Discrete Applied Mathematics 161 (2013) 2772–2777

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Note

On the existence of retransmission permutation arrays

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ARTICLE INFO

Article history:
Received 21 July 2012
Received in revised form 3 February 2013
Accepted 22 April 2013
Available online 18 May 2013

Keywords:
Retransmission permutation array
Latin rectangle
Latin square

ABSTRACT

We investigate retransmission permutation arrays (RPAs) that are motivated by applications in overlapping channel transmissions. An RPA is an $n \times n$ array in which each row is a permutation of $\{1, \ldots, n\}$, and for $1 \leqslant i \leqslant n$, all n symbols occur in each $i \times \lceil \frac{n}{i} \rceil$ rectangle in specified corners of the array. The array has types 1, 2, 3 and 4 if the stated property holds in the top left, top right, bottom left and bottom right corners, respectively. It is called latin if it is a latin square. We show that for all positive integers n, there exists a type-1, 2, 3, 4 RPA(n) and a type-1, 2 latin RPA(n).

1. Introduction

Collisions in overlapping OFDM channels are becoming an increasingly major problem in the deployment of high-speed wireless networks, especially when the number of orthogonal channels is limited (e.g., [1,2,4,6,9]). In an OFDM network, each channel is allocated a set of consecutive subcarriers, and a message is transmitted by assigning one bit to each subcarrier. Two channels overlap when they contain common subcarriers. When two messages are transmitted in overlapping channels, only the bits assigned to the subcarriers contained by both channels collide; the bits in other subcarriers are clean and can be collected. However, the non-colliding subcarriers do not contain complete message information. The consequence is that messages will need to be retransmitted in order to get all bits successfully.

Retransmission permutation arrays were introduced by Li et al. [8] in an attempt to resolve such overlapping channel retransmission problems. The main idea is to use a different assignment of bits to subcarriers in each retransmission in order that the number of retransmissions is optimized for overlapping channels with any number of colliding subcarriers. Since each assignment can be viewed as a permutation of the set of bits, the schedule of assignments can be arranged as a permutation array. This motivated Dinitz et al. [5] to study the following new type of combinatorial structures.

A *type* 1 *retransmission permutation array of order n*, denoted *type*-1 RPA(n), is an $n \times n$ array A, in which each cell contains a symbol from the set $[n] = \{1, \ldots, n\}$, such that the following properties are satisfied:

(i) every row of A is a permutation of the n symbols, and
(ii) for $1 \leqslant i \leqslant n$, the $i \times \lceil \frac{n}{i} \rceil$ rectangle in the upper left hand corner of A contains all n symbols.

There are variations of the above definition. If property (ii) is modified so it instead holds for rectangles in the upper right, lower left, or lower right hand corner of A, then we say that A is, respectively, a *type*-2, *type*-3 or *type*-4 RPA(n).

An array that is simultaneously a type-1 RPA(n) and a type-2 RPA(n) is referred to as a type-1, 2 RPA(n). This notation will be generalized in the obvious way to arrays that satisfy different combinations of the variations of property (ii). The different types of RPAs are closely related, as shown in [5]. The same paper showed that, for all positive integers n, there exists a type-1, 2 RPA(n), a type-1, 3 RPA(n) and a type-1, 4 RPA(n). The following is an example of a type-1, 2, 3, 4 RPA(11).

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http://dx.doi.org/10.1016/j.dam.2013.04.017

Example 1.1. A type-1, 2, 3, 4 RPA(11), where the cells in the $i \times \lceil \frac{11}{i} \rceil$ rectangle, $1 \leqslant i \leqslant 11$, in the four corners are shaded.

6	2	3	4	5	1	7	8	9	10	11
7	11	8	9	10	6	2	3	4	1	5
10	5	1	11	3	4	8	9	6	7	2
4	9	11	1	7	5	6	2	10	3	8
3	8	5	7	6	10	1	11	2	4	9
1	7	10	2	9	8	3	4	5	11	6
9	4	6	10	11	2	5	1	7	8	3
8	3	2	6	4	9	11	7	1	5	10
2	1	7	5	8	3	10	6	11	9	4
5	6	4	3	1	11	9	10	8	2	7
11	10	9	8	2	7	4	5	3	6	1

An infinite family of type-1, 2, 3, 4 RPAs exists by [5]:

Theorem 1.2. *For all positive even integers n, there exists a type*-1, 2, 3, 4 RPA(n).

