Mesh Parameterizations

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Outline

• Definition
• Tutte’s barycentric mapping
• Least squares conformal maps (LSCM, ASAP)
• Angle-Based Flattening (ABF)
  • ABF++, LABF
• As-rigid-as-possible (ARAP)
  • Simplex Assembly
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Definition

• A function that puts input surface in one-to-one correspondence with a 2D domain.

• Parameterization of a Triangulated Surface
  • all \((u_i, v_i)\) coordinates associated with each vertex \(v_i = (x_i, y_i, z_i)^T\)
Definition

• Build a local coordinate system on input triangle $t$.

• The mapping is piecewise linear.

• $J_t$ is $2 \times 2$.

$$\begin{bmatrix} u_j - u_i & u_k - u_i \\ v_j - v_i & v_k - v_i \end{bmatrix} \begin{bmatrix} x_j - x_i & x_k - x_i \\ y_j - y_i & y_k - y_i \end{bmatrix}^{-1} \begin{bmatrix} u_i \\ v_i \end{bmatrix} = x_j - x_i$$

$$f_t(x) = J_t x + b_t$$
Definition

- $J_t$ is the Jacobian of $f_t(x)$.

\[
J_t = \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y}
\end{pmatrix} = \nabla u
\]

\[
= \frac{1}{A_t} \begin{pmatrix}
y_j - y_k & y_k - y_i & y_i - y_j \\
x_k - x_j & x_i - x_k & x_j - x_i
\end{pmatrix}
\begin{pmatrix}
u_i \\
u_j \\
u_k
\end{pmatrix}
\]

\[
f_t(x) = J_t x + b_t
\]
Constraints

• Bijective
  • The image of the surface in parameter space does not self-intersect.
  • The intersection of any two triangles in parameter space is either a common edge, a common vertex, or empty.
Constraints

- Inversion-free
  - The orientation of each triangle is positive.
Constraints

• Locally injective
  • The orientation of each triangle is positive $\rightarrow \det J > 0$.
  • For boundary vertex, the mapping is locally bijective $\rightarrow \theta(v) > 2\pi$. 

\[ \theta(v) < 2\pi \quad \theta(v) > 2\pi \]
Constraints

• Low distortion
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Barycentric Mapping

• One of the most widely used methods.

Given a triangulated surface homeomorphic to a disk, if the \((u, v)\) coordinates at the boundary vertices lie on a convex polygon in order, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the \((u, v)\) coordinates form a valid parameterization (without self-intersections, bijective).
Barycentric Mapping

- Homeomorphic to a disk.
- A convex polygon
  - circle, square,......
- A convex combination
  - $\omega_{ij} > 0$
  - Uniform Laplacian, mean value coordinate
- Solver: linear equation.
Mean value coordinates

- Our aim is to study sets of weights \( \lambda_1, \ldots, \lambda_k \geq 0 \) such that
  \[
  \sum_{i=1}^{k} \lambda_i v_i = v_0 \\
  \sum_{i=1}^{k} \lambda_i = 1
  \]

\( v_i \) is on 2D.
Proposition

• The weights

\[
\lambda_i = \frac{\omega_i}{\sum_{i=1}^{k} \omega_i}, \\
\omega_i = \frac{\tan \frac{\alpha_{i-1}}{2} + \tan \frac{\alpha_i}{2}}{\|v_i - v_0\|}
\]

are the valid weights.

Proof: substitution. ???

Come from the mean value theorem for harmonic functions. ???
Mean value coordinates

• The input mesh is a spatial one.
  • $v_i \in R^3$
  • the mean value coordinates can be applied directly.
  • compute the coordinates directly form the spatial angle.

