, each of which contains every point of the design exactly α times. When $\alpha = 1$, a $(1, \alpha)$-resolvable H-design of type g^n is simply called a resolvable H-design and denoted by RH(g^n), for which the general existence problem has been determined leaving mainly the case of $g \equiv 0 \pmod{12}$ open. When $\alpha = 2$, a (1, 2)-RH(1^n) is usually called a (1, 2)-resolvable Steiner quadruple system of order n, for which the existence problem is far from complete. In this paper, we consider these two outstanding problems. First, we prove that an RH(12^n) exists for all $n \geq 4$ with a small number of possible exceptions. Next, we give a near complete solution to the existence problem of (1, 2)-resolvable H-designs with group size 2. As a consequence, we obtain a near complete solution to the above two open problems.

Keywords $(1, \alpha)$-resolvable · Candelabra t-systems · H-designs · H-frames · Steiner systems

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1 Introduction

Let v be a non-negative integer, t be a positive integer and K be a set of positive integers. A *group divisible t-design* of order v with block sizes from K, denoted by GDD(t, K, v), is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

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(1) X is a set of v elements (called *points*);
(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of nonempty subsets (called *groups*) of X which partition X;
(3) $\mathcal{B}$ is a family of transverses (called *blocks*) of $\mathcal{G}$, each of cardinality from K, where a *transverse* is a subset of X intersects any given group in at most one point;
(4) every t-element transverse T of $\mathcal{G}$ is contained in a unique block.

The *type* of the GDD(t, K, v) is defined as the list $(|G||G \in \mathcal{G})$. If a GDD has n_i groups of size g_i, $1 \le i \le r$, then we use the notation $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ to denote the group type. Mills in [8] used H(n, g, k, t) design to denote the GDD(t, k, ng) of type g^n. In this paper, we use H$(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r})$ to denote the GDD$(3, 4, \sum n_i g_i)$ of type $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ for short.

For the existence of H(g^n), Mills [8] showed that for $n > 3, n \neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for $n = 5$, an H(g^5) exists if g is divisible by 4 or 6. Recently, Ji [5] improved these results by showing that an H(g^5) exists whenever g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$. When $g = 1$, an H(1^n) is normally called a *Steiner quadruple system* of order n, denoted by SQS(n). It is well known that an SQS(n) exists if and only if $n \equiv 2$ or $4 \pmod 6$ [1].

An H(g^n) is said to be $(1, \alpha)$*-resolvable*, denoted by $(1, \alpha)$-RH(g^n), if its block set can be partitioned into parts (called α*-parallel classes*), such that each point of the design occurs in exactly α blocks in each part.

When $\alpha = 1$, an α-parallel class is normally called a *parallel class* and the design is said to be *resolvable*. An RH(1^n) is usually called a *resolvable Steiner quadruple system* of order n, denoted by RSQS(n), which exists if and only if $n \equiv 4$ or $8 \pmod{12}$ [2,6]. For the general existence problem of resolvable H-designs, Zhang and Ge [10,11] recently studied the existence of RH(g^n) when $g \in \{2, 3, 4, 6\}$. In [11], the authors concluded that the existence problem of RH(g^n) for all admissible parameters was reduced to the two open cases of $n \equiv 3 \pmod 6$ when $g = 12$ and $n \in \{73, 149\}$ when $g = 4$. We summarize these results as follows:

Theorem 1.1 *The necessary conditions* $gn \equiv 0 \pmod 4$, $g(n-1)(n-2) \equiv 0 \pmod 3$ *and* $n \geq 4$ *for the existence of an RH*(g^n) *are sufficient for each* $g \equiv 1, 2, 3, 5, 6, 7, 9, 10, 11 \pmod{12}$*, and also sufficient for each* $g \equiv 4, 8 \pmod{12}$ *with two possible exceptions* $n = 73, 149$.

A $(1, 2)$-RH(1^n) is usually called a $(1, 2)$*-resolvable Steiner quadruple system* of order n, denoted by RSQS$(1, 2, n)$, of which the necessary conditions for the existence are $n \equiv 2$ or $10 \pmod{12}$. Hartman and Phelps [4] posed a question that: Whether the necessary conditions for the existence of an RSQS$(1, 2, n)$ are also sufficient? Recently, Meng, Ji and Du [7] proved the nonexistence of an RSQS$(1, 2, 10)$ and the existence of an RSQS$(1, 2, n)$ when $n \equiv 74 \pmod{96}$. They also established a connection between an RSQS$(1, 2, 2n)$ and a $(1, 2)$-RH(2^n) as follows.

Theorem 1.2 *If there exists a* $(1, 2)$*-RH*(2^n) *and n is odd, then there exists an RSQS*$(1, 2, 2n)$.

Motivated by the above two theorems, in this paper, we will investigate the existence of resolvable H-designs with group size 12 and $(1, 2)$-resolvable H-designs with group size 2. The remainder of this paper is organized as follows. In Sect. 2, we will describe several recursive constructions for $(1, \alpha)$-resolvable H-designs based on the theory of uniformly resolvable candelabra systems and $(1, \alpha)$-resolvable H-frames. In Sect. 3, we first establish a tripling construction for resolvable H-designs with group size 12, then combining with the recursive methods in Sect. 2, we give an almost complete solution to the existence problem of an RH(12^n). In Sect. 4, we provide a product construction and two tripling constructions for $(1, 2)$-resolvable H-designs with group size 2, then combining with several initial designs, we give an almost complete solution to the existence problem of a (1, 2)-RH(2^n). As a consequence, we prove that the necessary conditions for the existence of an RSQS$(1, 2, n)$ are also sufficient with a finite number of possible exceptions, which will be addressed in Sect. 5.

2 Standard recursive constructions

In this section, we shall describe several recursive constructions for $(1, \alpha)$-resolvable H-designs.

Lemma 2.1 (Weighting construction) *Suppose that there exists a* $(1, \alpha)$*-RH*(g^n). *Then there is a* $(1, \alpha)$*-RH*$((mg)^n)$ *for any positive integer* m.

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given $(1, \alpha)$-RH(g^n) with group set $\mathcal{G} = \{G_0, \ldots, G_{n-1}\}$ and block set $\mathcal{B}$, which has a resolution $P(i)$, $1 \le i \le (n-1)(n-2)g^2/6\alpha$. For each positive integer m, we will construct a $(1, \alpha)$-RH$((mg)^n)$ on $X \times Z_m$ with groups $G_i \times Z_m$, $0 \le i \le n-1$ as follows.

For each block $B \in \mathcal{B}$, construct an RH(m^4) on $B \times Z_m$ with group set $\{\{x\} \times Z_m : x \in B\}$ and block set $\mathcal{A}_B$, which has a resolution $P_B(k)$, $1 \le k \le m^2$. Such a design exists by [6]. Let $\mathcal{B}' = \cup_{B \in \mathcal{B}} \mathcal{A}_B$. Then $\mathcal{B}'$ is the block set of the desired $(1, \alpha)$-RH$((mg)^n)$, which has a resolution $Q_{i,k} = \cup_{B \in P(i)} P_B(k)$ with $1 \le i \le (n-1)(n-2)g^2/6\alpha$ and $1 \le k \le m^2$. □

Let s be a non-negative integer. A *candelabra* t*-system* (or t-CS) of order v and block sizes from K, denoted by CS(t, K, v), is a quadruple $(X, S, \mathcal{G}, \mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements;
(2) S is an s-subset (called the *stem* of the *candelabra*) of X;
(3) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of $X \backslash S$, which partition $X \backslash S$;
(4) $\mathcal{A}$ is a collection of subsets of X, each of cardinality from K;
(5) every t-subset T of X with $|T \cap (S \cup G_i)| < t$, for all i, is contained in a unique block of $\mathcal{A}$, and no t-subset of $S \cup G_i$, for any i, is contained in any block of $\mathcal{A}$.

The *group type* of a t-CS $(X, S, \mathcal{G}, \mathcal{A})$ is defined as the list $(|G||G \in \mathcal{G} : |S|)$. If a t-CS has n_i groups of size g_i, $1 \le i \le r$, and stem size s, then we use the notation $(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$ to denote the group type. A candelabra system with $t = 3$ and $K = \{4\}$ is called a *candelabra quadruple system* and denoted by CQS$(g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r} : s)$.

A CS$(t, K, v)(X, S, \mathcal{G}, \mathcal{A})$ is said to be *resolvable*, denoted by RCS(t, K, v), if the block set $\mathcal{A}$ can be partitioned into several parts, each being a partition on X (called a *parallel class*) or a partition on $X \backslash (G \cup S)$ for some $G \in \mathcal{G}$ (called a *partial parallel class*). An RCS(t, K, v) is called *uniform*, denoted by URCS(t, K, v) if all the blocks in each resolution class have the same size. If $K = \{4\}$, it is denoted by RCQS, for which the number of parallel classes on X is $((\sum_{G \in \mathcal{G}} |G|)^2 - \sum_{G \in \mathcal{G}} |G|^2)/6$ and the number of partial parallel classes on $X \backslash (G \cup S)$ is $|G|(|G| + 2|S| - 3)/6$ for each $G \in \mathcal{G}$.

Lemma 2.2 [7] *For each integer* $n \ge 2$, *there exists an RCQS*$(3^{(2^{2n}-1)/3} : 1)$.

For non-negative integers q, g, k and t, an $H(q, g, k, t)$ *frame* (as in [3]), denoted by HF(q, g, k, t), is a quadruple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of qg points;
2. $\mathcal{G} = \{G_1, G_2, \ldots, G_q\}$ is an equipartition of X into q groups;

3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i = \{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its size; and
4. $\mathcal{B}$ is a set of k-element transverses of $\mathcal{G}$ with the property that every t-element transverse of $\mathcal{G}$, which is not a t-element transverse of any hole $F_i \in \mathcal{F}$ is contained in precisely one block of $\mathcal{B}$, and no block contains a t-element transverse of any hole.

If an HF$(q, g, 4, 3)$ has n holes of size $m+s$, which intersect on a common hole of size s, then we denote such a design by HF$(m^n : s)$ with group size g, or shortly by HF$_g(m^n : s)$. If an HF$(q, g, 4, 3)$ has only one hole of size s, then we call it an *incomplete H-design* of type $(g^q : g^s)$, denoted by IH$(g^q : g^s)$.

An HF$_g(m^n : s)$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with $\mathcal{F} = \{F_i : 0 \le i \le n\}$ and F_0 the common hole of size s is said to be $(1, \alpha)$-*resolvable*, denoted by $(1, \alpha)$-RHF$_g(m^n : s)$, if its block set can be partitioned into $(nmg^2(m+2s-3)+n(n-1)(mg)^2)/6\alpha$ parts with the following properties:

(1) For each hole F_i, $1 \le i \le n$, there are exactly $mg^2(m+2s-3)/6\alpha$ parts, each being a partial α-parallel class of $X\backslash(\bigcup_{G\in F_i} G)$;
(2) There are $n(n-1)(mg)^2/6\alpha$ parts, each being an α-parallel class on X.

An IH$(g^{m+s} : g^s)$ $(X, \mathcal{G}, \mathcal{B}, F)$ with the only hole F of size s is said to be $(1, \alpha)$-*resolvable*, denoted by $(1, \alpha)$-IRH$(g^{m+s} : g^s)$, if its block set can be partitioned into $(m+s-1)(m+s-2)g^2/6\alpha$ parts, $(s-1)(s-2)g^2/6\alpha$ of which are partial α-parallel classes of $X\backslash(\bigcup_{G\in F} G)$, and $m(m+2s-3)g^2/6\alpha$ of which are α-parallel classes on X. When $\alpha = 1$, a $(1, \alpha)$-RHF$_g(m^n : s)$ and a $(1, \alpha)$-IRH$(g^{m+s} : g^s)$ are simply denoted by RHF$_g(m^n : s)$ and IRH$(g^{m+s} : g^s)$, respectively.

The constructions given below are simple extensions of [11, Theorems 2.4 and 2.5].

Theorem 2.3 *Suppose that* $(X, S, \Gamma, \mathcal{A})$ *is a* 3-*CS*$(m^n : s)$ *and* $\infty \in S$. *Let* $K_1 = \{|A| : \infty \in A \in \mathcal{A}\}$ *and* $K_2 = \{|A| : \infty \notin A \in \mathcal{A}\}$. *If there exists an HF*$_g(t^{k_1-1} : a)$ *for each* $k_1 \in K_1$ *and an H*$((gt)^{k_2})$ *for each* $k_2 \in K_2$, *then there exists an HF*$_g((tm)^n : t(s-1)+a)$. *Furthermore, if the* 3-*CS*$(m^n : s)$ *is uniformly resolvable, and each of the HF*$_g(t^{k_1-1} : a)$ *and the H*$((gt)^{k_2})$ *is* $(1, \alpha)$-*resolvable,* $k_1 \in K_1$ *and* $k_2 \in K_2$, *then the resultant HF*$_g((tm)^n : t(s-1)+a)$ *is also* $(1, \alpha)$-*resolvable.*

Proof Suppose $(X, S, \Gamma, \mathcal{A})$ is the given URCS$(m^n : s)$, where $\Gamma = \{G_1, \ldots, G_n\}$ and $\mathcal{A}$ has a resolution $\mathcal{A} = (\bigcup_{1\le i\le n} \mathcal{Q}_i) \bigcup \mathcal{Q}$ with each member of $\mathcal{Q}_i$ being a partial parallel class on $X\backslash(G_i \cup S)$ and each member of $\mathcal{Q}$ being a parallel class on X. Define $G'_{x,j} = \{x\}\times\{j\}\times Z_g$. Let $X' = ((X\backslash\{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$, $\mathcal{G}' = \{G'_{x,j} : x \in X\backslash\{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$, $\mathcal{F} = \{F_i : 0 \le i \le n\}$, where $F_0 = \{G'_{x,j} : x \in S\backslash\{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and $F_i = \{G'_{x,j} : x \in G_i, j \in Z_t\} \cup F_0$ for $1 \le i \le n$. We will construct a $(1, \alpha)$-RHF$_g((tm)^n : t(s-1)+a)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}$.

For each $B \in \mathcal{A}$ and $\infty \in B$, construct a $(1, \alpha)$-RHF$_g(t^{|B|-1} : a)$ on $X'_B = ((B\backslash\{\infty\}) \times Z_t \times Z_g) \cup (\{\infty\} \times Z_a \times Z_g)$ with group set $\mathcal{G}'_B = \{G'_{x,j} : x \in B\backslash\{\infty\}, j \in Z_t\} \cup \{G'_{\infty,j} : j \in Z_a\}$ and hole set $\mathcal{F}_B = \{F_x : x \in B\}$, where $F_x = \{G'_{x,j} : j \in Z_t\} \cup F_\infty$ with $F_\infty = \{G'_{\infty,j} : j \in Z_a\}$ being the common hole of size a. Denote its block set by $\mathcal{C}_B$, which has a resolution $\{\mathcal{C}_B(x, j) : x \in B\backslash\{\infty\}, 1 \le j \le tg^2(t+2a-3)/6\alpha\} \cup \{\mathcal{C}_B(l) : 1 \le l \le (|B|-1)(|B|-2)(tg)^2/6\alpha\}$ with each $\mathcal{C}_B(x, j)$ being a partial α-parallel class on $X'_B\backslash(\bigcup_{G\in F_x} G)$ and each $\mathcal{C}_B(l)$ being an α-parallel class on X'_B.

For each $B \in \mathcal{A}$ and $\infty \notin B$, construct a $(1, \alpha)$-RH$((gt)^{|B|})$ on $X'_B = B \times Z_t \times Z_g$ with group set $\mathcal{G}'_B = \{\{x\} \times Z_t \times Z_g : x \in B\}$ and block set $\mathcal{C}_B$, which can be partitioned into α-parallel classes $\mathcal{C}_B(l)$, $1 \le l \le (|B|-1)(|B|-2)(tg)^2/6\alpha$.

Then $\mathcal{A}' = \bigcup_{B\in\mathcal{A}} \mathcal{C}_B$ is the block set of the required design. We need to partition the blocks into resolution classes.

For each member $Q \in \mathcal{Q}_i$, $1 \le i \le n$, suppose its block size is k_Q. Then $P_Q(l) = \bigcup_{B\in Q} \mathcal{C}_B(l)$ is a partial α-parallel class of $X' \backslash (\bigcup_{G\in F_i} G)$ for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6\alpha$.

For each $x \in \bigcup_{G\in F_i} G$, $1 \le i \le n$, $P_{x,j} = \bigcup_{B\in\mathcal{A},\infty\in B} \mathcal{C}_B(x, j)$ is a partial α-parallel class of $X' \backslash (\bigcup_{G\in F_i} G)$ for $1 \le j \le tg^2(t + 2a - 3)/6\alpha$.

For each member $Q \in \mathcal{Q}$, suppose its block size is k_Q. Then $P'_Q(l) = \bigcup_{B\in Q} \mathcal{C}_B(l)$ is an α-parallel class of X' for $1 \le l \le (k_Q - 1)(k_Q - 2)(tg)^2/6\alpha$.

Thus we obtain a $(1, \alpha)$-RHF$_g((tm)^n : t(s - 1) + a)$. □

Theorem 2.4 *Suppose that there exists a $(1, \alpha)$-RHF$_g(m^n : s)$. If there exists a $(1, \alpha)$-IRH$(g^{m+s} : g^s)$, then there exists a $(1, \alpha)$-IRH$(g^{mn+s} : g^{m+s})$. Furthermore, if there is a $(1, \alpha)$-RH(g^{m+s}), then there is a $(1, \alpha)$-RH(g^{mn+s}).*

3 Resolvable H-designs with group size 12

In this section, we shall study the existence of resolvable H-designs with group size 12.

Theorem 3.1 (Product construction I) [11, Theorem 2.6] *If there exist both an RH(g^m) and an RH(g^n), then there exists an RH(g^{mn}) and an IRH$(g^{mn} : g^n)$.*

A regular graph (V, E) of degree k is said to have a *one-factorization* if the edge set E can be partitioned into k parts $E = F_1|F_2|\ldots|F_k$ so that each F_i is a partition of the vertex set V into pairs. The parts F_i are called *one-factors*.

For $x \in Z_n$, we define $|x|$ by

$$|x| = \begin{cases} x, & \text{if } 0 \le x \le n/2, \\ -x, & \text{if } n/2 < x < n. \end{cases}$$

For $n \ge 2$ and $L \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$, define G$(n, L)$ be the regular graph with vertex set Z_n and edge set E given by $\{x, y\} \in E$ if and only if $|x - y| \in L$.

The following lemma was stated by Stern and Lenz in [9].

Lemma 3.2 *Let $L \subseteq \{1, 2, \ldots, n\}$. Then $G(2n, L)$ has a one-factorization if and only if $2n/gcd(j, 2n)$ is even for some $j \in L$.*

The concept of a resolvable B_4-pairing has been defined in [11] to establish the tripling construction for resolvable H-designs with group size 4. Here, for resolvable H-designs with group size 12, a concept of B_{12}-pairing is given below.

For non-negative integers n and s, a B_{12}-pairing, $B_{12}(n, s)$ consists of four subsets D, R_0, R_1, R_2 of $Z_{12(n+s)}$ and three subsets PR_0, PR_1, PR_2 of $Z_{12(n+s)} \times Z_{12(n+s)}$ with the following properties for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

  (a) $|D| = 12s$, $|R_i| = 4n$,
  (b) $D = -D$.

