Maximal Resolvable Packings and Minimal Resolvable Coverings of Triples by Quadruples

Xiande Zhang, Gennian Ge
Department of Mathematics, Zhejiang University, Hangzhou 310027, Zhejiang, P. R. China, E-mail: xdzhangzju@163.com; gnge@zju.edu.cn

Received February 9, 2009; revised May 6, 2009

Published online 24 June 2009 in Wiley InterScience (www.interscience.wiley.com).
DOI 10.1002/jcd.20234

Abstract: Determination of maximal resolvable packing number and minimal resolvable covering number is a fundamental problem in designs theory. In this article, we investigate the existence of maximal resolvable packings of triples by quadruples of order v (MRPQS(v)) and minimal resolvable coverings of triples by quadruples of order v (MRCQS(v)). We show that an MRPQS(v) (MRCQS(v)) with the number of blocks meeting the upper (lower) bound exists if and only if $v \equiv 0 \pmod 4$. As a byproduct, we also show that a uniformly resolvable Steiner system URS$(3, \{4, 6\}, \{r_4, r_6\}, v)$ with $r_6 \leq 1$ exists if and only if $v \equiv 0 \pmod 4$. All of these results are obtained by the approach of establishing a new existence result on RH(6^{2n}) for all $n \geq 2$. © 2009 Wiley Periodicals, Inc. J Combin Designs 18: 209–223, 2010

1991 MSC: *05B40*
Keywords: *MRCQS; MRPQS; RB-pairing; RG; RH; RHF*

1. INTRODUCTION

A *packing quadruple system* (*covering quadruple system*, *respectively*) of order v, denoted by PQS(v) (CQS(v)) is a pair $(X, \mathcal{B})$, where X is a set of cardinality n and $\mathcal{B}$ is a set of 4-subsets of X such that every 3-subset of X is contained in at most one

Contract grant sponsor: National Outstanding Youth Science Foundation of China; Contract grant number: 10825103; Contract grant sponsor: National Natural Science Foundation of China; Contract grant number: 10771193; Contract grant sponsor: Specialized Research Fund for the Doctoral Program of Higher Education; Contract grant sponsor: Program for New Century Excellent Talents in University; Contract grant sponsor: Zhejiang Provincial Natural Science Foundation of China.

(at least one) block of $\mathcal{B}$. Note that in this article the notation of CQS does not denote a candelabra quadruple system.

A PQS(v) (CQS(v)) $(X,\mathcal{B})$ is called *maximal* (*minimal*), denoted by MPQS(v) (MCQS(v)), if there does not exists any PQS(v) (CQS(v)) $(X,\mathcal{A})$ with $|\mathcal{A}|>|\mathcal{B}|$ ($|\mathcal{A}|<|\mathcal{B}|$). We denote by $p(v)$ ($c(v)$) the number of blocks in an MPQS(v) (MCQS(v)).

The Johnson bound [12] $j(v)$ for the packing numbers is given by

$$p(v)\le j(v)=\begin{cases}\left\lfloor \frac{v}{4}\left\lfloor \frac{v-1}{3}\left\lfloor \frac{v-2}{2}\right\rfloor\right\rfloor\right\rfloor, & v\not\equiv 0 \pmod 6,\\ \left\lfloor \frac{v}{4}\left\lfloor \frac{v-1}{3}\left\lfloor \frac{v-2}{2}\right\rfloor\right\rfloor-1\right\rfloor, & v\equiv 0 \pmod 6.\end{cases}$$

Here, $\lfloor x\rfloor$ denotes the largest integer not greater than x.

When $v\equiv 2,4 \pmod 6$, Hanani [3] showed that $p(v)=j(v)$ by constructing a PQS(v) with the property that each triple is contained in exactly one block. Such a design is called a *Steiner quadruple system* of order v and denoted by SQS(v). Deleting a point and all blocks containing it from an SQS($v+1$) yields that $p(v)=j(v)$ for $v\equiv 1,3 \pmod 6$. Brouwer [1] showed that $p(v)=j(v)$ for all $v\equiv 0 \pmod 6$. Recently, Ji [8] showed that the last packing number for $v\equiv 5 \pmod 6$ is equal to Johnson bound with 21 undecided values $v=6k+5, k\in\{m: m \text{ is odd, } 3\le m\le 35, m\ne 17,21\}\cup\{45,47,75,77,79,159\}$.

The Schönheim bound [14] $s(v)$ for the covering numbers is given by

$$c(v)\ge s(v)=\left\lceil \frac{v}{4}\left\lceil \frac{v-1}{3}\left\lceil \frac{v-2}{2}\right\rceil\right\rceil\right\rceil.$$

Here, $\lceil x\rceil$ denotes the smallest integer not less than x.

Mills [15] has shown that $c(v)=s(v)$ for all $v\not\equiv 7 \pmod{12}$. Kalbfleisch and Stanton [13] and Swift [19] have noted that $c(7)=s(7)+1$. Mills [16] has also proved that $c(499)=s(499)$. Hartman et al. [6] have shown that $c(v)=s(v)$ for all $v\ge 52423$. Recently, Ji [9] proved that $c(v)=s(v)$ for all $v\equiv 7 \pmod{12}$ with an exception $v=7$ and possible exceptions of $v=12k+7, k\in\{1,2,3,4,5,7,8,9,10,11,12,16,21,23,25,29\}$.

A PQS(v) (CQS(v)) $(X,\mathcal{B})$ is called *resolvable*, denoted by RPQS(v) (RCQS(v)), if $\mathcal{B}$ can be partitioned into *parallel classes*, each of which partitions the set X.

An RPQS(v) (RCQS(v)) $(X,\mathcal{B})$ is called *maximal* (*minimal*), denoted by MRPQS(v) (MRCQS(v)), if there does not exist any RPQS(v) (RCQS(v)) $(X,\mathcal{A})$ with $|\mathcal{A}|>|\mathcal{B}|$ ($|\mathcal{A}|<|\mathcal{B}|$). It is easy to see that the necessary condition for the existence of an MRPQS(v) (MRCQS(v)) is $v\equiv 0 \pmod 4$. In 1986, Hartman [5] proved that there exist RPQS(v)s (RCQS(v)s) which have the property that each triple is contained in exactly one block with 23 possible exceptions, where $v\equiv 4,8 \pmod{12}$. Such a design is called a *resolvable Steiner quadruple system* of order v and denoted by RSQS(v). Recently, Ji and Zhu [11] constructed RSQS(v)s for the last 23 undecided orders. The results are summarized as follows.

Theorem 1.1. *There exists an RSQS*(v) *if and only if* $v\equiv 4,8 \pmod{12}$.

Maximal resolvable packings and minimal resolvable coverings with strength $t=2$ are fundamental problems in combinatorial designs theory (see, for examples, [2, 7]). It is natural and interesting to consider the corresponding problems for strength $t=3$. For the existence of MRPQS(v) and MRCQS(v), we need only to consider the case

$v \equiv 0 \pmod{12}$ by Theorem 1.1, since an RSQS(v) is simply both an MRPQS(v) and an MRCQS(v). In this article, we will focus on the investigation of the existence of MRPQS(v) and MRCQS(v) with $v=12t$ for all $t \geq 1$. Denote by $p'(v)$ $(c'(v))$ the number of blocks in an MRPQS(v) (MRCQS(v)). Since $v=12t$, it is easy to check that $p'(v) \leq 3t(24t^2-6t-1)$ and $c'(v) \geq 3t(24t^2-6t+1)$. In the remainder of this article, when we talk about an MRPQS(v) (MRCQS(v)) we will mean the RPQS(v) (RCQS(v)) with the number of blocks meeting the previous upper (lower) bound for $p'(v)$ $(c'(v))$.

