Super-Simple Resolvable Balanced Incomplete Block Designs with Block Size 4 and Index 2

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Abstract: The necessary conditions for the existence of a super-simple resolvable balanced incomplete block design on v points with block size $k = 4$ and index $\lambda = 2$, are that $v \geq 16$ and $v \equiv 4 \pmod{12}$. These conditions are shown to be sufficient. © 2006 Wiley Periodicals, Inc. J Combin Designs 15: 341–356, 2007

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1. INTRODUCTION

A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. $\mathcal{G}$ is a partition of a set X (of *points*) into subsets called *groups*;
2. $\mathcal{B}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point;
3. Every pair of points from distinct groups occurs in exactly λ blocks.

The *group type* (or *type*) of the GDD is the multiset $\{|G| : G \in \mathcal{G}\}$. We shall use an "*exponential*" notation to describe types: so type $t_1^{u_1} \cdots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$, in the multiset.

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A GDD with block sizes from a positive integer set K is called a (K, λ)-GDD. When $K=\{k\}$, we simply write k for K. When $\lambda = 1$, we simply write K-GDD for a (K, λ)-GDD. A (k, λ)-GDD with group type 1^v $(k < v)$ is called a *balanced incomplete block design*, denoted by (v, k, λ)-BIBD.

A design is called *simple* if it contains no repeated blocks. A design is said to be *super-simple* if the intersection of any two blocks has at most two elements. When $k = 3$, a super-simple design is just a simple design. When $\lambda = 1$, the designs are necessarily super-simple. In this article, when we talk about super-simple BIBDs, we usually mean the case $k \geq 4$ and $\lambda > 1$.

The term super-simple designs was introduced by Gronau and Mullin in [24] (see the survey paper [9]). The existence of super-simple designs is an interesting extremal problem in itself, but there are also some useful applications. Such designs are used in the construction of coverings [8], in construction of new designs [7], and in the construction of superimposed codes [26].

For the existence of super-simple $(v, 4, \lambda)$-BIBDs, the necessary conditions are known to be sufficient for $\lambda = 2, 3, 4, 6$ (see [12]). Gronau and Mullin [24] solved the case for $\lambda = 2$, and the corrected proof appeared in [25]. The $\lambda = 3$ case was solved by Chen [10]. The $\lambda = 4$ case was solved independently by Adams, Bryant, and Khodkar [6] and Chen [11]. The case of $\lambda = 6$ was solved by Chen, Cao, and Wei [12]. A recent survey on super-simple $(v, 4, \lambda)$-BIBDs with $v \leq 32$ appeared in [9]. We summarize these known results in the following theorem.

Theorem 1.1 [24,25,10,6,11,12]**.** *A super-simple* $(v, 4, \lambda)$*-BIBD exists for* $\lambda = 2, 3, 4, 6$ *if and only if the following conditions are satisfied:*

1. $\lambda = 2$, $v \equiv 1 \pmod 3$ *and* $v \geq 7$;
2. $\lambda = 3$, $v \equiv 0, 1 \pmod 4$ *and* $v \geq 8$;
3. $\lambda = 4$, $v \equiv 1 \pmod 3$ *and* $v \geq 10$;
4. $\lambda = 6$, $v \geq 14$.

For the existence of super-simple $(v, 5, \lambda)$-BIBDs, the necessary conditions for $\lambda = 2, 4, 5$ are proved to be sufficient with a finite number of possible exceptions. Gronau, Kreher, and Ling [23] solved the case of $\lambda = 2$ with 11 unsettled values. Recently, 9 of these values were removed by Abel and Bennett [1], and Chen and Wei [13]. The $\lambda = 4, 5$ cases were solved by Chen and Wei [13,14]. We summarize these known results for $k = 5$ in the following theorem.

Theorem 1.2 [23,1,13,14]**.** *A super-simple* $(v, 5, \lambda)$*-BIBD exists for* $\lambda = 2, 4, 5$ *if and only if the following conditions are satisfied:*

1. $\lambda = 2$, $v \equiv 1, 5 \pmod{10}$, $v \neq 5, 15$, *except possibly when* $v \in \{115, 135\}$;
2. $\lambda = 4$, $v \equiv 0, 1 \pmod 5$, $v \geq 15$;
3. $\lambda = 5$, $v \equiv 1 \pmod 4$ *and* $v \geq 17$, *except possibly when* $v = 21$.

A GDD or a BIBD is said to be *resolvable* if its blocks can be partitioned into parallel classes each of which spans the set of points. We denote them by (K, λ)-RGDD or (v, k, λ)-RBIBD.

It is well known that the following are the necessary conditions for the existence of a super-simple (v, k, λ)-RBIBD:

1. $v \geq (k-2)\lambda + 2$;
2. $\lambda(v-1) \equiv 0 \pmod{k-1}$;
3. $v \equiv 0 \pmod{k}$.

The existence result on super-simple (v, k, λ)-RBIBDs for $k = 4$ and $\lambda = 3$ has been established by Ge and Lam [19], which is restated in the following theorem.

Theorem 1.3 [19]. *The necessary conditions for the existence of a super-simple $(v, 4, 3)$-RBIBD, that is, $v \geq 8$ and $v \equiv 0 \pmod 4$, are also sufficient except for $v = 12$.*

In this article we investigate the existence of super-simple $(v, 4, 2)$-RBIBDs, for which the necessary conditions reduce to $v \equiv 4 \pmod{12}$. It is easy to see that there exists no super-simple $(4, 4, 2)$-RBIBD. So we only need to consider the case of $v \geq 16$. We shall prove the following main result.

Theorem 1.4. *The necessary conditions for the existence of a super-simple $(v, 4, 2)$-RBIBD, namely, $v \equiv 4 \pmod{12}$ and $v \geq 16$, are also sufficient.*

The article is organized as follows. In Section 2, we shall introduce a special class of k-frames, that is, k-frames with a GP-fixed automorphism, which can be used to construct super-simple $(k, 2)$-frames. Some recursive constructions will also be listed there. In Section 3, partitionable skew Room frames will be employed to construct 4-frames with a GP-fixed automorphism, and the existence of a new class of partitionable skew Room frames will be established. In Section 4, some ingredient super-simple RBIBDs will be constructed directly by computer search. The crucial class of super-simple (4, 2)-frames of type 12^n will be constructed in Section 5. Some remarks and conclusion will be given in the last section.

2. RECURSIVE CONSTRUCTIONS

To describe our recursive constructions, we need the following auxiliary designs.

