6. Equations Governing the Motion of a Fluid (2)

Batchelor's Chap. 3, § 3.5-3.7

Chapter 3. Equations Governing the Motion of a Fluid

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Bernoulli's theorem (1)

$$\frac{D}{Dt}\left(\tfrac{1}{2}\mathbf{u}^2+E\right)=u_iF_i+\frac{1}{\rho}\frac{\partial(u_i\sigma_{ij})}{\partial x_j}+\frac{1}{\rho}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right).$$

假设 $\mathbf{F}=-\nabla\Psi,\ \Psi=\Psi(\mathbf{x})$ 保守体积力,势能定常

$$\Rightarrow\ u_iF_i=-u_i\frac{\partial\Psi}{\partial x_i}=-\frac{D\Psi}{Dt}$$

$$-\frac{1}{\rho}\frac{\partial(u_ip\delta_{ij})}{\partial x_j},\ =\frac{p}{\rho^2}\frac{D\rho}{Dt}-\frac{u_i}{\rho}\frac{\partial p}{\partial x_i}=-\frac{D(p/\rho)}{Dt}+\frac{1}{\rho}\frac{\partial p}{\partial t},$$

Bernoulli (1738)
对于定常不可压缩流首先得到！

无剪切粘性、无热传导，定常压力 (动力学压力)

$$\Rightarrow\ \frac{D}{Dt}\left(\tfrac{1}{2}\mathbf{u}^2+E+\frac{p}{\rho}+\Psi\right)=0\ \Rightarrow\ H=\tfrac{1}{2}q^2+E+\frac{p}{\rho}+\Psi$$ 总能

沿迹线不变（定常流沿流线不变）

Bernoulli's theorem (2)

- 流管推导方法

截面能量对流流率 $(\tfrac{1}{2}q^2+E+\Psi)q\rho\,\delta A,$

截面压力做功功率 $pq\,\delta A.$

定常流，两截面间能量为常值；边界无粘性、无热传导，则

$$\left(\tfrac{1}{2}q^2+E+\frac{p}{\rho}+\Psi\right)q\rho\,\delta A$$ 沿流管为常数

$q\rho\,\delta A$ 沿流管也为常数（质量守恒）

⇒ **Bernoulli**定理: $\left(\tfrac{1}{2}q^2+E+\frac{p}{\rho}+\Psi\right)$ 沿流管(流线)为常数

Bernoulli's theorem (3)

- Bernoulli定理成立的条件要求等熵吗？
  - ◆常把等熵作为Bernoulli定理成立的必要条件，但并不严格正确！
    - 只要求边界上无剪切粘性、无热传导，内部粘性仍会导致不等熵过程，如过激波的定常绝热流关系
    - 体积粘性也会导致熵增（不过，无剪切粘性、无热传导的流体仍具有体积粘性是十分少见的！）
  - ◆实用的角度，可以认为定常、等熵流是Bernoulli定理成立的条件，但实际上是充分的（+保守）

Alternative forms of Bernoulli theorem

$I = E + \frac{p}{\rho}$ $\quad H = \frac{1}{2}q^2 + I + \Psi.$

气体势能的变化常可忽略 ⟹ $H = \frac{1}{2}q^2 + I = const.$ 驻点焓

$\frac{D}{Dt}\left(E+\frac{p}{\rho}\right) = T\frac{DS}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt}.$ 等熵时 $= \frac{D}{Dt}\int\frac{c^2}{\rho}d\rho,\quad c^2 = (\partial p/\partial\rho)_S$

⟹ $H = \frac{1}{2}q^2 + \int\frac{c^2}{\rho}d\rho + \Psi = const.$ 知道压强和密度等熵关系时使用

或等熵不可压缩时 $\frac{D}{Dt}\left(E+\frac{p}{\rho}\right) = T\frac{DS}{Dt} + \frac{1}{\rho}\frac{Dp}{Dt} = \frac{D}{Dt}\int\frac{dp}{\rho}$

⟹ $H = \int\frac{dp}{\rho} + \Psi + \frac{q^2}{2} = const.$ 沿迹线/流线，见正压流体动量方程积分

Special forms of Bernoulli theorem

- **Incompressible fluid/flow**

无粘、无热传导（等熵） $DE/Dt = 0.$

$$H = \frac{1}{2}q^2 + \frac{p}{\rho} + \Psi = const.$$

- **Perfect gas**

$$p = (c_p - c_v)\rho T,$$

$$H = \frac{1}{2}q^2 + \int c_p\,dT + \Psi. = const.$$

量热完全气体（比热为常数）

$$H = \frac{1}{2}q^2 + c_p T + \Psi = \frac{1}{2}q^2 + \frac{c^2}{\gamma-1} + \Psi. = const.$$