(2) *Partitioning conditions*

  (a) PR_i is a partition of R_i into pairs, thus $|PR_i| = 2n$,
  (b) $Z_{12(n+s)} = D \cup R_0 \cup R_1 \cup R_2$.

(3) *Pairing conditions*
Let $L_i = \{|x-y| : \{x, y\} \in PR_i\}$ and $N = \{n+s, 2(n+s), \ldots, 6(n+s)\}$,

(a) $N \cap L_i = \emptyset$,
(b) $|L_i| = 2n$,
(c) the complement G_i of the graph $G(12(n+s), L_i \cup N)$ has a one-factorization.

Let $S_0, S_1, S_2, \overline{R}_0, \overline{R}_1, \overline{R}_2$ be subsets of $Z_{12(n+s)}$ and PS_0, PS_1, PS_2 be subsets of $Z_{12(n+s)} \times Z_{12(n+s)}$. A B_{12}-pairing $B_{12}(n, s)$ with $D, R_i, PR_i, i \in \{0, 1, 2\}$, is said to be *resolvable*, denoted by $RB_{12}(n, s)$, if the following properties are satisfied for each $i \in \{0, 1, 2\}$:

(1) *Cardinality and symmetry conditions*

(c) $|S_i| = 4n, |\overline{R}_i| = 2n$.

(2) *Partitioning conditions*

(c) PS_i is a partition of S_i into pairs, thus $|PS_i| = 2n$,
(d) $Z_{12(n+s)} = D \cup R_i \cup S_i \cup \overline{R}_{i+1} \cup -\overline{R}_{i-1}$.

(3) *Pairing conditions*
Let $O_i = \{|x-y| : \{x, y\} \in PS_i\}$,

(d) $N \cap O_i = \emptyset$,
(e) $|O_i| = 2n$, $L_i \cap O_i = \emptyset$, and all members of O_i are odd,
(f) the complement G_i' of the graph $G(12(n+s), L_i \cup O_i \cup N)$ has a one-factorization.

We have the following theorem as in [10,11].

Theorem 3.3 *If there exists a $B_{12}(n, s)$, then there exists an $HF_{12}((n+s)^3 : s)$. Furthermore, if the $B_{12}(n, s)$ is resolvable, then the $HF_{12}((n+s)^3 : s)$ is resolvable. Moreover, if $k(n+s) \in D$ for all k, $0 \le k \le 11$, then the resultant $RHF_{12}((n+s)^3 : s)$ has a sub-design $RH(12^4)$.*

In order to construct an $RB_{12}(n, s)$, as in [11], we shall construct a special $B_{12}(n, s)$ on $Z_{12(n+s)}$, D, R_i, PR_i, with extra subsets A_i of $Z_{12(n+s)}$ and PA_i of $Z_{12(n+s)} \times Z_{12(n+s)}$ satisfying the following conditions for each $i \in \{0, 1, 2\}$:

(1) $R_i = -R_i$, $A_i \subset R_i$, $|A_i| = 2n$,
(2) PA_i is a partition of A_i into pairs. Let $O_i' = \{|x-y| : \{x, y\} \in PA_i\}$,

(a) $|O_i'| = n$, all O_0', O_1', O_2' are disjoint and of odd members,
(b) $(\cup_{i=0}^2 O_i') \bigcap (N \bigcup (\cup_{i=0}^2 L_i)) = \emptyset$.

Thus the $B_{12}(n, s)$ is resolvable with $S_0 = A_1 \cup A_2$, $S_1 = A_0 \cup (-A_2)$, $S_2 = (-A_0) \cup (-A_1)$, $PS_0 = PA_1 \cup PA_2$, $PS_1 = PA_0 \cup (-PA_2)$, $PS_2 = (-PA_0) \cup (-PA_1)$, $\overline{R}_0 = -(R_0 \backslash A_0)$, $\overline{R}_1 = R_1 \backslash A_1$ and $\overline{R}_2 = -(R_2 \backslash A_2)$.

The lemma below gives the construction of $RB_{12}(n, s)$ for any $n \ge 0$ and $s \ge 1$, where we list the components D, PR_i, PA_i for short or D, PR_i, PS_i, $\overline{R}_i$ for full, $i \in \{0, 1, 2\}$.

Lemma 3.4 *There exists an $RB_{12}(n, s)$.*

Proof When $n = 0$, we take $D = Z_{12(n+s)}$ and $R_i = S_i = \overline{R}_i = \emptyset$. When $n > 0$, $s > 0$, the desired $RB_{12}(n, s)$ is constructed directly as follows:

(1) For s odd and n even, let

$D=\{(n+s)j:0\le j\le 11\}\cup\{(n+s)i+j:0\le i\le 11, n/2+1\le j\le n/2+s-1\}$,
$PR_0=\{\{j,-j\}:1\le j\le n/2$ or $5(n+s)+1\le j\le 5(n+s)+n/2$ or $n+s+1\le j\le n+s+n/2$ or $n+s+n/2+s\le j\le 2(n+s)-1\}$,
$PR_1=\{\{j,-j\}:2(n+s)+1\le j\le 2(n+s)+n/2$ or $2(n+s)+n/2+s\le j\le 3(n+s)-1$ or $4(n+s)+1\le j\le 4(n+s)+n/2$ or $4(n+s)+n/2+s\le j\le 5(n+s)-1\}$,
$PR_2=\{\{j,-j\}:3(n+s)+1\le j\le 3(n+s)+n/2$ or $3(n+s)+n/2+s\le j\le 4(n+s)-1$ or $n/2+s\le j\le n+s-1$ or $5(n+s)+n/2+s\le j\le 6(n+s)-1\}$.
$PA_0=\{\{j,7(n+s)-j\}:1\le j\le n/2\}\cup\{\{5(n+s)+j,2(n+s)-j\}:1\le j\le n/2\}$,
$PA_1=\{\{2(n+s)+j,5(n+s)-j\}:1\le j\le n/2\}\cup\{\{4(n+s)+j,3(n+s)-j\}:1\le j\le n/2\}$,
$PA_2=\{\{3(n+s)+j,4(n+s)-j\}:1\le j\le n/2\}\cup\{\{n+s-j,8(n+s)+j:1\le j\le n/2\}\}$.

(2) For s even and n odd,

(2.1) $n\ge 3$, let
$D=\{(n+s)j:0\le j\le 11\}\cup\{(n+s)i+j:0\le i\le 11, 1\le j\le (s-2)/2$ or $n+s-(s-2)/2\le j\le n+s-1\}\cup\{(n+s)i+(s-2)/2+1, (n+s)i'-(s-2)/2-1: i=0,1,2,6,7,8, i'=4,5,6,10,11,12\}$,
$PR_0=\{\{j,-j\}:s/2+1\le j\le n+s/2$ or $3(n+s)+s/2\le j\le 3(n+s)+n+s/2-1\}$,
$PR_1=\{\{j,-j\}:n+s+s/2+1\le j\le n+s+n+s/2$ or $5(n+s)+s/2\le j\le 5(n+s)+n+s/2-1\}$,
$PR_2=\{\{j,-j\}:2(n+s)+s/2+1\le j\le 2(n+s)+n+s/2$ or $4(n+s)+s/2\le j\le 4(n+s)+n+s/2-1\}$.
$PA_0=\{\{s/2+j,n+s/2-j\}:1\le j\le (n-1)/2\}\cup\{\{3(n+s)+s/2-1+j,9(n+s)-s/2-j\}:1\le j\le (n-1)/2\}\cup\{\{n+s/2,3(n+s)+n+s/2-2\}\}$,
$PA_1=\{\{(n+s)+s/2+j,5(n+s)+n+s/2-j\}:1\le j\le n\}$,
$PA_2=\{\{2(n+s)+s/2+j,4(n+s)+n+s/2-j\}:1\le j\le n\}$.

(2.2) $n=1$, let
$D=Z_{12(s+1)}\backslash\{\pm((n+s)i+s/2+1):0\le i\le 5\}$,
$PR_0=\{\{s/2+1,11(s+1)+s/2\},\{s+1+s/2+1,10(s+1)+s/2\}\}$,
$PR_1=\{\{2(s+1)+s/2+1,9(s+1)+s/2\},\{3(s+1+s/2+1,8(s+1)+s/2\}\}$,
$PR_2=\{\{4(s+1)+s/2+1,7(s+1)+s/2\},\{5(s+1)+s/2+1,6(s+1)+s/2\}\}$.
$PA_0=\{\{s/2+1,10(s+1)+s/2\}\}$,
$PA_1=\{\{2(s+1)+s/2+1,8(s+1)+s/2\}\}$,
$PA_2=\{\{4(s+1)+s/2+1,6(s+1)+s/2\}\}$.

(3) For s even and n even,

(3.1) $n\ge 4$, let
$D=\{(n+s)j:0\le j\le 11\}\cup\{(n+s)i+j:0\le i\le 11, n/2+1\le j\le n/2+s-1\}$,
$PR_0=\{\{j,-j\}:1\le j\le n/2$ or $n/2+s\le j\le n+s-1$ or $2(n+s)+1\le j\le 2(n+s)+n/2$ or $2(n+s)+n/2+s\le j\le 3(n+s)-1\}$,
$PR_1=\{\{j,-j\}:n+s+1\le j\le n+s+n/2$ or $n+s+n/2+s\le j\le 2(n+s)-1$ or $5(n+s)+1\le j\le 5(n+s)+n/2$ or $5(n+s)+n/2+s\le j\le 6(n+s)-1\}$,

$PR_2 = \{\{j, -j\} : 3(n+s)+1 \le j \le 3(n+s)+n/2$ or $3(n+s)+n/2+s \le j \le 4(n+s)-1$ or $4(n+s)+1 \le j \le 4(n+s)+n/2$ or $4(n+s)+n/2+s \le j \le 5(n+s)-1\}$.
$PA_0 = \{\{j, 12(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{2(n+s)+j, 10(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{n/2, 11(n+s)+n/2-1\}, \{2(n+s)+n/2, 9(n+s)+n/2-1\}\}$,
$PA_1 = \{\{(n+s)+j, 11(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{5(n+s)+j, 7(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{(n+s)+n/2, 10(n+s)+n/2-1\}, \{5(n+s)+n/2, 6(n+s)+n/2-1\}\}$,
$PA_2 = \{\{3(n+s)+j, 9(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{4(n+s)+j, 8(n+s)-1-j\} : 1 \le j \le n/2-1\} \cup \{\{3(n+s)+n/2, 8(n+s)+n/2-1\}, \{4(n+s)+n/2, 7(n+s)+n/2-1\}\}$.

(3.2) $s > 2$ even and $n = 2$, let
$D = \{(2+s)i+j : 0 \le i \le 2$ or $9 \le i \le 11, 0 \le j \le s+1\} \cup \{(2+s)i+j : 3 \le i \le 8, 0 \le j \le s/2-2$ or $j = s/2+1$ or $s/2+4 \le j \le s+1\}$,
$PR_0 = \{\{j, -j\} : 3(s+2)+s/2-1 \le j \le 3(s+2)+s/2$ or $3(s+2)+s/2+2 \le j \le 3(s+2)+s/2+3\}$,
$PR_1 = \{\{j, -j\} : 4(s+2)+s/2-1 \le j \le 4(s+2)+s/2$ or $4(s+2)+s/2+2 \le j \le 4(s+2)+s/2+3\}$,
$PR_2 = \{\{j, -j\} : 5(s+2)+s/2-1 \le j \le 5(s+2)+s/2$ or $5(s+2)+s/2+2 \le j \le 5(s+2)+s/2+3\}$.
$PA_0 = \{\{3(s+2)+s/2-1, 3(s+2)+s/2+2\}, \{3(s+2)+s/2, 9(s+2)-s/2-3\}\}$,
$PA_1 = \{\{4(s+2)+s/2-1, 8(s+2)-s/2-2\}, \{4(s+2)+s/2+2, 8(s+2)-s/2-3\}\}$,
$PA_2 = \{\{5(s+2)+s/2, 7(s+2)-s/2+1\}, \{5(s+2)+s/2+2, 5(s+2)+s/2+3\}\}$.

(3.3) For $s = 2$ and $n = 2$, let
$D = \{4i+j : 0 \le i \le 2$ or $9 \le i \le 11, 0 \le j \le 3\} \cup \{24\} \setminus \{36\}$,
$PR_0 = \{\{12, 34\}, \{13, 36\}, \{14, 35\}, \{15, 33\}\}$,
$PR_1 = \{\{16, 30\}, \{17, 32\}, \{18, 31\}, \{19, 29\}\}$,
$PR_2 = \{\{20, 26\}, \{21, 28\}, \{22, 27\}, \{23, 25\}\}$.
$PA_0=\{\{13,36\}, \{14,35\}\}$, $PA_1=\{\{17,32\}, \{18,31\}\}$, $PA_2=\{\{21,28\},\{22,27\}\}$.

(4) For s odd and n odd,

(4.1) $s \ge 3$ odd and $n \ge 5$ odd, let
$D = \{(n+s)j, (n+s)j+(n+s)/2 : 0 \le j \le 11\} \cup \{(n+s)i+j : 0 \le i \le 11, 1 \le j \le (s-3)/2$ or $n+s-(s-3)/2 \le j \le n+s-1\} \cup \{(n+s)i+(s-3)/2+1, (n+s)i'-(s-3)/2-1 : i = 0, 1, 2, 6, 7, 8, i' = 4, 5, 6, 10, 11, 12\}$,
$PR_0 = \{\{j, -j\} : (s-3)/2+2 \le j \le (n+s)/2-1$ or $(n+s)/2+1 \le j \le n+s-(s-3)/2-1$ or $3(n+s)+(s-3)/2+1 \le j \le 3(n+s)+(n+s)/2-1$ or $3(n+s)+(n+s)/2+1 \le j \le 4(n+s)-(s-3)/2-2\}$,
$PR_1 = \{\{j, -j\} : n+s+(s-3)/2+2 \le j \le n+s+(n+s)/2-1$ or $n+s+(n+s)/2+1 \le j \le 2(n+s)-(s-3)/2-1$ or $5(n+s)+(s-3)/2+1 \le j \le 5(n+s)+(n+s)/2-1$ or $5(n+s)+(n+s)/2+1 \le j \le 6(n+s)-(s-3)/2-2\}$,
$PR_2 = \{\{j, -j\} : 2(n+s)+(s-3)/2+2 \le j \le 2(n+s)+(n+s)/2-1$ or $2(n+s)+(n+s)/2+1 \le j \le 3(n+s)-(s-3)/2-1$ or $4(n+s)+(s-3)/2+1 \le j \le 4(n+s)+(n+s)/2-1$ or $4(n+s)+(n+s)/2+1 \le j \le 5(n+s)-(s-3)/2-2\}$.

$PA_0 = \{\{(s-3)/2+1+j, 12(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{3(n+s)+(s-3)/2+j, 9(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{(n+s)/2-1, 12(n+s)-(n+s)/2-2\}, \{3(n+s)+(n+s)/2-1, 9(n+s)-(n+s)/2-2\}\}$,
$PA_1 = \{\{(n+s)+(s-3)/2+1+j, 11(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{5(n+s)+(s-3)/2+j, 7(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{(n+s)+(n+s)/2-1, 11(n+s)-(n+s)/2-2\}, \{5(n+s)+(n+s)/2-1, 7(n+s)-(n+s)/2-2\}\}$,
$PA_2 = \{\{2(n+s)+(s-3)/2+1+j, 10(n+s)-(s-3)/2-2-j\} : 1 \le j \le (n-3)/2\} \cup \{\{4(n+s)+(s-3)/2+j, 8(n+s)-(s-3)/2-1-j\} : 1 \le j \le (n-1)/2\} \cup \{\{2(n+s)+(n+s)/2-1, 10(n+s)-(n+s)/2-2\}, \{4(n+s)+(n+s)/2-1, 8(n+s)-(n+s)/2-2\}\}$.

(4.2) $s = 1$ and $n \equiv 1 \pmod 4$ and $n \ge 5$, let
$D = \{(n+1)i : 0 \le i \le 11\}$,
$PR_0 = \{\{j, -j-1\} : 1 \le j \le (n+1)/2-1$ or $(n+1)+1 \le j \le (n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)/2+1 \le j \le n$ or $(n+1)+(n+1)/2+1 \le j \le 2(n+1)-1\} \cup \{\{(n+1)/2, 12(n+1)-1\}, \{(n+1)+(n+1)/2, 11(n+1)-1\}\}$,
$PR_1 = \{\{j, -j-1\} : 2(n+1)+1 \le j \le 2(n+1)+(n+1)/2-1$ or $4(n+1)+1 \le j \le 4(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 2(n+1)+(n+1)/2+1 \le j \le 3(n+1)-1$ or $4(n+1)+(n+1)/2+1 \le j \le 5(n+1)-1\} \cup \{\{2(n+1)+(n+1)/2, 10(n+1)-1\}, \{4(n+1)+(n+1)/2, 8(n+1)-1\}\}$,
$PR_2 = \{\{j, -j-1\} : 3(n+1)+1 \le j \le 3(n+1)+(n+1)/2-1$ or $5(n+1)+1 \le j \le 5(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 3(n+1)+(n+1)/2+1 \le j \le 4(n+1)-1$ or $5(n+1)+(n+1)/2+1 \le j \le 6(n+1)-1\} \cup \{\{3(n+1)+(n+1)/2, 9(n+1)-1\}, \{5(n+1)+(n+1)/2, 7(n+1)-1\}\}$.
$PA_0 = \{\{j, -j-1\} : 1 \le j \le (n+1)/2-1$ or $(n+1)+1 \le j \le (n+1)+(n+1)/2-1\} \cup \{\{n-1, n\}\}$,
$PA_1 = \{\{j, -j-1\} : 2(n+1)+1 \le j \le 2(n+1)+(n+1)/2-1$ or $4(n+1)+1 \le j \le 4(n+1)+(n+1)/2-1\} \cup \{\{2(n+1)+(n+1/2+1, 4(n+1)+(n+1)/2\}\}$,
$PA_2 = \{\{j, -j-1\} : 3(n+1)+1 \le j \le 3(n+1)+(n+1)/2-1$ or $5(n+1)+1 \le j \le 5(n+1)+(n+1)/2-1\} \cup \{\{3(n+1)+(n+1)/2, 5(n+1)+(n+1)/2+1\}\}$.