A retransmission permutation array A is called *latin* if every column of A is a permutation of the n symbols. A latin retransmission permutation array (LRPA) of order n is in fact a *latin square*, where each symbol occurs exactly once in each row and column. Note that the type-1, 2, 3, 4 RPA(11) in Example 1.1 is actually latin. For small orders $n \in \{1, 2, \ldots, 9, 10, 12, 14, 16, 36\}$, a type-1, 2, 3, 4 LRPA(n) was constructed in [5]. Finding general constructions for LRPAs seems to be quite difficult. However, it was conjectured in [5] that a type-1, 2, 3, 4 LRPA(n) exists for all positive integers n.

The main contribution of this paper is new constructions for RPAs and LRPAs. In particular, in Section 2 we strengthen Theorem 1.2 by showing that a type-1, 2, 3, 4 RPA(n) exists for all positive integers n. Further, in Section 3 we prove that a type-1, 2 LRPA(n) exists for all positive integers n, which is the first known infinite family of latin RPAs.

2. Type-1, 2, 3, 4 RPAs of odd orders

Theorem 2.7 of [5] gave an effective way to construct a type-1, 2, 3, 4 RPA of even order from a type-1, 2 RPA. In this section, we will use a similar idea to construct type-1, 2, 3, 4 RPAs of odd order by imposing some additional structure on the type-1, 2 RPAs. Note that Dinitz et al. [5, Figure 3] gave an algorithm to construct type-1, 2 RPAs of odd orders. We adapt their algorithm to construct the desired type-1, 2 RPAs with extra conditions. To make the paper more self-contained, we now define some terminology and notation used in the algorithm.

For the time being, we assume that n is odd. An $r \times \lceil \frac{n}{r} \rceil$ rectangle is called *basic* if it does not contain an $r' \times \lceil \frac{n}{r'} \rceil$ rectangle where $r' < r$ and $\lceil \frac{n}{r} \rceil = \lceil \frac{n}{r'} \rceil$. When verifying property (ii) in the definition of RPAs, it suffices to consider only basic rectangles. A basic rectangle R of size $r \times \lceil \frac{n}{r} \rceil$ is called *canonical* if there is a partition $n = a_1 + \cdots + a_r$ where $\lceil \frac{n}{r} \rceil = a_1 \geqslant \cdots \geqslant a_r > 0$, such that every symbol in $[n]$ occurs exactly once in the union, over $1 \leqslant i \leqslant r$, of the first a_i cells of row i.

Example 2.1. A type-1, 2, 3, 4 LRPA(13). The basic rectangles in the upper left hand corner have dimensions 1×13, 2×7, 3×5, 4×4, 5×3, 7×2 and 13×1. The shaded symbols show that these basic rectangles are canonical.

7	2	3	4	5	6	1	8	9	10	11	12	13
8	13	12	11	10	9	7	5	4	3	2	1	6
9	6	1	2	3	4	10	11	12	13	7	8	5
10	5	6	1	2	3	11	7	13	12	8	9	4
11	4	5	6	1	2	12	13	8	7	9	10	3
12	3	4	5	7	13	9	1	6	8	10	11	2
1	8	9	12	6	10	2	3	11	5	4	13	7
2	11	10	13	8	7	5	4	1	9	6	3	12
3	10	7	9	13	8	6	12	2	1	5	4	11
4	9	13	8	11	12	3	2	5	6	1	7	10
5	1	8	7	12	11	4	10	3	2	13	6	9
6	7	2	3	4	1	13	9	10	11	12	5	8
13	12	11	10	9	5	8	6	7	4	3	2	1

Denote the number of basic rectangles by b. For $1 \leqslant k \leqslant b$, let the k-th basic rectangle be R_k and suppose it has dimensions $r_k \times c_k$. For $2 \leqslant k \leqslant b - 2$, a canonical basic rectangle R_k is called *sum-free* if the following two conditions are satisfied.

(a) For any two symbols x, y in the same row of R_k, it holds that $x + y \neq n + 1$.
(b) For any two symbols x, y in the last $c_k - c_{k+1}$ columns of R_k, it holds that $x + y \neq n + 1$.