If $(X, \mathcal{G}, \mathcal{B})$ is a (k, λ)-GDD and $G \in \mathcal{G}$, then we say that a set $P \subset \mathcal{B}$ of blocks is a *holey parallel class* with *hole* G provided that P consists of $(|X| - |G|)/k$ disjoint blocks that partition $X \setminus G$. If we can partition the set of blocks $\mathcal{B}$ into a set $\mathcal{P}$ of holey parallel classes, then we say that $(X, \mathcal{G}, \mathcal{B})$ is a *(k, λ)-frame*.

The *group-type* (or *type*) of the frame is the multiset $\{|G| : G \in \mathcal{G}\}$. As with GDDs we shall use an "*exponential*" notation to describe group-type.

A *transversal design* (TD) TD(k, n) is a GDD of group type n^k and block size k. A resolvable TD(k, n) (denoted by RTD(k, n)) is equivalent to a TD$(k + 1, n)$. It is well known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. In this article, we mainly employ the following known results on TDs.

Lemma 2.1 [4]. *An RTD$(4, n)$ exists for all $n \geq 4$ except for $n = 6$ and possibly excepting $n = 10$.*

An *incomplete balanced incomplete block design* (IBIBD) with block size k and index λ is a triple $(X, H, \mathcal{B})$ which satisfies the following properties:

1. H is a subset of X (of *points*) called the *hole*;
2. $\mathcal{B}$ is a collection of k-subsets of X, called *blocks*, such that any block shares at most one common point with the hole;
3. Every pair of points from X is either in H or in exactly λ blocks but not in both.

We denote this design by (k, λ)-IBIBD(v, h), where v is size of the point set X and h is size of the hole set H. When $H = \emptyset$, a (k, λ)-IBIBD(v, h) is just a (v, k, λ)-BIBD.

A (k, λ)-IBIBD(v, h) is said to be *resolvable* if its blocks can be partitioned into parallel classes and partial parallel classes, called *holey parallel classes*, the latter partitioning $X \setminus H$. We denote it by (k, λ)-IRBIBD(v, h).

To obtain the main results, we shall use the following basic constructions, for which proofs can be found in [18]. Here, we just need to do the routine check for the super-simple property.

Construction 2.2 (*Weighting I*). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a GDD with index unity, and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X. *Suppose that for each block* $B \in \mathcal{B}$, *there exists a super-simple* (k, λ)*-frame of type* $\{w(x) : x \in B\}$. *Then there is a super-simple* (k, λ)*-frame of type* $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.

Construction 2.3 (*Weighting II*). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a super-simple GDD with index* λ, *and let* $w : X \to Z^+ \cup \{0\}$ *be a weight function on* X. *Suppose that for each block* $B \in \mathcal{B}$, *there exists a* k*-frame of type* $\{w(x) : x \in B\}$. *Then there is a super-simple* (k, λ)*-frame of type* $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.

Construction 2.4 (*Generalized frame constructions*). *Suppose there is a super-simple* (k, λ)*-frame with type* $T = \{t_i : i = 1, 2, \ldots, n\}$. *Let* $b > 0$. *If there exists a super-simple* (k, λ)*-IRBIBD*$(t_i + b, b)$ *for* $i = 1, 2, \ldots, n - 1$, *then there exists a super-simple* (k, λ)*-IRBIBD*$(u + b, t_n + b)$ *where* $u = \sum_{i=1}^{n} t_i$. *Furthermore, if a super-simple* $(t_n + b, k, \lambda)$*-RBIBD exists, then a super-simple* $(u + b, k, \lambda)$*-RBIBD also exists.*

In this article the main construction we use is Construction 2.4. If we have got a super-simple $(16, 4, 2)$-RBIBD and a super-simple $(4, 2)$-IRBIBD$(12 + 4, 4)$ as ingredient designs, then using Construction 2.4, we only need a super-simple $(4, 2)$-frame of type 12^n to obtain a super-simple $(12n + 4, 4, 2)$-RBIBD. To get the super-simple $(4, 2)$-frames, we consider a special class of 4-frames.

Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a k-frame, $\mathcal{P}$ is the set of holey parallel classes. Let σ be a permutation on X. If the block set, the group set and the set of holey parallel classes are fixed under the action of σ, that is, $\forall\ B \in \mathcal{B}$, $\sigma(B) \in \mathcal{B}$, $\forall\ G \in \mathcal{G}$, $\sigma(G) \in \mathcal{G}$, and $\forall\ P \in \mathcal{P}$, $\sigma(P) = \{\sigma(B) | B \in P\} \in \mathcal{P}$, then σ is called an *automorphism* of $(X, \mathcal{G}, \mathcal{B})$. Furthermore, if the automorphism σ keeps each of the groups as well as each of the holey parallel classes fixed, we say that the automorphism is *GP-fixed*. k-frames with a GP-fixed automorphism that decomposes each group into h cycles of length 2 play an essential role in the construction of super-simple $(k, 2)$-frames, which is stated in the following theorem.

Theorem 2.5. *Suppose there exists a* k*-frame of type* $(2h)^n$ *with a GP-fixed automorphism that decomposes each group into* h *cycles of length* 2, *then there exists a super-simple* $(k, 2)$*-frame of type* h^n.

Proof. Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a k-frame of type $(2h)^n$ with a GP-fixed automorphism σ that decomposes each group into h cycles of length 2, $\mathcal{P}$ is the set of holey parallel classes. Let X' be a maximal subset of X such that if $x \in X'$ then $\sigma(x) \in X \setminus X'$. Hence, $|X'| = hn$.

For the set X', we define a new group set $\mathcal{G}' = \{G' = G \cap X' : G \in \mathcal{G}\}$, and a new block set $\mathcal{B}' = \{B' = (B \cap X') \cup (\sigma(B) \cap X') : B, \sigma(B) \in \mathcal{B}\}$; then $|G'| = h$ for any $G' \in \mathcal{G}'$, and $|B'| = k$ for any $B' \in \mathcal{B}'$. We will show that $(X', \mathcal{G}', \mathcal{B}')$ is the desired design.

First, we prove that $(X', \mathcal{G}', \mathcal{B}')$ is a $(k, 2)$-GDD of type h^n. It is only to be checked that every pair of points from distinct groups occurs in exactly 2 blocks. For any pair $\{x, y\}$ with $x \in G'_1 \in \mathcal{G}'$ and $y \in G'_2 \in \mathcal{G}'$, we have four distinct blocks $B_1, \sigma(B_1), B_2, \sigma(B_2)$ such that $\{x, y\} \subset B_1$, $\{\sigma(x), \sigma(y)\} \subset \sigma(B_1)$, $\{x, \sigma(y)\} \subset B_2$, $\{\sigma(x), y\} \subset \sigma(B_2)$. Then we get $B'_1 = (B_1 \cap X') \cup (\sigma(B_1) \cap X')$, $B'_2 = (B_2 \cap X') \cup (\sigma(B_2) \cap X')$ such that $\{x, y\} \subset B'_1$, $\{x, y\} \subset B'_2$.

