形式化方法导引

第 2 章 经典数理逻辑-问题定义

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本章内容

- Define verification
  - $\mathcal{M} \models \phi$
- A method of define $\mathcal{M}$ and ϕ: Logics
  - Propositional logic
  - Predicate logic
  - Higher-order logic

Verifier —— Logic

回顾: 定义: Verifier

A *verifier* for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c \rangle \text{ for some string } c \}.$$

回顾: 验证过程

(1) 构建模型 w. **(2)** 设计规约 A. **(3)** (手动或自动) 构建证明 c
(4) 使用验证器 V, 输入 c, 输出是否$w \in A$

Verifier —— Logic

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定义: Verification in Logics

Most logics used in the design, specification and verification of computer systems fundamentally deal with a *satisfaction relation*:

$$\mathcal{M} \models \phi$$

- $\mathcal{M}$ is some sort of situation or *model* of a system
- ϕ is a *specification*, a formula of that logic, expressing what should be true in situation $\mathcal{M}$.
- At the *heart* of this set-up is that one can often *specify and implement algorithms* for computing $\models$.

Verifier —— Logic

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下一个问题:

- 问: 如何统一化定义 $\mathcal{M}$ 和 ϕ? 答: *Logics*
- 问: 如何支持 $\models$ 和 algorithms? 答: *Rules*

定义 $\mathcal{M}$ 和 ϕ? Logics

回顾: 如何定义一个问题? – 问题 3

Given a set S, a machine M, and $x \in S$, compute whether $x \in L(M)$.

- M is a *machine*, e.g., *finite automaton*.
- $L(M)$ is the *language* of M.

Define a special group of languages: Logics

- Propositional logic (命题逻辑)
- Predicate logic (谓词逻辑)
  - a.k.a., First-order Logic (一阶逻辑)
- Higher-Order Logic (高阶逻辑)

定义: Propositions (命题)

Declarative sentences which one can argue as being *true* or *false*, e.g.,

- The sum of the numbers 3 and 5 equals 8.
- Jane reacted violently to Jack's accusations.
- Every even natural number >2 is the sum of two prime numbers.
- All Martians like pepperoni on their pizza.

- 问题: 过于繁杂...
  - 解决方法: 从原子开始组建...

定义: Atomic Propositions (原子命题)

Propositions which is *indecomposable*, e.g.,

- The number 5 is even.

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定义: Atomic Propositions (原子命题)

Propositions which is *indecomposable*, e.g.,

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The logics we intend to design are *symbolic* in nature. We translate a certain sufficiently large subset of all English declarative sentences into strings of symbols. This gives us a *compressed* but still *complete* encoding of declarative sentences and allows us to concentrate on the mere mechanics of our argumentation. This is important since specifications of systems or software are sequences of such declarative sentences. It further opens up the possibility of *automatic manipulation of such specifications*, a job that computers just love to. Our strategy is to consider certain declarative sentences as being atomic, or indecomposable.

定义 $\mathcal{M}$ 和 ϕ? Logics

定义: Propositions (命题)

Declarative sentences which one can argue as being *true* or *false*.

定义: Atomic Propositions (原子命题)

Propositions which is *indecomposable*.

Symbols representing Atomic Propositions

We assign certain distinct symbols $p, q, r, ...$, or sometimes $p_1, p_2, p_3, ...$ to each of these atomic sentences

Symbols representing Logical Operators

$\{\neg, \vee, \wedge, \rightarrow\}$

- We can then code up more complex sentences in a compositional way

定义 $\mathcal{M}$ 和 ϕ? Logics

Preparation: given the following atomic sentences

- p: 'I won the lottery last week.'
- q: 'I purchased a lottery ticket.'
- r: 'I won last week's sweepstakes.'

Definition of the Logical Operators

$\neg$: *Negation*. $\neg p$ denotes *negation of* p

- i.e., it is *not true* that I won the lottery last week.

$\vee$: *Disjunction* (析取). $p \vee r$ denotes *at least one* of $\{p, r\}$ is true.

- i.e., I won the lottery last week, or I won last week's sweepstakes

$\wedge$: *Conjunction* (合取). $p \wedge r$ denotes *both* p and r are true.

- i.e., Last week I won the lottery and the sweepstakes.

$\rightarrow$: *Implication* (蕴含). $p \rightarrow q$ denotes q is a *logical consequence* of p.