(4.3) $s = 1$ and $n \equiv 3 \pmod 4$ and $n \ge 5$, let
$D = \{(n+1)i : 0 \le i \le 11\}$,
$PR_0 = \{\{j, -j-1\} : 2 \le j \le (n+1)/2-1$ or $3(n+1)+2 \le j \le 3(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)/2+1 \le j \le n$ or $3(n+1)+(n+1)/2+1 \le j \le 4(n+1)-1$ or $j = 1, 3(n+1)+1\} \cup \{\{(n+1)/2, 12(n+1)-2\}, \{3(n+1)+(n+1)/2, 9(n+1)-2\}\}$,
$PR_1 = \{\{j, -j-1\} : (n+1)+2 \le j \le (n+1)+(n+1)/2-1$ or $5(n+1)+2 \le j \le 5(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : (n+1)+(n+1)/2+1 \le j \le 2(n+1)-1$ or $5(n+1)+(n+1)/2+1 \le j \le 6(n+1)-1\} \cup \{\{(n+1)+(n+1)/2, 11(n+1)-2\}, \{5(n+1)+(n+1)/2, 7(n+1)-2\}\}$,
$PR_2 = \{\{j, -j-1\} : 2(n+1)+2 \le j \le 2(n+1)+(n+1)/2-1$ or $4(n+1)+2 \le j \le 4(n+1)+(n+1)/2-1\} \cup \{\{j, -j\} : 2(n+1)+(n+1)/2+1 \le j \le 3(n+1)-1$ or $4(n+1)+(n+1)/2+1 \le j \le 5(n+1)-1\} \cup \{\{2(n+1)+(n+1)/2, 10(n+1)-2\}, \{4(n+1)+(n+1)/2, 8(n+1)-2\}\}$.
$PA_0 = \{\{j, -j-1\} : 2 \le j \le (n+1)/2-1$ or $3(n+1)+2 \le j \le 3(n+1)+(n+1)/2-1\} \cup \{\{n-6+J, n-j\} : 0 \le j \le 1\} \cup \{\{1, 4(n+1)-2\}\}$,

$PA_1 = \{\{j, -j-1\} : (n+1)+2 \leq j \leq (n+1)+(n+1)/2-1$ or $5(n+1)+2 \leq j \leq 5(n+1)+(n+1)/2-1\} \cup \{\{(n+1)+(n+1)/2+j, 5(n+1)+(n+1)/2+3-j\} : 0 \leq j \leq 2\}$,
$PA_2 = \{\{j, -j-1\} : 2(n+1)+2 \leq j \leq 2(n+1)+(n+1)/2-1$ or $4(n+1)+2 \leq j \leq 4(n+1)+(n+1)/2-1\} \cup \{\{2(n+1)+(n+1)/2+j, 4(n+1)+(n+1)/2+3-j\} : 0 \leq j \leq 2\}$.

(4.4) For $s > 1$ odd and $n = 3$, let
$D = \{(n+s)j : 0 \leq j \leq 11\} \cup \{(n+s)i + j : 0 \leq i \leq 11, 3 \leq j \leq s\} \cup \{(n+s)i + j : (i, j) \in \{(0, 1), (0, 2), (1, 1), (1, 2), (2, 1), (2, 2), (9, s+1), (9, s+2), (10, s+1), (10, s+2), (11, s+1), (11, s+2)\}\}$,
$PR_0 = \{\{j, -j\} : s+1 \leq j \leq s+2$ or $3(3+s)+1 \leq j \leq 3(n+s)+2$ or $3(3+s)+s+1 \leq j \leq 3(n+s)+s+2\}$,
$PR_1 = \{\{j, -j\} : (3+s)+s+1 \leq j \leq (3+s)+s+2$ or $5(3+s)+1 \leq j \leq 5(n+s)+2$ or $5(3+s)+s+1 \leq j \leq 5(n+s)+s+2\}$,
$PR_2 = \{\{j, -j\} : 2(3+s)+s+1 \leq j \leq 2(3+s)+s+2$ or $4(3+s)+1 \leq j \leq 4(n+s)+2$ or $4(3+s)+s+1 \leq j \leq 4(n+s)+s+2\}$.
$PA_0 = \{\{s+1, 3(s+3)+s+2\}, \{s+2, 3(s+3)+s+1\}, \{3(s+3)+1, 3(s+3)+2\}\}$,
$PA_1 = \{\{(s+3)+s+1, 5(s+3)+s+2\}, \{(s+3)+s+2, 5(s+3)+s+1\}, \{5(s+3)+1, 10(s+3)+2\}\}$,
$PA_2 = \{\{2(s+3)+s+1, 4(s+3)+s+2\}, \{2(s+3)+s+2, 4(s+3)+s+1\}, \{4(s+3)+2, 9(s+3)+1\}\}$.

(4.5) For $s = 1$ and $n = 3$, let
$D = \{4i + j : 0 \leq i \leq 2$ or $9 \leq i \leq 11, j = 0, 2\} \cup \{24\} \backslash \{36\}$,
$PR_0 = \{\{1, 47\}, \{3, 45\}, \{12, 34\}, \{13, 36\}, \{14, 35\}, \{15, 33\}\}$,
$PR_1 = \{\{9, 39\}, \{11, 37\}, \{16, 30\}, \{17, 32\}, \{18, 31\}, \{19, 29\}\}$,
$PR_2 = \{\{5, 43\}, \{7, 41\}, \{20, 26\}, \{21, 28\}, \{22, 27\}, \{23, 25\}\}$.
$PA_0 = \{\{13, 36\}, \{14, 35\}, \{3, 12\}\}$, $PA_1 = \{\{17, 32\}, \{18, 31\}, \{11, 30\}\}$,
$PA_2 = \{\{21, 28\}, \{22, 27\}, \{20, 23\}\}$.

(4.6) For $s > 3$ odd and $n = 1$, let
$D = Z_{12(s+1)} \backslash \{\pm((n+s)i + j) : 0 \leq i \leq 2, 1 \leq j \leq 2\}$,
$PR_0 = \{\{1, 11(s+1)+s\}, \{2, 11(s+1)+s-1\}\}$,
$PR_1 = \{\{(s+1)+1, 10(s+1)+s\}, \{(s+1)+2, 10(s+1)+s-1\}\}$,
$PR_2 = \{\{2(s+1)+1, 9(s+1)+s\}, \{2(s+1)+2, 9(s+1)+s-1\}\}$.
$PA_0 = \{\{1, 2\}\}$, $PA_1 = \{\{(s+1)+1, 10(s+1)+s-1\}\}$,
$PA_2 = \{\{2(s+1)+1, 9(s+1)+s-1\}\}$.

(4.7) For $s = 3$ and $n = 1$, let
$D = Z_{48} \backslash \{1, 2, 5, 6, 9, 10, 38, 39, 42, 43, 46, 47\}$,
$PR_0 = \{\{5, 6\}, \{10, 39\}\}$, $PR_1 = \{\{42, 43\}, \{9, 46\}\}$, $PR_2 = \{\{2, 47\}, \{1, 38\}\}$,
$PS_0 = \{\{9, 46\}, \{2, 47\}\}$, $PS_1 = \{\{5, 38\}, \{1, 10\}\}$, $PS_2 = \{\{6, 39\}, \{42, 43\}\}$,
$\overline{R}_0 = \{9, 46\}$, $\overline{R}_1 = \{38, 43\}$, $\overline{R}_2 = \{6, 47\}$.

(4.8) For $s = 1$ and $n = 1$, let
$D = \{0, 1, 2, 3, 4, 5, 12, 19, 20, 21, 22, 23\}$,
$PR_0 = \{\{6, 7\}, \{8, 11\}\}$, $PR_1 = \{\{9, 10\}, \{13, 18\}\}$, $PR_2 = \{\{14, 17\}, \{15, 16\}\}$,
$PS_0 = \{\{9, 16\}, \{10, 15\}\}$, $PS_1 = \{\{8, 15\}, \{11, 14\}\}$, $PS_2 = \{\{6, 13\}, \{9, 18\}\}$,
$\overline{R}_0 = \{7, 8\}$, $\overline{R}_1 = \{13, 14\}$, $\overline{R}_2 = \{6, 7\}$. □

Combining Theorem 3.3 and Lemma 3.4, we obtain the following theorem.

Theorem 3.5 *Suppose that $n \geq 0$ and $s \geq 1$. There exists an $RHF_{12}((n+s)^3 : s)$. When $(n, s) \notin \{(1, 1), (2, 2), (3, 1)\}$, there exists an $RHF_{12}((n+s)^3 : s)$ having a sub-design $RH(12^4)$.*

Consequently, we have our first tripling construction for resolvable H-designs with group size 12 as follows.

Corollary 3.6 (Tripling construction I) *Let $n \geq 0$ and $s \geq 1$. If there exists an $IRH(12^n : 12^s)$, then there exist both an $IRH(12^{3n-2s} : 12^n)$ and an $IRH(12^{3n-2s} : 12^s)$. Furthermore, if there exists an $RH(12^n)$ or an $RH(12^s)$, then there exists an $RH(12^{3n-2s})$, as well as an $IRH(12^{3n-2s} : 12^4)$ when $(n, s) \notin \{(3, 1), (5, 1), (6, 2)\}$.*

Lemma 3.7 *There exists an $RH(12^n)$ for each $n \equiv 0, 1, 2, 4, 5 \pmod 6$ and $n \geq 4$.*

Proof For each $n \equiv 1$ or $2 \pmod 3$, $n \geq 4$ and $n \in \{73, 149\}$, an $RH(12^n)$ can be obtained by applying the Weighting construction with an $RH(4^n)$ in Theorem 1.1 and $m = 3$. For each $n \equiv 0 \pmod 6$ and $n \geq 4$, an $RH(12^n)$ can be obtained by applying the Weighting construction with an $RH(6^n)$ in Theorem 1.1 and $m = 2$.

For the design $RH(12^{73})$, it can be constructed by applying Tripling construction I with $(n, s) = (25, 1)$. For the design $RH(12^{149})$, it can be obtained by applying Tripling construction I with $(n, s) = (51, 2)$. Here, the $IRH(12^{51} : 12^2)$ exists by Tripling construction I with $(n, s) = (23, 9)$ and the $IRH(12^{23} : 12^9)$ exists by Tripling construction I with $(n, s) = (9, 2)$, where an $RH(12^9)$ is constructed in the Appendix. □

Lemma 3.8 *There exists an $RHF_{12}(3^5 : 2)$ and an $IRH(12^n : 12^s)$ for each $(n, s) \in \{(13, 5), (17, 7)\}$.*

Proof Let $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ be an $RHF_4(3^5 : 2)$. Such a design was constructed in [11]. Let $X' = X \times Z_3$, $\mathcal{G}' = \{G \times Z_3 : G \in \mathcal{G}\}$ and $\mathcal{F}' = \{\{G \times Z_3 : G \in F\} : F \in \mathcal{F}\}$. For each block $B \in \mathcal{B}$, construct an $RH(3^4)$ on $B \times Z_3$ with block set $\mathcal{A}_B$. Then $\bigcup_{B \in \mathcal{B}} \mathcal{A}_B$ is the block set of an $RHF_{12}(3^5 : 2)$ on X' with group set $\mathcal{G}'$ and hole set $\mathcal{F}'$.

An $IRH(12^{13} : 12^5)$ and an $IRH(12^{17} : 12^7)$ can be obtained by applying the Tripling construction I with $(n, s) = (5, 1)$ and $(7, 2)$, respectively. □

Lemma 3.9 *There exists an $IRH(12^n:12^s)$ for all $n \equiv 35 \pmod{36}$ and $s \in \{1, 2, 4, 5, 6, 7, 11, 17\}$.*

Proof For each $n = 36m - 1$, $m \geq 1$, start from a URCS$(3, \{4, 6\}, 12m)$ of type $(1^{12m-1} : 1)$, which is obtained from an $RG(6^{2m})$ (see [11]). Applying Theorem 2.3 with an $RHF_{12}(3^{k-1} : 2)$ and an $RH(36^k)$ with $k \in \{4, 6\}$, we get an $RHF_{12}(3^{12m-1} : 2)$. Applying Theorem 2.4 with an $RH(12^5)$, we get an $RH(12^n)$ and an $IRH(12^n : 12^5)$. Here, the input designs $RHF_{12}(3^{k-1} : 2)$ with $k \in \{4, 6\}$ are from Theorem 3.5 and Lemma 3.8. The designs with a hole of sizes 1 or 2 are actually an $RH(12^n)$. The designs with a hole of sizes 11 or 17 exist since we input an $RHF_{12}(3^{k-1} : 2)$ with $k \in \{4, 6\}$, respectively when applying Theorem 2.3. The design with a hole of size 7 exists since there exists an $IRH(12^{17} : 12^7)$ by Lemma 3.8. For each $s \in \{4, 6\}$, the design with a hole of size s exists since the input design $RH(36^s)$ with a subdesign $RH(12^s)$ exists. □

As a corollary of the Tripling construction I, we obtain

Theorem 3.10 *If there exists a constant $M \geq 7$, such that for any odd integer n in the range $M \leq n < 3M$, there exists an IRH$(12^n : 12^6)$, then for all odd integer $n \geq M$, there exists an IRH$(12^n : 12^6)$.*

Proof It is clear that the existence of an IRH$(12^n : 12^6)$ implies the existence of an IRH$(12^n : 12^s)$ for all $s \in \{1, 2, 6\}$. We proceed the proof by induction. Let n be an odd integer and $n \geq 3M$. Assume that for all odd n' in the range $M \leq n' < n$, there exists an IRH$(12^{n'} : 12^6)$. Write $n = 3m - 2 \cdot s$, where $s = 1, 6, 2$ when $n \equiv 1, 3, 5 \pmod 6$, respectively. It is simple to check that m is odd and $M \leq m < n$. Then applying Tripling construction I gives the conclusion. □

Lemma 3.11 *For each odd integer $n \geq 5$ and $n \notin \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 189, 213, 231, 243, 321, 681\}$, there exists an RH$(12^n)$.*

Proof Let L be the poset of pairs (n, s) such that an IRH$(12^n : 12^s)$ is known. For every two pairs (n, s) and (n', s'), define $(n, s) \prec (n', s')$ if $n < n'$ or, $n = n'$ and $s < s'$. We will compute the output of the Tripling construction I and the Product construction I by a computer programme, which involves the following steps:

- Step 1: Initialize L. Let $L = \{(5, 1), (5, 2), (7, 1), (7, 2), (9, 1), (9, 2), (13, 1), (13, 2), (13, 5)\} \cup \{(n, s)$: there exists an IRH$(12^n : 12^s)$ in Lemma 3.9$\}$. Sort L in ascending order. Let (n, s) be the smallest pair in L.
- Step 2: Check whether (n, s) satisfies Tripling construction I's condition, i.e., $(n, s) \notin \{(3, 1), (5, 1), (6, 2)\}$. If not, go to Step 3. If yes, update L by adding pairs $(3n - 2s, n)$, $(3n - 2s, 4)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Sort the updated L in ascending order, then go to Step 3.
- Step 3: Apply the Product construction I. For each m such that $(m, 1) \in L$, update L by adding pairs (mn, n), (mn, m) and (mn, k) for all k such that (n, k) or $(m, k) \in L$. Sort the updated L in ascending order. Let (n, s) be the next smallest pair in the updated L, then go to Step 2.

The programme was run with $n < 2000$ and $s \leq 64$, and produced two results as follows:

- Result 1: For all odd n and $4 \leq n < 1102$, there exists an RH(12^n) with eighteen possible exceptions $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 189, 213, 231, 243, 321, 681\}$,
- Result 2: There exists an IRH$(12^n : 12^6)$ for all odd n in the range $1102 \leq n < 3306$.

By Theorem 3.10, there exists an IRH$(12^n : 12^6)$ for all odd $n \geq 1102$. Hence there exists an RH(12^n) by Theorem 2.4. This completes the proof. □

Lemma 3.12 *There exists an RH(12^n) for each $n \in \{189, 681\}$.*

Proof For $n = 189$, start from an RCQS$(3^5 : 1)$. Applying Theorem 2.3 with an RHF$_{12}(12^3 : 9)$ and an RH(144^4), we get an RHF$_{12}(36^5 : 9)$. Applying Theorem 2.4 with an IRH$(12^{45} : 12^9)$, we get the desired RH(12^{189}). Here, the RHF$_{12}(12^3 : 9)$ exists by Theorem 3.5. The IRH$(12^{45} : 12^9)$ can be obtained by applying the Product construction with an RH(12^5) and an RH(12^9).

For $n = 681$, start from an RCQS$(1^7 : 1)$. Applying Theorem 2.3 with an RHF$_{12}(97^3 : 2)$ from Theorem 3.5 and an RH$((12 \times 97)^4)$, we get an RHF$_{12}(97^7 : 2)$. Applying Theorem 2.4 with an RH(12^{99}), we get an RH(12^{681}). □

Combining Lemmas 3.7, 3.11 and 3.12, we obtain the main result in this section.

Theorem 3.13 *The necessary conditions for the existence of an RH*(12^n) *are also sufficient except possibly when* $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 213, 231, 243, 321\}$.

4 (1, 2)-resolvable H-designs with group size 2

In this section, we shall investigate the existence of a $(1, 2)$-RH(2^n), for which the necessary condition is $n \equiv 1, 2 \pmod 3$. First, we have the following lemma.

Lemma 4.1 *If there exists an RH*(g^n) *and g is even, then there exists a* $(1, 2)$-*RH*(g^n).

Proof The number of parallel classes of an RH(g^n) is $(n-1)(n-2)g^2/6$, which is even when g is even. Thus we can obtain exactly $(n-1)(n-2)g^2/12$ 2-parallel classes. This completes the proof. □

As a corollary of Lemma 4.1, we have the following result.

Corollary 4.2 *There exists a* $(1, 2)$-*RH*(2^n) *for each* $n \equiv 2, 4 \pmod 6$.

Lemma 4.3 [7] *There exists a* $(1, 2)$-*RH*(2^{13}).

Now, we give direct constructions for several designs with small orders.

Lemma 4.4 *There exists a* $(1, 2)$-*RH*(2^n) *for each* $n \in \{7, 25\}$.

Proof For each given $n \in \{7, 25\}$, let the point set be Z_{2n} and the group set be $\{\{j, j+n\} : j = 0, 1, \ldots, n-1\}$. We list below the base blocks of a $(1, 2)$-RH(2^n) which are developed by $\alpha = (0\ 1\ 2 \ldots n-2\ n-1)(n\ n+1\ n+2 \ldots 2n-2\ 2n-1)$. Here, every n consecutive blocks in the first m rows form a 2-parallel class. Each of the remaining blocks consists of two members in every cycle of α, hence gives a 2-parallel class when developed by α.