Crocco relation (1)

- 定常等熵流Bernoulli常数H如何在不同流线间变化？
  - ◆一般不能确定，与流动如何建立有关
  - ◆但可以期望得到H和S在不同流线间变化的相容关系！

$$T\nabla S = \nabla E + p\nabla(1/\rho).$$

⟹ $\nabla H = T\nabla S + \nabla(\frac{1}{2}q^2 + \Psi) + \frac{1}{\rho}\nabla p.$ $\quad H = \frac{1}{2}q^2 + E + \frac{p}{\rho} + \Psi$

This relation is quite general, except for the assumption that $\mathbf{F} = -\nabla\Psi$.

定常、无粘流的动量方程 $\rho\mathbf{u}.\nabla\mathbf{u} = -\rho\nabla\Psi - \nabla p,$

结合 $\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla q^2 - \mathbf{u}.\nabla\mathbf{u}$ ⟹ $\mathbf{u}\times\boldsymbol{\omega} = \nabla(\frac{1}{2}q^2 + \Psi) + \frac{1}{\rho}\nabla p.$

⟹ $\boxed{\nabla H = T\nabla S + \mathbf{u}\times\boldsymbol{\omega},}$ Crocco 关系(1937)

Crocco relation (2)

$$\nabla H = T\nabla S + \mathbf{u}\times\boldsymbol{\omega},$$

- Crocco关系和Bernoulli定理的比较
  - Crocco关系用到热力学关系和动量方程，而Bernoulli定理为能量方程的积分
  - Crocco关系成立的条件：无粘、定常、保守
  - Bernoulli定理成立的条件：无粘无热传导、定常、保守
- 均熵等价均能的条件
  - 无旋流
  - 速度场和涡量场处处平行(Beltrami flow, helical flow)
- 均熵流 $\nabla H = \mathbf{u}\times\boldsymbol{\omega}$ ⟹ 动量方程的空间积分
  - H沿Lamb面为常数　　定常、均熵、无粘、保守
    - H沿涡线或流线为常数

Crocco relation(3)

- 非定常Crocco关系：成立条件为无粘、保守

前面由热力学关系＋保守体力 ⟹ $\nabla H = T\nabla S + \nabla(\tfrac{1}{2}q^2+\Psi) + \frac{1}{\rho}\nabla p.$

无粘、保守体力的动量方程 $\dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\dfrac{1}{\rho}\nabla p - \nabla\Psi$

Lamb形式 $\dfrac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = -\dfrac{1}{\rho}\nabla p - \nabla\Psi - \nabla(\dfrac{\mathbf{u}^2}{2})$

⟹ $\dfrac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = T\nabla S - \nabla H$　比较定常形式 $\nabla H = T\nabla S + \mathbf{u}\times\boldsymbol{\omega},$

动量方程的特殊形式，满足一定的条件有空间积分！

前面已得到定常、均熵时的积分

正压流体动量方程的积分

$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Psi - \nabla(\frac{\mathbf{u}^2}{2})$$

- **Barotropic fluid** $\nabla\rho \,//\, \nabla p \Rightarrow \rho = \rho(p)$ ⟹ $\dfrac{1}{\rho}\nabla p = \nabla\int\dfrac{dp}{\rho}$

⟹ $\dfrac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega}\times\mathbf{u} = -\nabla H,\ H = \int\dfrac{dp}{\rho} + \Psi + \dfrac{q^2}{2}$

定常、正压、无粘、保守

$$H = \int\frac{dp}{\rho} + \Psi + \frac{q^2}{2} = const.$$

Lamb面内成立

Bernoulli积分

无旋、正压、无粘、保守

$$\frac{\partial\phi}{\partial t} + \int\frac{dp}{\rho} + \Psi + \frac{q^2}{2} = c(t) \to const.$$

全场成立

Cauchy-Lagrange积分

旋转坐标系下动量方程的积分(1)

$\boldsymbol{\Omega} = const.$　虚拟力 $-2\boldsymbol{\Omega}\times\mathbf{u} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x})$　加到动量方程中

科氏力沿转轴和流线方向分量均为零

离心力为保守力

$$-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{x}) = \frac{1}{2}\nabla(\boldsymbol{\Omega}\times\mathbf{x})^2$$