The article is organized as follows. In Section 2, we show that the existence of both an MRPQS$(12t)$ and an MRCQS$(12t)$ can be implied by that of an RH(6^{2t}) (For the definition of RH(g^n), see Section 2). We give a product construction and a tripling construction for RH(g^n) in Section 3, and use them to present a complete solution to the existence problem of an RH(6^{2t}) for any integer $t \geq 2$ in Section 4. We state our main result in Section 5.

2. CONSTRUCTING MRPQS AND MRCQS BY RH

In this section, we show that the existence of both MRPQS$(12t)$ and MRCQS$(12t)$ can be implied by that of RH(6^{2t}). We need the following concept.

A regular graph (V,E) of degree k is said to have a *one-factorization* if the edge set E can be partitioned into k parts $E=F_1|F_2|\cdots|F_k$ so that each F_i is a partition of the vertex set V into pairs. The parts F_i are called *one-factors*.

For $x \in Z_n$, we define $|x|$ by

$$|x|=\begin{cases} x & \text{if } 0 \leq x \leq n/2, \\ -x & \text{if } n/2 < x < n. \end{cases}$$

For $n \geq 2$ and $L \subseteq \{1,2,\ldots,\lfloor n/2 \rfloor\}$, define G$(n,L)$ to be the regular graph with vertex set Z_n and edge set E given by $\{x,y\} \in E$ if and only if $|x-y| \in L$.

The following lemma is proved by Stern and Lenz in [18].

Lemma 2.1. *Let $L \subseteq \{1,2,\ldots,n\}$. Then $G(2n,L)$ has a one-factorization if and only if $2n/gcd(j,2n)$ is even for some $j \in L$.*

First, we prove the existence of both an MRPQS(12) and an MRCQS(12).

Lemma 2.2. *There exist both an MRPQS(12) and an MRCQS(12).*

Proof. It is easy to check that $p'(12) \leq 51$ and $c'(12) \geq 57$.

Let $X=Z_6 \times Z_2$ with two subsets $A=Z_6 \times \{0\}$ and $B=Z_6 \times \{1\}$. It is easy to check that $F_1=\{\{0,1\},\{2,4\},\{3,5\}\}$, $F_2=\{\{4,5\},\{0,2\},\{1,3\}\}$, $F_3=\{\{0,3\},\{2,5\},\{1,4\}\}$, $F_4=\{\{2,3\},\{0,4\},\{1,5\}\}$ and $F_5=\{\{0,5\},\{1,2\},\{3,4\}\}$ form a one-factorization of the complete graph on Z_6. Let $f_{i,k}^j=\{(x,j),(y,j)\}$, where $\{x,y\}$ is the kth member of F_i, $1 \leq k \leq 3$, $1 \leq i \leq 5$ and $j \in Z_2$. Then $\{F_i^j=\{f_{i,k}^j : 1 \leq k \leq 3\} : 1 \leq i \leq 5\}$ is a one-factorization of the complete graph on $Z_6 \times \{j\}$ for each $j \in Z_2$. We will construct an MRCQS(12) and an MRPQS(12) on X as follows.

For MRPQS(12), $\{f_{i,k}^0 \cup f_{i,k+l}^1 : 1 \leq k \leq 3\}$ with $(i,l) \in (\{2,3,4,5\} \times Z_3) \cup (\{1\} \times (Z_3 \setminus \{0\}))$ are the first 14 parallel classes. Next, $\{f_{1,s}^0 \cup f_{1,1+s}^0, f_{1,s}^1 \cup f_{1,1+s}^1, f_{1,2+s}^0 \cup f_{1,2+s}^1\}$ for $s \in Z_3$ are the last 3 parallel classes. It is clear that all the blocks in these 17 parallel classes form an MRPQS(12).

For MRCQS(12), $\{f^0_{i,k}\cup f^1_{i,k+l}:1\le k\le 3\}$ with $(i,l)\in(\{3,5\}\times Z_3)\cup(\{1,2,4\}\times(Z_3\setminus\{0\}))$ are the first 12 parallel classes. Next, $\{f^0_{i,1}\cup f^1_{i,1}:i=1,2,4\}$ and $\{f^0_{i,1}\cup f^0_{i,1+l}, f^1_{i,1}\cup f^1_{i,1+l}, f^0_{i,1+l'}\cup f^1_{i,1+l'}\}$ for $i\in\{1,2,4\}$ and $(l,l')\in\{(1,2),(2,1)\}$ are the last 7 parallel classes. It is clear that all the blocks in these 19 parallel classes form an MRCQS(12). □

Let v be a non-negative integer, t be a positive integer and K be a set of positive integers. A *G-design* of order v with block sizes from K, denoted by $G(t,K,v)$, is a quadruple $(X,\mathcal{G},\mathcal{A})$ that satisfies the following properties:

(1) X is a set of v elements (called *points*);
(3) $\mathcal{G}=\{G_1,G_2,\ldots\}$ is a set of nonempty subsets (called *groups* or *branches*) of X, which partition X;
(4) $\mathcal{A}$ is a family of subsets (called *blocks*) of X, each of cardinality from K;
(5) every t-subset T of X with $|T\cap G_i|<t$, for all i, is contained in a unique block, and no t-subset of G_i, for any i, is contained in any block.

The *type* of the G(t,K,v) is defined as the list $(|G||G\in\mathcal{G})$. In this article, we denote a $G(3,\{4\},v)$ of type g^n by $G(g^n)$ for short.

A $G(g^n)$ is said to be *resolvable*, denoted by $RG(g^n)$, if the block set can be partitioned into parallel classes.

Lemma 2.3. *If there exists an $RG(12^t)$, then there exist both an $MRPQS(12t)$ and an $MRCQS(12t)$.*

Proof. Suppose that the RG(12^t) is based on X with $|X|=12t$. It is easy to check that an RG(12^t) contains $18t(4t+3)(t-1)$ blocks. Adjoining these $18t(4t+3)(t-1)$ blocks with t disjoint MRPQS(12)s based on the t different groups of the RG(12^t), we obtain $3t(24t^2-6t-1)$ blocks which cover the triples of X at most once. Hence, we have an MRPQS$(12t)$. Similarly, we can obtain an MRCQS$(12t)$. □

Let v be a non-negative integer, t be a positive integer and K be a set of positive integers. A *group divisible t-design* of order v with block sizes from K, denoted by GDD(t,K,v), is a triple $(X,\mathcal{G},\mathcal{B})$ such that

(1) X is a set of v elements;
(2) $\mathcal{G}=\{G_1,G_2,\ldots\}$ is a set of nonempty subsets of X, which partition X;
(3) $\mathcal{B}$ is a family of subsets of X, each of cardinality from K such that each block intersects any given group in at most one point;
(4) every t-subset T of X from t distinct groups is contained in a unique block.