Second, we check the resolvability. Denote $t = h(n-1)/k$, and suppose $P \in \mathcal{P}$, $P = \{B_1, B_2, \ldots, B_t, \sigma(B_1), \sigma(B_2), \ldots, \sigma(B_t)\}$. Then we obtain $P' = \{B'_1, B'_2, \ldots, B'_t\}$, where $B'_i = (B_i \cap X') \cup (\sigma(B_i) \cap X')$ for $1 \le i \le t$ satisfying

$$\begin{aligned}\bigcup_{i=1}^{t} B'_i &= \bigcup_{i=1}^{t}((B_i \cap X') \cup (\sigma(B_i) \cap X')) = \bigcup_{i=1}^{t}((B_i \cup \sigma(B_i)) \cap X') \\ &= \left(\bigcup_{i=1}^{t}(B_i \cup \sigma(B_i))\right) \cap X' = (X \setminus G) \cap X' = X' \setminus G'.\end{aligned}$$

Hence we get the conclusion that P' is a holey parallel class.

Finally, we check the super-simplicity. Suppose there are two blocks B'_1 and B'_2 sharing at least three common points. Suppose $\{x, y, z\} \subset B'_1 \cap B'_2$, and $\{x, y, z\} \subset B_1$, where B'_1 is obtained from B_1 and $\sigma(B_1)$, B'_2 is obtained from B_2 and $\sigma(B_2)$. We can conclude that there exist two blocks of $\{B_1, B_2, \sigma(B_1), \sigma(B_2)\}$ sharing at least two common points, which contradicts to the fact that $(X, \mathcal{G}, \mathcal{B})$ is a k-frame. ■

Similar to the proof of Theorem 2.5, we can obtain the following lemma.

Lemma 2.6. *Suppose there is a k-RGDD of type* 2^n *with a GP-fixed automorphism that decomposes each group into one cycle of length* 2. *Then there is a super-simple* $(n, k, 2)$-*RBIBD.*

Before closing this section, we provide the following two ingredient designs.

Lemma 2.7. *There exists a super-simple* (16, 4, 2)-*RBIBD.*

Proof. It is easy to check that the super-simple 2-resolvable (16, 4, 2)-design presented in [23, Lemma 2.9] is also resolvable. Colbourn also gave a solution in [15]. ■

Lemma 2.8. *There exists a super-simple* (4, 2)-*IRBIBD*(12 + 4, 4).

Proof. Let the point set be $X = \{1, 2, \ldots, 16\}$ and the hole set be $H = \{1, 2, 3, 4\}$. The required blocks are listed below:

{5, 6, 7, 8},	{9, 10, 11, 12},	{13, 14, 15, 16},	
{5, 6, 9, 10},	{7, 8, 13, 14},	{11, 12, 15, 16},	
{1, 5, 7, 11},	{2, 6, 8, 12},	{3, 9, 13, 15},	{4, 10, 14, 16},
{1, 5, 12, 14},	{2, 6, 11, 13},	{3, 7, 10, 16},	{4, 8, 9, 15},
{1, 6, 9, 16},	{2, 5, 10, 15},	{3, 7, 12, 14},	{4, 8, 11, 13},
{1, 6, 14, 15},	{2, 5, 13, 16},	{3, 8, 9, 12},	{4, 7, 10, 11},
{1, 7, 9, 13},	{2, 8, 10, 14},	{3, 5, 11, 15},	{4, 6, 12, 16},
{1, 10, 12, 13},	{2, 9, 11, 14},	{3, 5, 8, 16},	{4, 6, 7, 15},
{1, 8, 10, 15},	{2, 7, 9, 16},	{3, 6, 11, 14},	{4, 5, 12, 13},
{1, 8, 11, 16},	{2, 7, 12, 15},	{3, 6, 10, 13},	{4, 5, 9, 14}.

Each of the first two rows gives a holey parallel class missing the hole H, and each of the remaining eight rows gives a parallel class. It is readily checked that the design is a super-simple IRBIBD. ■

3. PARTITIONABLE SKEW ROOM FRAMES

To construct 4-frames with GP-fixed automorphisms, we need the concept of skew Room frames.

Let X be a set, and let $\{H_1, \dots, H_n\}$ be a partition of X. An $\{H_1, \dots, H_n\}$-*Room frame* is an $|X| \times |X|$ array, F, indexed by X, which satisfies the properties:

1. every cell either is empty or contains an unordered pair of symbols of X,
2. the subarrays H_k^2 are empty, for $1 \le k \le n$ (these subarrays are referred to as *holes*),
3. each symbol of $X \setminus H_k$ occurs precisely once in row (or column) r, where $r \in H_k$,
4. the pairs occurring in F are precisely those $\{i, j\}$ where $(i, j) \in X^2 \setminus \cup_{k=1}^{n} H_k^2$.

A *skew Room frame* is a Room frame in which cell (i, j) is occupied if and only if cell (j, i) is empty.

The *type* of an $\{H_1, \dots, H_n\}$-Room frame F will be the multiset $\{|H_1|, \dots, |H_n|\}$. We will say that F has type $t_1^{u_1} \cdots t_k^{u_k}$ provided there are u_j H_i's of cardinality t_j, for $1 \le j \le k$.

From a skew Room frame of type h^n, one can get a 4-GDD of type $(6h)^n$ [29]. The 4-GDD is based on $H_i \times Z_6$, $1 \le i \le n$. The block set $\mathcal{B}$ contains all blocks $\{(a, j), (b, j), (c, 1 + j), (r, 4 + j)\}$, where $j \in Z_6$, $\{a, b\} \in F$, $\{a, b\}$ occurs in column c and row r.

If all the quadruples (a, b, c, r) can be partitioned into sets such that each set forms a partition of $X \setminus H_i$ for some i, and each H_i corresponds to $2h$ of the sets, we call the skew Room frame *partitionable*.

Skew Room frames have played an important role in the constructions of BIBDs and GDDs with block size four [29] and the resolution of the existence problem for weakly 3-chromatic BIBDs with block size four [30]. Partitionable skew Room frames were first introduced by Colbourn, Stinson, and Zhu in [17] to construct 4-frames. Here, we restate their construction as follows.

Lemma 3.1 [17]. *If there is a partitionable skew Room frame of type h^n, then there exists a 4-frame of type $(6h)^n$.*

The following is a simple observation based on the Colbourn–Stinson–Zhu construction.

Lemma 3.2. *Suppose there exists a partitionable skew Room frame of type h^n. Then there exists a* 4*-frame of type* $(6h)^n$ *with a GP-fixed automorphism* σ *that decomposes each group into* $3h$ *cycles of length* 2*. Moreover, there exists a super-simple* $(4, 2)$*-frame of type* $(3h)^n$.

