Morphing

Xiao-Ming Fu
Outlines

• Definition
• Angle, length, area, volume, and curvature
  • Example-Driven Deformations Based on Discrete Shells
• Affine transformation
  • As-Rigid-As-Possible Shape Interpolation
• Data-driven morphing
  • A Data-Driven Approach to Realistic Shape Morphing
  • Data-Driven Shape Interpolation and Morphing Editing
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Definition

• Morphing is a special effect in motion pictures and animations that changes (or morphs) one image or shape into another through a seamless transition.
Definition

• Problem: Given $M^0$, $M^1$, and $t$, how to compute the shape $M^t$?
  • $t \in [0,1]$, interpolation
  • $t \not\in [0,1]$, extrapolation
Requirements

- **Look naturally and intuitively**
- Symmetry
- Smooth vertex paths
- Bounded distortion / low distortion
- Foldover-free
- Large deformation
- ......
Some methods

• First interpolate some values/metrics, then reconstruct the shape.

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Interpolation

Angle, length, and volume

\[ l_e^t = (1 - t)l_e^0 + tl_e^1 \]
\[ \theta_e^t = (1 - t)\theta_e^0 + t\theta_e^1 \]
\[ V^t = (1 - t)V^0 + tV^1 \]

\( l_e \): edge length
\( \theta_e \): dihedral angles
\( V \): volume
\[ V = \frac{1}{6} \sum_{f_{i,j,k}} (x_i \times x_j) \cdot x_k \]
Reconstruction

• A mesh with prescribed edge lengths and dihedral angles does not exist.

\[
E_l = \frac{1}{2} \sum_e (l_e - l_e^t)^2 \\
E_a = \frac{1}{2} \sum_e (\theta_e - \theta_e^t)^2 \\
E_v = \frac{1}{2} \sum_e (V_e - V_e^t)^2 \\
E = \lambda E_l + \mu E_b + \nu E_v
\]
Figure 5: Interpolation and extrapolation of the yellow example poses. The blending weights are 0, 0.35, 0.65, 1.0, and 1.25.

Figure 6: Interpolation of an adaptively meshed and strongly twisted helix with blending weights 0, 0.25, 0.5, 0.75, 1.0.
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Interpolation

• How to define $A(t)$ reasonably?

• Simplest solution:
  \[ A(t) = (1 - t)I + tA \]

• More elaborate approaches:
  • Singular value decomposition
    \[ A = U\Sigma V^T \]
    \[ A(t) = U(t)((1 - t)I + t\Sigma)V^T(t) \]
  • Polar decomposition
    \[ A = U\Sigma V^T = UV^T V\Sigma V^T = RS \]
    \[ A(t) = R(t)((1 - t)I + tS) \]
\begin{align*}
A(t) &= (1 - t)I + tA \\
A(t) &= U(t)((1 - t)I + t\Sigma)V^T(t) \\
A(t) &= U(t)((1 - t)I + t\Sigma)V^T(t) \text{ with subtracting } 2\pi \\
A(t) &= R(t)((1 - t)I + tS)
\end{align*}
Reconstruction

• Least squares:

\[ E = \sum_f \| J - A(t) \|^2_F \]

Figure 12: Morph between photographs of an elephant and a giraffe.
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Data-driven approach

• Problem:
  • Input: a database with various models belonging to the same category and containing identical connectivity
  • Given source and target models, how to utilize the database to generate the morphing?
Two stages

- Offline stage
  - Analyze the model database to form local shape spaces that better characterize the plausible distribution of models in the category.

- Online stage
  - When the source and target models are given, we find reference models in the local shape spaces and use them to guide the as-rigid-as-possible shape morphing.
More details

• Offline stage
  • Define distance between pairs of models

• Online stage
  • Find a minimal distance path connecting the source and target models
  • In-between reference models, do as-rigid-as-possible shape interpolation.
Distance Measure

\[ \bar{d}(M_i, M_j) = \sqrt{\frac{\sum_{k=1}^{n} \| v_k^i - v_k^j \|^2}{n}} \]

\( v_k^i \): the \( k^{th} \) vertex of the \( i^{th} \) model \((M_i)\).

\( n \): the vertex number of the model

**Pre-alignment:** align models in a database using rigid transforms with the known correspondences.
Local Shape Spaces

• The existing database may be small.

• Upsampling:
  • Find models which are close enough and use them to interpolate models and add them to the collection.
Morphing

- Path Optimization
  - Shortest path (see more complex algorithm in the paper)
- Interpolation

\[ E_k = \sum_{i=1}^{N_R} \left( \sum_{j \in \Omega(i)} w_{ij} \left( \| \hat{v}_k^i - \hat{v}_k^j \| - R_k^i (v_k^i - v_k^j) \|^2 + \gamma \| \hat{v}_k^i - v_k^i \|^2 \right) + \|
\]

\[ w_k(t) = \exp(-\varepsilon |t - t_k|) \text{ where } t_k = \frac{k-1}{N_R-1} \]

Solver: Local/global

The number of models on the generated path.
Point set registration

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Point set registration

• The process of finding a spatial transformation that aligns two point sets.
• The purpose of finding such a transformation includes merging multiple data sets into a globally consistent model, and mapping a new measurement to a known data set to identify features or to estimate its pose.
Problem

• Input: two finite size point sets \( \{P, Q\} \), which contain \( M \) and \( N \) points.
• Output: a transformation to be applied to the moving “model” point set \( P \) such that the difference between \( P \) and the static “scene” set \( Q \) is minimized.