$n = 7$: $m = 2$
{10, 13, 8, 9} {0, 1, 2, 4} {0, 8, 9, 11} {3, 5, 6, 7} {2, 3, 12, 13} {4, 6, 10, 12}
{1, 5, 7, 11}
{4, 5, 9, 10} {3, 6, 8, 11} {2, 4, 12, 7}
$n = 25$: $m = 25$
{25, 7, 45, 47} {25, 27, 10, 45} {26, 3, 4, 10} {26, 11, 14, 20} {27, 12, 18, 24} {28, 11, 13, 18}
{31, 22, 28, 29} {30, 44, 3, 29} {30, 8, 16, 2} {33, 41, 19, 47} {39, 6, 33, 12} {31, 15, 4, 5}
{32, 21, 22, 6} {34, 16, 0, 8} {36, 13, 40, 32} {38, 17, 34, 36} {40, 20, 35, 39} {49, 9, 37, 43}
{35, 43, 46, 24} {38, 44, 23, 9} {19, 1, 14, 41} {21, 23, 0, 42} {7, 17, 48, 5} {1, 42, 48, 49}
{2, 37, 15, 46}
{25, 4, 34, 38} {25, 7, 14, 21} {26, 31, 8, 38} {26, 29, 41, 44} {29, 39, 45, 27} {30, 32, 40, 0}
{27, 33, 43, 20} {28, 30, 31, 37} {37, 22, 28, 34} {32, 40, 42, 47} {35, 49, 33, 9} {47, 39, 41, 45}
{36, 43, 44, 35} {48, 49, 10, 36} {42, 46, 4, 9} {18, 20, 46, 23} {0, 3, 15, 18} {1, 3, 11, 21}
{16, 24, 2, 5} {13, 5, 7, 11} {24, 8, 22, 48} {12, 15, 16, 19} {1, 6, 12, 17} {13, 17, 2, 6}
{10, 14, 19, 23}
{25, 28, 36, 47} {25, 34, 48, 49} {27, 16, 42, 26} {26, 33, 42, 0} {27, 28, 39, 41} {29, 32, 33, 36}
{29, 34, 40, 45} {30, 10, 41, 21} {43, 31, 35, 40} {47, 35, 38, 43} {44, 48, 3, 32} {38, 39, 18, 44}
{1, 9, 16, 49} {7, 17, 23, 30} {1, 7, 17, 19} {12, 14, 15, 46} {24, 15, 21, 22} {24, 2, 10, 46}
{11, 37, 20, 4} {5, 12, 13, 4} {19, 20, 6, 8} {23, 3, 18, 22} {8, 13, 0, 5} {14, 2, 31, 11}
{9, 37, 45, 6}

{25, 29, 39, 43} {25, 32, 39, 21} {26, 41, 46, 47} {26, 31, 43, 23} {0, 2, 40, 22} {0, 8, 10, 40}
{1, 27, 8, 20} {3, 17, 1, 2} {7, 11, 3, 30} {5, 18, 22, 27} {7, 20, 23, 28} {9, 24, 29, 5}
{4, 34, 11, 16} {15, 48, 30, 38} {6, 33, 46, 28} {16, 42, 48, 35} {18, 33, 13, 14} {19, 49, 36, 41}
{32, 9, 35, 37} {12, 17, 44, 49} {4, 34, 15, 45} {24, 31, 44, 21} {10, 37, 38, 19} {6, 14, 42, 45}
{13, 47, 36, 12}
{0, 31, 41, 43} {0, 31, 14, 42} {1, 27, 35, 44} {1, 27, 13, 14} {2, 15, 43, 23} {2, 32, 47, 26}
{3, 29, 8, 9} {3, 38, 19, 25} {5, 11, 44, 47} {6, 45, 21, 30} {4, 11, 45, 28} {4, 6, 40, 42}
{12, 13, 49, 25} {5, 33, 9, 37} {18, 24, 28, 39} {20, 48, 10, 38} {22, 32, 35, 16} {10, 24, 33, 34}
{23, 40, 17, 46} {22, 36, 37, 21} {19, 46, 48, 15} {8, 39, 18, 29} {12, 16, 26, 30} {7, 36, 41, 20}
{34, 17, 49, 7}
{0, 4, 37, 42} {0, 4, 38, 41} {0, 1, 30, 31} {0, 7, 39, 46} {0, 27, 3, 30} {0, 2, 35, 45}
{0, 35, 41, 22} {0, 33, 35, 15} {0, 32, 33, 24} {0, 32, 16, 49} {0, 26, 37, 14} {0, 27, 36, 13}
{0, 29, 46, 23} {0, 3, 29, 32} {0, 28, 29, 7} {0, 5, 36, 41} {0, 30, 36, 16} {0, 29, 37, 17}
{0, 29, 38, 16} {0, 32, 14, 46} {25, 5, 32, 12} {25, 3, 40, 18} {25, 35, 16, 23} {25, 6, 16, 43}
{25, 2, 15, 47} {25, 1, 7, 44} {25, 6, 39, 17} {25, 1, 9, 43} {25, 7, 8, 49} {25, 1, 12, 39}
{25, 2, 11, 38} {25, 1, 37, 13} {25, 28, 4, 7} {25, 3, 4, 32} {25, 30, 11, 16} {25, 5, 11, 41}
{25, 38, 17, 22} {25, 4, 12, 42} {25, 6, 35, 21} {25, 38, 16, 21} {25, 4, 13, 41} {25, 29, 14, 18}
{25, 1, 30, 6} {25, 15, 20, 46} {25, 5, 17, 47} {25, 27, 3, 5} {25, 33, 15, 23} {25, 32, 20, 22}
{25, 27, 15, 22} {25, 1, 32, 19} {25, 41, 22, 23} {25, 31, 14, 17} {25, 1, 34, 18} {25, 27, 4, 21}
{25, 27, 11, 13} {25, 29, 21, 23} {25, 3, 29, 7} {25, 30, 20, 24} {25, 29, 13, 16} □

Lemma 4.5 *There exists a* $(1, 2)$*-RH*(2^n) *for each* $n \in \{11, 19, 23, 29\}$.

Proof For each given $n \in \{11, 19, 23, 29\}$, let the point set be Z_{2n} and the group set be $\{\{j, j+n\} : j = 0, 1, \ldots, n-1\}$. The desired $(1, 2)$-RH(2^n) is obtained by adding d modulo $2n$ to the base blocks as follows. Here, every n consecutive blocks in the first m rows form a 2-parallel class on Z_{2n}. Each of the remaining blocks consists of two odd and two even members, hence gives a 2-parallel class when developed by adding 2 modulo Z_{2n} and in total of $\frac{2}{d}$ such classes when developed by adding d modulo Z_{2n}.

$n = 11:$ $d = 2$ $m = 4$
{1, 9, 10, 11} {7, 13, 15, 19} {1, 5, 9, 17} {2, 6, 8, 10} {14, 15, 16, 2} {3, 4, 6, 16}
{4, 12, 14, 20} {8, 12, 18, 0} {17, 19, 21, 5} {20, 7, 11, 13} {18, 21, 0, 3}
{5, 12, 14, 18} {3, 13, 15, 16} {1, 3, 6, 8} {5, 9, 10, 14} {2, 4, 9, 11} {17, 20, 21, 2}
{12, 16, 17, 21} {0, 6, 13, 19} {20, 1, 4, 7} {0, 7, 8, 15} {10, 11, 18, 19}
{3, 8, 13, 20} {4, 11, 14, 21} {0, 3, 12, 15} {1, 6, 15, 20} {3, 4, 9, 10} {0, 1, 6, 7}
{8, 10, 11, 13} {3, 12, 17, 20}
$n = 19:$ $d = 2$ $m = 20$
{0, 6, 24, 30} {2, 8, 30, 35} {2, 8, 19, 24} {4, 12, 15, 26} {14, 17, 27, 31} {11, 14, 27, 29}
{3, 26, 34, 0} {3, 18, 29, 1} {4, 19, 20, 34} {12, 20, 28, 33} {7, 35, 37, 5} {23, 32, 36, 18}
{25, 28, 31, 37} {16, 17, 23, 9} {7, 32, 22, 10} {6, 13, 21, 5} {36, 10, 22, 1} {9, 11, 25, 15}
{13, 21, 33, 16}
{10, 26, 35, 36} {3, 11, 25, 32} {4, 11, 22, 32} {15, 23, 24, 27} {0, 1, 13, 27} {5, 8, 15, 29}
{0, 4, 16, 20} {14, 18, 20, 17} {14, 18, 19, 16} {28, 8, 29, 30} {26, 9, 3, 31} {7, 28, 5, 19}
{7, 36, 37, 31} {35, 22, 34, 2} {17, 30, 37, 33} {9, 21, 23, 24} {21, 33, 6, 1} {10, 12, 34, 25}
{2, 6, 12, 13}
{7, 16, 34, 24} {28, 19, 26, 10} {8, 26, 6, 3} {7, 13, 27, 31} {9, 30, 13, 17} {32, 30, 17, 24}
{21, 1, 23, 34} {32, 5, 6, 0} {29, 9, 16, 37} {14, 4, 12, 2} {8, 36, 12, 29} {25, 20, 11, 33}
{18, 4, 27, 0} {14, 15, 22, 18} {3, 36, 37, 1} {20, 28, 2, 23} {33, 35, 19, 10} {11, 15, 21, 22}
{31, 35, 25, 5}

{0, 7, 11, 27} {35, 18, 29, 13} {22, 12, 19, 36} {6, 18, 13, 20} {2, 35, 31, 37} {6, 19, 30, 32}
{11, 15, 37, 27} {17, 2, 8, 0} {25, 20, 15, 5} {22, 33, 23, 21} {14, 25, 5, 7} {10, 12, 26, 16}
{30, 34, 24, 4} {16, 34, 3, 9} {26, 33, 10, 17} {9, 24, 3, 4} {23, 28, 32, 21} {29, 36, 14, 1}
{8, 31, 28, 1}
{0, 8, 26, 5} {1, 9, 27, 6} {0, 8, 9, 37} {6, 15, 32, 33} {3, 4, 28, 37} {4, 5, 14, 1}
{20, 7, 12, 19} {10, 14, 23, 35} {34, 7, 10, 21} {26, 16, 3, 11} {18, 29, 20, 31} {27, 18, 23, 30}
{32, 33, 16, 17} {25, 22, 12, 11} {30, 13, 24, 21} {28, 2, 13, 17} {25, 36, 34, 19} {22, 29, 24, 31}
{2, 35, 36, 15}
{22, 25, 32, 37} {0, 7, 32, 17} {0, 11, 34, 5} {28, 34, 19, 37} {0, 1, 14, 15} {33, 8, 5, 16}
{11, 28, 34, 27}
$n = 23:$ $d = 2$ $m = 20$
{1, 2, 4, 6} {1, 2, 5, 15} {4, 5, 10, 38} {7, 8, 14, 16} {6, 7, 17, 27} {8, 9, 21, 25}
{10, 12, 16, 40} {13, 15, 19, 9} {11, 14, 20, 26} {17, 20, 29, 13} {18, 21, 36, 28} {19, 22, 40, 0}
{24, 27, 11, 41} {30, 33, 23, 35} {26, 32, 44, 24} {3, 12, 39, 37} {34, 43, 41, 39} {42, 22, 28, 18}
{38, 31, 42, 30} {32, 36, 44, 0} {33, 37, 45, 25} {3, 34, 35, 23} {29, 45, 31, 43}
{3, 9, 21, 37} {9, 18, 36, 8} {2, 11, 10, 32} {11, 20, 28, 0} {4, 13, 29, 19} {0, 18, 8, 40}
{13, 31, 21, 23} {31, 12, 20, 28} {29, 10, 45, 39} {6, 33, 30, 4} {35, 16, 40, 2} {12, 39, 33, 27}
{24, 5, 7, 23} {14, 22, 38, 42} {37, 26, 6, 30} {15, 1, 19, 43} {35, 14, 43, 17} {22, 1, 34, 44}
{42, 25, 32, 16} {7, 36, 26, 38} {5, 17, 41, 27} {3, 44, 34, 24} {25, 15, 41, 45}
{27, 35, 5, 11} {35, 24, 2, 26} {37, 26, 39, 21} {2, 37, 28, 42} {4, 39, 21, 3} {6, 41, 1, 3}
{4, 28, 30, 42} {45, 23, 25, 43} {17, 30, 10, 36} {19, 32, 25, 17} {6, 19, 38, 34} {33, 0, 32, 12}
{14, 27, 45, 5} {16, 9, 31, 7} {40, 33, 41, 13} {36, 22, 40, 10} {29, 8, 20, 24} {31, 14, 9, 23}
{18, 1, 15, 29} {22, 34, 12, 18} {43, 38, 8, 44} {20, 15, 11, 7} {13, 44, 0, 16}
{0, 13, 5, 43} {13, 39, 45, 7} {39, 32, 18, 4} {43, 36, 15, 37} {41, 34, 38, 24} {25, 4, 8, 12}
{0, 25, 45, 19} {6, 31, 9, 17} {29, 12, 24, 36} {2, 31, 11, 35} {1, 42, 27, 23} {6, 1, 22, 20}
{8, 3, 35, 15} {26, 16, 42, 14} {9, 40, 10, 26} {18, 3, 20, 14} {28, 29, 32, 33} {41, 44, 34, 23}
{28, 17, 21, 2} {37, 30, 38, 33} {19, 44, 22, 7} {21, 16, 27, 30} {10, 5, 11, 40}
{0, 31, 19, 7} {0, 31, 35, 21} {2, 18, 4, 12} {2, 3, 4, 5} {5, 6, 11, 26} {7, 8, 15, 42}
{12, 13, 21, 6} {9, 10, 22, 3} {8, 11, 14, 17} {10, 13, 22, 25} {15, 18, 33, 32} {17, 20, 41, 30}
{16, 19, 43, 44} {16, 25, 34, 43} {36, 45, 26, 35} {38, 1, 27, 30} {29, 42, 41, 24} {1, 40, 37, 32}
{38, 33, 28, 23} {39, 34, 20, 23} {44, 29, 14, 45} {9, 40, 27, 36} {39, 24, 28, 37}
{9, 18, 17, 14} {9, 18, 35, 2} {9, 18, 34, 1} {0, 27, 8, 35} {0, 27, 16, 43} {27, 8, 5, 42}
{27, 8, 13, 6} {0, 27, 13, 22} {27, 8, 10, 3} {0, 35, 24, 13} {0, 35, 2, 37} {35, 24, 15, 34}
{35, 24, 39, 18} {35, 24, 30, 9} {0, 13, 26, 39} {0, 13, 6, 19} {13, 26, 45, 10} {0, 13, 25, 14}
{13, 26, 44, 27} {0, 39, 32, 25} {0, 39, 18, 11} {39, 32, 43, 30} {0, 39, 29, 42} {0, 25, 4, 29}
{0, 25, 8, 33} {25, 4, 37, 44} {25, 4, 41, 26} {0, 25, 41, 34} {0, 29, 12, 41} {0, 29, 24, 7}
{29, 12, 19, 40} {29, 12, 31, 32} {0, 29, 31, 10} {29, 12, 38, 39} {0, 41, 26, 21} {41, 36, 11, 28}
{0, 31, 32, 17} {31, 16, 33, 38} {0, 31, 3, 44}
$n = 29:$ $d = 1$ $m = 20$
{0, 5, 13, 15} {0, 35, 33, 47} {2, 15, 1, 41} {1, 34, 52, 42} {5, 4, 14, 2} {3, 54, 8, 40}
{3, 12, 38, 30} {6, 7, 23, 49} {6, 13, 9, 17} {4, 53, 25, 23} {12, 7, 43, 29} {8, 31, 51, 11}
{9, 54, 20, 30} {10, 35, 29, 41} {10, 11, 49, 55} {14, 21, 55, 39} {25, 16, 22, 26} {37, 32, 16, 44}
{52, 17, 21, 43} {31, 18, 46, 26} {53, 20, 42, 18} {27, 48, 57, 37} {24, 36, 44, 57} {38, 28, 50, 56}
{24, 56, 32, 36} {51, 19, 47, 45} {50, 48, 28, 46} {33, 34, 39, 19} {40, 45, 22, 27}
{0, 19, 52, 56} {0, 17, 16, 44} {1, 4, 55, 19} {1, 32, 37, 13} {2, 45, 22, 28} {2, 13, 26, 10}
{3, 21, 33, 38} {4, 14, 40, 17} {6, 32, 30, 5} {3, 11, 55, 54} {7, 5, 23, 16} {10, 54, 6, 15}
{9, 23, 27, 7} {8, 48, 18, 52} {11, 57, 37, 21} {20, 12, 18, 46} {12, 14, 56, 20} {15, 27, 31, 39}
{9, 53, 29, 39} {28, 38, 22, 48} {51, 42, 24, 34} {46, 41, 31, 43} {44, 45, 47, 33} {8, 51, 35, 29}
{36, 57, 35, 25} {41, 26, 50, 24} {50, 36, 53, 43} {47, 40, 49, 42} {49, 25, 34, 30}
{0, 8, 30, 16} {0, 18, 24, 36} {1, 8, 22, 40} {1, 24, 12, 38} {2, 47, 21, 29} {2, 27, 19, 17}
{3, 10, 29, 33} {3, 52, 11, 39} {11, 6, 9, 31} {4, 27, 48, 28} {4, 49, 22, 56} {5, 30, 15, 21}
{13, 14, 25, 9} {5, 16, 20, 36} {6, 25, 53, 49} {15, 32, 54, 26} {7, 10, 48, 26} {17, 38, 14, 34}

{20, 51, 57, 23}{28, 35, 12, 46}{19, 42, 41, 45}{57, 44, 37, 7} {50, 23, 43, 31}{42, 53, 47, 39}
{44, 37, 13, 56}{46, 51, 35, 54}{34, 52, 55, 43}{45, 55, 18, 50}{41, 40, 33, 32}
{0, 49, 4, 10} {0, 53, 28, 12} {1, 26, 35, 57} {1, 20, 36, 38} {2, 19, 15, 29} {2, 5, 35, 17}
{5, 40, 44, 3} {3, 16, 44, 47} {7, 40, 4, 25} {8, 7, 45, 18} {6, 15, 21, 32} {17, 42, 8, 11}
{11, 12, 6, 27} {13, 20, 36, 9} {22, 13, 9, 52} {21, 16, 46, 57}{22, 45, 23, 42}{27, 14, 34, 51}
{18, 26, 37, 51}{31, 29, 48, 30}{37, 50, 25, 38}{10, 19, 24, 33}{28, 56, 31, 55}{33, 55, 54, 48}
{50, 30, 23, 39}{41, 47, 52, 24}{39, 53, 14, 34}{56, 46, 49, 43}{41, 43, 54, 32}
{0, 12, 43, 35} {0, 26, 11, 13} {0, 35, 48, 25} {0, 33, 32, 7} {0, 4, 17, 37} {0, 42, 19, 55}
{0, 40, 57, 23} {0, 46, 9, 25} {0, 32, 5, 1} {0, 50, 35, 7} □

Lemma 4.6 *There exists a* $(1, 2)$-*IRH*$(2^{11} : 2^5)$.

Proof Let the point set be Z_{22}, the group set be $\{\{j, j+11\} : j = 0, 1, \dots, 10\}$ and the hole set be $\{\{j, j+11\} : j = 6, 7, \dots, 10\}$. Firstly, we give the four partial 2-parallel classes missing the hole as follows. Let P consist of the following six blocks.

{1, 3, 15, 16} {2, 11, 12, 14} {4, 5, 11, 13} {0, 2, 5, 12} {0, 1, 14, 15} {3, 4, 13, 16}

Let $\beta = (0)(1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)(11)(12\ 13\ 14\ 15\ 16)(17\ 18\ 19\ 20\ 21)$. Then $\beta^i(P)$ for $i = 1, 2, 3, 4$ form the four partial 2-parallel classes.

Secondly, we list six 2-parallel classes on Z_{22} as follows. Here the eleven blocks of every two consecutive rows (the ith and the $(i+1)$th rows for $i = 1, 3, 5, \dots, 11$) form a 2-parallel class.