Following a similar strategy to [5], we work on $R_1, R_2, \ldots, R_b$ in turn. At the point at which we start working on R_k for $3 \leqslant k \leqslant b-1$, it should be true that R_{k-1} is a sum-free canonical basic rectangle (although R_{k-1} may subsequently be changed). We then copy the entries in $R_{k-1} \setminus R_k$ into empty cells in R_k, in a way that ensures R_k is canonical and satisfies Condition (a). Our ability to do this is guaranteed by [5, Lemma 4.6]. Next, we perform *simple exchange* (SE) *operations* to ensure that R_k is sum-free. An SE operation swaps the contents of two non-empty cells within the same row. It is clear that SE operations within R_k cannot affect whether $R_1, \ldots, R_k$ satisfy Condition (a). Details of how we achieve Condition (b) will be given below. Once the top left hand corner has been settled, in each row except the first we will "reflect" the filled entries to produce basic rectangles in the top right corner. By construction, $R_2, \ldots, R_b$ will all satisfy Condition (a), which guarantees that the reflection does not repeat any symbol within a row, so it is then trivial to complete the array to a type-1, 2 RPA.

As just mentioned, the crucial step is to show that SE operations can modify a given canonical basic rectangle R_k to make it sum-free. Partition R_k into two parts $\mathcal{D}_k$ and $\mathcal{E}_k$, where $\mathcal{D}_k$ consists of the cells in the first c_{k+1} columns of R_k and $\mathcal{E}_k$ consists of the cells in the remaining columns of R_k. Define a graph G_k over the n non-empty cells of R_k, where two cells are adjacent if the symbols in the two cells sum to $n+1$. Note that G_k consists of $(n-1)/2$ disjoint edges, and no two cells in a row form an edge if R_k satisfies Condition (a). We say an edge of G_k is of *type A* if both endpoints are in $\mathcal{E}_k$, of *type B* if exactly one endpoint is in $\mathcal{E}_k$ and of *type C* if no endpoint is in $\mathcal{E}_k$. In order to satisfy Condition (b), we will remove type A edges from G_k by SE operations. The description thus far mirrors the algorithm used in [5]. The main difference in our approach is that some additional conditions will be imposed on the RPA, which we now construct.

Lemma 2.2. *Let $n \geqslant 15$ be an odd integer and $h = (n+1)/2$. There exists a type-1, 2 RPA(n), $A = (a_{i,j})$, with the following properties:*

- *$a_{1,1} = h$, $a_{h,1} = 1$ and $a_{h,n} = h$;*
- *$a_{i,n} \neq 1$ for $1 \leqslant i \leqslant h$;*
- *A contains a canonical basic rectangle of size $h \times 2$ in the upper left and right corners.*

Proof. Since $n \geqslant 15$, we have $c_2 - c_3 \geqslant 2$ and $r_{b-1} - r_{b-2} \geqslant 2$. For $1 \leqslant k \leqslant b$, we will write the symbols from $[n]$ into R_k as described before the lemma.

When $k = 1$, fill in R_1 of size $1 \times n$, from left to right with symbols $1, 2, \ldots, n$. Then interchange the symbols 1 and h.

When $k = 2$, fill in the first $h-1$ cells of row 2 from left to right with the symbols $h+1, n, h+2, h+3, \ldots, n-1$. It is easy to check that R_2 is a sum-free canonical basic rectangle.

When $k = 3$, copy the symbols in the cells in $\mathcal{E}_2$ into the left most cells of row 3. Since symbol 1 appears in $\mathcal{E}_2$ and $|\mathcal{E}_2| \geqslant 3$, we can set $a_{3,3} = 1$. It is obvious that R_3 is canonical and satisfies Condition (a).

For $3 \leqslant k \leqslant b-2$, suppose we have a canonical basic rectangle R_k that satisfies Condition (a). Let a_k denote the number of type A edges in G_k. If $a_k = 0$ then R_k is sum-free, so copy the symbols from $\mathcal{E}_k$ into the empty cells of R_{k+1} such that R_{k+1} is canonical and satisfies Condition (a). This is guaranteed to be possible by [5, Lemma 4.6] and the remark following it. If $a_k > 0$, we will show that a_k can be reduced by at least one by applying SE operations to cells other than $(1,1)$, $(2,2)$ and $(3,3)$. Therefore a sequence of SE operations will reduce a_k to 0 with $a_{1,1} = h$, $a_{2,2} = n$ and $a_{3,3} = 1$ fixed.