- i.e., If I won the lottery last week, then I purchased a lottery ticket.

回顾: 自动机

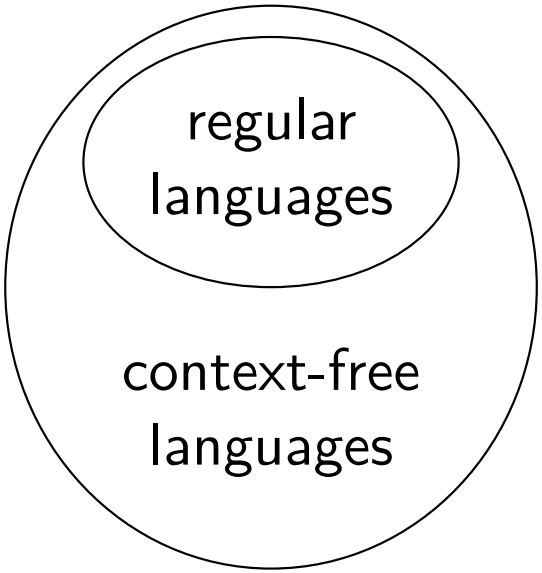
如何定义一个问题? – 问题 3

Given a set S, a machine M, and $x \in S$, compute whether $x \in L(M)$.

- M is a *machine*, e.g., *finite automaton*.
- $L(M)$ is the *language* of M.

M 的类型?

- regular languages
  - 例: 正则表达式匹配、词法分析
- *context-free languages*
  - 例: 语法分析



Define *Propositional logic* using a *context-free language*

- Backus-Naur Form (BNF) (巴科斯范式)

定义: Propositional Logic in *BNF*

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi)$$

where p stands for any atomic proposition and each occurrence of ϕ to the right of $::=$ stands for any already constructed formula.

Well-formed formula, 例:

$$(((\neg p) \wedge q) \rightarrow (p \wedge (q \vee (\neg r))))$$

Not well-formed formula, 例:

$$(\neg)() \vee pq \rightarrow$$

定义: Propositional Logic in *BNF*

$$\phi ::= p \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi)$$

where p stands for any atomic proposition and each occurrence of ϕ to the right of $::=$ stands for any already constructed formula.

Well-formed formula, 例:

$$(((\neg p) \wedge q) \to (p \wedge (q \vee (\neg r))))$$

Not well-formed formula, 例:

$$(\neg)() \vee pq \to$$

定义: The *well-formed* formulas of propositional logic are those which we obtain by using the construction rules below, and only those, finitely many times:

- atom: Every propositional atom $p, q, r, ...$ and $p_1, p_2, p_3, ...$ is a well-formed formula.
- $\neg$: If ϕ is a well-formed formula, then so is $(\neg\phi)$.
- $\wedge$: If ϕ and ψ are well-formed formulas, then so is $(\phi \wedge \psi)$.
- $\vee$: If ϕ and ψ are well-formed formulas, then so is $(\phi \vee \psi)$.
- $\to$: If ϕ and ψ are well-formed formulas, then so is $(\phi \to \psi)$.

- 问题: 怎样利用 atomic propositions 和 logical operators 来计算 ϕ?
  - 基本方法: 使用 Bool 真值表

定义: T and F

- The set of truth values contains two elements **T** and **F**, where **T** represents 'true'and **F** represents 'false'.
- A *valuation* or *model* of a formula ϕ is an assignment of each propositional atom in ϕ to a truth value.

ϕ	ψ	$\phi \wedge \psi$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

ϕ	ψ	$\phi \vee \psi$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

ϕ	ψ	$\phi \rightarrow \psi$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

ϕ	$\neg\phi$
T	**F**
F	**T**

$\top$
T

$\bot$
F

例：真值表

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$\phi = (p \to \neg q) \to (q \vee \neg p)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

回到主题: 对于如下 $\mathcal{M}$ 和 ϕ, 是否满足 $\mathcal{M} \models \phi$?

- $\mathcal{M} = \{p, q\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - Yes, 即 $\mathcal{M} \models \phi$
- $\mathcal{M} = \{p, \neg q\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - No, 即 $\mathcal{M} \not\models \phi$
- $\mathcal{M} = \{p\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - 无法确定，原因：公理不完备 (建模出现问题)
- $\mathcal{M} = \{p, \neg p\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - 公理存在矛盾 (建模出现问题)

定义 $\mathcal{M}$ 和 ϕ? Logics

回到主题: 对于如下 $\mathcal{M}$ 和 ϕ, 是否满足 $\mathcal{M} \vDash \phi$?