The *type* of the GDD(t,K,v) is defined as the list $(|G||G\in\mathcal{G})$. Mills used H(n,g,k,t) to denote the GDD(t,k,ng) of type g^n, and he proved the existence for H$(n,g,4,3)$, which we simply write H(g^n) for short. In [17], he showed that for $n>3$, $n\neq 5$, an H(g^n) exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for $n=5$, an H(g^5) exists if g is divisible by 4 or 6. Recently, L. Ji [10] improved the result by showing that an H(g^5) exists whenever g is even, $g\neq 2$, and $g\not\equiv 10,26 \pmod{48}$.

An H(g^n) is said to be *resolvable*, denoted by RH(g^n), if the block set can be partitioned into parallel classes.

Lemma 2.4. *If there exists an* $RH(g^{2t})$ *with* g *even, then there exist both an* $RG((2g)^t)$ *and an* $RG(g^{2t})$.

Proof. Let $(X, \mathcal{G}, \mathcal{B})$ be the given RH(g^{2t}), where $\mathcal{G}=\{G_0, \ldots, G_{2t-1}\}$. Let $\mathcal{F}=\{F_1, \ldots, F_{2t-1}\}$ be a one-factorization of the complete graph on Z_{2t}.

For $0 \le i \le 2t-1$, let $\mathcal{F}^i=\{F_1^i, \ldots, F_{g-1}^i\}$ be a one-factorization of the complete graph on G_i. Let $F_j^i=\{f_j^i(0), \ldots, f_j^i(g/2-1)\}$. For all n, j, k, $1 \le n \le 2t-2$, $1 \le j \le g-1$, $0 \le k \le g/2-1$, it is easy to see that

$$\{f_j^x(l) \cup f_j^y(l+k) : 0 \le l \le g/2-1, \{x, y\} \in F_n\}$$

is a partition of X. Denote the set of all these blocks by $\mathcal{A}$.

Then it is easy to check that $(X, \mathcal{G}', \mathcal{A} \cup \mathcal{B})$ is an RG($(2g)^t$) with group set $\mathcal{G}'=\{G_x \cup G_y : \{x, y\} \in F_{2t-1}\}$.

Furthermore, if we adjoin the parallel classes formed by $\{f_j^x(l) \cup f_j^y(l+k) : 0 \le l \le g/2-1, \{x, y\} \in F_n\}$ into the RG($(2g)^t$), where $n=2t-1$, $1 \le j \le g-1$, $0 \le k \le g/2-1$, then we obtain an RG(g^{2t}) with group set $\mathcal{G}$. □

Lemma 2.5. *If there exists an RH*(6^{2t}), *then there exist both an MRPQS*($12t$) *and an MRCQS*($12t$).

Proof. Combining Lemmas 2.3 and 2.4, the conclusion then follows. □

3. PRODUCT AND TRIPLING CONSTRUCTIONS FOR RH(g^n)

In this section, we give the product construction and tripling construction for RH(g^n). In [11], we have the following results for RHs.

Lemma 3.1 (Ji and Zhu [11]). *There exists an RH*(g^4) *for any positive integer* g.

Lemma 3.2 (Ji and Zhu [11]). *Suppose that there exists an RH*(g^u). *Then there is an RH*($(mg)^u$) *for any positive integer* m.

For non-negative integers q, g, k, and t, an H(q, g, k, t) *frame* (as in [6]) is an ordered four-tuple $(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the following properties:

1. X is a set of qg points;
2. $\mathcal{G}=\{G_1, G_2, \ldots, G_q\}$ is an equipartition of X into q groups;
3. $\mathcal{F}$ is a family $\{F_i\}$ of subsets of $\mathcal{G}$ called *holes*, which is closed under intersections. Hence each hole $F_i \in \mathcal{F}$ is of the form $F_i=\{G_{i_1}, G_{i_2}, \ldots, G_{i_s}\}$, and if F_i and F_j are holes then $F_i \cap F_j$ is also a hole. The number of groups in a hole is its size; and
4. $\mathcal{B}$ is a set of k-element transverses (called *blocks*) of $\mathcal{G}$ with the property that every t-element transverse of $\mathcal{G}$, which is not a t-element transverse of any hole $F_i \in \mathcal{F}$, is contained in precisely one block, and no block contains a t-element transverse of any hole, where a *transverse* is a subset of X that meets each G_i in at most one point.

In this article, an H(q, g, k, t) frame is shortly denoted by HF(q, g, k, t). If an HF$(q, g, 4, 3)$ has n holes of size $m+s$, which intersect on a common hole of size s, then such a design will be denoted by HF($m^n:s$) with group size g. If an HF($q,g,4,3$) has only one hole of size s, then we call it an *incomplete H-design* of type $(g^q:g^s)$, denoted by IH($g^q:g^s$).

An HF($m^n:s$) $(X,\mathcal{G},\mathcal{B},\mathcal{F})$ with group size g, $\mathcal{F}=\{F_i:0\le i\le n\}$ and F_0 the common hole of size s is said to be *resolvable*, denoted by RHF($m^n:s$), if the block set can be partitioned into $(nmg^2(m+2s-3)+n(n-1)(mg)^2)/6$ parts with the following properties:

(1) For each hole F_i, $1\le i\le n$, there are exactly $mg^2(m+2s-3)/6$ parts, each being a partition of $X\setminus F_i^*$ (called a *partial parallel class*), where $F_i^*=\bigcup_{G\in F_i}G$;
(2) There are $n(n-1)(mg)^2/6$ parts, each being a parallel class on X.

An IH($g^{m+s}:g^s$) $(X,\mathcal{G},\mathcal{B},F)$ with the only hole F of size s is said to be *resolvable*, denoted by IRH($g^{m+s}:g^s$), if the block set can be partitioned into $(m+s-1)(m+s-2)g^2/6$ parts, $(s-1)(s-2)g^2/6$ of which being partitions of $X\setminus(\bigcup_{G\in F}G)$, and $m(m+2s-3)g^2/6$ of which being parallel classes on X.

The following lemma is simple, but useful.

Lemma 3.3. *Suppose that there exist both an IRH($g^{m+s}:g^s$) and an RH(g^s). Then there exists an RH(g^{m+s}).*

Lemma 3.4. *Suppose that there exists an RHF($m^n:s$) with group size g. If there exists an IRH($g^{m+s}:g^s$), then there exist both an IRH($g^{mn+s}:g^{m+s}$) and an IRH($g^{mn+s}:g^s$).*

Proof. Let $(X,\mathcal{G},\mathcal{B},\mathcal{F})$ be the given RHF($m^n:s$), $\mathcal{F}=\{F_k:0\le k\le n\}$ and F_0 be the common hole of size s. Let $F_k^*=\bigcup_{G\in F_k}G$, $0\le k\le n$. Then the block set $\mathcal{B}$ has a partition $\{P(k,j):1\le k\le n, 1\le j\le mg^2(m+2s-3)/6\}\cup\{P'(i):1\le i\le n(n-1)(mg)^2/6\}$ such that (1) for each pair (k,j), $1\le k\le n$ and $1\le j\le mg^2(m+2s-3)/6$, $P(k,j)$ is a partition of $X\setminus F_k^*$; (2) for each i, $1\le i\le n(n-1)(mg)^2/6$, $P'(i)$ is a parallel class on X.