Now we deal with the case when $a_k > 0$. For convenience, let F be the set of cells $\{(2,2), (3,3)\}$. Let x_1y_1 be a type A edge with cells x_1, y_1 in rows i_1, i_2 respectively. Let I denote the set of cells in rows i_1 and i_2. If there is a type C edge x_2y_2 where cell $x_2 \in I$ and $\{x_2, y_2\} \cap F = \emptyset$, then stop. Otherwise, all edges with one endpoint in $(I \cap \mathcal{D}_k) \setminus F$ are of type B. However,

$$|I \cap \mathcal{D}_k \setminus F| - |I \cap \mathcal{E}_k \setminus \{x_1, y_1\}| \geqslant |I \cap \mathcal{D}_k| - |I \cap \mathcal{E}_k| > 0.$$

The last inequality holds by [5, Lemma 4.2]. Therefore, one of these type B edges has an endpoint in $\mathcal{E}_k \setminus I$. Denote the edge by x_2y_2, where $x_2 \in (I \cap \mathcal{D}_k) \setminus F$ and $y_2 \in \mathcal{E}_k \setminus I$ is in a row, say i_3. Here $y_2 \notin F$ since $(2,2)(3,3)$ is an edge of G_k. Add cells in row i_3 to I. If there is a type C edge x_3y_3 where $x_3 \in I$ and $\{x_3, y_3\} \cap F = \emptyset$, then stop. If not, since

$$|I \cap \mathcal{D}_k \setminus (F \cup \{x_2\})| - |I \cap \mathcal{E}_k \setminus \{x_1, y_1, y_2\}| \geqslant |I \cap \mathcal{D}_k| - |I \cap \mathcal{E}_k| > 0,$$

there is a type B edge x_3y_3, where $\{x_3, y_3\} \cap F = \emptyset$, $x_3 \in I \cap \mathcal{D}_k$ and $y_3 \in \mathcal{E}_k \setminus I$ is in a row, say i_4. Then add cells in row i_4 to I and continue this process until we find a type C edge x_jy_j where $x_j \in I \cap \mathcal{D}_k$ and $\{x_j, y_j\} \cap F = \emptyset$, which must happen since $|I|$ is bounded by n^2 and thus cannot grow indefinitely. Now, imitating [5, Lemma 4.5], we find a sequence $e_1, \ldots, e_\ell$ of edges from among $\{x_1y_1, \ldots, x_jy_j\}$. We start with $e_1 = x_jy_j$, which is of type C. Then for $1 < i < \ell$ the type B edge e_i is chosen so that its vertex in $\mathcal{E}_k$ is in the same row as the vertex of e_{i-1} in $\mathcal{D}_k \cap I$. Eventually we reach a type B edge $e_{\ell-1}$ with a vertex in row i_1 or row i_2. This determines ℓ, and we finish with $e_\ell = x_1y_1$, which is of type A. Crucially, this construction allows us to perform SE operations to swap the content in one end of e_i with that in an end of e_{i+1} for $1 \leqslant i < \ell$. These operations remove the type A edge x_1y_1 (all the affected edges are now of type B), thereby reducing a_k by one. Whenever an SE operation is performed on two cells in row 1, say $(1,i)$ and $(1,j)$ with $1 < i < j < h$, we also perform an SE operation on cells $(1, n+1-i)$ and $(1, n+1-j)$. Thus after a sequence of SE operations, we can change a canonical basic rectangle R_k to a sum-free rectangle with $a_{2,2} = n$ and $a_{3,3} = 1$ fixed. Note that $a_{1,1} = h$ is fixed since $(1,1)$ is an isolated vertex in G_k, for $3 \leqslant k \leqslant b-2$.

When $k = b-1$, we can let $a_{h,1} = 1$ since symbol 1 is in a cell in $\mathcal{E}_{b-2}$. When $k = b$, copy the symbols from $\mathcal{E}_{b-1}$ into the empty cells in R_b.