Proof. Suppose $(X \times Z_6, \mathcal{G}, \mathcal{B})$ is a 4-frame obtained from a partitionable skew Room frame, where $\mathcal{G} = \{H_i \times Z_6 : 1 \leq i \leq n\}$. We define an automorphism on $X \times Z_6$, $\sigma : (x, i) \rightarrow (x, i + 3 \mod 6)$. It is easy to check that σ is a GP-fixed automorphism of order 2 on $X \times Z_6$, which decomposes each group into $3h$ cycles of length 2. The second assertion comes from Theorem 2.5. ■

In the remainder of this section, we shall concentrate on the constructions of partitionable skew Room frames of type 4^n, which will provide us the super-simple (4, 2)-frames of type 12^n by Lemma 3.2. First, we need the following recursive constructions, which are simple modifications of the constructions for skew Room frames [31].

Theorem 3.3 (*Inflation Construction*). *Suppose there is a partitionable skew Room frame of type* $t_1^{u_1} t_2^{u_2} \ldots t_k^{u_k}$*, and suppose also that* $m \neq 2, 3, 6,$ *or* 10*. Then there exists a partitionable skew Room frame of type* $(mt_1)^{u_1}(mt_2)^{u_2} \ldots (mt_k)^{u_k}$.

Theorem 3.4 (*Wilson's Fundamental Construction*). *Let* $(X, \mathcal{G}, \mathcal{B})$ *be a GDD with index unity, and let* $w : X \rightarrow Z^+ \cup \{0\}$ *be a weight function on* X*. Suppose that for each block* $B \in \mathcal{B}$*, there exists a partitionable skew Room frame of type* $\{w(x) : x \in B\}$*. Then there is a partitionable skew Room frame of type* $\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}$.

Define $PSF_4 = \{n :$ there exists a partitionable skew Room frame of type $4^n\}$. Then the following corollary of Theorem 3.4 says that the set PSF_4 is PBD-closed, which was similarly stated in [27].

Corollary 3.5. *Suppose there is an* (n, PSF_4)*-PBD. Then* $n \in PSF_4$.

Proof. The hypothesized PBD can be thought of as a GDD in which every group has size 1. Give every point weight 4 and apply Theorem 3.4. ■

Room frames of type h^n are often constructed using an abelian group of order hn, which can be found in [17]. Let G be an abelian group, written additively, and let H be a subgroup of G. Denote $g = |G|$, $h = |H|$, and suppose that $g - h$ is even. A *frame starter* in $G \setminus H$ is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (g - h)/2\}$ satisfying

1. $\bigcup_{1 \leq i \leq (g-h)/2} (\{s_i\} \cup \{t_i\}) = G \setminus H$, and
2. $\bigcup_{1 \leq i \leq (g-h)/2} \{\pm(s_i - t_i)\} = G \setminus H$.

An *adder* for S is an injection $A : S \rightarrow G \setminus H$ such that

$$\bigcup_{1 \leq i \leq (g-h)/2} (\{s_i + a_i\} \cup \{t_i + a_i\}) = G \setminus H,$$

where $a_i = A(s_i, t_i)$, $1 \leq i \leq (g - h)/2$. An adder A is *skew* if, further,

$$\bigcup_{1 \leq i \leq (g-h)/2} (\{a_i\} \cup \{-a_i\}) = G \setminus H.$$

From a starter S and a skew adder A, we can construct a skew Room frame F in which the cell $(j, -a_i + j)$ is occupied by $\{s_i + j, t_i + j\}$ for $1 \le i \le (g-h)/2$ and any $j \in G$. To obtain a 4-frame, it suffices to partition the quadruples: $(s_i + j, t_i + j, -a_i + j, j)$.

Below are direct constructions for some partitionable skew Room frames.

Lemma 3.6. *There exists a partitionable skew Room frame of type* 4^5.

Proof. Using a computer program we found a starter S and a skew adder A of type 4^5 as follows.

$G = Z_{20}$ and $H = \{0, 5, 10, 15\}$.

$S = \{\{1, 2\}, \{3, 6\}, \{4, 8\}, \{7, 16\}, \{9, 17\}, \{11, 18\}, \{12, 14\}, \{13, 19\}\}$.

$A = \{2, 6, 13, 12, 9, 16, 19, 3\}$.

Translate the initial quadruples:

1,	2,	−2,	0	—add 1—	2,	3,	19,	1
3,	6,	−6,	0	—add 3—	6,	9,	17,	3
4,	8,	−13,	0	—add 4—	8,	12,	11,	4
7,	16,	−12,	0	—add 1—	8,	17,	9,	1
9,	17,	−9,	0	—add 2—	11,	19,	13,	2
11,	18,	−16,	0	—add 3—	14,	1,	7,	3
12,	14,	−19,	0	—add 2—	14,	16,	3,	2
13,	19,	−3,	0	—add 4—	17,	3,	1,	4

It is obvious that each quadruple on the right covers the four non-zero residues modulo 5, and hence will give one partition of $G \setminus H$ by adding $k \pmod{20}$, where $k \in H$. Altogether we get eight partitions of $G \setminus H$. Under the action of G, we get further partitions so that the skew Room frame is partitionable. ■

Lemma 3.7. *There exists a partitionable skew Room frame of type* 4^6.

Proof. Using a computer program we found a starter S and a skew adder A of type 4^6 as follows.

$G = Z_{24}$ and $H = \{0, 6, 12, 18\}$.

$S = \{\{1, 2\}, \{3, 5\}, \{4, 9\}, \{7, 16\}, \{8, 21\}, \{10, 14\}, \{11, 19\}, \{13, 23\}, \{15, 22\}, \{17, 20\}\}$.

$A = \{13, 2, 17, 4, 1, 9, 21, 14, 19, 8\}$.

Translate the initial quadruples:

1,	2,	−13,	0	—add	2—	3,	4,	13,	2
4,	9,	−17,	0	—add	10—	14,	19,	17,	10
7,	16,	−4,	0	—add	9—	16,	1,	5,	9
11,	19,	−21,	0	—add	20—	7,	15,	23,	20
13,	23,	−14,	0	—add	22—	11,	21,	8,	22
3,	5,	−2,	0	—add	5—	8,	10,	3,	5
8,	21,	−1,	0	—add	23—	7,	20,	22,	23
10,	14,	−9,	0	—add	1—	11,	15,	16,	1
15,	22,	−19,	0	—add	4—	19,	2,	9,	4
17,	20,	−8,	0	—add	21—	14,	17,	13,	21

It is evident that the first 5 quadruples on the right form a partition of $G \setminus H$, so do the last 5 quadruples. Hence this is a partitionable skew Room frame of type 4^6. ■

Lemma 3.8. *There exists a partitionable skew Room frame of type* 4^7.