For $1\le k\le n-1$, construct an IRH($g^{m+s}:g^s$) on F_k^* with group set F_k and hole F_0. Denote the set of blocks by $\mathcal{A}_k$. Then there are $(m+s-1)(m+s-2)g^2/6$ parts $Q(k,j)$, such that for $1\le j\le m(m+2s-3)g^2/6$, $Q(k,j)$ is a partition of F_k^*; for $m(m+2s-3)g^2/6<j\le(m+s-1)(m+s-2)g^2/6$, each $Q(k,j)$ is a partition of $F_k^*\setminus F_0^*$. Then each $P(k,j)\cup Q(k,j)$ with $1\le k\le n-1, 1\le j\le mg^2(m+2s-3)/6$ forms a parallel class on X. Each $\bigcup_{1\le k\le n-1}Q(k,j)$ with $m(m+2s-3)g^2/6<j\le(m+s-1)(m+s-2)g^2/6$ forms a partition of $X\setminus F_n^*$. So the resulting design is an IRH($g^{mn+s}:g^{m+s}$).

Furthermore, if we construct an IRH($g^{m+s}:g^s$) on F_n^* with group set F_n and hole F_0, then we obtain an IRH($g^{mn+s}:g^s$). □

Lemma 3.5 (Product construction). *Suppose that there exist both an RH(g^{2u}) and an RH(g^{2t}). Then there exists an RH(g^{2ut}).*

Proof. Let$(X,\mathcal{G},\mathcal{B})$ be the given RH(g^{2u}), where $\mathcal{G}=\{G_0,\ldots,G_{2u-1}\}$. Let $\mathcal{F}=\{F_1,\ldots,F_{2u-1}\}$ be a one-factorization of the complete graph on Z_{2u}. Applying Lemma 3.2, we construct an RH($(tg)^{2u}$) on $X'=X\times Z_t$ with the group set $\mathcal{G}'=\{G_i'=G_i\times Z_t:0\le i\le 2u-1\}$ and a resolution of the block set $\mathcal{A}$, $P_1\mid P_2\mid\cdots\mid P_s$, where $s=(2u-1)(2u-2)(tg)^2/6$.

Since an RH(g^{2t}) exists, gt is even. For $0\le i\le 2u-1$, let $\mathcal{F}^i=\{F_1^i,\dots,F_{g(t-1)}^i\}$ be a one-factorization of the complete multiple-graph on $G_i\times Z_t$ with t parts $\{G_i\times\{l\}: l\in Z_t\}$. For any $\{a,b\}\in F_m^x$ and $\{c,d\}\in F_m^y$, construct a block $\{a,b,c,d\}$, where $1\le m\le g(t-1)$ and $\{x,y\}\in F_n$ with $2\le n\le 2u-1$. Denote the set of all these blocks by $\mathcal{A}'$. Here, for any fixed r, $0\le r\le tg/2-1$, the blocks $\{a,b,c,d\}$ with $\{a,b\}$ being the kth edge of F_m^x, $\{c,d\}$ being the $(k+r)$th edge of F_m^y with $1\le k\le tg/2$ form a partition of the set $G'_x\cup G'_y$. Hence, for each $\{x,y\}\in F_n$, we can obtain $g(t-1)\cdot tg/2$ parts each of which partitions $G'_x\cup G'_y$. In total, we can get $(2u-2)\cdot g(t-1)\cdot tg/2$ parallel classes.

For $1\le k\le u$, let the kth edge of F_1 be $\{x,y\}$. Construct an RH(g^{2t}) on $G'_x\cup G'_y$ with group set $\{G_x\times\{l\}, G_y\times\{l\}: l\in Z_t\}$. Denote its block set by $\mathcal{C}_k$, which can be partitioned into parallel classes $Q(k,1),\dots,Q(k,(2t-1)(2t-2)g^2/6)$. Let $\mathcal{C}=\bigcup_{1\le k\le u}\mathcal{C}_k$. Here, for each fixed j, $1\le j\le (2t-1)(2t-2)g^2/6$, $\bigcup_{1\le k\le u} Q(k,j)$ forms a parallel class.

Let $\mathcal{G}''=\{G_i\times\{l\}: 0\le i\le 2u-1, l\in Z_t\}$; it is easy to check that $(X',\mathcal{G}'',\mathcal{A}\cup\mathcal{A}'\cup\mathcal{C})$ is an H(g^{2ut}). By the construction, the number of parallel classes is $(2u-1)(2u-2)(tg)^2/6+(2u-2)\cdot g(t-1)\cdot tg/2+(2t-1)(2t-2)g^2/6=(2ut-1)(2ut-2)g^2/6$. Hence, the resulting H-design is resolvable. □

The tripling construction has played an important role in the construction of resolvable Steiner quadruple systems [6]. In the remainder of this section, we generalize this construction to the case for RHs (for the case of H-design, see [4]).

For non-negative integers n and s, a *B-pairing* $B(n,s)$ consists of four subsets D, R_0, R_1, R_2 of $Z_{6(n+s)}$ and three subsets PR_0, PR_1, PR_2 of $Z_{6(n+s)}\times Z_{6(n+s)}$ with the following properties for each $i\in Z_3$:

(1) *Cardinality and symmetry conditions*
  (a) $|D|=6s$, $|R_i|=2n$,
  (b) $D=-D$.
(2) *Partitioning conditions*
  (a) PR_i is a partition of R_i into pairs, thus $|PR_i|=n$,
  (b) $Z_{6(n+s)}=D\cup R_0\cup R_1\cup R_2$.
(3) *Pairing conditions*
  Let $L_i=\{|x-y|:\{x,y\}\in PR_i\}$ and $N=\{n+s, 2(n+s), 3(n+s)\}$,
  (a) $N\cap L_i=\emptyset$,
  (b) $|L_i|=n$,
  (c) the complement G_i of the graph $G(6(n+s), L_i\cup N)$ has a one-factorization.

Let $S_0,S_1,S_2,\overline{R}_0,\overline{R}_1,\overline{R}_2$ be subsets of $Z_{6(2n+s)}$ and PS_0,PS_1,PS_2 be subsets of $Z_{6(2n+s)}\times Z_{6(2n+s)}$. A B-pairing $B(2n,s)$ with D,R_0,R_1,R_2,PR_0,PR_1 and PR_2 is said to be *resolvable*, denoted by $RB(2n,s)$, if the following properties are satisfied for each $i\in Z_3$:

(1) *Cardinality and symmetry conditions*
  (c) $|S_i|=4n$, $|\overline{R}_i|=2n$.
(2) *Partitioning conditions*
  (c) PS_i is a partition of S_i into pairs, thus $|PS_i|=2n$,
  (d) $Z_{6(2n+s)}=D\cup R_i\cup S_i\cup\overline{R}_{i+1}\cup -\overline{R}_{i-1}$.
(3) *Pairing conditions*

Let $O_i = \{|x-y| : \{x,y\} \in PS_i\}$,
(d) $N \cap O_i = \emptyset$,
(e) $|O_i| = 2n$, $L_i \cap O_i = \emptyset$, and all members of O_i are odd,
(f) the complement G_i' of the graph $G(6(2n+s), L_i \cup O_i \cup N)$ has a one-factorization.