At last, define the reflection map $\pi : [n] \to [n]$ as follows:

$$\pi(i) = \begin{cases} n+1-i & \text{if } i \notin \{1, h\}, \\ h & \text{if } i = 1, \\ n & \text{if } i = h. \end{cases} \quad (1)$$

For each non-empty cell (i, j) in rows $2, \ldots, n$ of A define $a_{i,n+1-j} = \pi(a_{i,j})$. Complete each row of A to a permutation of the n symbols to get the desired type-1, 2 RPA(n). Note that $a_{2,2} = n$, so that $a_{i,1} \neq n$ and hence $a_{i,n} \neq 1$ for $1 \leqslant i \leqslant h$. □

Lemma 2.3. *For all odd integers n, there is a type-*1, 2, 3, 4 RPA(n).

Proof. For $n \leqslant 13$, a type-1, 2, 3, 4 RPA(n) exists by [5] and Examples 1.1 and 2.1. For $n \geqslant 15$, let $A = (a_{i,j})$ be the type-1, 2 RPA(n) on $[n]$ obtained in Lemma 2.2. Let $X = \{a_{2,1}, \ldots, a_{h-1,1}\}$, $Y = [n] \setminus (X \cup \{1, h\})$, $Z = \{a_{1,n}, \ldots, a_{h-1,n}\}$ and $W = [n] \setminus (Z \cup \{1, h\})$. Let $\sigma : [n] \to [n]$ be any bijection satisfying $\sigma(X) = W$, $\sigma(Z) = Y$, $\sigma(1) = h$ and $\sigma(h) = 1$ (such a bijection must exist because $|X| = h-2 = |W|$, $|Z| = h-1 = |Y|$ and $|X \cap Z| = |X| - |X \cap W| = |W| - |X \cap W| = |W \cap Y|$). Define an $n \times n$ array $B = (b_{i,j})$ as follows:

$$b_{i,j} = \begin{cases} a_{i,j} & \text{if } i \leqslant h, \\ \sigma(a_{n+1-i,n+1-j}) & \text{if } i > h. \end{cases}$$

It can be checked that B is a type-1, 2, 3, 4 RPA(n). Most of the verifications are straightforward. The only tricky points are to ensure that the $n \times 1$ and $h \times 2$ basic rectangles in the lower left and lower right corners of B contain all n symbols. The $n \times 1$ basic rectangles in the lower left and lower right corners of B contain $X \cup \{h, 1\} \cup Y$, $Z \cup W \cup \{1, h\}$ respectively, which are the whole set $[n]$. Let U be the $h \times 2$ basic rectangle in the upper left corner of A. The $h \times 2$ basic rectangle in the lower right corner of B, say V, contains the symbol h together with the image of the first $h - 1$ rows of U under the bijection σ. Because U is canonical, it follows that the first $h - 1$ rows of U contain all the symbols in $[n] \setminus \{1\}$, and hence the last $h - 1$ rows of V contain all the symbols in $[n] \setminus \{h\}$. A similar argument applies to the $h \times 2$ basic rectangle in the lower left corner of B. □

Combining Lemma 2.3 and Theorem 1.2, we get the following result.

Theorem 2.4. *For all positive integers n, there exists a type-*1, 2, 3, 4 RPA(n).

3. Type-1, 2 LRPAs

In this section, we prove the existence of a type-1, 2 LRPA(n) for all positive integers n. First we need some terminology for dealing with latin squares. A *partial latin square* of order n is an $n \times n$ array with each cell filled or empty, such that each symbol occurs at most once in each row and column. A partial latin square is *completable* if it can be completed to a latin square. A row (column) in a partial latin square is *completable* if it can be extended to another partial latin square by filling the row (column). A *latin rectangle* is a rectangular array in which each symbol in a given set of symbols occurs exactly once in each row and at most once in each column. It is convenient to also use the phrase "latin rectangle" to describe a partial latin square in which some rows are completely filled and the other rows are completely empty.

The following famous theorem is due to Hall [7].