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  - Yes, 即 $\mathcal{M} \vDash \phi$
- $\mathcal{M} = \{p, \neg q\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - No, 即 $\mathcal{M} \vDash \neg\phi$
- $\mathcal{M} = \{p\}$, $\phi = (p \to \neg q) \to (q \vee \neg p)$
  - 无法确定，原因：公理不完备 (建模出现问题)
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定义: Semantic entailment relation

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to **T**, ψ evaluates to **T** as well, we say that

$$\phi_1, \phi_2, \ldots, \phi_n \vDash \psi$$

holds and call $\vDash$ the *semantic* entailment relation.

例：真值表

p	q	$\neg p$	$\neg q$	$p \to \neg q$	$q \vee \neg p$	$\phi = (p \to \neg q) \to (q \vee \neg p)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

复杂度?

- 若 $\mathcal{M}$ 原子命题的个数为 n, 判定所需时间为 $O(2^n)$.
- 怎么办?

定义 $\mathcal{M}$ 和 ϕ? Logics

问题: Consider the declarative sentence:

- *Every* student is younger than *some* instructor.
  - How to define when there are 1,000,000,000 students?
  - Moreover, how to specify an instructor for each student?

解决方法: Design a richer *language* (*logic*):

- Predicate Logic (谓词逻辑), a.k.a, *First-order Logic (一阶逻辑)*
  - *Inherit* Propositional Logic
  - Introduce *Predicate*
    - $S(andy)$ to denote that Andy is a student.
    - $I(paul)$ to say that Paul is an instructor.
    - $Y(andy, paul)$ could mean that Andy is younger than Paul.
    - The symbols S, I and Y are called *predicates*.
  - Introduce *quantifiers* $\forall$ and $\exists$
    - $\forall$: for all, $\exists$: there exists

答案: $\forall x\ (S(x) \rightarrow (\exists y\ (\ I(y) \wedge Y(x, y)\ )))$

定义 $\mathcal{M}$ 和 ϕ? Logics
2. First-order Logic | Introduction

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We begin this second chapter by pointing out the limitations of propositional logic with respect to encoding declarative sentences. Propositional logic dealt quite satisfactorily with sentence components like not, and, or and if ... then, but the logical aspects of natural and artificial languages are much richer than that. What can we do with modifiers like there exists ..., all ..., among ... and only ...? Here, propositional logic shows clear limitations and the desire to express more subtle declarative sentences led to the design of predicate logic, which is also called first-order logic.

In propositional logic, we could identify this assertion with a propositional atom p. However, that fails to reflect the finer logical structure of this sentence. What is this statement about? Well, it is about being a student, being an instructor and being younger than somebody else. These are all properties of some sort, so we would like to have a mechanism for expressing them together with their logical relationships and dependences.

定义: Term

- Any variable is a term.
- If $c \in \mathcal{F}$ is a nullary function, then c is a term.
- If $t_1, t_2, ..., t_n$ are terms and $f \in \mathcal{F}$ has arity $n > 0$, then $f(t_1, t_2, ..., t_n)$ is a term.
- Nothing else is a term.

定义: Term in *BNF*

$$t ::= x \mid c \mid f(t, ..., t)$$

where x ranges over a set of variables var, c over nullary function symbols in $\mathcal{F}$, and f over those elements of $\mathcal{F}$ with arity $n > 0$.

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A predicate vocabulary consists of three sets: a set of predicate symbols P, a set of function symbols $\mathcal{F}$ and a set of constant symbols C. Each predicate symbol and each function symbol comes with an arity, the number of arguments it expects. In fact, constants can be thought of as functions which don't take any arguments (and we even drop the argument brackets) —therefore, constants live in the set $\mathcal{F}$ together with the 'true'functions which do take arguments. From now on, we will drop the set C, since it is convenient to do so, and stipulate that constants are 0-arity, so-called *nullary*, functions.

定义: Term in *BNF*

$$t ::= x \mid c \mid f(t, ..., t)$$

where x ranges over a set of variables var, c over nullary function symbols in $\mathcal{F}$, and f over those elements of $\mathcal{F}$ with arity $n > 0$.