Proof. Using a computer program we found a starter S and a skew adder A of type 4^7 as follows.

$G = Z_{28}$ and $H = \{0, 7, 14, 21\}$.

$S = \{\{1, 2\}, \{3, 5\}, \{4, 8\}, \{6, 9\}, \{10, 19\}, \{11, 23\}, \{12, 22\}, \{13, 26\}, \{15, 20\},$
$\{16, 27\}, \{17, 25\}, \{18, 24\}\}.$

$A = \{8, 27, 9, 17, 10, 13, 3, 5, 24, 6, 2, 16\}.$

Translate the initial quadruples:

1,	2,	20,	0	—add	4—	5,	6,	24,	4
3,	5,	1,	0	—add	8—	11,	13,	9,	8
13,	26,	23,	0	—add	3—	16,	1,	26,	3
4,	8,	19,	0	—add	12—	16,	20,	3,	12
6,	9,	11,	0	—add	4—	10,	13,	15,	4
12,	22,	25,	0	—add	11—	23,	5,	8,	11
10,	19,	18,	0	—add	13—	23,	4,	3,	13
11,	23,	15,	0	—add	1—	12,	24,	16,	1
17,	25,	26,	0	—add	8—	25,	5,	6,	8
15,	20,	4,	0	—add	2—	17,	22,	6,	2
16,	27,	22,	0	—add	10—	26,	9,	4,	10
18,	24,	12,	0	—add	1—	19,	25,	13,	1

It is easily seen that the first 3 quadruples on the right cover the residues modulo 14 except $\{0, 7\}$, and hence give one partition of $G \setminus H$ by adding +14 (mod 28) to each element of these 3 quadruples, so do the second 3 quadruples, the third 3 quadruples and the last 3 quadruples. Hence this is a partitionable skew Room frame of type 4^7. ■

Lemma 3.9. *There exists a partitionable skew Room frame of type* 4^9.

Proof. Using a computer program, we found a starter S and a skew adder A of type 4^9 as follows.

$G = Z_{36}$ and $H = \{0, 9, 18, 27\}$.

$$S = \{\{1,2\},\{3,5\},\{4,7\},\{6,10\},\{8,13\},\{11,21\},\{12,24\},\{14,28\},\{15,26\},$$
$$\{16,31\},\{17,34\},\{19,35\},\{20,33\},\{22,30\},\{23,29\},\{25,32\}\}.$$
$$A = \{1,2,6,15,16,5,10,14,4,19,3,13,11,29,24,8\}.$$

Translate the initial quadruples:

1,	2,	35,	0	—add 6—	7,	8,	5,	6
17,	34,	33,	0	—add 4—	21,	2,	1,	4
3,	5,	34,	0	—add 3—	6,	8,	1,	3
23,	29,	12,	0	—add 2—	25,	31,	14,	2
4,	7,	30,	0	—add 4—	8,	11,	34,	4
25,	32,	28,	0	—add 5—	30,	1,	33,	5
6,	10,	21,	0	—add 5—	11,	15,	26,	5
12,	24,	26,	0	—add 4—	16,	28,	30,	4
8,	13,	20,	0	—add 8—	16,	21,	28,	8
11,	21,	31,	0	—add 2—	13,	23,	33,	2
14,	28,	22,	0	—add 7—	21,	35,	29,	7
16,	31,	17,	0	—add 6—	22,	1,	23,	6
15,	26,	32,	0	—add 5—	20,	31,	1,	5
19,	35,	23,	0	—add 7—	26,	6,	30,	7
20,	33,	25,	0	—add 6—	26,	3,	31,	6
22,	30,	7,	0	—add 7—	29,	1,	14,	7

It is easily seen that for each $i = 1, 2, \ldots, 8$, the $(2i-1)$-th and $2i$-th quadruples cover the eight non-zero residues modulo 9, and hence give one partition of $G \setminus H$. Altogether we get eight partitions of $G \setminus H$. ■

Lemma 3.10. *There exists a partitionable skew Room frame of type* 1^{33}.

Proof. Let $G = Z_{33}$ and $H = \{0\}$. A starter and a skew adder of type 1^{33} are given as follows:

$$S = \{\{1,2\},\{3,5\},\{4,7\},\{6,10\},\{8,13\},\{9,20\},\{11,23\},\{12,25\},\{14,24\},$$
$$\{15,30\},\{16,32\},\{17,31\},\{18,26\},\{19,28\},\{21,27\},\{22,29\}\}.$$
$$A = \{1,2,6,13,16,11,26,9,3,15,12,8,29,23,5,19\}.$$

Translate the initial quadruples:

1,	2,	−1,	0	—add 2—	3,	4,	1,	2
3,	5,	−2,	0	—add 8—	11,	13,	6,	8
11,	23,	−26,	0	—add 20—	31,	10,	27,	20
12,	25,	−9,	0	—add 26—	5,	18,	17,	26
16,	32,	−12,	0	—add 24—	7,	23,	12,	24
17,	31,	−8,	0	—add 30—	14,	28,	22,	30
21,	27,	−5,	0	—add 21—	9,	15,	16,	21
4,	7,	−6,	0	—add 25—	29,	32,	19,	25
6,	10,	−13,	0	—add 1—	7,	11,	21,	1
8,	13,	−16,	0	—add 12—	20,	25,	29,	12
9,	20,	−11,	0	—add 30—	6,	17,	19,	30

14,	24,	−3,	0	—add	18—	32,	9,	15,	18
15,	30,	−15,	0	—add	8—	23,	5,	26,	8
18,	26,	−29,	0	—add	10—	28,	3,	14,	10
19,	28,	−23,	0	—add	27—	13,	22,	4,	27
22,	29,	−19,	0	—add	2—	24,	31,	16,	2

It is easily seen that each of the first and the last 8 quadruples gives one partition of $G \setminus H$. Under the action of group G, we get other partitions so that the skew Room frame is partitionable. ■

Lemma 3.11 [16]. $B(\{5, 6, 7, 8, 9\}) = N \setminus \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34\}$.

Lemma 3.12. *There exists a partitionable skew Room frame of type 4^n for each $n \geq 5$ except possibly $n \in \{10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 27, 28, 32, 34\}$.*

Proof. From [17, Lemma 2.6], there exists a partitionable skew Room frame of type 4^8. Combining Lemmas 3.6–3.9, we get a partitionable skew Room frame of type 4^n for each $n \in \{5, 6, 7, 8, 9\}$. Applying Corollary 3.5 and Lemma 3.11, we obtain a partitionable skew Room frame of type 4^n for each $n \in N \setminus \{10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 23, 24, 27, 28, 29, 32, 33, 34\}$.