{1, 3, 15, 16} {0, 1, 8, 9} {7, 8, 11, 12} {4, 14, 18, 21}{4, 13, 21, 17}{3, 9, 16, 18}
{2, 10, 14, 19}{0, 2, 20, 17} {5, 7, 12, 19} {10, 6, 11, 15}{5, 6, 13, 20}
{2, 11, 12, 14}{4, 8, 11, 20} {6, 10, 13, 16}{2, 10, 15, 20}{1, 9, 13, 18} {3, 6, 16, 21}
{5, 15, 19, 17}{4, 7, 12, 17} {0, 5, 7, 8} {0, 3, 21, 18} {1, 9, 14, 19}
{4, 5, 11, 13} {0, 4, 17, 19} {0, 2, 9, 10} {7, 6, 14, 12} {3, 7, 15, 21} {1, 10, 11, 17}
{9, 8, 16, 14} {1, 16, 20, 18}{3, 6, 15, 20} {5, 8, 13, 18} {2, 12, 21, 19}
{0, 2, 5, 12} {2, 6, 14, 20} {3, 10, 15, 17}{0, 1, 19, 21} {8, 9, 11, 13} {4, 6, 16, 18}
{5, 14, 17, 18}{3, 7, 11, 19} {10, 9, 12, 15}{1, 8, 13, 20} {4, 7, 16, 21}
{0, 1, 14, 15} {2, 16, 19, 20}{3, 13, 17, 20}{1, 7, 14, 21} {5, 9, 11, 21} {0, 3, 10, 6}
{8, 7, 15, 13} {4, 8, 16, 17} {2, 6, 11, 18} {4, 10, 12, 19}{5, 9, 12, 18}
{3, 4, 13, 16} {6, 7, 11, 16} {2, 9, 14, 21} {5, 8, 12, 17} {0, 5, 18, 20} {3, 12, 20, 21}
{2, 8, 15, 17} {0, 4, 6, 7} {9, 10, 11, 14}{1, 15, 18, 19}{1, 10, 13, 19}

Thirdly, the remaining twenty 2-parallel classes on Z_{22} are obtained by applying the automorphism β to the following four initial 2-parallel classes. Here the eleven blocks of every two consecutive rows (the ith and the $(i+1)$th rows for $i = 1, 3, 5, 7$) form an initial 2-parallel class.

{2, 7, 8, 16} {4, 6, 8, 13} {4, 9, 16, 19} {10, 12, 14, 18}{11, 14, 17, 20}{0, 15, 12, 18}
{2, 6, 15, 21} {5, 1, 3, 17} {5, 10, 11, 19} {0, 1, 7, 20} {3, 9, 13, 21}
{13, 14, 16, 18}{1, 8, 11, 18} {0, 7, 12, 21} {7, 12, 13, 16} {2, 8, 10, 11} {3, 9, 15, 19}
{1, 15, 20, 17} {4, 5, 20, 17} {2, 4, 6, 19} {3, 5, 9, 10} {0, 6, 14, 21}
{1, 3, 6, 19} {2, 6, 10, 12} {5, 10, 7, 15} {3, 4, 17, 18} {5, 8, 14, 20} {0, 9, 16, 21}
{1, 4, 11, 19} {2, 12, 17, 18} {13, 14, 21, 20}{0, 9, 7, 13} {8, 11, 15, 16}
{10, 13, 15, 18}{16, 12, 17, 20}{11, 16, 20, 21}{1, 8, 14, 17} {2, 7, 15, 19} {6, 9, 12, 13}
{0, 1, 5, 10} {0, 2, 18, 19} {8, 11, 14, 21} {4, 5, 3, 6} {3, 4, 9, 7} □

Lemma 4.7 (Product construction II) *Let p and q be odd integers. If there exist both a* $(1, 2)$-*RH*(2^p) *and a* $(1, 2)$-*RH*(2^q)*, then there exists a* $(1, 2)$-*RH*(2^{pq}) *and a* $(1, 2)$-*IRH*$(2^{pq} : 2^q)$.

Proof Let $(X, \mathcal{G}, \mathcal{B})$ be the given $(1, 2)$-RH(2^p), where $\mathcal{G} = \{G_0, \ldots, G_{p-1}\}$. By the Weighting construction, we can construct a $(1, 2)$-RH$((2q)^p)$ on $X' = X \times Z_q$ with group set $\mathcal{G}' = \{G'_i = G_i \times Z_q : 0 \leq i \leq p-1\}$ and block set $\mathcal{A}$.

For each $i, 0 \leq i \leq p-1$, construct a $(1, 2)$-RH(2^q) on $G_i \times Z_q$ with group set $\{G_i \times \{l\} : l \in Z_q\}$ and block set $\mathcal{C}_i$, which has a resolution $P_i(k)$, $1 \leq k \leq (q-1)(q-2)/3$.

For each $i, 0 \leq i \leq p-1$, let $\mathcal{F}^i = \{F^i_1, \ldots, F^i_{2(q-1)}\}$ be a one-factorization of the complete multiple-graph on $G_i \times Z_q$ with partite set $\{G_i \times \{l\} : l \in Z_q\}$. Let

$$\mathcal{D} = \{\{a, b, c, d\} : \{a, b\} \in F^i_j, \{c, d\} \in F^{i'}_j, 0 \leq i \neq i' \leq p-1, 1 \leq j \leq 2(q-1)\},$$

then $\mathcal{B}' = \mathcal{A} \cup (\cup_{i=0}^{p-1} \mathcal{C}_i) \cup \mathcal{D}$ is the block set of an H(2^{pq}) on the group set $\mathcal{G}'' = \{G_i \times \{l\} : l \in Z_q, 0 \leq i \leq p-1\}$. It is clear that $\cup_{i=0}^{p-1} \mathcal{C}_i$ has a resolution $Q(k) = \cup_{i=0}^{p-1} P_i(k)$, $1 \leq k \leq (q-1)(q-2)/3$. It remains to show that $\mathcal{D}$ can be partitioned into 2-parallel classes.

Let $\mathcal{F}' = \{F'_1, \ldots, F'_{(p-1)/2}\}$ be a partition of the edge set of the complete graph on Z_p, such that each point of Z_p is contained in exactly two edges of F'_i, $1 \leq i \leq (p-1)/2$. For each j and k, $1 \leq j \leq (p-1)/2$, $0 \leq k \leq q-1$, let

$$\mathcal{D}_{j,k} = \{\{a, b, c, d\} : \{a, b\} \text{ is the } m\text{th member of } F^i_j,$$
$$\{c, d\} \text{ is the } (m+k)\text{th member of } F^{i'}_j, 1 \leq m \leq q, \{i, i'\} \in F'_j\}.$$

It is clear that each $\mathcal{D}_{j,k}$ is a 2-parallel class of X'. Since $\mathcal{D} = \cup_{1 \leq j \leq (p-1)/2, 0 \leq k \leq q-1} \mathcal{D}_{j,k}$, the desired H$(2^{pq})$ is $(1, 2)$-resolvable.

For each $i, 0 \leq i \leq p-1$, $\mathcal{B}' \backslash \mathcal{C}_i$ is the block set of an incomplete design $(1, 2)$-IRH$(2^{pq} : 2^q)$ on X' with group set $\mathcal{G}''$ and hole set $\{G_i \times \{l\} : l \in Z_q\}$. □

Theorem 4.8 (Tripling construction II) *If there exists a* $(1, 2)$-*RHF*$_2(g^3 : s)$*, then there exists a* $(1, 2)$-*RHF*$_2((3g)^3 : s)$.

Proof Start with a CQS$(3^3 : 1)$ (as in [6]) on $Z_9 \cup \{\infty\}$ with groups $G_i = \{i, i+3, i+6\}$, $0 \leq i \leq 2$ and stem $\{\infty\}$, whose block set $\mathcal{B}$ is generated by the following 9 base blocks under the automorphism group $\langle(0\ 3\ 6)(1\ 4\ 7)(2\ 5\ 8)(\infty)\rangle$.

$\mathcal{A}_\infty$: $\{0, 1, 2, \infty\}$, $\{0, 4, 8, \infty\}$, $\{0, 5, 7, \infty\}$,
$\mathcal{A}_1$: $\{1, 3, 2, 6\}$, $\{1, 3, 5, 7\}$, $\{2, 6, 5, 7\}$,
$\mathcal{A}_2$: $\{4, 7, 5, 8\}$, $\{3, 6, 5, 8\}$, $\{3, 6, 4, 7\}$.

View each base block as an ordered quadruple given above so that each block $B \in \mathcal{B}$ is ordered.

We will construct a $(1, 2)$-RHF$_2((3g)^3 : s)$ on $X = (Z_9 \times Z_2 \times Z_g) \cup (\{\infty\} \times Z_2 \times Z_s)$ with groups $G(x, j) = \{x\} \times Z_2 \times \{j\}$, $x \in Z_9$, $j \in Z_g$, and $G(\infty, j) = \{\infty\} \times Z_2 \times \{j\}$, $j \in Z_s$, and three holes $F_i = \{G(i, j), G(i+3, j), G(i+6, j) : j \in Z_g\} \cup S$, $0 \leq i \leq 2$, which intersect on a common hole $S = \{G(\infty, j) : j \in Z_s\}$.

For each block $B \in \mathcal{B}$ containing ∞, construct a $(1, 2)$-RHF$_2(g^3 : s)$ on $X_B = ((B \backslash \{\infty\}) \times Z_2 \times Z_g) \cup (\{\infty\} \times Z_2 \times Z_s)$ with group set $\{G(x, j) : x \in B \backslash \{\infty\}, j \in Z_g\} \cup S$, three holes $\{G(x, j) : j \in Z_g\} \cup S$, $x \in B \backslash \{\infty\}$ and a common hole S. Denote its block set by $\mathcal{A}_B$, which has a resolution $\{P_B(x, l) : x \in B \backslash \{\infty\}, 1 \leq l \leq g(g+2s-3)/3\} \cup \{P_B(m) : 1 \leq m \leq 2g^2\}$ such that each $P_B(x, l)$ is a partial 2-parallel class on $(B \backslash \{\infty, x\}) \times Z_2 \times Z_g$ and each $P_B(m)$ is a 2-parallel class on X_B.

For each block $B \in \mathcal{B}$ and $\infty \notin B$, we shall construct an $\mathrm{RH}((2g)^4)$ on $B \times Z_2 \times Z_g$ with groups $\{x\} \times Z_2 \times Z_g,\ x \in B$. Denote its block set by $\mathcal{C}_B$ and resolution classes by $Q_B(m)$, $1 \le m \le 4g^2$.

Let $\mathcal{D} = (\cup_{B\in\mathcal{B},\infty\notin B}\mathcal{C}_B) \cup (\cup_{B\in\mathcal{B},\infty\in B}\mathcal{A}_B)$. By Theorem 2.3, $\mathcal{D}$ is the block set of an $\mathrm{HF}_2((3g)^3 : s)$. It remains to show the resolvability. This $\mathrm{HF}_2((3g)^3 : s)$ should be partitioned into $18g^2$ 2-parallel classes on X and $g(3g + 2s - 3)$ partial 2-parallel classes on $(Z_9\backslash G_i) \times Z_2 \times Z_g$ for each $i, 0 \le i \le 2$.

For each $i, 0 \le i \le 2$, let $P(i, x, l) = \cup_{B\in\mathcal{B},\{x,\infty\}\subset B} P_B(x, l)$, $1 \le l \le g(g + 2s - 3)/3$, $x \in G_i$. Then each $P(i, x, l)$ is a partial 2-parallel class on $(Z_9\backslash G_i) \times Z_2 \times Z_g$. The other $2g^2$ partial 2-parallel classes on $(Z_9\backslash G_i) \times Z_2 \times Z_g$ can be obtained as follows. Denote the three base blocks of $\mathcal{A}_2$ by B_0, B_1, B_2 in order. For $0 \le i \le 2$, let $\mathcal{B}_i = \{3j + B_i : 0 \le j \le 2\}$, then each $\mathcal{B}_i$ is a partial 2-parallel class on $Z_9\backslash G_i$. Let $P'(i, m) = \cup_{B\in\mathcal{B}_i} Q_B(m)$, $2g^2 + 1 \le m \le 4g^2$. Then each $P'(i, m)$ is a partial 2-parallel class on $(Z_9\backslash G_i) \times Z_2 \times Z_g$.

Now we give the required $18g^2$ 2-parallel classes on X. Denote the three base blocks of $\mathcal{A}_1$ by A_0, A_1, A_2 in order. Let $D_0 = A_0$, $D_1 = A_1 + 3 = \{4, 6, 8, 1\}$ and $D_2 = A_2 + 6 = \{8, 3, 2, 4\}$. Further, let

$$\mathcal{A}(0, 0) = \{\{0, 4, 8, \infty\}, A_0, A_1, A_2\},$$
$$\mathcal{A}(1, 0) = \{\{0, 1, 2, \infty\}, B_0, B_1, B_2\},$$
$$\mathcal{A}(2, 0) = \{\{0, 5, 7, \infty\}, D_0, D_1, D_2\}$$

and $\mathcal{A}(i, j) = \{3j + B : B \in \mathcal{A}(i, 0)\}$ for $0 \le i, j \le 2$.

For each $i = 1, 2$, $j = 0, 1, 2$ and $1 \le m \le 2g^2$, let $P''(i, j, m) = (\cup_{B\in\mathcal{A}(i,j),\infty\in B} P_B(m)) \cup (\cup_{B\in\mathcal{A}(i,j),\infty\notin B} Q_B(m))$. For each $j = 0, 1, 2$ and $1 \le m \le 2g^2$, let $P''(0, j, m) = (\cup_{B\in\mathcal{A}(0,j),\infty\in B} P_B(m)) \cup (\cup_{B\in\mathcal{A}(0,j),\infty\notin B} Q_B(m + 2g^2))$. Then each $P''(i, j, m)$ is a 2-parallel class on X.

So $\mathcal{D}$ has the resolution $\{P(i, x, l) : 0 \le i \le 2, x \in G_i, 1 \le l \le g(g + 2s - 3)/3\} \cup \{P'(i, m) : 0 \le i \le 2, 2g^2 + 1 \le m \le 4g^2\} \cup \{P''(i, j, m) : 0 \le i, j \le 2, 1 \le m \le 2g^2\}$, and the $\mathrm{HF}_2((3g)^3 : s)$ is (1, 2)-resolvable. □

To give the third tripling construction for (1, 2)-resolvable H-designs with group size 2, we adapt the concept of resolvable B-pairing as follows. For non-negative integers n and s, a *resolvable simple pairing*, $RP(2n + 1, 2s)$ consists of subsets $D, R_i, S_i, \overline{R}_i, i \in \{0, 1, 2\}$ of $Z_{12n+6+2s}$ and subsets $PR_i, PS_i, i \in \{0, 1, 2\}$ of $Z_{12n+6+2s} \times Z_{12n+6+2s}$ having the following properties:

(1) *Cardinality and symmetry conditions*

(a) $|D| = 2s, |R_i| = |S_i| = 4n + 2, |\overline{R}_i| = 2n + 1, i \in \{0, 1, 2\}$,
(b) $D = -D$.

(2) *Partitioning conditions*

(a) PR_i and PS_i are partitions of R_i and S_i into pairs, respectively. Thus $|PR_i| = |PS_i| = 2n + 1, i \in \{0, 1, 2\}$,
(b) $Z_{12n+6+2s} = D|R_0|R_1|R_2$,
(c) $Z_{12n+6+2s} \cup Z_{12n+6+2s}$
$= D \cup (D - \epsilon) \cup R_0 \cup (R_0 - \epsilon) \cup S_0 \cup (S_0 - \epsilon) \cup \overline{R}_2 \cup (\overline{R}_2 - \epsilon) \cup (-\overline{R}_1) \cup (-\overline{R}_1 - \epsilon)$
$= D \cup (D - \epsilon) \cup R_1 \cup (R_1 - \epsilon) \cup S_1 \cup (S_1 - \epsilon) \cup \overline{R}_0 \cup (\overline{R}_0 - \epsilon) \cup (-\overline{R}_2) \cup (-\overline{R}_2 - \epsilon)$
$= D \cup (D - \epsilon) \cup R_2 \cup (R_2 - \epsilon) \cup S_2 \cup (S_2 + 6n + 3 + s) \cup \overline{R}_1 \cup (\overline{R}_1 - \epsilon) \cup (-\overline{R}_0) \cup$
$(-\overline{R}_0 - \epsilon)$, where ϵ is some odd integer.

(3) *Pairing conditions*
Let $L_i = \{|x - y| : \{x, y\} \in PR_i\}$ and $O_i = \{|x - y| : \{x, y\} \in PS_i\}$, $i \in \{0, 1, 2\}$,

(a) $6n + 3 + s \notin (L_i \cup O_i)$, $i \in \{0, 1, 2\}$,
(b) $|L_i| = |O_i| = 2n + 1$, $L_i \cap O_i = \emptyset$, and all members of O_i are odd, $i \in \{0, 1, 2\}$,
(c) the complement G_i of the graph $G(12n + 6 + 2s, L_i \cup O_i \cup \{6n + 3 + s\})$ has a one-factorization.

Lemma 4.9 *For each pair of integers $k \geq 0$ and $s > 0$, there exists an RP$(2k + 1, 2s)$.*

Proof For each given pair of integers $k \geq 0$ and $s > 0$, we construct the desired RP$(2k+1, 2s)$ directly, which are separated into the following four cases:

(1) For $k \geq 1$ and $s = 2m$, take $\epsilon = 6k + 2m + 3$ and let

$D = \{(6k + 2m + 3)i + j : 0 \leq i \leq 1, 0 \leq j \leq m - 1 \text{ or } 6k + m + 4 \leq j \leq 6k + 2m + 2\} \cup \{m, 12k + 3m + 6\}$,
$PR_0 = \{\{m + 2 + j, m + 4k + 2 - j\} : 0 \leq j \leq 2k - 1\} \cup \{\{m + 1, m + 2k + 2\}\}$,
$PR_1 = \{\{m + 4k + 3 + j, 8k + 3m + 3 - j\} : 0 \leq j \leq 2k\}$,
$PR_2 = \{\{8k + 3m + 4 + j, 12k + 3m + 4 - j\} : 0 \leq j \leq 2k - 1\} \cup \{\{10k + 3m + 4, 12k + 3m + 5\}\}$,
$PS_0 = \{\{m + 4k + 3 + j, 10k + 3m + 4 - j\} : 0 \leq j \leq 2k\}$,
$PS_1 = \{\{8k + 3m + 4 + j, 12k + 3m + 5 - j\} : 0 \leq j \leq 2k\}$,
$PS_2 = \{\{6k+3m+4+j, 8k+3m+3-j\} : 0 \leq j \leq k-1\} \cup \{\{m+j, 8k+3m+3-j\} : 0 \leq j \leq k - 1\} \cup P$, where $P = \{\{m + k, m + 4k + 2\}\}$ when k is odd and $P = \{\{7k + 3m + 3, m + 4k + 2\}\}$ when k is even.
$\overline{R}_0 = \{m + 2k + 2 + j : 0 \leq j \leq 2k\}$,
$\overline{R}_1 = \{m + 4k + 3 + j : 0 \leq j \leq 2k\}$,
$\overline{R}_2 = \{10k + 3m + 5 + j : 0 \leq j \leq 2k\}$.