Figure 3. Comparisons from left to right:
(3a) Triangulation, (3b) Tutte, (3c) shape-preserving, (3d) mean value
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Conformal mapping

• Conformal mappings locally correspond to similarities

**Figure 5.8.** A conformal parameterization transforms a small circle into a small circle, i.e., it is locally a similarity transform. (Image taken from [Hormann et al. 07]. ©2007 ACM, Inc. Included here by permission.)
Similar transform

• 2D case: for one triangle \( t \)

• \( J_t = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = s \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \)

• \[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \]

• \[ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \]

• Cauchy-Riemann Equations.

\[ J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \]
Least squares conformal maps (LSCM, ASAP)

• Energy

\[ E_{LSCM} = \sum_t A_t \left( \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right) \]

• measure non-conformality
• It is invariant with respect to arbitrary translations and rotations.
• \( E_{LSCM} \) does not have a unique minimizer.
• Fixing at least two vertices. Significantly affect the results.
Implementation – Homework 3

• Least squares
• https://en.wikipedia.org/wiki/Least_squares
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Angle-Based Flattening (ABF)

- Key observation: the parameter space is a 2D triangulation, uniquely defined by all the angles at the corners of the triangles.
  - Find angles instead of \((u_i, v_i)\) coordinates.
  - Use angles to reconstruct the resulting parameterization.
- Optimization goal:
  \[
  E_{ABF} = \sum_t \sum_{i=1}^{3} \omega_i^t (\alpha_i^t - \beta_i^t)^2
  \]
  \(\beta_i^t\): Optimal angles for \(\alpha_i^t\).
  \(\omega_i^t = (\beta_i^t)^{-2}\).
  
  \[
  \beta_i^t = \begin{cases} 
  \frac{\tilde{\beta}_i^t \cdot 2\pi}{\sum_i \tilde{\beta}_i^t}, & \text{Interior vertex} \\
  \tilde{\beta}_i^t, & \text{Boundary vertex}
  \end{cases}
  \]
Constraints

• Positive resulting angles:
  \[ \alpha_i^t > 0 \]

• The three triangle angles have to sum to \( \pi \):
  \[ \alpha_i^t + \alpha_2^t + \alpha_3^t = \pi \]

• For each internal vertex the incident angles have to sum to \( 2\pi \):
  \[ \sum_{t \in \Omega(v)} \alpha_k^t = 2\pi \]

• Reconstruction constraints:
  \[ \prod_{t \in \Omega(v)} \sin \alpha_{k\oplus 1}^t = \prod_{t \in \Omega(v)} \sin \alpha_{k\ominus 1}^t \]
Linear ABF

• Reconstruction constraints are nonlinear and hard to solve.
• Initial estimation + estimation error
  • $\alpha^t_i = \gamma^t_i + e^t_i$

\[
\log \left( \prod_{t \in \Omega(v)} \sin \alpha_{k \oplus 1}^t \right) = \log \left( \prod_{t \in \Omega(v)} \sin \alpha_{k \oplus 1}^t \right)
\]

\[
\sum_{t \in \Omega(v)} \log(\sin \alpha_{k \oplus 1}^t) = \sum_{t \in \Omega(v)} \log(\sin \alpha_{k \oplus 1}^t)
\]

• Taylor expansion:

\[
\log(\sin \alpha_{k \oplus 1}^t) = \log(\sin \gamma_{k \oplus 1}^t + e_{k \oplus 1}^t)
= \log(\sin \gamma_{k \oplus 1}^t) + e_{k \oplus 1}^t \cot \gamma_{k \oplus 1}^t + \cdots
\]

It is linear with estimation error.
Solver

• Set $\gamma_i^t = \beta_i^t$

• Problem:

$$\min_{e} E_{ABF} = \sum_{t} \sum_{i=1}^{3} \omega_i^t (e_i^t)^2$$

subject to

$$Ae = b$$

$$\Rightarrow$$

$$\begin{pmatrix} D & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$$\Rightarrow$$

$$e = D^{-1}A^T (AD^{-1}A^T)^{-1}b$$
Reconstruct parameterization

• Greed method.
  • constructs the triangles one by one using a depth-first traversal.