Theorem 3.1. *Any $r \times n$ latin rectangle is completable to an $n \times n$ latin square.*

Define $\Xi_{k,n,\ell}$ to be the set of $n \times n$ partial latin squares with the first k rows filled, exactly ℓ filled cells in row $k + 1$, and no other filled cells. Then define $f_{k,n}$ to be the maximum integer $\ell < n$ such that all partial latin squares in $\Xi_{k,n,\ell}$ are completable. Note that, by Theorem 3.1, partial latin squares in $\Xi_{k,n,\ell}$ are completable if and only if row $k+1$ is completable. Informally, $f_{k,n}$ is the number of "free-choices" that you have when extending a $k \times n$ latin rectangle by one row. Due to Theorem 3.1 the value of $f_{k,n}$ is well-defined, and we have $f_{k,n} \geqslant 1$ for $0 < k < n - 1$ and $f_{0,n} = f_{n-1,n} = n - 1$. It has been shown in [3] that we often have many free choices, particularly when extending "thin" latin rectangles:

Theorem 3.2.

$$f_{k,n} = \begin{cases} n - 2k & 1 \leqslant k < \frac{1}{2}n, \\ 1 & \frac{1}{2}n \leqslant k \leqslant n - 2. \end{cases}$$

The values of $f_{k,n}$ were determined in [3] by applying the classical Frobenius–König Theorem. We restate the main idea of the proof below in Lemma 3.3, since we will use it to construct type-1, 2 LRPAs. For an $n \times n$ matrix $M = (m_{ij})$, the *permanent* of M, per M, is defined by

$$\text{per } M = \sum_{\tau} m_{1\tau(1)} \cdots m_{n\tau(n)},$$

where the sum is over all permutations τ of $[n]$.

Suppose $S \in \Xi_{k,n,\ell}$ and let R be the $k \times n$ latin rectangle formed by the first k rows of S. Define a (0, 1)-matrix $M(R)$ by assigning the (i, j)-entry to be 1 if and only if symbol i does not appear in column j of R. Let $M'(S)$ be the submatrix of $M(R)$ obtained by deleting the ℓ rows and ℓ columns that correspond to, respectively, the symbols and columns of the filled entries in row $k + 1$ of S. We have:

Lemma 3.3. *S is not completable if and only if $M'(S)$ contains an $r \times s$ matrix of zeros where $r + s = n - l + 1$.*

Proof. It is well known that the number of extensions of R to a $(k + 1) \times n$ latin rectangle is per $M(R)$. By the same logic, the number of extensions of S to a $(k + 1) \times n$ latin rectangle is the permanent of the $(n - l) \times (n - l)$ submatrix $M'(S)$. In particular, S is not completable if and only if per $M'(S) = 0$. By the Frobenius–König Theorem, per $M'(S) = 0$ if and only if $M'(S)$ contains an $r \times s$ zero submatrix such that $r + s = n - l + 1$. □

We will apply Theorem 3.2 and Lemma 3.3 repeatedly in our construction of LRPAs. We will also utilize the construction for a type-1, 2 RPA(n) given by [5], but first we need to rectify a small glitch in the algorithm for odd n. If used exactly as described in [5], it is plausible that when building the second row, an SE operation may swap the entry in the middle of the first row with something to its left. This would then be followed by a 'mirror' SE operation that interchanged the new entry in the middle of the first row with something to its right, and that would mean that R_2 no longer contains every symbol. This problem is easily avoided. For example, we can specify that the second row begins $n, h + 1, h + 2, \ldots, n - 1$ and then it is easily checked that no SE operations are required, because R_2 is already sum-free. Beginning in this way, the symbol in the middle of the first row will remain a 1 through all subsequent steps, and the symbol that is eventually written in the middle of the second row will be an h.

Theorem 3.4. *For all positive integers n, there exists a type-*1, 2 LRPA(n).

Proof. For $n < 14$, a type-1, 2 LRPA(n) exists by [5] and Examples 1.1 and 2.1, so we may assume $n \geqslant 14$.

Consider the construction for a type-1, 2 RPA(n) given in [5], modified slightly as above. If we pause this construction at the completion of R_{b-1}, then we have a partial latin square in which each of the basic rectangles $R_1, R_2, \ldots, R_{b-1}$ in the top left corner satisfies Condition (a) and contains one copy of every symbol. Moreover, we can complete the second row and the first $b - 1$ basic rectangles in the top right corner exactly as in [5]. The only extra requirement that we have here is that no symbol should be repeated within any column. This property is immediate when creating canonical basic rectangles and doing SE operations inside them, since canonical basic rectangles contain only one copy of each symbol. Also, there is no problem in the middle of the first two rows, given the comments before this theorem.