For each $n \in \{13, 17, 29, 33\}$, we have a partitionable skew Room frame of type 1^n by [17, Lemmas 2.4 and 2.5] and Lemma 3.10. Applying Theorem 3.3 with $m = 4$ gives the desired partitionable skew Room frames of type 4^n. ■

4. DIRECT CONSTRUCTIONS

For most of our direct constructions, we adapt the familiar difference method, where a finite abelian group is used to generate the set of blocks for a given design. Thus, instead of listing all the blocks of the design, we list a set of base blocks and generate the others by an additive group and perhaps some further automorphisms. If G is the additive group under consideration, then we shall adapt the following convention:

$$\text{dev } \mathbf{B} = \{B + g : B \in \mathbf{B} \text{ and } g \in G\},$$

where $\mathbf{B}$ is the collection of base blocks of the design.

Lemma 4.1. *Let $k \geq 4$. Suppose a GDD on Z_v is obtained cyclicly from the base blocks $\mathbf{B} = \{B_i = \{b_{i1}, b_{i2}, \ldots, b_{ik}\} : 1 \leq i \leq b'\}$ without short orbit, then the GDD is super-simple if and only if the difference pairs*

$$\{b_{iy} - b_{ix}, b_{iz} - b_{ix}\},\ \{b_{ix} - b_{iy}, b_{iz} - b_{iy}\},\ \{b_{ix} - b_{iz}, b_{iy} - b_{iz}\},$$

are distinct for all possible triples $\{b_{ix}, b_{iy}, b_{iz}\} \subset B_i$ with $1 \leq i \leq b'$. Here, all the operations are performed modulo v.

Proof. We first check the sufficiency. Suppose that the GDD is not super-simple, and there are two blocks intersecting at three points $\{a, b, c\}$, which are translates of $\{b_{ix}, b_{iy}, b_{iz}\} \subset B_i$ and $\{b_{ju}, b_{jv}, b_{jw}\} \subset B_j$ with $B_i, B_j \in \mathbf{B}$. Then $|\{b_{ix}, b_{iy}, b_{iz}\} \cup \{b_{ju}, b_{jv}, b_{jw}\}| \geq 4$ and the difference pairs of $\{b_{ix}, b_{iy}, b_{iz}\}$ and $\{b_{ju}, b_{jv}, b_{jw}\}$ are exactly the same, which leads to a contradiction.

Conversely, suppose that the GDD is super-simple. Suppose that there are two difference pairs $\{b_{iy} - b_{ix}, b_{iz} - b_{ix}\} = \{b_{jv} - b_{ju}, b_{jw} - b_{ju}\}$ with $\{b_{ix}, b_{iy}, b_{iz}\} \subset B_i$ and $\{b_{ju}, b_{jv}, b_{jw}\} \subset B_j$. According to the fact that each of the base blocks forms a full orbit, we have $|(B_i - b_{ix}) \cap (B_j - b_{ju})| \geq 3$, which contradicts to the super-simplicity. ■

Lemma 4.2. *There exists a super-simple* (4, 2)*-frame of type* 3^5.

Proof. Let the point set be $G = Z_{15}$, and let the group set be $\{\{j, j+5, j+10\} : j = 0, 1, \ldots, 4\}$. We first construct a base block:

$$\{1, 3, 4, 12\}.$$

Multiplying each point in the above base block by 7 modulo 15, we get another base block

$$\{7, 6, 13, 9\}.$$

By Lemma 4.1, it is readily checked that these 2 base blocks give a super-simple (4, 2)-frame of type 3^5 as required. ■

Since a (4, 2)-frame of type 12^n can exist only when $n \geq 5$, we can not use Construction 2.4 to get a super-simple $(12n + 4, 4, 2)$-RBIBD for $n = 2, 3, 4$. These designs can be obtained directly by a computer search.

Lemma 4.3. *There exists a* 4*-RGDD of type* 2^{28} *with a GP-fixed automorphism that decomposes each group into one cycle of length* 2*, and hence a super-simple* (28, 4, 2)*-RBIBD.*

Proof. First, we construct a 4-RGDD of type 2^{28}. Let the point set be $Z_{28} \times \{0, 1\}$, and let the group set be $\{\{(i, j), (i + 14, j)\} : i = 0, 1, 2, \ldots, 13; j = 0, 1\}$. Below are the required base blocks, which are to be developed by (+1 mod 28, −).

{(0,0), (1,0), (0,1), (3,1) },	{(1,0), (4,0), (2,1), (9,1)},	{(0,0), (2,0), (17,0), (21,0)},
{(7,1), (9,1), (13,1), (26,1)},	{(1,0), (6,0), (15,1), (25,1)},	{(4,0), (26,0), (8,1), (20,1)},
{(5,0), (25,0), (17,1), (18,1)},	{(8,0), (24,0), (14,1), (19,1)},	{(9,0), (27,0), (16,1), (24,1)}.

Here, the first 2 blocks (in the first row) give two parallel classes by adding +2 (mod 28) to the first component. The remaining 7 blocks give a parallel class by adding +14 (mod 28) to the first component.

Define an automorphism on $Z_{28} \times \{0, 1\}$, $\sigma : (i, j) \to (i + 14 \bmod 28, j)$. It is easy to check that σ is a GP-fixed automorphism of order 2 that decomposes each group into one cycle of length 2. Using Lemma 2.6, there exists a super-simple (28, 4, 2)-RBIBD. ■

Lemma 4.4. *There exists a* 4*-RGDD of type* 2^{40} *with a GP-fixed automorphism that decomposes each group into one cycle of length* 2*, and hence a super-simple* (40, 4, 2)*-RBIBD.*

Proof. It is shown in [21, Lemma 3.5] that there exists a 4-RGDD of type 2^{40} on the point set $Z_{40} \times \{0, 1\}$ and group set $\{\{(i, j), (i + 20, j)\} : i = 0, 1, 2, \ldots, 19; j = 0, 1\}$. We define an automorphism on the point set, $\sigma : (i, j) \to (i + 20 \bmod 40, j)$. It is easy to check that σ is a GP-fixed automorphism of order 2 that decomposes each group into one cycle of length 2. By Lemma 2.6, there exists a super-simple (40, 4, 2)-RBIBD. ■

Lemma 4.5. *There exists a super-simple* (52, 4, 2)*-RBIBD.*

Proof. First, we give two (52, 4, 1)-RBIBDs $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$. Let the point set be $X = Z_4 \times Z_{13}$. The left-hand column and the right-hand column below give the required base blocks of the two RBIBDs.