• Least squares method.
  • an angle based least squares formulation which solves a set of linear equations relating angles to coordinates.
Greed method

• Choose a mesh edge \( e^1 = (v^1_a, v^1_b) \).
• Project \( v^1_a \) to \((0,0,0)\) and \( v^1_b \) to \((\|e^1\|, 0,0)\).
• Push \( e^1 \) on the stack \( S \).
• While \( S \) not empty, pop an edge \( e = (v_a, v_b) \). For each face \( f_i = (v_a, v_b, v_c) \) containing \( e \):
  • If \( f_i \) is marked as set, continue.
  • If \( v_c \) is not projected, compute its position based on \( v_a, v_b \) and the face angles of \( f_i \).
  • Mark \( f_i \) as set, push edge \((v_b, v_c)\) and \((v_a, v_c)\) on the stack.
• Accumulate numerical error.
Least squares method

• The ratio of triangle edge lengths $\|\overrightarrow{P_1P_3}\|$ and $\|\overrightarrow{P_1P_2}\|$ is

$$\frac{\|\overrightarrow{P_1P_3}\|}{\|\overrightarrow{P_1P_2}\|} = \frac{\sin \alpha_2}{\sin \alpha_3}$$

$\Rightarrow$

$$\overrightarrow{P_1P_3} = \frac{\sin \alpha_2}{\sin \alpha_3} \begin{pmatrix} \cos \alpha_1 & -\sin \alpha_1 \end{pmatrix} \overrightarrow{P_1P_2}$$

• Thus for each triangle, given the position of two vertices and the angles, the position of the third vertex can be uniquely derived.

  • greedy method.
Least squares method

∀$t = (j, k, j)$, $M^t(P_k - P_j) + P_j - P_l = 0$

$M^t = \frac{\sin \alpha_k}{\sin \alpha_l} \begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix}$

1. Two equations per triangle for the $x$ and $y$ coordinates of the vertices.

2. The angles of a planar triangulation define it uniquely up to rigid transformation and global scaling.
   - Introduce four constraints which eliminate these degrees of freedom.
   - Fix two vertices sharing a common edge.
Least squares method

• Choose one edge $e^1 = (v^1_a, v^1_b)$.
• Project $v^1_a$ to (0,0,0) and $v^1_b$ to ($\|e^1\|$, 0,0).

• Solve following energy to compute positions of other vertices:

$$E = \sum_t \left\| M^t (P_k - P_j) + P_j - P_l \right\|^2$$
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As-rigid-as-possible method

Paper: A Local/Global Approach to Mesh Parameterization

Figure 1: Parameterization of the Gargoyle model using (a) our As-Similar-As-Possible (ASAP) procedure, (b) As-Rigid-As-Possible (ARAP) procedure, (c) Linear ABF [ZLS07], (d) inverse curvature approach [YKL*08], and (e) curvature prescription approach [BCGB08]. The pink lines are the seams of the closed mesh when cut to a disk.
Distortion type

• Three common distortion types:
  • Isometric mapping: rotation + translation
  • Conformal mapping: similarity + translation
  • Area-preserving mapping: area-preserving + translation
  • Conformal + Area-preserving $\iff$ Isometric
Singular values

• Isometric mapping
  • $J_t \implies$ rotation matrix
  • $\sigma_1 = \sigma_2 = 1$

• Conformal mapping
  • $J_t \implies$ similar matrix
  • $\sigma_1 = \sigma_2$

• Area-preserving mapping
  • $\det J_t = 1$
  • $\sigma_1 \sigma_2 = 1$

$f_t(x) = J_t x + b_t$

$J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$

$\sigma_1, \sigma_2$ are the two singular values of $J_t$. 
Goal

\[ E(u, L) = \sum_t A_t \| J_t - L_t \|^2_F \]

\( L_t \): target transformation
- Isometric mapping: rotation matrix
- Conformal mapping: similar matrix

\textbf{Variables:}
- 2D parameterization coordinate
- Target transformation

\textbf{How to optimize?}

\[ f_t(x) = J_t x + b_t \]

\[ J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} \]

\( \sigma_1, \sigma_2 \) are the two singular values of \( J_t \).
General Local/Global Approach

• Alternatively optimization
  • Local step:
    • Fix 2D parameterization coordinates, optimize target transformations.
  • Global step:
    • Fix target transformations, optimize 2D parameterization coordinates.

• **Global step**:
  • Quadratic energy
  • Linear system
  • Eigen

\[
E(u, L) = \sum_{t} A_t \| J_t - L_t \|_F^2
\]
**Local step:** Procrustes analysis

- Approximate one $2 \times 2$ matrix $J_t$ as best we can by another $2 \times 2$ matrix $L_t$.