Now, we give two examples of resolvable B-pairings.

Example 1. Let $n=1, s=1$; we construct an RB$(2,1)$ on Z_{18} as follows:

$D = \{0,3,6,9,12,15\}$.
$PR_0 = \{\{1,17\},\{4,14\}\}$, $PR_1 = \{\{2,16\},\{5,13\}\}$, $PR_2 = \{\{7,11\},\{8,10\}\}$.
$PS_0 = \{\{2,13\},\{7,8\}\}$, $PS_1 = \{\{1,14\},\{10,11\}\}$, $PS_2 = \{\{4,17\},\{5,16\}\}$.
$\overline{R}_0 = \{1,14\}$, $\overline{R}_1 = \{5,16\}$, $\overline{R}_2 = \{7,8\}$.

Example 2. Let $n=2, s=2$; we construct an RB$(4,2)$ on Z_{36} as follows:

$D = \{0,6,12,18,24,30,3,9,15,21,27,33\}$.
$PR_0 = \{\{1,35\},\{2,34\},\{4,32\},\{5,31\}\}$, $PR_1 = \{\{7,29\},\{8,28\},\{16,20\},\{17,19\}\}$,
$PR_2 = \{\{10,26\},\{11,25\},\{13,23\},\{14,22\}\}$.
$PS_0 = \{\{8,17\},\{7,20\},\{10,25\},\{11,22\}\}$, $PS_1 = \{\{2,5\},\{1,32\},\{26,11\},\{25,14\}\}$,
$PS_2 = \{\{34,31\},\{35,4\},\{28,19\},\{29,16\}\}$.
$\overline{R}_0 = \{1,2,32,5\}$, $\overline{R}_1 = \{29,28,16,19\}$, $\overline{R}_2 = \{10,23,13,22\}$.

Theorem 3.6. *If there exists an $RB(2n,s)$, then there exists an RHF$((2n+s)^3 : s)$ of group size* 6.

Proof. Let $X = (Z_{6(2n+s)} \times Z_3) \cup \{\infty_0, \infty_1, \ldots, \infty_{6s-1}\}$. Define the groups $G(i,j) = \{(k(2n+s)+i, j) : k \in Z_6\}$, $i \in Z_{2n+s}$, $j \in Z_3$, $G(\infty, j) = \{\infty_{ks+j} : k \in Z_6\}$, $j \in Z_s$ and the holes $\mathcal{F} = \{F_0, F_1, F_2, F_3\}$ with $F_0 = \{G(\infty, j) : j \in Z_s\}$ and $F_{1+j} = F_0 \cup \{G(i,j) : i \in Z_{2n+s}\}$, $j = 0,1,2$.

For $i \in Z_3$, let $D, R_i, \overline{R}_i, S_i, PR_i, PS_i$ be a resolvable B-pairing $RB(2n,s)$. Let $F_i^{2k-1} | F_i^{2k}$ be a one-factorization of the graph $G(12n+6s, \{m\})$, where m is the kth member of O_i for $1 \le k \le 2n$. Let $F_i^{4n+1} | F_i^{4n+2} | \cdots | F_i^{8n+6s-6}$ be a one-factorization of the complement of the graph $G(6(2n+s), L_i \cup O_i \cup N)$. Then it is natural that $F_i^1 | F_i^2 | \cdots | F_i^{8n+6s-6}$ is a one-factorization of the complement of the graph $G(6(2n+s), L_i \cup N)$.

We construct an HF$(X, \mathcal{G}, \mathcal{B}, \mathcal{F})$ with the block set $\mathcal{B}$ consisting of the following three parts:

(1) $\{\infty_j, (a,0), (b,1), (c,2)\}$, where $a+b+c \equiv d \pmod{6(2n+s)}$, d is the jth member of D and $0 \le j < 6s$.
(2) $\{(a+q, i), (a+t, i), (b, i+1), (c, i+2)\}$, where $a+b+c \equiv 0 \pmod{6(2n+s)}$, $\{q,t\} \in PR_i$ and $i \in Z_3$.
(3) $\{(a,i), (b,i), (c,i+1), (d,i+1)\}$, where $\{a,b\} \in F_i^j$ and $\{c,d\} \in F_{i+1}^j$, $i \in Z_3$ and $j = 1,2,\ldots,8n+6s-6$.

Now, we partition them into (partial) parallel classes.

First, we give the partial parallel classes. Define $F_i^j = \{F_i^j(k) : 0 \le k \le 6n+3s-1\}$. For each $i \in Z_3$, the $6(2n+s)(2n+3s-3)$ partial parallel classes missing the hole F_i

are defined as follows:

$$P_i(j,k)=\{\{(a,i+1),(b,i+1),(c,i+2),(d,i+2)\}:\{a,b\}=F^j_{i+1}(m),\{c,d\}=F^j_{i+2}(m+k),$$
$$0\le m\le 6n+3s-1\}\text{ where }4n+1\le j\le 8n+6s-6\text{ and }0\le k\le 6n+3s-1.$$

It is clear that each $P_i(j,k)$ forms a partition of $X\setminus(\bigcup_{G\in F_i}G)$.

Next, we give the $(6(2n+s))^2$ complete parallel classes. For each (a,b,c) such that $a+b+c\equiv 0 \pmod{6(2n+s)}$, let $P(a,b,c)$ be comprised of $6s$ blocks from Part (δ), $6n$ blocks from Part (ρ) and $3n$ blocks from Part (ϕ) as follows:

Part (δ): $\{\{\infty_j,(a+d,0),(b-d,1),(c+d,2)\}: d$ is the jth member of $D, 0\le j<6s\}$.

Part (ρ):

$$\{(a+q,0),(a+t,0),(b-u,1),(c+u,2)\}\quad\text{for } i=0,$$

$$\{(a+u,0),(b+q,1),(b+t,1),(c-u,2)\}\quad\text{for } i=1,$$

$$\{(a-u,0),(b+u,1),(c+q,2),(c+t,2)\}\quad\text{for } i=2,$$

where $\{q,t\}$ is the jth pair in PR_i and u is the jth member of $\overline{R}_i$, $1\le j\le 2n$.

Part (ϕ): To select the blocks of Part (ϕ), let $PA_i|PB_i$ be a partition of PS_i into parts of size n. Then the blocks of Part (ϕ) are all those of the forms:

$$\{(a+s,0),(a+s',0),(b+t,1),(b+t',1)\},$$

$$\{(b+u,1),(b+u',1),(c+w,2),(c+w',2)\},$$

$$\{(c+y,2),(c+y',2),(a+z,0),(a+z',0)\},$$

where the pairs $\{s,s'\},\{t,t'\},\{u,u'\},\{w,w'\},\{y,y'\}$ and $\{z,z'\}$ are the jth $(1\le j\le n)$ pairs selected from the sets PA_i, PB_i according to the parities of a,b and c as follows:

(i) If a,b and c are all even, then $\{s,s'\}\in PA_0,\{t,t'\}\in PA_1,\{u,u'\}\in PB_1,\{w,w'\}\in PB_2,\{y,y'\}\in PA_2,\{z,z'\}\in PB_0$.