$\{(0,6),(1,6),(2,6),(3,6)\}$, mod $(-,13)$,	$\{(0,12),(1,12),(2,12),(3,12)\}$, mod $(-,13)$,
$\{(0,0),(1,1),(0,4),(1,8)\}$, mod $(4,13)$,	$\{(0,0),(1,1),(0,4),(1,8)\}$, mod $(4,13)$,
$\{(0,9),(2,11),(0,12),(0,7)\}$, mod $(4,13)$,	$\{(0,7),(2,9),(2,11),(2,6)\}$, mod $(4,13)$,
$\{(3,10),(0,2),(1,5),(0,3)\}$, mod $(4,13)$,	$\{(0,10),(1,2),(2,5),(1,3)\}$, mod $(4,13)$,
$\{(0,0),(2,6),(1,7),(3,4)\}$, mod $(4,13)$,	$\{(0,0),(2,6),(1,7),(3,4)\}$, mod $(4,13)$.

These base blocks are developed as follows. We first look at the blocks listed in the left-hand column. Add +1 (mod 4) to the first component of the 2nd, 3rd, and 4th blocks to obtain 12 blocks. These 12 blocks, together with the first block in this column, form an initial parallel class. The fifth block gives another initial parallel class by adding +1 (mod 13) to the second component. Adding +1 (mod 13) to the second component for each element of the first initial parallel class and adding +1 (mod 4) to the first component of the second initial parallel class give the desired (52, 4, 1)-RBIBD $(X, \mathcal{B}_1)$. Similarly, we can obtain $(X, \mathcal{B}_2)$ from the base blocks on the right-hand column.

Now, we define two maps from the set $Z_4 \times Z_{13}$ to the set Z_{52}:

$$f_1 : (i, j) \longrightarrow (13i + j) \pmod{52},$$

$$f_2 : (i, j) \longrightarrow 17(i + 4j) \pmod{52}.$$

It is readily checked that $(Z_{52}, f_1(\mathcal{B}_1) \cup f_2(\mathcal{B}_2))$ is the super-simple (52, 4, 2)-RBIBD as required. ■

5. RESULTS OBTAINED BY RECURSION

In this section, we deal mainly with the existence of the crucial super-simple (4, 2)-frames of type 12^n and super-simple $(12n + 4, 4, 2)$-RBIBDs.

Lemma 5.1. *There exists a super-simple* (4, 2)*-frame of type* 12^n *for each* $n \geq 5$ *except possibly* $n \in \{10, 11, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 27, 28, 32, 34\}$.

Proof. Combining Lemmas 3.2 and 3.12, we obtain super-simple (4, 2)-frames as desired. ■

Lemma 5.2. *There exists a super-simple* (4, 2)*-frame of type* 3^n *for each* $n \in \{13, 17\}$.

Proof. For each $n \in \{13, 17\}$, there exists a partitionable skew Room frame of type 1^n from [17, Lemmas 2.4 and 2.5]. Applying Lemma 3.2, we obtain the desired super-simple (4, 2)-frames of type 3^n. ■

Lemma 5.3. *There exists a super-simple* (4, 2)*-frame of type* 12^n *for each* $n \in \{10, 11, 15, 16, 20\}$.

Proof. For each $n \in \{10, 11, 15, 16, 20\}$, start from a (5, 1)-GDD of type 4^n coming from [22], and apply Construction 2.2 with weight 3 to obtain the (4, 2)-frames as desired. Here, we need the input super-simple (4, 2)-frame of type 3^5, which comes from Lemma 4.2. ■

Lemma 5.4. *There exists a super-simple* (4, 2)*-frame of type* 12^{12}.

Proof. There exists a 4-frame of type 24^{12} in [20, Lemma 3.2], which is based on the point set $Z_{48} \times Z_6$ with the group set $\{\{j, j+12, j+24, j+36\} \times Z_6 : j = 0, 1, \ldots, 11\}$. Define an automorphism σ on the point set $Z_{48} \times Z_6$, $\sigma : (i, j) \to (i, j+3 \bmod 6)$. It is easy to check that the design is a 4-frame with a GP-fixed automorphism of order 2. Applying Theorem 2.5, we obtain the desired super-simple (4, 2)-frame. ■

Lemma 5.5 *There exists a super-simple* (4, 2)*-frame of type* 12^n *for each* $n \in \{18, 23, 28\}$.

Proof. For each $n \in \{18, 23, 28\}$, we have a $(4n+1, \{5, 13^*\}, 1)$-PBD by [28, Theorem 2.7]. Remove one point outside of the block of size 13 to obtain a {5, 13}-GDD of type 4^n. Apply Construction 2.2 with weight 3 to obtain the (4, 2)-frames as desired. Here, we need the input super-simple (4, 2)-frames of types 3^5 and 3^{13}, which come from Lemmas 4.2 and 5.2, respectively. ■

Lemma 5.6. *There exists a super-simple* (4, 2)*-frame of type* 12^n *for each* $n \in \{22, 24, 27, 32, 34\}$.

Proof. The proof is similar to that of Lemma 5.5. Here, for each $n \in \{22, 24, 27, 32, 34\}$, we need a $(4n+1, \{5, 17^*\}, 1)$-PBD, which comes from [5]. ■

Combining Lemmas 2.7, 2.8, 5.1, 5.3–5.6 and applying Construction 2.4 with $b = 4$, we have the following.

Lemma 5.7. *For each* $n \geq 5$ *and* $n \notin \{14, 19\}$, *there exist both a super-simple* (4, 2)*-frame of type* 12^n *and a super-simple* $(12n+4, 4, 2)$*-RBIBD.*

To get the super-simple $(12n+4, 4, 2)$-RBIBDs with $n \in \{14, 19\}$, we need the following concept.

Let S be a set and $\mathcal{H} = \{S_1, S_2, \ldots, S_n\}$ be a set of subsets of S. A *holey Steiner pentagon system* having hole set $\mathcal{H}$ is a triple $(S, \mathcal{H}, \mathcal{B})$ where $\mathcal{B}$ is a collection of pentagons, satisfying the following properties:

1. Two vertices from a same hole S_i do not occur together in any pentagon of $\mathcal{B}$.
2. Two vertices from different holes S_i and S_j $(i \neq j)$ are joined by a path of length 1 in exactly one pentagon of $\mathcal{B}$, and also by a path of length 2 in exactly one pentagon of $\mathcal{B}$.

The order of the system is $|S|$.

If $\mathcal{H} = \{S_1, S_2, \ldots, S_n\}$ is a partition of S, then we simply denote the system by $HSPS(T)$, where T is the type and is defined to be the multiset $\{|S_i| : 1 \leq i \leq n\}$. The S_i are called *holes*. We shall use an "*exponential*" notation to describe types: so type $t_1^{u_1} \cdots t_k^{u_k}$ denotes u_i occurrences of t_i, $1 \leq i \leq k$, in the multiset.