- $d(J_t, L_t) = \|J_t - L_t\|^2_F = \text{trace}\left((J_t - L_t)^T(J_t - L_t)\right)$

- Minimize $d(J_t, L_t)$ through Singular Value Decomposition (SVD)
  - $J_t = U\Sigma V^T$, $\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$
  - Signed SVD: $U$ and $V$ are rotation matrix, $\sigma_2$ maybe negative
  - Best rotation: $UV^T$
  - Best similar matrix: $U \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} V^T$, $s = \frac{\sigma_1 + \sigma_2}{2}$
Local/Global Approach summary

**Figure 2:** Parameterizing a mesh by aligning locally flattened triangles. (Left) Original 3D mesh; (middle) flattened triangles; (right) 2D parameterization.
Connection to singular values

• Conformal

$$E(u) = \sum_t A_t \left( \sigma_t^1 - \sigma_t^2 \right)^2$$

• Isometric

$$E(u) = \sum_t A_t \left( (\sigma_t^1 - 1)^2 + (\sigma_t^2 - 1)^2 \right)$$

$f_t(x) = J_t x + b_t$

$\sigma_t^1, \sigma_t^2$ are the two singular values of $J_t$. 

$$E(u, L) = \sum_t A_t \| J_t - L_t \|^2_F$$
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Information

• Computing Inversion-Free Mappings by Simplex Assembly
• ACM Transactions on Graphics(SIGGRAPH Asia) 35(6), 2016.
• http://staff.ustc.edu.cn/~fuxm/projects/SimplexAssembly/index.html
Affine transformation

Key observation: the parameter space is a 2D triangulation, uniquely defined by all the AFFINE TRANSFORMATIONS on the triangles.

Edge assembly constraints:

$$A_i(v_a - v_b) = A_j(v_a - v_b)$$
Key idea

• disassembly + assembly
  • Treat affine transformation as variables
  • Unconstrained optimization

(a)  (b)  (c)  (d)
Distortion control

Conformal: \( d_i^c = \begin{cases} 
\frac{1}{2} \|A_i\|_F \|A_i^{-1}\|_F, & d = 2 \\
\frac{1}{8} \left( \|A_i\|^2_F \|A_i^{-1}\|^2_F - 1 \right), & d = 3 
\end{cases} \)

Volumetric: \( d_i^{vol} = \frac{1}{2} \left( \det(A_i) + \frac{1}{\det(A_i)} \right) \)

Isometric: \( d_i^{iso} = 0.5 \cdot (d_i^c + d_i^{vol}) \)

Barrier function on distortion:

1. The type of distortion and distortion bound \( K \) are given:
\[
E_C^* = \sum_{i=1}^{N} \frac{e^{s \cdot d_i^*}}{K - d_i^*}
\]

2. The type of distortion is not specified or distortion bound \( K = \infty \):
\[
E_C^* = \sum_{i=1}^{N} e^{s \cdot d_i^*}
\]
Unconstrained optimization problem

Disassembly: project initial $A_i^0$ into feasible space.

$\min_{A_1,\ldots,A_N, T_1,\ldots,T_N} \lambda E_{assembly} + E_C + \mu E_m$

$\lambda_{k+1} = \min \left( \lambda_{\min} \cdot \max \left( \frac{E_{C,k} + \mu E_{m,k}}{E_{assembly,k}}, 1 \right), \lambda_{\max} \right)$

1. $E_{assembly}$ dominates the energy, approach zero;
2. $\lambda_{\max}$: avoid large distortion.

$E_{assembly}$: summation of squares of edge, assembly constraints.

$E_C$: Barrier function on distortion

$E_m$: users’ designed energy
Optimal bound

- Use the current maximal distortion as the bound for the next round of minimization.
Locally injective mapping

• Requirements for locally injective mapping on triangle mesh:
  • 1. inversion-free;
  • 2. the sum of triangle angles $\theta_v$ around boundary vertex $v$ is less than $2\pi$.

• A barrier term:
  $$ E_\theta = \sum_{v \in \partial M} \frac{1}{2\pi - \theta_v} $$

[Lipman 2012] [Fu et al. 2012] [Schuller et al. 2013] [Kovalsky et al. 2012] Ours without $E_\theta$ Ours with $E_\theta$