(ii) If just a is even, then $\{s,s'\}\in PB_0,\{t,t'\}\in PB_1,\{u,u'\}\in PA_1,\{w,w'\}\in PB_2,\{y,y'\}\in PA_2,\{z,z'\}\in PA_0$.

(iii) If just b is even, then $\{s,s'\}\in PB_0,\{t,t'\}\in PB_1,\{u,u'\}\in PA_1,\{w,w'\}\in PA_2,\{y,y'\}\in PB_2,\{z,z'\}\in PA_0$.

(iv) If just c is even, then $\{s,s'\}\in PA_0,\{t,t'\}\in PA_1,\{u,u'\}\in PB_1,\{w,w'\}\in PA_2,\{y,y'\}\in PB_2,\{z,z'\}\in PB_0$.

It is clear that each $P(a,b,c)$ forms a partition of X. Note that for all (a,b,c) such that $a+b+c\equiv 0 \pmod{6(2n+s)}$, the blocks of Part (ϕ) cover all the blocks of the form $\{(x,i+1),(y,i+1),(z,i+2),(w,i+2)\}$, where $\{x,y\}\in F^j_{i+1}$ and $\{z,w\}\in F^{j'}_{i+2}$ such that $1\le j,j'\le 4n$ and $\{j,j'\}$ is an appropriate pair, $i\in Z_3$. □

Lemma 3.7. *Suppose that $n\ge 0$ and $s\ge 1$. An $RB(2n,s)$ exists for each n whenever s is odd, and for each $n\ne 1$ whenever s is even.*

TABLE I.

0	$2n+s$	$2(2n+s)$	$3(2n+s)$	$4(2n+s)$	$5(2n+s)$
1 $\vdots$ n	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n+1$ $\vdots$ $n+s-1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n+s$ $\vdots$ $2n+s-1$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

D	D	D	D	D	D
R_0 S_1 $\overline{R}_0$	R_0 $-\overline{R}_0$ S_2	S_0 $\overline{R}_2$ R_2	$-\overline{R}_2$ S_1 R_2	S_0 R_1 $-\overline{R}_1$	$\overline{R}_1$ R_1 S_2
D	D	D	D	D	D
S_0 R_1 $-\overline{R}_1$	$\overline{R}_1$ R_1 S_2	S_0 $\overline{R}_2$ R_2	$-\overline{R}_2$ S_1 R_2	R_0 S_1 $\overline{R}_0$	R_0 $-\overline{R}_0$ S_2

Proof. When $n=0$, we take $D=Z_{6(2n+s)}$ and $R_i=S_i=\overline{R_i}=\emptyset$. When $n>0$, the desired $RB(2n,s)$ is constructed directly as follows:

For s odd, let

$$
\begin{aligned}
&D=\{(2n+s)j:0\le j\le 5\}\cup\{(2n+s)i+j:0\le i\le 5, n+1\le j\le n+s-1\},\\
&PR_0=\{\{j,-j\}:1\le j\le n \text{ or } (2n+s)+1\le j\le (2n+s)+n\},\\
&PR_1=\{\{j,-j\}:n+s\le j\le 2n+s-1 \text{ or } (2n+s)+n+s\le j\le 2(2n+s)-1\},\\
&PR_2=\{\{j,-j\}:2(2n+s)+1\le j\le 2(2n+s)+n \text{ or } 2(2n+s)+n+s\le j\le 3(2n+s)-1\},\\
&PS_0=\{\{j,-j-(2n+s)\}:n+s\le j\le 2n+s-1 \text{ or } 2(2n+s)+1\le j\le 2(2n+s)+n\},\\
&PS_1=\{\{j,-j-(2n+s)\}:1\le j\le n\}\cup\{\{j+(2n+s),-j\}:2(2n+s)+1\le j\le 2(2n+s)+n\},\\
&PS_2=\{\{j+(2n+s),-j\}:1\le j\le n \text{ or } n+s\le j\le 2n+s-1\},\\
&\overline{R}_0=\{j:1\le j\le n \text{ or } 4(2n+s)+n+s\le j\le 5(2n+s)-1\},\\
&\overline{R}_1=\{j:(2n+s)+n+s\le j\le 2(2n+s)-1 \text{ or } 5(2n+s)+1\le j\le 5(2n+s)+n\},\\
&\overline{R}_2=\{j:2(2n+s)+1\le j\le 2(2n+s)+n \text{ or } 2(2n+s)+n+s\le j\le 3(2n+s)-1\}.
\end{aligned}
$$

We display the above sets in Table I, which is separated into two parts: the upper part is an arrangement of the elements of $Z_{6(2n+s)}$ and the lower one is an arrangement of these sets correspondingly.

For s even and $n>1$, let

$$
\begin{aligned}
&D=\{(2n+s)j:0\le j\le 5\}\cup\{(2n+s)i+j:0\le i\le 5, n+1\le j\le n+s-1\},\\
&PR_0=\{\{j,-j\}:1\le j\le n \text{ or } n+s\le j\le 2n+s-1\},\\
&PR_1=\{\{j,-j\}:(2n+s)+1\le j\le (2n+s)+n \text{ or } 2(2n+s)+n+s\le j\le 3(2n+s)-1\},\\
&PR_2=\{\{j,-j\}:(2n+s)+n+s\le j\le 2(2n+s)-1 \text{ or } 2(2n+s)+1\le j\le 2(2n+s)+n\}.
\end{aligned}
$$

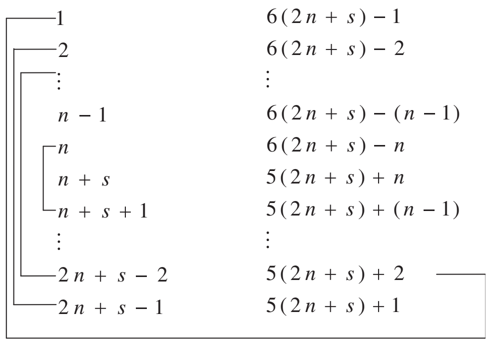

FIGURE 1. PR_0 and PA_0.

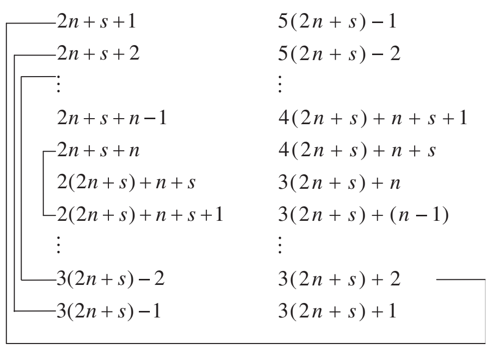

FIGURE 2. PR_1 and PA_1.

Let

$$PA_0=\{\{j,2n+s+1-j\}:2\le j\le n\}\cup\{\{1,5(2n+s)+2\}\},$$
$$PA_1=\{\{2n+s+j,3(2n+s)+1-j\}:2\le j\le n\}\cup\{\{2n+s+1,3(2n+s)+2\}\},$$
$$PA_2=\{\{2(2n+s)-j,4(2n+s)-1+j\}:2\le j\le n\}\cup\{\{2(2n+s)-1,4(2n+s)-2\}\}.$$

We illustrate the above sets in Figures 1–3.