⟹ $\dfrac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega}+2\boldsymbol{\Omega})\times\mathbf{u} = -\dfrac{1}{\rho}\nabla p - \nabla(\dfrac{q^2}{2} + \Psi - \dfrac{1}{2}(\boldsymbol{\Omega}\times\mathbf{x})^2)$　无粘、保守

两点修正即可：$\boldsymbol{\omega} \Rightarrow \boldsymbol{\omega} + 2\boldsymbol{\Omega},\quad \Psi \Rightarrow \Psi - \dfrac{1}{2}(\boldsymbol{\Omega}\times\mathbf{x})^2$

前述Crocco关系、动量方程的积分仍成立（修正之后）

例如 $\dfrac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega}+2\boldsymbol{\Omega})\times\mathbf{u} = T\nabla S - \nabla H$　$H = \tfrac{1}{2}q^2 + E + \dfrac{p}{\rho} + \Psi - \dfrac{1}{2}(\boldsymbol{\Omega}\times\mathbf{x})^2$

定常、均熵时沿流线有积分

旋转坐标系下动量方程的积分(2)

- 无粘、保守、正压流体

$$\frac{\partial \mathbf{u}}{\partial t} + (\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \mathbf{u} = -\nabla\left(\frac{q^2}{2} + \int \frac{dp}{\rho} + \Psi - \frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{x})^2\right)$$

定常、正压、无粘、保守

$$H = \int \frac{dp}{\rho} + \Psi - \frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{x})^2 + \frac{q^2}{2} = const.$$

沿相对流线成立

相对系中无旋、正压、无粘、保守

$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \Psi - \frac{1}{2}(\boldsymbol{\Omega} \times \mathbf{x})^2 + \frac{q^2}{2} = c(t) \to const.$$

沿相对流线、转轴均成立

Constancy of H in one-dimensional steady flow

$$\rho\frac{DH}{Dt} = \rho u\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\left(\tfrac{4}{3}\mu u\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right),$$

$$\frac{\partial}{\partial x}(\rho u) = 0.$$

$$\Longrightarrow \quad \rho u[H]_{x_1}^{x_2} = \left[\tfrac{4}{3}\mu u\frac{\partial u}{\partial x} + k\frac{\partial T}{\partial x}\right]_{x_1}^{x_2},$$

只要两点的速度梯度和温度梯度为零，则H不变

激波前后H也是不变的，虽然熵是要变的！

$$\rho u[S]_{x_1}^{x_2} = \left[\frac{k}{T}\frac{\partial T}{\partial x}\right]_{x_1}^{x_2} + \int_{x_1}^{x_2}\left\{\frac{\kappa\Delta^2}{T} + \frac{\rho\Phi}{T} + \frac{k}{T^2}\left(\frac{\partial T}{\partial x}\right)^2\right\}dx,$$

Complete set of equations

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla.\mathbf{u} = 0.$$

$$\rho\frac{Du_i}{Dt} = \rho F_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\{2\mu(e_{ij} - \tfrac{1}{3}\Delta\,\delta_{ij})\},$$

$$T\frac{DS}{Dt} = c_p\frac{DT}{Dt} - \frac{\beta T}{\rho}\frac{Dp}{Dt} = \Phi + \frac{1}{\rho}\frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right), \quad \beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

$$\mu \equiv \mu(\rho, T), \quad k \equiv k(\rho, T).$$

$$f(p, \rho, T) = 0.$$

Isentropic flow

$DS/Dt = 0;$ $\quad c_p\frac{DT}{Dt} = \frac{\beta T}{\rho}\frac{Dp}{Dt},$ set μ and k equal to zero

$$\frac{1}{\rho c^2}\frac{Dp}{Dt} + \nabla.\mathbf{u} = 0, \quad c^2 = (\partial p/\partial \rho)_S$$

$$\rho\frac{D\mathbf{u}}{Dt} = \rho\mathbf{F} - \nabla p,$$

$$\rho \equiv \rho(p, S),$$

均熵更容易处理

c的物理意义？

平衡态 $\rho_0\mathbf{F} = \nabla p_0.$

线性小扰动方程组

$$\frac{1}{\rho_0 c_0^2}\frac{\partial p_1}{\partial t} + \nabla.\mathbf{u} = 0, \quad \rho_0\frac{\partial \mathbf{u}}{\partial t} = \rho_1\mathbf{F} - \nabla p_1,$$

$$\Longrightarrow \quad \frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} = \nabla^2 p_1 - \rho_1\nabla.\mathbf{F} - \frac{\mathbf{F}.\nabla p_1}{c_0^2}.$$

重力场中，右端第2项为零；第3项通常为小量=>声波方程

Conditions for incompressible flow (velocity to be solenoidal) (1)