• The mapping may consist of a rigid or non-rigid transformation.
  • Rigid registration: translation and rotation
  • Non-rigid registration: affine transformations or any nonlinear transformation

For example: Spline
Challenges

• No correspondences.
• Noisy point cloud.
Iterative closest point (ICP)

https://en.wikipedia.org/wiki/Iterative_closest_point

1. \( \forall p_i \in P \), match the closest point in \( Q \), denoted as \( q_i \)

2. Estimate the rigid transformation that aligns the corresponding points as much as possible.

3. Iterate above two steps.
Estimation of rigid transformation

• Error:

\[ E(P, Q) = \sum_{(p_i, q_i)} \| p_i - q_i \|^2 \]

Compute rotation \( R \) and translation \( t \):

\[ E(P, Q) = \sum_{(p_i, q_i)} \| Rp_i + t - q_i \|^2 \]
Analytical solution

• Define $\mu_p = \frac{1}{n} \sum_{i=1}^{n} p_i$, $\mu_q = \frac{1}{n} \sum_{i=1}^{n} q_i$

$E(P, Q) = \sum_{i=1}^{n} \|Rp_i + t - q_i\|^2$

$= \sum_{i=1}^{n} \|Rp_i + t - q_i + R\mu_p - \mu_q - R\mu_p + \mu_q\|^2$

$= \sum_{i=1}^{n} \|R(p_i - \mu_p) - (q_i - \mu_q) + t + R\mu_p - \mu_q\|^2$
Analytical solution

\[ E(P_i, Q) = \sum_{i=1}^{n} \left\| R(p_i - \mu_p) - (q_i - \mu_q) \right\|^2 + \left\| t + R\mu_p - \mu_q \right\|^2 + 2(t + R\mu_p - \mu_q)^T (R(p_i - \mu_p) - (q_i - \mu_q)) \]

Since:

\[ \sum_{i=1}^{n} 2(t + R\mu_p - \mu_q)^T (R(p_i - \mu_p) - (q_i - \mu_q)) = 2(t + R\mu_p - \mu_q)^T \left( \sum_{i=1}^{n} R(p_i - \mu_p) - \sum_{i=1}^{n} (q_i - \mu_q) \right) = 0 \]
Analytical solution

\[ E(P, Q) = \sum_{i=1}^{n} \| R(p_i - \mu_p) - (q_i - \mu_q) \|^2 + \| t + R\mu_p - \mu_q \|^2 \]

No matter what \( R \) is got, set \( t = -R\mu_p + \mu_q \).

Thus,

\[ E(P, Q) = \sum_{i=1}^{n} \| R(p_i - \mu_p) - (q_i - \mu_q) \|^2 \]

\[ = \sum_{i=1}^{n} (p_i - \mu_p)^T R^T R (p_i - \mu_p) + \| (q_i - \mu_q) \|^2 - 2(q_i - \mu_q)^T R(p_i - \mu_p) \]

\[ = \sum_{i=1}^{n} (p_i - \mu_p)^T (p_i - \mu_p) + \| (q_i - \mu_q) \|^2 - 2(q_i - \mu_q)^T R(p_i - \mu_p) \]
Analytical solution

$$\arg \min_{R} E(P, Q)$$

$$= \arg \min_{R} \sum_{i=1}^{n} (p_i - \mu_P)^T (p_i - \mu_P) + \| (q_i - \mu_Q) \|^2 - 2(q_i - \mu_Q)^T R(p_i$$
Analytical solution

• If $M$ is a positive-symmetric-definite matrix then for any orthogonal $R$, $\text{tr}(M) > \text{tr}(RM)$.
• Proof: Set $M = AA^T$

$$\text{tr}(RM) = \text{tr}(RAA^T) = \text{tr}(A^T RA) = \sum a_i^T (Ra_i)$$

Schwarz inequality: $a_i^T (Ra_i) \leq \sqrt{a_i^T a_i (a_i R^T R a_i)} = a_i^T a_i = \text{tr}(M)$
Analytical solution

• Denote $H = \sum_{i=1}^{n} \left( (p_i - \mu_p)(q_i - \mu_q)^T \right) = U\Sigma V^T$.

Solve $\arg \max \, tr(2RH)$.

Set $X = VU^T$,

Then, $XH = V\Sigma V^T$

For any orthonormal matrix $B,$

$$\text{tr}(XH) \geq \text{tr}(BXH)$$

Thus,

$$VU^T = X = \arg \max_R tr(2RH)$$