(2) For $k = 0$ and $s = 2m$, take $\epsilon = 2m + 3$ and let

$D = \{(2m+3)i+j : 0 \leq i \leq 1, 0 \leq j \leq m-1 \text{ or } m+4 \leq j \leq 2m+2\} \cup \{m, 3m+6\}$,
$PR_0 = \{\{m + 1, 3m + 5\}\}$, $PR_1 = \{\{m + 3, 3m + 3\}\}$, $PR_2 = \{\{3m + 4, m + 2\}\}$,
$PS_0 = \{\{m + 3, 3m + 4\}\}$, $PS_1 = \{\{m + 1, m + 2\}\}$, $PS_2 = \{\{m + 2, 3m + 3\}\}$,
$\overline{R}_0 = \{m + 2\}$, $\overline{R}_1 = \{m + 3\}$, $\overline{R}_2 = \{3m + 5\}$.

(3) For $k \geq 1$ and $s = 2m + 1$, take $\epsilon = 1$ and let

$D = \{(6k+2m+4)i+j : 0 \leq i \leq 1, 0 \leq j \leq m \text{ or } 6k+m+4 \leq j \leq 6k+2m+3\}$,
$PR_0 = \{\{m + 4k + 3 + j, 8k + 3m + 5 - j\} : 0 \leq j \leq 2k\}$,
$PR_1 = \{\{m + 2 + j, m + 4k + 2 - j\} : 0 \leq j \leq 2k - 1\} \cup \{\{m + 1, m + 2k + 2\}\}$,
$PR_2 = \{\{8k + 3m + 6 + j, 12k + 3m + 6 - j\} : 0 \leq j \leq 2k - 1\} \cup \{\{10k + 3m + 6, 12k + 3m + 7\}\}$,
$PS_0 = \{\{m + 2k + 2 + j, 12k + 3m + 7 - j\} : 0 \leq j \leq 2k\}$,
$PS_1 = \{\{4k + m + 3 + j, 10k + 3m + 6 - j\} : 0 \leq j \leq 2k\}$,
$PS_2 = \{\{6k+3m+5+j, 8k+3m+4-j\} : 0 \leq j \leq k-1\} \cup \{\{m+j, 8k+3m+5-j\} : 0 \leq j \leq k\}$,
$\overline{R}_0 = \{3m + 6k + 5 + j : 0 \leq j \leq 2k\}$,
$\overline{R}_1 = \{m + 2k + 2 + j : 0 \leq j \leq 2k\}$,
$\overline{R}_2 = \{m + 1 + j : 0 \leq j \leq 2k\}$.

(4) For $k = 0$ and $s = 2m + 1$, take $\epsilon = 1$ and let

$D = \{(2m + 4)i + j : 0 \le i \le 1, 0 \le j \le m$ or $m + 4 \le j \le 2m + 3\}$,
$PR_0 = \{\{m + 3, 3m + 5\}\}$, $PR_1 = \{\{m + 1, m + 2\}\}$, $PR_2 = \{\{3m + 6, 3m + 7\}\}$,
$PS_0 = \{\{m + 2, 3m + 7\}\}$, $PS_1 = \{\{m + 3, 3m + 6\}\}$, $PS_2 = \{\{m, 3m + 5\}\}$,
$\overline{R}_0 = \{3m + 5\}$, $\overline{R}_1 = \{m + 2\}$, $\overline{R}_2 = \{m + 1\}$. □

Lemma 4.10 *There exists a* $(1, 2)$-*RHF*$_2((6n + 3 + s)^3 : s)$ *for each* $n \ge 0$ *and* $s \ge 1$. *Furthermore, there exists a* $(1, 2)$-*RHF*$_2((6n + 3 + s)^3 : s)$ *having a subdesign* $(1, 2)$-*RH*(2^4) *when s is even.*

Proof Let $X = (Z_{12n+6+2s} \times Z_3) \cup \{\infty_0, \infty_1, \ldots, \infty_{2s-1}\}$. Define the groups $G(i, j) = \{(k(6n + s + 3) + i, j) : k = 0, 1\}$, $i = 0, 1, \ldots, 6n + s + 2$, $j \in Z_3$, $G(\infty, j) = \{\infty_{ks+j} : k = 0, 1\}$, $j = 0, 1, \ldots, s - 1$ and the hole set $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ with $F_0 = \{G(\infty, j) : j = 0, 1, \ldots, s - 1\}$ and $F_{1+j} = F_0 \cup \{G(i, j) : i = 0, 1, \ldots, 6n + s + 2\}$, $j = 0, 1, 2$.

Let $D, R_i, \overline{R}_i, S_i, PR_i, PS_i, i = 0, 1, 2$ be an $RP(2n + 1, 2s)$, which exists by Lemma 4.9. Let $F_i^{2k-1}|F_i^{2k}$ be a one-factorization of the graph $G(12n + 6 + 2s, \{m\})$, where m is the k-th member of O_i for $1 \le k \le 2n + 1$. Let $F_i^{4n+3}|F_i^{4n+4}|\ldots|F_i^{8n+2s+2}$ be a one-factorization of the complement of the graph $G(12n + 6 + 2s, L_i \cup O_i \cup \{6n + s + 3\})$. Then it is natural that $F_i^1|F_i^2|\ldots|F_i^{8n+2s+2}$ is a one-factorization of the complement of the graph $G(12n + 6 + 2s, L_i \cup \{6n + s + 3\})$.

We construct an HF$_2((6n + 3 + s)^3 : s)$ $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the block set $\mathcal{B}$ consisting of the following three parts:

(1) $\{\infty_j, (a, 0), (b, 1), (c, 2)\}$, where $a + b + c \equiv d \pmod{12n + 6 + 2s}$, d is the j-th member of D and $0 \le j < 2s$.
(2) $\{(a + q, i), (a + t, i), (b, i + 1), (c, i + 2)\}$, where $a + b + c \equiv 0 \pmod{12n + 6 + 2s}$, $\{q, t\} \in PR_i$ and $i \in Z_3$.
(3) $\{(a, i), (b, i), (c, i + 1), (d, i + 1)\}$, where $\{a, b\} \in F_i^j$ and $\{c, d\} \in F_{i+1}^j$, $i \in Z_3$ and $j = 1, 2, \ldots, 8n + 2s + 2$.

Now, we partition them into partial 2-parallel classes or 2-parallel classes.

First, we give the partial 2-parallel classes. Define $F_i^j = \{F_i^j(k) : 0 \le k \le 6n + s + 2\}$. For each $i \in Z_3$, let

$$\begin{aligned} P_i(j, k) = \{&\{(a, i + 1), (b, i + 1), (c, i + 2), (d, i + 2)\} : \{a, b\} = F_{i+1}^j(m), \{c, d\} \\ &= F_{i+2}^j(m + k), 0 \le m \le 6n + s + 2\}, \text{ where} \\ &4n + 3 \le j \le 8n + 2s + 2 \text{ and } 0 \le k \le 6n + s + 2. \end{aligned}$$

It is clear that $P_i(j, k) \cup P_i(12n + 2s + 5 - j, k)$ with $4n + 3 \le j \le 6n + s + 2$, $0 \le k \le 6n + s + 2$ form the $(2n + s)(6n + s + 3)$ partial 2-parallel classes missing the hole F_i.

Next, we give the $2(6n + 3 + s)^2$ complete 2-parallel classes. For each (a, b, c) such that $a + b + c \equiv 0 \pmod{12n + 6 + 2s}$ and b is even, let $P(a, b, c)$ be comprised of $4s$ blocks from Part (δ), $6(2n + 1)$ blocks from Part (ρ) and $6n + 3$ blocks from Part (ϕ) as follows:

Part (δ):

$$\{\infty_j, (a + d, 0), (b - d, 1), (c + d, 2)\},$$
$$\{\infty_j, (a + d, 0), (b - d + \epsilon, 1), (c + d - \epsilon, 2)\}, d \text{ is the } j\text{-th member of } D, 0 \le j < 2s.$$

Part (ρ):

$$\{(a+q,0),(a+t,0),(b+u,1),(c-u,2)\} \text{ and}$$
$$\{(a+q,0),(a+t,0),(b+u+\epsilon,1),(c-u-\epsilon,2)\} \text{ for } i=0;$$
$$\{(a-u,0),(b+q,1),(b+t,1),(c+u,2)\} \text{ and}$$
$$\{(a-u,0),(b+q+\epsilon,1),(b+t+\epsilon,1),(c+u-\epsilon,2)\} \text{ for } i=1;$$
$$\{(a+u,0),(b-u,1),(c+q,2),(c+t,2)\} \text{ and}$$
$$\{(a+u,0),(b-u+\epsilon,1),(c+q-\epsilon,2),(c+t-\epsilon,2)\} \text{ for } i=2;$$

where $\{q,t\}$ is the j-th pair in PR_i and u is the j-th member of $\overline{R}_i$, $1 \le j \le 2n+1$.

Part (ϕ): If a is even, then the blocks of Part (ϕ) are all those of the forms:

$$\{(a+v,0),(a+v',0),(b+t,1),(b+t',1)\},$$
$$\{(b+t+\epsilon,1),(b+t'+\epsilon,1),(c+w,2),(c+w',2)\},$$
$$\{(c+w+6n+3+s,2),(c+w'+6n+3+s,2),(a+v,0),(a+v',0)\}.$$

If a is odd, then the blocks of Part (ϕ) are all those of the forms:

$$\{(a+v,0),(a+v',0),(b+t+\epsilon,1),(b+t'+\epsilon,1)\},$$
$$\{(b+t,1),(b+t',1),(c+w,2),(c+w',2)\},$$
$$\{(c+w+6n+3+s,2),(c+w'+6n+3+s,2),(a+v,0),(a+v',0)\}.$$

The pairs $\{v,v'\}$, $\{t,t'\}$ and $\{w,w'\}$ above are the j-th ($1 \le j \le 2n+1$) members of PS_0, PS_1 and PS_2, respectively.

Thus the above $\text{HF}_2((6n+3+s)^3 : s)$ is (1, 2)-resolvable. By Lemma 4.9, $\{0, 6n+s+3\} \subset D$ and $\epsilon = 6n+s+3$ when s is even. Without loss of generality we may assume 0 and $6n+s+3$ are the 0th and sth elements of D, respectively. Let

$$\delta_0 = \{\{\infty_{ks}, (a+d)_0, (b-d)_1, (c+d)_2\} : a+b+c \equiv 0 \pmod{12n+6+2s}, a,b,c \in \{0, 6n+s+3\}, d \text{ is the } (ks)\text{th member of } D \text{ and } 0 \le k \le 1\}.$$

Note that $\delta_0 \subset \delta$ and δ_0 forms the block set of an RH(2^4) with the group set $\{\{0_i, (6n+s+3)_i\} : i \in \{0,1,2\}\} \cup \{\{\infty_0, \infty_s\}\}$ and parallel classes $\{\{\infty_{(i+j+k+g)s}, ((i+g)(6n+s+3))_0, ((j+g)(6n+s+3))_1, ((k+g)(6n+s+3))_2\} : g = 0, 1\}, i+j+k \equiv 0 \pmod 2$, which is a (1, 2)-RH($2^4$) indeed. Hence, the (1, 2)-$\text{RHF}_2((6n+s+3)^3 : s)$ contains a subdesign (1, 2)-RH(2^4) when s is even. □

As a consequence, we have our third tripling construction for (1, 2)-resolvable H-designs with group size 2.

Corollary 4.11 (Tripling construction III) *Let $n \equiv 2s+3 \pmod 6$ and $s \ge 1$. If there exists a (1, 2)-IRH($2^n : 2^s$), then there exist both a (1, 2)-IRH($2^{3n-2s} : 2^n$) and a (1, 2)-IRH($2^{3n-2s} : 2^s$). Furthermore, if there exists a (1, 2)-RH(2^n) or a (1, 2)-RH(2^s), then there exists a (1, 2)-RH(2^{3n-2s}), as well as a (1, 2)-IRH($2^{3n-2s} : 2^4$) when s is even.*

Lemma 4.12 *There exists a (1, 2)-IRH($2^{17} : 2^s$) for each $s \in \{1, 2, 4, 7\}$, a (1, 2)-$RHF_2(12^k : 1)$ and a (1, 2)-$RHF_2(15^k : 2)$ for each $k \in \{3, 5\}$.*

Proof Applying the Tripling construction III with $(n, s) = (7, 2)$, there exists a (1, 2)-RH(2^{17}), a (1, 2)-IRH($2^{17} : 2^7$) and a (1, 2)-IRH($2^{17} : 2^4$). The designs with a hole of sizes 1 or 2 are actually a (1, 2)-RH(2^{17}). By Lemma 4.10, there exists a (1, 2)-$\text{RHF}_2(4^3 : 1)$ and a $(1, 2)$-RHF$_2(5^3 : 2)$. Then there exists a $(1, 2)$-RHF$_2(12^3 : 1)$ and a $(1, 2)$-RHF$_2(15^3 : 2)$ by the Tripling construction II.

For the existence of a $(1, 2)$-RHF$_2(12^5 : 1)$ and a $(1, 2)$-RHF$_2(15^5 : 2)$, start from an RCQS$(3^5 : 1)$. Then applying Theorem 2.3 with a $(1, 2)$-RHF$_2(4^3 : 1)$ and a $(1, 2)$-RH(8^4), or a $(1, 2)$-RHF$_2(5^3 : 2)$ and a $(1, 2)$-RH(10^4) gives the desired $(1, 2)$-RHF$_2(12^5 : 1)$ and $(1, 2)$-RHF$_2(15^5 : 2)$, respectively. □

Lemma 4.13 *There exists a* $(1, 2)$*-IRH*$(2^n : 2^s)$ *for each* $n \equiv 17, 37 \pmod{60}$ *and* $s \in \{1, 2, 4, 7, 17\}$.

Proof For each $n = 60m + 17$ or $60m + 37, m \geq 0$, start from an RCQS$(1^{(n-2)/5} : 1)$ obtained from an RSQS$((n + 3)/5)$. Applying Theorem 2.3 with a $(1, 2)$-RHF$_2(5^3 : 2)$ and a $(1, 2)$-RH(10^4), we get a $(1, 2)$-RHF$_2(5^{(n-2)/5} : 2)$. Applying Theorem 2.4 with a $(1, 2)$-RH(2^7), we get a $(1, 2)$-RH(2^n). The design with a hole of size 17 exists since we input a $(1, 2)$-RHF$_2(5^3 : 2)$ when applying Theorem 2.3. For each $s \in \{1, 2, 4, 7\}$, there exists a $(1, 2)$-IRH$(2^n : 2^s)$ since a $(1, 2)$-IRH$(2^{17} : 2^s)$ exists by Lemma 4.12. □

Lemma 4.14 *There exists a* $(1, 2)$*-IRH*$(2^n{:}2^s)$ *for each* $n \equiv 37 \pmod{48}$ *and* $s \in \{1, 2, 4, 13, 37, 61\}$.

Proof For each $n = 48m + 37, m \geq 0$, start from a URCS$(3, \{4, 6\}, 4m + 4)$ of type $(1^{4m+3} : 1)$, which is obtained from a URS$(4m + 4)$ (see [11]). Applying Theorem 2.3 with a $(1, 2)$-RHF$_2(12^{k-1} : 1)$ and a $(1, 2)$-RH(24^k) with $k \in \{4, 6\}$, we get a $(1, 2)$-RHF$_2(12^{4m+3} : 1)$. Applying Theorem 2.4 with a $(1, 2)$-RH(2^{13}), we get a $(1, 2)$-RH(2^n) and a $(1, 2)$-IRH$(2^n : 2^{13})$. The designs with a hole of sizes 1 or 2 are actually a $(1, 2)$-RH(2^n). The designs with a hole of sizes 37 or 61 exist since we input a $(1, 2)$-RHF$_2(12^3 : 1)$ and a $(1, 2)$-RHF$_2(12^5 : 1)$ when applying Theorem 2.3. The design with a hole of size 4 exists since the input design $(1, 2)$-RH(24^4) exists, which contains a subdesign $(1, 2)$-RH(2^4). □

Lemma 4.15 *There exists a* $(1, 2)$*-IRH*$(2^n : 2^s)$ *for each* $n \equiv 47 \pmod{60}$ *and* $s \in \{1, 2, 4, 7, 17, 47\}$.

Proof For each $n = 60m + 47, m \geq 0$, start from a URCS$(3, \{4, 6\}, 4m + 4)$ of type $(1^{4m+3} : 1)$. Applying Theorem 2.3 with a $(1, 2)$-RHF$_2(15^{k-1} : 2)$ and a $(1, 2)$-RH(30^k) with $k \in \{4, 6\}$, we get a $(1, 2)$-RHF$_2(15^{4m+3} : 2)$. Applying Theorem 2.4 with a $(1, 2)$-RH(2^{17}), we get a $(1, 2)$-RH(2^n) and a $(1, 2)$-IRH$(2^n : 2^{17})$. For each $s \in \{1, 2, 4, 7\}$, a $(1, 2)$-IRH$(2^n : 2^s)$ exists since there exists a $(1, 2)$-IRH$(2^{17} : 2^s)$ by Lemma 4.12. The design with a hole of size 47 exists since we input a $(1, 2)$-RHF$_2(15^3 : 2)$ when applying Theorem 2.3. □

As a consequence of the Tripling construction III, we obtain

Theorem 4.16 *If there exists a constant* $M \geq 7$*, such that for any integer* $n \equiv 1$ *or* $5 \pmod{6}$ *in the range* $M \leq n < 3M$*, there exists a* $(1, 2)$*-IRH*$(2^n : 2^s)$ *with* $s \in \{5, 17\}$*, then for all integer* $n \geq M$ *and* $n \equiv 1$ *or* $5 \pmod{6}$*, there exists a* $(1, 2)$*-IRH*$(2^n : 2^s)$ *with* $s \in \{5, 17\}$.

Proof It is clear that the existence of a $(1, 2)$-IRH$(2^n : 2^{17})$ implies the existence of a $(1, 2)$-IRH$(2^n : 2^s)$ for all $s \in \{1, 2, 4, 7, 17\}$ by Lemma 4.12. We proceed the proof by induction. Let n be an integer, $n \equiv 1$ or $5 \pmod{6}$ and $n \geq 3M$. Assume that for all $n' \equiv 1$ or $5 \pmod{6}$ in the range $M \leq n' < n$, there exists a $(1, 2)$-IRH$(2^{n'} : 2^{s'})$ with $s' \in \{5, 17\}$. Write $n = 3m - 2 \cdot s$, where $s = 7, 17, 4, 5, 1, 2$ when $n \equiv 1, 5, 7, 11, 13, 17 \pmod{18}$, respectively. It is simple to check that $m \equiv 1$ or $5 \pmod{6}$ and $M \leq m < n$. Then applying Tripling construction III gives the conclusion. □

Lemma 4.17 *For each integer* $n \equiv 1$ *or* $5 \pmod 6$, $n \geq 7$ *and* $n \notin \{41, 55, 59, 61, 65, 73, 115, 131, 149, 173, 179, 185, 193, 215, 235, 281, 311, 335, 379, 389, 451, 491, 505, 509, 515, 535, 541, 545, 547, 599, 671, 829, 839, 929, 1103, 1121, 1153, 1315, 1415, 1459, 1465, 1531, 1535, 1571, 1601\}$*, there exists a* $(1, 2)$*-RH*(2^n).