$\frac{D\rho}{Dt}=0$, or $\nabla\cdot\mathbf{u}=0$ 要求 $|\nabla.\mathbf{u}| \ll U/L$, $\left|\frac{1}{\rho}\frac{D\rho}{Dt}\right| \ll \frac{U}{L}$.

$$\frac{Dp}{Dt}=c^2\frac{D\rho}{Dt}+\left(\frac{\partial p}{\partial S}\right)_\rho\frac{DS}{Dt}. \Longrightarrow \left|\frac{1}{\rho c^2}\frac{Dp}{Dt}-\frac{1}{\rho c^2}\left(\frac{\partial p}{\partial S}\right)_\rho\frac{DS}{Dt}\right| \ll \frac{U}{L}.$$

I. $\left|\frac{1}{\rho c^2}\frac{Dp}{Dt}\right| \ll \frac{U}{L}$ *the fluid is behaving as if it were incompressible.*

更重要!

暂不考虑熵变，借助无粘动量方程得 $\left|\frac{1}{\rho c^2}\frac{\partial p}{\partial t}-\frac{1}{2c^2}\frac{Dq^2}{Dt}+\frac{\mathbf{u}.\mathbf{F}}{c^2}\right| \ll \frac{U}{L}$,

第2项中的对流项而来的条件 $\Longrightarrow$ $U^2/c^2 \ll 1.$ 即 $M^2 \ll 1$

第1项的条件 $\Longrightarrow$ $\frac{n^2L^2}{c^2} \ll 1.$ 即 $M^2St^2 \ll 1$

第2项的非定常项导致更弱的条件 $M^2St \ll 1$

声波是可压缩的!

涉及的无量纲参数

$Ma=U/c$, Mach number

$St=nL/U$, Strouhal number

$Fr=U^2/gL$, Froude number

$\mathrm{Re}=\rho UL/\mu$, Reynolds number

$Kn=\lambda/L\approx M/\mathrm{Re}$, Knudsen number

Conditions for incompressible flow (velocity to be solenoidal) (2)

重力的影响 $\Longrightarrow$ $\frac{gL}{c^2} \ll 1.$ 即 $M^2/Fr \ll 1$

垂直高度：大气12km，水200km以内即可满足

粘性力的影响 $\Longrightarrow$ $M^2/\mathrm{Re}=M\bullet Kn \ll 1$ 易满足

熵变条件 $\Longrightarrow$

$$\frac{\beta U^2}{c_p}\frac{\mu}{\rho LU} \ll 1, \quad \beta\theta\frac{\kappa}{LU} \ll 1,$$

耗散条件最易满足

热传导条件也容易满足

air and water at 15 °C and one atmosphere pressure

$L=1$ cm, $U=10$ cm/sec, $\theta=10$ °C:

	$\frac{\beta U^2}{c_p}\frac{\mu}{\rho LU}$	$\beta\theta\frac{\kappa}{LU}$
air	5×10^{-10}	7×10^{-4}
water	4×10^{-13}	3×10^{-7}

Summary of the lecture

- Bernoulli定理（能量积分）：
  - 条件：定常、无粘、无热传导、保守；沿流(迹)线成立
  - 其它形式；特殊形式（不可压、完全气体）
- Crocco关系：无粘、保守
  - 定常、非定常
- 动量积分
  - 定常、均熵、无粘、保守；沿Lamb面成立
  - 定常、正压、无粘、保守；沿Lamb面成立
  - 无旋、正压、无粘、保守；全场成立
- 旋转坐标系中的动量积分
- 完备方程组
- 不可压缩流假设成立的条件
- 一维定常流的等H性；等熵流与声波方程

Concluding remarks to Chap. 1-3

- 建立了完备的流体力学方程组！还有什么？
  - 力学和物理过程，预言流动性质与知道控制方程不同
  - 与环境有关，依赖于定解条件，流动特性变化很大
  - 只是搭建了流体运动分析的舞台！
- 理论流体力学是如何分析的？（往往只研究特殊情况）
  - 理论流体力学的各分支主要是对个别力学过程的研究
    - 气体动力学：主要研究胀压过程对流动的影响
    - 不可压缩粘性流：主要研究剪切过程对流动的影响
      - 润滑理论、边界层等
    - 自然对流：浮力的影响
    - 流动稳定性：不引入新的力学过程，但有合适的方法
- **Batchelor**的后四章重点：惯性、剪切粘性起主要作用的流动

End of Lecture 6

Readings: Batchelor's Chap.3, §3.5-3.7