It is known in [3] that the existence of an $HSPS$ of type T implies the existence of a (5, 2)-GDD of type T. We shall call an $HSPS$ of type T super-simple if its underlying (5, 2)-GDD of type T is super-simple; that is, any two blocks of the GDD intersect in at most two points.

Lemma 5.8. *There exists a super-simple* $(12n+4, 4, 2)$*-RBIBD for each* $n \in \{14, 19\}$.

Proof. By [1,2], we have super-simple *HSPSs* of types $2^{12}4^1$ and $2^{17}4^1$, hence super-simple (5, 2)-GDDs of types $2^{12}4^1$ and $2^{17}4^1$. From [28, Theorem 3.8] there exists a 4-frame of type 6^5. Apply Construction 2.3 with weight 6 to obtain super-simple (4, 2)-frames of types $12^{12}24^1$ and $12^{17}24^1$. Adjoin 4 infinite points and apply Construction 2.4 to obtain the desired $(12n + 4, 4, 2)$-RBIBD. Here, we need a super-simple (4, 2)-IRBIBD$(12 + 4, 4)$ and a super-simple (28, 4, 2)-RBIBD as input designs, which come from Lemmas 2.8 and 4.3, respectively. ■

6. CONCLUDING REMARKS

Now, we are in a position to prove our main result which is restated as follows.

Theorem 6.1. *The necessary conditions for the existence of a super-simple* $(v, 4, 2)$*-RBIBD, namely,* $v \equiv 4 \pmod{12}$ *and* $v \geq 16$*, are also sufficient.*

Proof. Combining Lemmas 2.7, 4.3–4.5, and 5.7–5.8, the conclusion then follows. ■

In [17], partitionable skew Room frames have been found useful in the constructions of 4-frames. In this article, the existence of partitionable skew Room frames has played an important role in the constructions of super-simple (4, 2)-frames, which are crucial in the recursive constructions. This is another demonstration for the significance of partitionable skew Room frames. Hence, the existence of partitionable skew Room frames will be an interesting topic for further investigation. We will report it in a future paper.

REFERENCES

[1] R. J. R. Abel and F. E. Bennett, Super-simple Steiner pentagon systems, preprint.

[2] R. J. R. Abel, F. E. Bennett, and G. Ge, Super-simple holey Steiner pentagon systems and related designs, preprint.

[3] R. J. R. Abel, F. E. Bennett, and H. Zhang, Holey Steiner pentagon systems, J Combin Des 7 (1999), 41–56.

[4] R. J. R. Abel, A. E. Brouwer, C. J. Colbourn, and J. H. Dinitz, Mutually orthogonal Latin squares, In C.R.C. Handbook of Combinatorial Designs, C. J. Colbourn and J. H. Dinitz (Editors), CRC Press, Boca Raton, FL, 1996, pp. 111–142.

[5] R. J. R. Abel, G. Ge, M. Greig, and A. C. H. Ling, Further results on $(v, 5, w^*, 1)$-PBDs, preprint.

[6] P. Adams, D. Bryant, and A. Khodkar, On the existence of super-simple designs with block size 4, Aequationes Math 52 (1996), 230–246.

[7] I. Bluskov, New designs, J Combin Math Combin Comput 23 (1997), 212–220.

[8] I. Bluskov and H. Hämäläinen, New upper bounds on the minimum size of covering designs, J Combin Des 6 (1998), 21–41.

[9] I. Bluskov and K. Heinrich, Super-simple designs with $v \leq 32$, J Statist Plann Inference 95 (2001), 121–131.

[10] K. Chen, On the existence of super-simple $(v, 4, 3)$-BIBDs, J Combin Math Combin Comput 17 (1995), 149–159.

[11] K. Chen, On the existence of super-simple $(v, 4, 4)$-BIBDs, J Statist Plann Inference 51 (1996), 339–350.

[12] K. Chen, Z. Cao, and R. Wei, Super-simple balanced incomplete block designs with block size 4 and index 6, J Statist Plann Inference 133 (2005), 537–554.

[13] K. Chen and R. Wei, Super-simple $(v, 5, 4)$ designs, preprint.

[14] K. Chen and R. Wei, Super-simple $(v, 5, 5)$ designs, Des Codes Cryptogr 39 (2006), 173–187.

[15] C. J. Colbourn, A Steiner 2-design with an automorphism fixing exactly $r + 2$ points, J Combin Des 7 (1999), 375–380.

[16] C. J. Colbourn and J. H. Dinitz, CRC handbook of combinatorial designs, CRC Press Inc., Boca Raton, 1996.

[17] C. J. Colbourn, D. R. Stinson, and L. Zhu, More frames with block size four, J Combin Math Combin Comput 23 (1997), 3–20.

[18] S. C. Furino, Y. Miao, and J. Yin, Frames and resolvable designs, CRC Press, Boca Raton, 1996.

[19] G. Ge and C. W. H. Lam, Super-simple resolvable balanced incomplete block designs with block size 4 and index 3, J Combin Des 12 (2004), 1–11.

[20] G. Ge, C. W. H. Lam, and A. C. H. Ling, Some new uniform frames with block size four and index one or three, J Combin Des 12 (2004), 112–122.

[21] G. Ge, C. W. H. Lam, A. C. H. Ling, and H. Shen, Resolvable maximum packings with quadruples, Des Codes Cryptogr 35 (2005), 287–302.

[22] G. Ge and A. C. H. Ling, Asymptotic results on the existence of 4-RGDDs and uniform 5-GDDs, J Combin Des 13 (2005), 222–237.

[23] H.-D. O. F. Gronau, D. L. Kreher, and A. C. H. Ling, Super-simple $(v, 5, 2)$-designs, Discrete Appl Math 138 (2004), 65–77.

[24] H.-D. O. F. Gronau and R. C. Mullin, On super-simple 2-$(v, 4, \lambda)$ designs, J Combin Math Combin Comput 11 (1992), 113–121.

[25] A. Khodkar, Various super-simple designs with block size four, Australas J Combin 9 (1994), 201–210.

[26] H. K. Kim and V. Lebedev, On optimal superimposed codes, J Combin Des 12 (2004), 79–91.

[27] R. C. Mullin, P. J. Schellenberg, S. A. Vanstone, and W. D. Wallis, On the existence of frames, Discrete Math 37 (1981), 79–104.

[28] R. Rees and D. R. Stinson, Frames with block size four, Canad J Math 44 (1992), 1030–1049.

[29] C. A. Rodger, Linear spaces with many small lines, Discrete Math 129 (1994), 167–180.

[30] C. A. Rodger, E. B. Wantland, K. Chen, and L. Zhu, Existence of certain skew Room frames with application to weakly 3-chromatic linear spaces, J Combin Des 2 (1994), 311–324.

[31] D. R. Stinson, Some constructions for frames, Room squares, and subsquares, Ars Combin 12 (1981), 229–267.