Proof As in Lemma 3.11, let L be the poset of pairs (n, s) such that there exists a $(1, 2)$-IRH$(2^n : 2^s)$. We will compute the output of the Tripling constructions II and III, and the Product construction II by a computer programme, which involves the following steps:

Step 1: Initialize L. Let $L = \{(7, 1), (7, 2), (11, 1), (11, 2), (11, 5), (13, 1), (13, 2), (19, 1), (19, 2), (23, 1), (23, 2), (25, 1), (25, 2), (29, 1), (29, 2)\} \cup \{(n, s) :$ there exists a $(1, 2)$-IRH$(12^n : 12^s)$ in Lemmas 4.13–4.15$\}$. Sort L in ascending order. Let (n, s) be the smallest pair in L.

Step 2: Check whether (n, s) satisfies Tripling construction III's condition, i.e., $n \equiv 2s + 3 \pmod 6$. If not, go to Step 3. If yes, update L by adding pairs $(3n - 2s, n)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Furthermore, add pair $(3n - 2s, 4)$ into L if s is even. Sort the updated L in ascending order, then go to Step 4.

Step 3: Check whether $n - s \equiv 0 \pmod 3$. If not, go to Step 4. If yes, write $n - s = 3^x \cdot t$, such that $t > s$ and $3 \nmid t$, or $s < t < 3s$ and $3|t$. Check whether $(t + s, s)$ satisfies the Tripling construction III's condition, i.e., $t \equiv s + 3 \pmod 6$. If not, go to Step 4. If yes, update L by adding pairs $(3n - 2s, n)$ and $(3n - 2s, k)$ for all k such that $(n, k) \in L$. Furthermore, add $(3n - 2s, 4)$ into L if s is even. Sort the updated L in ascending order, then go to Step 4.

Step 4: Apply the Product construction II. For each m such that $(m, 1) \in L$, update L by adding pairs (mn, n), (mn, m) and (mn, k) for all k such that (n, k) or $(m, k) \in L$. Sort the updated L in ascending order. Let (n, s) be the next smallest pair in the updated L, then go to Step 2.

The programme was run with $n < 7700$ and $s \leq 64$, and produced two results as follows:

Result 1: For all n, $n \equiv 1$ or $5 \pmod 6$ and $4 \leq n < 4289$, there exists a $(1, 2)$-RH(2^n) with possible exceptions $n \in \{41, 55, 59, 61, 65, 73, 115, 131, 149, 173, 179, 185, 193, 215, 235, 281, 311, 335, 379, 389, 451, 491, 505, 509, 515, 535, 541, 545, 547, 599, 671, 829, 839, 929, 1103, 1121, 1153, 1315, 1415, 1459, 1465, 1531, 1535, 1571, 1601\}$.

Result 2: There exists a $(1, 2)$-IRH$(2^n : 2^s)$ with $s \in \{5, 17\}$ for all n, $n \equiv 1$ or $5 \pmod 6$ in the range $4289 \leq n < 12867$.

By Theorem 4.16, there exists a $(1, 2)$-IRH$(2^n : 2^s)$ with $s \in \{5, 17\}$ for all $n \geq 4289$ and $n \equiv 1$ or $5 \pmod 6$. Hence there exists a $(1, 2)$-RH(2^n) by Theorem 2.4. This completes the proof. □

Lemma 4.18 *There exists a* $(1, 2)$*-RH*(2^n) *for each* $n \in \{311, 451, 671, 1531\}$.

Proof For each $n \in \{311, 671\}$, start from an RCQS$(1^{(n-1)/10} : 1)$ which is obtained from an RSQS$((n - 1)/10 + 1)$. Applying Theorem 2.3 with a $(1, 2)$-RHF$_2(10^3 : 1)$ and a $(1, 2)$-RH(20^4), we get a $(1, 2)$-RHF$_2(10^{(n-1)/10} : 1)$. Applying Theorem 2.4 with a $(1, 2)$-RH(2^{11}), we get the desired $(1, 2)$-RH(2^n). Here, the $(1, 2)$-RHF$_2(10^3 : 1)$ exists by Lemma 4.10.

For $n = 451$, start from an RCQS$(1^{15} : 1)$. Applying Theorem 2.3 with a $(1, 2)$-RHF$_2(30^3 : 1)$ and a $(1, 2)$-RH(60^4), we get a $(1, 2)$-RHF$_2(30^{15} : 1)$. Applying Theorem 2.4

with a $(1,2)$-RH(2^{31}), we get a $(1,2)$-RH(2^{451}). Here, the $(1,2)$-RHF$_2(30^3 : 1)$ exists by applying the Tripling construction II with a $(1,2)$-RHF$_2(10^3 : 1)$.

For $n = 1531$, start from an RCQS$(1^{139} : 1)$. Applying Theorem 2.3 with a $(1,2)$-RHF$_2(11^3 : 2)$ and a $(1,2)$-RH(22^4), we get a $(1,2)$-RHF$_2(11^{139} : 2)$. Applying Theorem 2.4 with a $(1,2)$-RH(2^{13}), we get a $(1,2)$-RH(2^{1531}). Here, the $(1,2)$-RHF$_2(11^3 : 2)$ exists by Lemma 4.10. □

Combining Lemmas 4.17 and 4.18, we get the main result in this section.

Lemma 4.19 *The necessary conditions $n \equiv 1$ or $2 \pmod 3$ for the existence of a $(1,2)$-RH(2^n) are also sufficient except for $n = 5$ and except possibly for* $n \in \{41, 55, 59, 61, 65, 73, 115, 131, 149, 173, 179, 185, 193, 215, 235, 281, 335, 379, 389, 491, 505, 509, 515, 535, 541, 545, 547, 599, 829, 839, 929, 1103, 1121, 1153, 1315, 1415, 1459, 1465, 1535, 1571, 1601\}$.

5 Conclusions

Combining Theorems 1.1 and 3.13, we have the following existence result for resolvable H-designs.

Theorem 5.1 *The necessary conditions $gn \equiv 0 \pmod 4$, $g(n-1)(n-2) \equiv 0 \pmod 3$ and $n \geq 4$ for the existence of a resolvable H-design of type g^n are also sufficient except possibly for $g \equiv 4, 8 \pmod{12}$ and $n = 73, 149$, or for $g \equiv 0 \pmod{12}$ and* $n \in \{15, 21, 27, 33, 39, 69, 75, 87, 105, 111, 129, 147, 213, 231, 243, 321\}$.

Combining Theorems 1.2 and 4.19, we have a near complete solution to the existence problem on RSQS$(1, 2, 2n)$, which was stated in [4].

Theorem 5.2 *The necessary conditions for the existence of an RSQS$(1,2,2n)$, i.e., $n \equiv 1$ or $5 \pmod 6$, are also sufficient except for $n = 5$ and possibly for* $n \in \{41, 55, 59, 61, 65, 73, 115, 131, 149, 173, 179, 185, 193, 215, 235, 281, 335, 379, 389, 491, 505, 509, 515, 535, 541, 545, 547, 599, 829, 839, 929, 1103, 1121, 1153, 1315, 1415, 1459, 1465, 1535, 1571, 1601\}$.

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Appendix: the existence of small RH-designs

Lemma 3 *There exists an RH(12^9).*

Proof In [8], Mills constructed an H(6^9) on $Z_{27} \times Z_2$ with group set $\{\{(m,0), (m+9,0), (m+18,0), (m,1), (m+9,1), (m+18,1)\} : m \in Z_9\}$ and the following 42 forms of blocks, where $m \in Z_{27}$, $a, b \in Z_2$.

$\{(m, a), (m+2, a), (m+5, b), (m+7, b)\}$, $\{(m, a), (m+5, b), (m+12, a+b), (m+20, a+b+1)\}$
$\{(m, a), (m+1, a), (m+5, b), (m+6, b)\}$, $\{(m, a), (m+5, b), (m+10, a+1), (m+21, a+b+1)\}$
$\{(m, a), (m+1, a), (m+7, b), (m+8, b)\}$, $\{(m, a), (m+5, b), (m+12, a+b+1), (m+17, a+1)\}$
$\{(m, a), (m+2, a), (m+8, b), (m+10, b)\}$, $\{(m, a), (m+6, b), (m+13, a+b+1), (m+19, a+1)\}$
$\{(m, a), (m+2, b), (m+12, b), (m+17, a)\}$, $\{(m, a), (m+1, b), (m+15, a+b+1), (m+17, a+1)\}$
$\{(m, a), (m+1, b), (m+2, a+b), (m+4, b)\}$, $\{(m, a), (m+1, b), (m+12, a+b+1), (m+14, b+1)\}$
$\{(m, a), (m+2, a+1), (m+5, b), (m+8, b)\}$, $\{(m, a), (m+2, a+1), (m+7, b), (m+23, a+b+1)\}$
$\{(m, a), (m+3, a), (m+7, b), (m+20, a+b)\}$, $\{(m, a), (m+2, a+1), (m+6, b), (m+21, a+b+1)\}$
$\{(m, a), (m+4, b), (m+8, a), (m+15, a+b)\}$, $\{(m, a), (m+4, b), (m+8, a+1), (m+16, a+b+1)\}$
$\{(m, a), (m+1, a+1), (m+5, b), (m+6, b+1)\}$, $\{(m, a), (m+3, a), (m+16, b), (m+23, a+b+1)\}$
$\{(m, a), (m+3, a), (m+11, b), (m+14, b+1)\}$, $\{(m, a), (m+4, b), (m+14, b+1), (m+21, a+b)\}$
$\{(m, a), (m+1, a+1), (m+7, b), (m+8, b+1)\}$, $\{(m, a), (m+5, b), (m+10, a), (m+16, a+b+1)\}$
$\{(m, a), (m+6, b), (m+12, a+b), (m+19, a)\}$, $\{(m, a), (m+2, a+1), (m+10, b), (m+24, a+b)\}$
$\{(m, a), (m+2, a+1), (m+19, b), (m+22, b)\}$, $\{(m, a), (m+4, b), (m+10, a+b), (m+20, a+b)\}$
$\{(m, a), (m+1, b), (m+11, b), (m+12, a+b)\}$, $\{(m, a), (m+3, a), (m+15, b), (m+19, a+b+1)\}$
$\{(m, a), (m+1, b), (m+11, b+1), (m+13, a)\}$, $\{(m, a), (m+1, b), (m+14, b), (m+16, a+b+1)\}$
$\{(m, a), (m+1, b), (m+3, a+1), (m+4, b+1)\}$, $\{(m, a), (m+3, a+1), (m+11, b), (m+15, a+b)\}$
$\{(m, a), (m+3, a+1), (m+8, b), (m+22, a+b)\}$, $\{(m, a), (m+1, b), (m+13, a+1), (m+15, a+b)\}$
$\{(m, a), (m+3, b), (m+6, b+1), (m+17, a+1)\}$, $\{(m, a), (m+5, b), (m+11, a+b), (m+19, a+b)\}$
$\{(m, a), (m+3, a+1), (m+7, b), (m+10, b+1)\}$, $\{(m, a), (m+3, b), (m+13, a+b+1), (m+17, a)\}$
$\{(m, a), (m+2, a), (m+6, b), (m+23, a+b+1)\}$, $\{(m, a), (m+1, b), (m+2, a+b+1), (m+25, b)\}$

For each form of blocks, taking $m = 0$ and $a, b \in Z_2$, we get four blocks. Thus we get 168 base blocks of the H(6^9) all together. Denote this set of base blocks by $\mathcal{B}''$, which are developed by $(+1 \; mod \; 27, -)$. Define a map $\phi : (x, y) \to 27y+x$, for each element $(x, y) \in Z_{27} \times Z_2$. Then $\mathcal{B}' = \phi(\mathcal{B}'')$ is the base block set of an H(6^9) on $I_{54} = \{0, 1, \ldots, 53\}$ with group set $\{\{m, m+9, \ldots, m+45\} : m \in Z_9\}$ and automorphism group $\langle(0\,1\ldots26)(27\,28\ldots53)\rangle$.

For each block $B = \{a, b, c, d\} \in \mathcal{B}'$, construct an H($2^4$) with group set $\{\{x, x+54\} : x \in B\}$ and block set $\mathcal{A}_B = \{\{a+54i, b+54(i+k), c+54j, d+54(j+k)\} : i, j, k \in Z_2\}$. Let $\mathcal{B} = \cup_{B\in\mathcal{B}'}\mathcal{A}_B$. It is clear that $\mathcal{B}$ is the set of base blocks of an H(12^9) on $I_{108} = \{0, 1, \ldots, 107\}$ with group set $\{\{m, m+9, \ldots, m+99\} : m \in Z_9\}$ and automorphism group $\langle\alpha\rangle$, where $\alpha = (0\,1\ldots26)(27\,28\ldots53)(54\,55\ldots80)(81\,82\ldots107)$. Now we need to show the resolution.

Note that there are several blocks in $\mathcal{B}$, each of which contains exactly one element in each cycle of α. We first list below some of these blocks, each of which gives a parallel class when developed under the automorphism group $\langle\alpha\rangle$.

{0, 28, 59, 87}	{0, 28, 61, 89}	{0, 29, 60, 102}	{0, 30, 61, 91}	{0, 28, 86, 60}
{0, 28, 88, 62}	{0, 28, 56, 106}	{0, 28, 84, 58}	{0, 28, 95, 70}	{0, 29, 93, 71}
{0, 32, 66, 98}	{0, 32, 93, 74}	{0, 29, 88, 77}	{0, 30, 60, 98}	{0, 32, 91, 75}
{0, 31, 89, 70}	{0, 30, 88, 64}	{27, 1, 59, 87}	{27, 1, 61, 89}	{27, 1, 83, 58}
{27, 1, 57, 85}	{27, 1, 65, 93}	{27, 1, 66, 95}	{27, 1, 67, 96}	{27, 5, 93, 74}
{27, 3, 67, 98}	{27, 3, 87, 71}	{27, 3, 65, 96}	{27, 3, 61, 91}	{27, 1, 88, 62}
{27, 2, 91, 78}	{27, 3, 89, 76}	{27, 3, 88, 64}	{54, 82, 5, 33}	{54, 82, 7, 35}
{54, 83, 7, 50}	{54, 82, 32, 6}	{54, 82, 34, 8}	{54, 82, 2, 52}	{54, 82, 30, 4}
{54, 82, 41, 16}	{54, 82, 15, 44}	{54, 83, 39, 17}	{54, 86, 12, 44}	{54, 86, 39, 20}
{54, 85, 8, 42}	{54, 83, 34, 23}	{54, 83, 33, 21}	{54, 84, 6, 44}	{54, 86, 37, 21}
{81, 55, 5, 33}	{81, 55, 7, 35}	{81, 55, 29, 4}	{81, 55, 11, 39}	{81, 55, 12, 41}
{81, 55, 13, 42}	{81, 56, 12, 44}	{81, 59, 39, 20}	{81, 56, 10, 51}	{81, 57, 33, 17}
{81, 59, 37, 16}	{81, 57, 11, 42}	{81, 57, 7, 37}	{81, 55, 32, 6}	{81, 55, 34, 8}

{81, 56, 37, 24}	{81, 57, 35, 22}	{81, 57, 38, 15}	{81, 57, 34, 10}	{0, 55, 30, 85}
{0, 59, 39, 98}	{0, 57, 33, 98}	{0, 59, 37, 102}	{0, 58, 35, 97}	{0, 55, 32, 87}
{0, 82, 41, 70}	{0, 86, 39, 74}	{0, 83, 34, 77}	{0, 56, 32, 88}	{0, 83, 33, 75}
{0, 56, 35, 91}	{0, 86, 37, 75}	{0, 84, 34, 64}	{0, 57, 34, 101}	{0, 61, 40, 98}
{0, 62, 41, 101}	{27, 82, 5, 60}	{27, 55, 5, 87}	{27, 82, 7, 62}	{27, 55, 7, 89}
{27, 55, 3, 85}	{27, 55, 11, 93}	{27, 55, 12, 95}	{27, 56, 12, 98}	{27, 83, 6, 77}
{27, 83, 8, 64}	{27, 57, 8, 103}	{27, 57, 13, 98}	{27, 57, 11, 96}	{27, 84, 15, 73}
{0, 67, 34, 98}	{27, 82, 13, 69}	{27, 84, 6, 71}	{27, 85, 10, 74}	{27, 86, 11, 73}
{54, 1, 84, 31}	{54, 1, 93, 41}	{54, 3, 87, 44}	{54, 5, 91, 48}	{54, 4, 89, 43}
{54, 28, 86, 6}	{54, 1, 88, 35}	{54, 28, 88, 8}	{54, 28, 84, 4}	{54, 28, 95, 16}
{54, 29, 93, 17}	{54, 32, 93, 20}	{54, 2, 86, 34}	{54, 29, 87, 21}	{54, 2, 89, 37}
{54, 32, 91, 21}	{54, 30, 88, 10}	{54, 3, 88, 47}	{54, 6, 94, 46}	{54, 7, 94, 44}
{81, 1, 59, 33}	{81, 28, 61, 8}	{81, 1, 61, 35}	{81, 1, 65, 39}	{81, 1, 66, 41}
{81, 1, 67, 42}	{81, 2, 66, 44}	{81, 29, 62, 10}	{81, 2, 64, 51}	{81, 3, 67, 44}
{81, 3, 65, 42}	{81, 30, 69, 19}	{81, 30, 70, 23}	{81, 3, 61, 37}	{54, 13, 87, 46}
{81, 28, 57, 4}	{81, 28, 67, 15}	{81, 30, 60, 17}	{81, 31, 64, 20}	{81, 32, 65, 19}
{0, 82, 59, 33}	{0, 82, 61, 35}	{0, 55, 93, 41}	{0, 55, 96, 44}	{0, 59, 93, 44}
{0, 57, 87, 44}	{0, 59, 91, 48}	{0, 84, 61, 37}	{0, 55, 86, 33}	{0, 55, 88, 35}
{0, 82, 56, 52}	{0, 82, 69, 44}	{0, 85, 62, 42}	{0, 56, 86, 34}	{0, 56, 89, 37}
{0, 84, 60, 44}	{0, 60, 94, 46}	{0, 61, 94, 44}	{0, 62, 95, 47}	{27, 82, 61, 8}
{27, 55, 83, 4}	{27, 83, 59, 7}	{27, 83, 60, 23}	{27, 83, 62, 10}	{27, 57, 87, 17}
{27, 59, 91, 16}	{27, 84, 69, 19}	{27, 84, 70, 23}	{27, 55, 86, 6}	{0, 67, 87, 46}
{27, 57, 89, 22}	{27, 86, 65, 19}	{27, 57, 92, 15}	{27, 57, 88, 10}	{54, 28, 5, 87}
{54, 28, 7, 89}	{54, 1, 30, 85}	{54, 1, 42, 98}	{54, 5, 39, 98}	{54, 29, 7, 104}
{54, 29, 6, 102}	{54, 3, 33, 98}	{54, 5, 37, 102}	{54, 4, 35, 97}	{54, 30, 7, 91}
{54, 1, 32, 87}	{54, 1, 34, 89}	{54, 28, 2, 106}	{54, 28, 15, 98}	{54, 32, 12, 98}
{54, 31, 8, 96}	{54, 2, 32, 88}	{54, 2, 35, 91}	{54, 30, 6, 98}	{54, 3, 34, 101}
{81, 28, 7, 62}	{81, 1, 29, 58}	{81, 5, 39, 74}	{81, 29, 5, 61}	{81, 29, 6, 77}
{81, 29, 8, 64}	{81, 3, 33, 71}	{81, 30, 15, 73}	{81, 30, 16, 77}	{81, 1, 32, 60}
{54, 13, 33, 100}	{54, 13, 34, 98}	{81, 1, 34, 62}	{81, 28, 3, 58}	{81, 28, 13, 69}
{81, 2, 37, 78}	{81, 3, 35, 76}	{81, 30, 6, 71}	{81, 31, 10, 74}	{81, 3, 38, 69}
{81, 3, 34, 64}	{27, 84, 16, 77}	{0, 67, 33, 100}	{27, 2, 64, 105}	{27, 3, 62, 103}
{54, 85, 35, 16}	{54, 84, 34, 10}	{0, 82, 34, 62}	{0, 82, 30, 58}	{54, 8, 95, 47}
{81, 28, 59, 6}	{0, 83, 61, 50}	{0, 83, 60, 48}	{0, 67, 88, 44}	{27, 82, 57, 4}
{54, 8, 41, 101}	{81, 28, 5, 60}			

Then we shift each of the remaining base blocks in $\mathcal{B}$ by a suitable automorphism α^i for some integer i. The result is listed below, where the blocks in each of the six consecutive rows, namely the ith, $(i+1)$th, ..., and $(i+5)$th rows for $i \in \{6k+1 : k = 0, 1, \ldots, 40\}$, form a parallel class.