Let

$$PS_0=PA_1\cup PA_2,\ PS_1=PA_0\cup(-PA_2),\ PS_2=(-PA_0)\cup(-PA_1),$$
$$\overline{R}_0=-(R_0\setminus A_0),\ \overline{R}_1=R_1\setminus A_1,\ \overline{R}_2=-(R_2\setminus A_2),\text{ where } A_i=\bigcup_{\{x,y\}\in PA_i}\{x,y\}.$$

It is readily checked that the above sets D, PR_0, PR_1, PR_2, PS_0, PS_1, PS_2, $\overline{R}_0$, $\overline{R}_1$, $\overline{R}_2$ form an $RB(2n,s)$ on $Z_{6(2n+s)}$. □

Combining Theorem 3.6 and Lemma 3.7, we obtain the following lemma.

Lemma 3.8. *Suppose that $n\ge 0$ and $s\ge 1$. There exists an RHF$((2n+s)^3:s)$ with group size 6 for each n whenever s is odd and for each $n\ne 1$ whenever s is even.*

$$\begin{array}{ll}
2n+s+n+s & 4(2n+s)+n \\
2n+s+n+s+1 & 4(2n+s)+n-1 \\
\vdots & \vdots \\
2(2n+s)-2 & 4(2n+s)+2 \\
2(2n+s)-1 & 4(2n+s)+1 \\
2(2n+s)+1 & 4(2n+s)-1 \\
2(2n+s)+2 & 4(2n+s)-2 \\
\vdots & \vdots \\
2(2n+s)+n-1 & 3(2n+s)+n+s+1 \\
2(2n+s)+n & 3(2n+s)+n+s
\end{array}$$

FIGURE 3. PR_2 and PA_2.

Combining Lemmas 3.4 and 3.8, we obtain the following theorem.

Theorem 3.9 (Tripling construction)**.** *Suppose $m \geq s \geq 1$ and $m-s \neq 2$ when s is even. If there exists an IRH$(6^{m+s}:6^s)$, then there exist both an IRH$(6^{3m+s}:6^s)$ and an IRH$(6^{3m+s}:6^{m+s})$.*

4. THE EXISTENCE OF RH(6^{2t}) FOR ALL INTEGERS $t \geq 2$

In this section, we shall give the existence of RH(6^{2t}) for all integers $t \geq 2$.

Lemma 4.1 (Ji and Zhu [11])**.** *There exists an RH(2^{2n}) for each $n \in \{5,7,13\}$, hence there exists an* RH(6^{2n}) *for each $n \in \{5,7,13\}$.*

Lemma 4.2. *There exists an RH(6^6).*

Proof. Let the point set be $G = Z_{36}$, and the group set be $\{\{j, j+6, \ldots, j+30\} : j = 0, 1, \ldots, 5\}$. We construct the base blocks as follows:

Part 1: $\{7,9,14,16\}$, $\{5,10,27,32\}$, $\{1,3,12,14\}$,
$\{2,5,16,19\}$, $\{1,11,20,30\}$, $\{1,4,11,14\}$.
Part 2: $\{0,2,4,9\}$, $\{14,21,22,30\}$, $\{6,11,34,3\}$,
$\{18,31,5,10\}$, $\{12,16,26,1\}$, $\{19,20,23,24\}$,
$\{25,32,33,35\}$, $\{7,27,28,29\}$, $\{8,13,15,17\}$.
Part 3: $\{11,16,27,32\}$, $\{1,14,17,34\}$, $\{2,13,21,30\}$,
$\{5,9,24,28\}$, $\{19,23,4,8\}$, $\{35,7,12,22\}$,
$\{18,20,3,10\}$, $\{26,31,6,15\}$, $\{0,25,29,33\}$.
Part 4: $\{18,32,33,34\}$, $\{1,5,16,27\}$, $\{19,29,9,10\}$,
$\{14,22,23,0\}$, $\{20,21,24,25\}$, $\{12,17,3,7\}$,
$\{2,11,15,31\}$, $\{8,30,35,4\}$, $\{6,13,26,28\}$.
Part 5: $\{9,22,23,31\}$, $\{17,25,26,33\}$, $\{29,0,16,20\}$,
$\{4,7,12,15\}$, $\{14,21,35,1\}$, $\{8,10,11,18\}$,
$\{5,13,24,32\}$, $\{2,27,30,34\}$, $\{19,28,3,6\}$.

Part 6: $\{0,2,5,15\}$, $\{9,30,4,7\}$, $\{29,31,12,14\}$,
$\{23,27,16,20\}$, $\{3,10,25,32\}$, $\{24,26,35,1\}$,
$\{11,13,21,28\}$, $\{6,19,22,33\}$, $\{17,18,34,8\}$.
Part 7: $\{1,10,14,33\}$, $\{4,11,19,32\}$, $\{31,9,12,26\}$,
$\{13,15,16,18\}$, $\{0,5,20,21\}$, $\{24,25,34,35\}$,
$\{2,17,27,28\}$, $\{22,23,3,6\}$, $\{7,8,29,30\}$.

Here, the base blocks are developed by $+2$ modulo 36. The elements of each block in Part 1 cover the residues modulo 4; hence, each block in Part 1 gives a parallel class when developed by $+4$ modulo 36. The elements of blocks in each of the other parts are different. Hence, each of these parts forms a parallel class. □

Lemma 4.3. *If there exists an RH(6^{2p}) for each prime p, then there exists an RH(6^{2n}) for each integer $n \geq 2$.*

Proof. For each integer $n \geq 2$, if n is a prime, then there exists an RH(6^{2n}) by assumption. Otherwise, let $n = p_1 \times n_1$, where p_1 is a prime. There exists an RH(6^{2p_1}) by assumption. If n_1 is also a prime, then there exists an RH(6^{2n_1}). Applying Lemma 3.5, we obtain an RH(6^{2n}). If n_1 is not a prime, let $n_1 = p_2 \times n_2$, we repeat the previous procedure. Since n is a finite number, this procedure will stop after finite steps. This completes the proof. □

Lemma 4.4. *There exists an RH(6^{2p}) for each prime p.*

Proof. There exist both an RH(6^4) and an RH(6^6) by Lemmas 3.1 and 4.2. We need only to consider the case of prime $p \geq 5$. Hence, $2p = 12k+10$ or $12k+14, k \geq 0$.

For $2p = 12k+10, 12k+14, k \geq 0$, the proof proceeds by induction. When $k = 0$, we have an RH(6^{2p}) for $p \in \{5,7\}$ by Lemma 4.1. For $k \geq 1$, suppose that there exists an RH(6^{2p}) for each prime p, $2p < 12k+10$. Then by the proof of Lemma 4.3, there exist both an RH$(6^{4(k+1)})$ and an RH(6^{4k+6}), as well as an IRH$(6^{4(k+1)}:6^1)$ and an IRH$(6^{4k+6}:6^2)$. For $2p = 12k+10$, apply Theorem 3.9 with $s = 1$ and $m = 4k+3$ to obtain an IRH$(6^{12k+10}:6^1)$. For $2p = 12k+14$, apply Theorem 3.9 with $s = 2$ and $m = 4k+4$ to obtain an IRH$(6^{12k+14}:6^2)$. This completes the proof. □

Theorem 4.5. *There exists an* RH(6^{2t}) *for any integer $t \geq 2$.*

Proof. Combining Lemmas 4.3 and 4.4, the conclusion then follows. □

5. CONCLUDING REMARKS

Now, we are in a position to state our main result of this article.