By the above method we obtain some partial latin square, P. We now fill in P, one row at a time. Suppose that we have completed k rows, where $2 \leqslant k < n/2$. Then P contains at most $2\lceil n/(k + 1)\rceil$ filled cells in row $k+1$. So, using Theorem 3.2, we can fill row $k + 1$ provided

$$n - 2k \geqslant 2\left\lceil \frac{n}{k+1} \right\rceil. \tag{2}$$

Checking directly, this condition holds when $k \in \{2, 3\}$, given that $n \geqslant 14$. So we may assume $k \geqslant 4$.

Suppose that $k \leqslant (n - 4)/3$. If (2) fails, then

$$n - 2k < 2\left\lceil \frac{n}{k+1} \right\rceil \leqslant \frac{2(n+k)}{k+1}.$$

Rearranging, we find that $2k(k + 2) > n(k - 1) \geqslant (3k + 4)(k - 1)$, which contradicts $k \geqslant 4$.

So we may assume that $k \geqslant n/3 - 1$. In this case, $2\lceil n/(k + 1)\rceil \leqslant 6$, so (2) holds for $k \leqslant \frac{1}{2}(n - 6)$.

Since $n \geqslant 14$, there are exactly four filled cells in row $k + 1$ of P if $\frac{1}{2}(n - 6) < k < \frac{1}{2}(n - 1)$, and exactly two filled cells if $k = \frac{1}{2}(n - 1)$. For $k \in \left\{\frac{1}{2}(n - 5), \frac{1}{2}(n - 4)\right\}$ this is sufficient, since $n - 2k \geqslant 4$.

If $k = \frac{1}{2}(n - 2)$, then let R be the $k \times n$ latin rectangle contained in P. In this case, $M(R)$ has exactly k zeros in each row and column. Denote the four distinct symbols in row $k + 1$ of P by a, b, c, d. Let M' be the submatrix obtained from $M(R)$ by deleting rows $\{a, b, c, d\}$ and columns $\{1, 2, n - 1, n\}$. Since each basic rectangle of size $\frac{n}{2} \times 2$ in the upper left and upper right corners covers all symbols exactly once, each row of M' has exactly $k - 2$ zeros. Thus any $r \times s$ zero submatrix of M' has $r \leqslant k$ and $s \leqslant k - 2$, meaning that $r + s \leqslant k + k - 2 = n - 4 < n - 3$. By Lemma 3.3, row $n/2$ is completable.

It remains to consider the possibility that $k \in \left\{\frac{1}{2}(n - 3), \frac{1}{2}(n - 1)\right\}$ and we cannot complete row $k + 1$. By Lemma 3.3, $M(R)$ contains an $r \times s$ zero submatrix with $r + s = n - l + 1 = 2k$. As $M(R)$ has only k zeros in each row and column, we must have $r = s = k$. This means that R contains a $k \times k$ latin subsquare, say C. Consider the first three columns of R. Either two of them are in C, or else two of them are outside of C. In either case there are two columns that between them contain at most $n - k$ distinct symbols. Hence the $\lceil n/3\rceil \times 3$ rectangle in the top left corner of R contains at most $n - k + \lceil n/3\rceil \leqslant n - \frac{1}{2}(n - 3) + \lceil n/3\rceil < n$ distinct symbols, using $n \geqslant 14$. This contradicts the choice of P.

We conclude that we can complete the first $\lceil n/2\rceil$ rows of P. Finally, we complete P to a latin square by Theorem 3.1 to get a type-1, 2 LRPA. □

4. Concluding remarks

The study of retransmission permutation arrays was motivated by a technique used to resolve overlapping channel transmissions. We completed the existence spectrum for type-1, 2, 3, 4 RPAs and constructed a type-1, 2 LRPA(n) for all positive integers n, which is the first known infinite family of latin RPAs. By rotating these examples, it is obvious that we obtain type-2, 4 LRPA(n), type-3, 4 LRPA(n) and type-1, 3 LRPA(n) for all positive integers n. However, the existence of type-1, 2, 3, 4 LRPA(n) for all n, conjectured in [5], remains open.

Acknowledgment

This research was supported by ARC grant DP1093320.

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