{0, 1, 5, 6}	{2, 30, 7, 35}	{3, 4, 10, 11}	{8, 36, 15, 43}	{12, 13, 14, 16}
{19, 20, 48, 17}	{21, 22, 51, 52}	{23, 24, 34, 9}	{25, 26, 37, 39}	{18, 46, 47, 49}
{27, 28, 32, 33}	{40, 41, 42, 38}	{54, 55, 59, 60}	{56, 84, 61, 89}	{57, 58, 64, 65}
{62, 90, 69, 97}	{66, 67, 68, 70}	{73, 74, 102, 71}	{75, 76, 105, 106}	{77, 78, 88, 63}
{80, 81, 91, 92}	{72, 107, 86, 93}	{83, 85, 95, 100}	{94, 96, 99, 101}	{44, 45, 103, 104}
{53, 82, 31, 87}	{50, 79, 29, 98}			
{0, 1, 11, 12}	{2, 3, 42, 17}	{4, 5, 18, 47}	{6, 7, 48, 50}	{8, 10, 20, 25}
{14, 19, 53, 31}	{9, 38, 16, 32}	{15, 44, 21, 36}	{23, 26, 29, 40}	{13, 41, 51, 52}
{24, 27, 35, 39}	{43, 45, 28, 33}	{54, 55, 65, 66}	{56, 57, 95, 97}	{58, 59, 98, 73}

{60, 61, 74, 103}	{62, 63, 104, 106}	{70, 75, 82, 87}	{64, 69, 76, 84}	{71, 100, 78, 94}
{72, 101, 77, 80}	{83, 88, 93, 99}	{86, 90, 67, 102}	{89, 92, 105, 85}	{22, 79, 37, 68}
{34, 91, 49, 107}	{81, 30, 96, 46}			
{0, 5, 12, 47}	{2, 6, 10, 17}	{9, 11, 14, 16}	{13, 42, 18, 21}	{1, 3, 7, 51}
{25, 27, 8, 22}	{4, 33, 23, 26}	{15, 20, 52, 36}	{24, 53, 29, 32}	{19, 48, 38, 41}
{30, 31, 44, 46}	{43, 45, 49, 39}	{54, 56, 66, 71}	{55, 59, 63, 70}	{60, 62, 65, 67}
{72, 74, 78, 95}	{58, 87, 64, 106}	{57, 86, 76, 79}	{80, 83, 61, 75}	{68, 73, 105, 89}
{69, 98, 88, 91}	{102, 103, 104, 100}	{92, 97, 77, 85}	{99, 101, 107, 82}	{93, 96, 28, 35}
{34, 90, 40, 84}	{37, 94, 50, 81}			
{0, 2, 8, 10}	{1, 31, 9, 23}	{3, 6, 43, 20}	{11, 15, 21, 4}	{7, 12, 17, 50}
{13, 18, 24, 5}	{22, 26, 30, 38}	{19, 47, 51, 25}	{14, 42, 16, 39}	{27, 28, 34, 35}
{46, 49, 32, 36}	{37, 40, 52, 29}	{54, 56, 62, 64}	{55, 84, 65, 79}	{57, 60, 97, 74}
{58, 61, 91, 102}	{66, 70, 76, 59}	{63, 68, 73, 106}	{69, 72, 80, 83}	{71, 101, 78, 81}
{77, 107, 85, 99}	{95, 98, 75, 88}	{86, 87, 93, 94}	{89, 90, 103, 105}	{44, 45, 100, 96}
{48, 104, 33, 92}	{67, 41, 53, 82}			
{0, 30, 11, 15}	{1, 4, 16, 47}	{2, 5, 13, 43}	{3, 6, 19, 53}	{7, 37, 14, 44}
{17, 20, 24, 10}	{8, 9, 40, 41}	{21, 22, 28, 29}	{18, 46, 52, 26}	{23, 51, 36, 38}
{25, 27, 35, 49}	{54, 59, 65, 73}	{56, 60, 91, 99}	{55, 85, 66, 70}	{61, 64, 76, 107}
{69, 72, 58, 92}	{62, 63, 94, 95}	{68, 96, 100, 74}	{79, 80, 86, 87}	{71, 77, 84, 90}
{93, 67, 104, 106}	{82, 57, 89, 105}	{31, 33, 97, 102}	{75, 81, 34, 42}	{83, 88, 12, 50}
{101, 48, 103, 45}	{39, 98, 78, 32}			
{0, 28, 30, 4}	{1, 29, 12, 14}	{3, 31, 15, 17}	{5, 33, 46, 21}	{7, 35, 22, 51}
{8, 37, 47, 25}	{6, 38, 18, 50}	{9, 41, 48, 2}	{11, 42, 19, 53}	{20, 49, 27, 16}
{24, 26, 32, 34}	{39, 40, 23, 52}	{54, 57, 61, 74}	{55, 83, 89, 63}	{56, 84, 85, 87}
{65, 93, 67, 90}	{58, 86, 88, 62}	{64, 92, 75, 77}	{66, 94, 78, 80}	{72,101,106,68}
{71, 100, 81, 95}	{59, 91, 97, 105}	{69, 99, 103, 79}	{96, 70, 98, 73}	{10, 13, 107, 60}
{43, 45, 102, 104}	{44, 76, 82, 36}			
{0, 2, 32, 34}	{3, 5, 36, 26}	{4, 33, 37, 25}	{8, 38, 43, 30}	{1, 31, 14, 18}
{9, 39, 15, 53}	{12, 44, 49, 6}	{24, 29, 7, 13}	{16, 48, 27, 35}	{11, 41, 45, 21}
{46, 20, 22, 50}	{54, 82, 94, 96}	{56, 84, 97, 72}	{55, 83, 70, 99}	{59, 88, 98, 76}
{57, 89, 69, 101}	{68, 100, 107, 61}	{71, 102, 79, 86}	{58, 60, 90, 92}	{74, 104, 80, 91}
{65, 73, 106, 85}	{103, 77, 64, 66}	{42, 17, 75, 63}	{67, 95, 51, 52}	{23, 78, 10, 93}
{47, 105, 28, 62}	{19, 81, 87, 40}			
{0, 31, 37, 47}	{1, 32, 36, 17}	{2, 5, 44, 21}	{4, 7, 42, 18}	{3, 6, 46, 26}
{9, 12, 43, 29}	{16, 20, 10, 30}	{8, 14, 48, 27}	{45, 19, 23, 51}	{41, 15, 25, 53}
{39, 13, 50, 52}	{34, 35, 49, 24}	{54, 83, 86, 89}	{55, 57, 88, 78}	{58, 87, 91, 79}
{59, 61, 94, 96}	{60, 90, 73, 77}	{62, 93, 99, 82}	{69, 101, 106, 63}	{67, 98, 102, 56}
{100, 74, 75, 71}	{92, 68, 76, 107}	{105, 81, 66, 70}	{64, 97, 104, 85}	{72, 103, 28, 38}
{11, 95, 22, 80}	{65, 40, 84, 33}			
{0, 6, 12, 19}	{1, 7, 21, 42}	{3, 10, 43, 47}	{5, 13, 46, 52}	{30, 31, 8, 9}
{34, 35, 14, 15}	{37, 11, 17, 45}	{48, 22, 50, 25}	{49, 23, 24, 20}	{28, 2, 16, 18}
{51, 27, 4, 44}	{33, 36, 40, 26}	{54, 86, 64, 70}	{55, 85, 93, 97}	{56, 59, 98, 75}
{57, 60, 95, 71}	{62, 65, 105, 58}	{69, 72, 103, 89}	{73, 77, 67, 87}	{68, 74, 61, 82}
{92, 66, 76, 104}	{96, 99, 80, 83}	{100, 101, 88, 63}	{91, 94, 102, 78}	{38, 41, 107, 84}
{79, 53, 81, 29}	{90, 39, 106, 32}			
{27, 1, 12, 41}	{28, 2, 14, 43}	{29, 3, 16, 18}	{30, 5, 15, 47}	{32, 10, 17, 22}
{33, 11, 45, 26}	{31, 8, 39, 46}	{44, 19, 24, 13}	{40, 42, 21, 23}	{53, 4, 9, 20}
{49, 25, 6, 37}	{54, 60, 66, 73}	{55, 62, 95, 99}	{85, 86, 63, 64}	{83, 57, 61, 89}

{87, 88, 67, 68}	{96, 70, 76, 104}	{98, 72, 74, 102}	{91, 65, 77, 106}	{105, 56, 90, 71}
{101, 78, 84, 94}	{97, 75, 107, 59}	{35, 38, 69, 82}	{52, 80, 36, 92}	{79, 0, 58, 48}
{100, 50, 81, 7}	{103, 51, 34, 93}			
{27, 29, 5, 7}	{28, 3, 6, 9}	{31, 33, 10, 0}	{37, 12, 16, 4}	{38, 13, 21, 35}
{43, 18, 8, 11}	{39, 15, 20, 34}	{47, 24, 30, 40}	{36, 14, 46, 25}	{45, 48, 2, 32}
{41, 42, 52, 26}	{81, 55, 66, 95}	{82, 56, 69, 71}	{83, 58, 68, 100}	{85, 63, 70, 75}
{84, 61, 92, 99}	{98, 73, 78, 67}	{94, 96, 72, 74}	{106, 54, 57, 60}	{59, 93, 101, 107}
{80, 90, 97, 103}	{104, 105, 88, 62}	{50, 51, 89, 64}	{102, 76, 23, 19}	{86, 91, 44, 22}
{1, 87, 65, 17}	{49, 77, 79, 53}			
{27, 3, 13, 44}	{28, 4, 34, 18}	{29, 7, 40, 48}	{31, 8, 12, 20}	{33, 36, 21, 25}
{47, 50, 9, 16}	{35, 11, 15, 45}	{6, 19, 39, 52}	{17, 51, 32, 38}	{41, 42, 53, 1}
{81, 83, 60, 77}	{82, 57, 61, 76}	{84, 86, 65, 67}	{88, 63, 71, 85}	{87, 62, 79, 55}
{94, 70, 75, 89}	{105, 54, 64, 95}	{90, 66, 96, 80}	{91, 69, 74, 58}	{92, 93, 97, 98}
{10, 14, 99, 107}	{2, 5, 72, 106}	{68, 73, 24, 30}	{102, 103, 26, 0}	{100, 101, 22, 23}
{49, 78, 59, 46}	{56, 37, 43, 104}			
{27, 1, 32, 6}	{0, 13, 34, 44}	{2, 35, 42, 50}	{4, 39, 45, 52}	{3, 40, 47, 53}
{31, 5, 38, 12}	{48, 49, 24, 25}	{28, 29, 14, 16}	{33, 37, 41, 21}	{51, 26, 43, 46}
{81, 59, 92, 100}	{83, 60, 64, 72}	{82, 85, 71, 78}	{86, 62, 66, 96}	{84, 58, 89, 63}
{55, 68, 88, 101}	{61, 74, 95, 105}	{99, 73, 106, 80}	{90, 91, 65, 94}	{104, 107, 56, 67}
{8, 9, 102, 77}	{54, 57, 7, 20}	{69, 103, 30, 36}	{15, 70, 76, 23}	{11, 93, 97, 17}
{18, 75, 87, 10}	{98, 19, 22, 79}			
{60, 9, 76, 29}	{96, 43, 84, 5}	{27, 86, 37, 97}	{32, 7, 91, 94}	{65, 12, 13, 69}
{1, 61, 67, 20}	{103, 107, 30, 10}	{3, 59, 8, 64}	{31, 63, 41, 74}	{106, 26, 90, 38}
{21, 77, 2, 58}	{70, 44, 50, 78}	{6, 88, 101, 22}	{25, 33, 93, 100}	{66, 72, 24, 4}
{95, 98, 81, 85}	{47, 105, 55, 36}	{52, 57, 62, 19}	{92, 15, 73, 0}	{39, 42, 28, 35}
{89, 40, 18, 83}	{102, 53, 87, 11}	{17, 46, 54, 68}	{48, 23, 80, 56}	{45, 104, 82, 34}
{75, 49, 51, 79}	{16, 71, 99, 14}			
{77, 106, 28, 31}	{19, 20, 85, 87}	{48, 76, 86, 34}	{94, 97, 53, 30}	{16, 44, 81, 82}
{13, 72, 24, 59}	{33, 61, 35, 64}	{70, 73, 27, 3}	{80, 2, 95, 18}	{50, 105, 91, 39}
{17, 23, 56, 63}	{9, 65, 69, 32}	{6, 43, 104, 83}	{107, 0, 67, 15}	{92, 41, 45, 58}
{55, 57, 7, 51}	{62, 36, 74, 22}	{98, 99, 46, 42}	{75, 79, 4, 14}	{21, 78, 88, 11}
{68, 96, 26, 1}	{84, 5, 103, 52}	{38, 40, 100, 102}	{8, 90, 37, 93}	{60, 10, 54, 47}
{66, 49, 29, 89}	{71, 101, 25, 12}			
{99, 100, 58, 60}	{98, 75, 25, 6}	{103, 106, 83, 69}	{81, 5, 66, 17}	{59, 64, 16, 24}
{87, 91, 70, 80}	{93, 14, 22, 90}	{72, 19, 56, 3}	{97, 102, 54, 62}	{76, 78, 28, 18}
{46, 51, 2, 40}	{30, 35, 42, 20}	{39, 67, 71, 45}	{38, 13, 44, 32}	{49, 77, 27, 55}
{86, 61, 11, 26}	{9, 92, 15, 84}	{89, 37, 101, 52}	{31, 36, 41, 47}	{65, 68, 0, 34}
{48, 105, 1, 95}	{21, 104, 94, 43}	{23, 79, 29, 73}	{12, 96, 74, 7}	{53, 82, 85, 33}
{50, 107, 88, 10}	{57, 4, 8, 63}			
{32, 61, 24, 54}	{5, 34, 91, 94}	{86, 6, 10, 92}	{88, 37, 47, 105}	{60, 9, 46, 77}
{100, 21, 80, 15}	{79, 28, 11, 69}	{102, 49, 87, 8}	{26, 81, 93, 41}	{82, 83, 43, 18}
{70, 107, 33, 39}	{30, 85, 35, 90}	{55, 7, 67, 20}	{59, 62, 48, 1}	{104, 0, 4, 66}
{14, 98, 22, 63}	{57, 65, 44, 50}	{13, 97, 101, 23}	{25, 3, 64, 99}	{42, 71, 103, 38}
{72, 73, 52, 53}	{76, 78, 27, 29}	{96, 17, 106, 12}	{2, 58, 68, 19}	{89, 36, 40, 95}
{84, 31, 45, 74}	{75, 51, 56, 16}			
{69, 21, 62, 29}	{76, 26, 16, 90}	{41, 42, 73, 74}	{27, 31, 10, 20}	{38, 39, 107, 55}
{65, 14, 80, 30}	{33, 34, 8, 37}	{32, 9, 67, 75}	{97, 45, 102, 50}	{72, 47, 52, 68}
{83, 87, 12, 22}	{106, 82, 40, 44}	{94, 43, 28, 86}	{11, 95, 103, 53}	{46, 24, 58, 63}
{1, 2, 60, 61}	{48, 78, 59, 36}	{35, 92, 51, 85}	{91, 66, 17, 6}	{96, 99, 84, 88}

{79, 0, 64, 15}	{101, 77, 81, 57}	{25, 3, 89, 100}	{105, 56, 7, 18}	{93, 98, 23, 4}
{71, 19, 104, 13}	{49, 54, 5, 70}			
{77, 2, 8, 69}	{26, 82, 92, 16}	{67, 18, 23, 83}	{21, 107, 4, 64}	{52, 55, 65, 42}
{89, 10, 95, 29}	{80, 57, 7, 14}	{78, 27, 62, 12}	{59, 60, 47, 49}	{5, 88, 15, 56}
{19, 102, 11, 68}	{38, 39, 99, 100}	{50, 53, 84, 70}	{96, 20, 25, 63}	{87, 61, 44, 46}
{98, 73, 51, 40}	{101, 48, 22, 105}	{3, 9, 97, 103}	{45, 104, 1, 93}	{66, 13, 106, 0}
{35, 90, 94, 41}	{76, 79, 6, 36}	{31, 34, 74, 54}	{17, 72, 58, 33}	{30, 85, 86, 28}
{75, 24, 37, 71}	{81, 32, 91, 43}			
{9, 37, 74, 76}	{18, 101, 25, 95}	{26, 2, 60, 73}	{19, 79, 66, 33}	{100, 50, 54, 35}
{63, 64, 11, 13}	{27, 55, 69, 17}	{40, 68, 80, 28}	{61, 89, 36, 38}	{47, 22, 93, 96}
{7, 92, 15, 103}	{94, 41, 78, 53}	{46, 51, 57, 65}	{52, 0, 86, 102}	{21, 49, 88, 90}
{56, 5, 44, 75}	{62, 12, 70, 23}	{30, 85, 99, 20}	{3, 6, 91, 104}	{24, 82, 34, 98}
{84, 4, 59, 1}	{29, 58, 48, 105}	{67, 42, 77, 10}	{16, 107, 87, 39}	{81, 31, 8, 97}
{83, 32, 45, 106}	{43, 71, 72, 14}			
{90, 92, 14, 16}	{69, 71, 0, 5}	{22, 104, 106, 26}	{20, 103, 30, 98}	{31, 6, 38, 27}
{82, 2, 4, 86}	{102, 76, 8, 10}	{66, 42, 101, 34}	{75, 50, 85, 45}	{99, 21, 107, 13}
{61, 37, 15, 56}	{3, 89, 96, 23}	{33, 35, 41, 43}	{52, 80, 64, 39}	{100, 47, 53, 81}
{1, 84, 65, 25}	{83, 88, 95, 73}	{62, 40, 72, 24}	{91, 93, 97, 87}	{48, 77, 55, 17}
{12, 67, 19, 74}	{18, 46, 57, 59}	{32, 7, 78, 54}	{58, 60, 9, 11}	{68, 70, 49, 51}
{105, 29, 63, 44}	{79, 28, 36, 94}			
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