Theorem 5.1. *An* MRPQS(v) (MRCQS(v)) *with the number of blocks meeting the upper* (*lower*) *bound exists if and only if* $v \equiv 0 \pmod 4$.

Proof. Combining Theorem 1.1, Lemma 2.2, Lemma 2.5, and Theorem 4.5, the conclusion then follows. □

A *t-wise balanced design* (tBD) of type t-(v, K, λ) is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of subsets of X (*blocks*) with the property that the size of every block is in the set K and every t-subset of X is contained in exactly λ blocks. A t-(v, K, λ) design is also denoted by $S_\lambda(t, K, v)$. If $\lambda = 1$, the notation $S(t, K, v)$ is often used and the design is a *Steiner system*.

An $S(t, K, v)$ $(X, \mathcal{B})$ is said to be *resolvable*, denoted by $RS(t, K, v)$, if the block set $\mathcal{B}$ can be partitioned into parallel classes. A parallel class is *uniform* if all blocks in the parallel class have the same size. A *uniformly resolvable Steiner system*, $\mathrm{URS}(t, K, R, v)$, is an $RS(t, K, v)$ such that all of the parallel classes are uniform, where R is a multiset with $|R| = |K|$ and for each $k \in K$ there corresponds a positive $r_k \in R$ such that there are exactly r_k parallel classes of size k.

It is easy to check that the necessary condition for the existence of a $\mathrm{URS}(3, \{4, 6\}, \{r_4, r_6\}, v)$ with $r_6 \leq 1$ is $v \equiv 0 \pmod 4$. For each $v \equiv 4, 8 \pmod{12}$, there exists an $\mathrm{RSQS}(v)$ which is a $\mathrm{URS}(3, \{4, 6\}, \{r_4, r_6\}, v)$ with $r_6 = 0$ indeed. For each $v \equiv 0 \pmod{12}$, there exists an $\mathrm{RG}(6^{v/6})$ by Lemma 2.4 and Theorem 4.5. If each group is regarded as a block, then all the groups form a parallel class, each block of which is of size 6. Hence, we obtain a $\mathrm{URS}(3, \{4, 6\}, \{r_4, r_6\}, v)$ with $r_6 = 1$. Thus, we obtain the following theorem.

Theorem 5.2. *There exists a* $\mathrm{URS}(3, \{4, 6\}, \{r_4, r_6\}, v)$ *with* $r_6 \leq 1$ *for each* $v \equiv 0 \pmod 4$.

In this article, the product and tripling constructions for $\mathrm{RH}(g^n)$ play an important role in the construction of $\mathrm{RH}(6^{2t})$, from which the maximal resolvable packings and minimal resolvable coverings are obtained. It is believed that these two constructions may also be useful in the investigation of the general existence problem of $\mathrm{RH}(g^n)$ for all admissible integers g and $n \geq 4$. According to the necessary conditions for the existence of an $\mathrm{RH}(g^n)$ and by Lemma 3.2, the general existence problem of $\mathrm{RH}(g^n)$ depends on the solution of the following six cases:

(1) $g = 1$ and $n \equiv 4, 8 \pmod{12}$,
(2) $g = 2$ and $n \equiv 2, 4 \pmod 6$,
(3) $g = 3$ and $n \equiv 0 \pmod 4$,
(4) $g = 4$ and $n \equiv 1, 2 \pmod 3$,
(5) $g = 6$ and $n \equiv 0 \pmod 2$,
(6) $g = 12$ and $n \in N$.

For Case (1), an $\mathrm{RH}(1^n)$ is actually an $\mathrm{RSQS}(n)$, which exists from [11]. For Case (5), the existence of $\mathrm{RH}(6^n)$ is proved in this article. Hence, the existence of $\mathrm{RH}(g^n)$ for Cases (2), (3), (4) and (6) will be an interesting topic for further investigation.

ACKNOWLEDGMENTS

The authors thank Professor Zhu Lie of Suzhou University for his important suggestions on this problem. Research was supported by the National Outstanding Youth Science Foundation of China under Grant No. 10825103, National Natural Science Foundation of China under Grant No. 10771193, Specialized Research Fund for the Doctoral Program of Higher Education, Program for New Century Excellent Talents in University, and Zhejiang Provincial Natural Science Foundation of China.

REFERENCES

[1] A. E. Brouwer, On the packing of quadruples without common triples, Ars Combin 5 (1978), 3–6.

[2] G. Ge, C. W. H. Lam, A. C. H. Ling, and H. Shen, Resolvable maximum packings with quadruples, Des Codes Cryptogr 35 (2005), 287–302.

[3] H. Hanani, On quadruple systems, Canad J Math 12 (1960), 145–157.

[4] A. Hartman, Tripling quadruple systems, Ars Combin 10 (1980), 255–309.

[5] A. Hartman, The existence of resolvable Steiner quadruple systems, J Combin Theory Ser A 44 (1987), 182–206.

[6] A. Hartman, W. H. Mills, and R. C. Mullin, Covering triples by quadruple: An asymptotic solution, J Combin Theory Ser A 41 (1986), 117–138.

[7] E. R. Lamken, W. H. Mills, and R. S. Rees, Resolvable minimum coverings with quadruples, J Combin Des 6 (1998), 431–450.

[8] L. Ji, Asymptotic determination of the last packing number of quadruples, Des Codes Cryptogr 38 (2006), 83–95.

[9] L. Ji, An improvement on covering triples by quadruples, J Combin Des 16 (2008), 231–243.

[10] L. Ji, An improvement on H design, J Combin Des 17 (2009), 25–35.

[11] L. Ji and L. Zhu, Resolvable Steiner quadruple systems for the last 23 orders, SIAM J Discrete Math 19 (2005), 420–430.

[12] S. M. Johnson, A new upper bound for error correcting codes, IRE Trans IT-8 (1962), 203–207.

[13] J. G. Kalbfleisch and R. G. Stanton, Maximal and minimal coverings of $(k-1)$-tuples by k-tuples, Pacific J Math 26 (1968), 131–140.

[14] H. Lenz, On the number of Steiner quadruple systems, Mitt Math Sem Giessen 169 (1985), 55–71.

[15] W. H. Mills, On the covering of triples by quadruples, Congr Numer 10 (1974), 563–581.

[16] W. H. Mills, A covering of triples by quadruples, Congr Numer 33 (1981), 253–260.

[17] W. H. Mills, On the existence of H designs, Congr Numer 79 (1990), 129–141.

[18] G. Stern and H. Lenz, Steiner triple systems with given subspaces, another proof of the Doyen–Wilson Theorem, Boll Un Mat Ital (5) 17-A (1980), 109–114.

[19] J. D. Swift, A generalized Steiner problem, Rend Mat 2 (1969), 563–569.