定义: First-order Logic in *BNF*

$$\phi ::= P(t_1, t_2, \ldots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\forall x\ \phi) \mid (\exists x\ \phi)$$

where $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, t_i are terms over $\mathcal{F}$ and x is a variable.

定义: Term in *BNF*

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where $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, t_i are terms over $\mathcal{F}$ and x is a variable.

定义: We define the set of formulas over (F, P) inductively, using the already defined set of terms over F:

- If $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, and if $t_1, t_2, ..., t_n$ are terms over $\mathcal{F}$, then $P(t_1, t_2, ..., t_n)$ is a formula.
- If ϕ is a formula, then so is $(\neg\phi)$.
- If ϕ and ψ are formulas, then so are $(\phi \wedge \psi)$, $(\phi \vee \psi)$ and $(\phi \to \psi)$.
- If ϕ is a formula and x is a variable, then $(\forall x\ \phi)$ and $(\exists x\ \phi)$ are formulas.
- Nothing else is a formula.

Recall that each occurrence of ϕ on the right-hand side of the $::=$ stands for any formula already constructed by these rules.

回到主题: 如何定义 $\mathcal{M}$?

定义: $\mathcal{M}$

Let $\mathcal{F}$ be a set of function symbols and $\mathcal{P}$ a set of predicate symbols, each symbol with a fixed number of required arguments. A *model* $\mathcal{M}$ of the pair $(\mathcal{F}, \mathcal{P})$ consists of the following set of data:

1. A non-empty set A, the universe of *concrete* values
2. for each nullary function symbol $f \in \mathcal{F}$, a *concrete* element $f^{\mathcal{M}}$ of A
3. for each $f \in \mathcal{F}$ with arity $n > 0$, a *concrete* function $f^{\mathcal{M}} : A^n \to A$ from A^n, the set of n-tuples over A, to A
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a subset $P^{\mathcal{M}} \subseteq A^n$ of n-tuples over A.

形式化方法导引

└定义 $\mathcal{M}$ 和 ϕ? Logics

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这里的定义看起来比较难懂。如果用类比的方法 (跟 C 语言类比)，就比较容易懂了：

- C 语言中，构建的.c 文件，其中 main() 函数就相当于 $\mathcal{M}$.
- $\mathcal{F}, \mathcal{P}$ 相当于声明, 比如 struct 结构体的声明
- A 相当于 enum, 同时包括 int32,long64 取值范围的设定
- $f^{\mathcal{M}}$, $P^{\mathcal{M}}$相当于 main() 中具体的语句

定义 $\mathcal{M}$ 和 ϕ? Logics

回到主题: 如何定义 $\mathcal{M}$?

例 (自动机):

Let $\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ and $\mathcal{P} = \{R, F\}$;

- i is a constant
- F a predicate symbol with one argument
- R a predicate symbol with two arguments

A model $\mathcal{M}$ may contain:

- A: a set of states of a computer program.
- $i^{\mathcal{M}}$: a designated initial state.
- $R^{\mathcal{M}}$: a state transition relation.
- $F^{\mathcal{M}}$: a set of final (accepting) states.

$\mathcal{M}$ 的实例:

- $A \stackrel{\text{def}}{=} \{a, b, c\}$
- $i^{\mathcal{M}} = a$
- $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$
- $F^{\mathcal{M}} = \{b, c\}$.

ϕ 的实例:

- $\exists y\ R(i, y)$
- $\neg F(i)$
- $\forall x \forall y \forall z\ (R(x, y) \wedge R(x, z) \rightarrow y = z)$
- $\forall x \exists y\ R(x, y)$

定义 $\mathcal{M}$ 和 ϕ? Logics
2. First-order Logic | Define $\mathcal{M}$
回到主题: 如何定义 $\mathcal{M}$?
例 (自动机):
Let $\mathcal{F} \stackrel{\text{def}}{=} \{i\}$ and $\mathcal{P} = \{R, F\}$;
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$\mathcal{M}$ 的实例:
- $A \stackrel{\text{def}}{=} \{a, b, c\}$
- $i^{\mathcal{M}} = a$
- $R^{\mathcal{M}} \stackrel{\text{def}}{=} \{(a, a), (a, b), (a, c), (b, c), (c, c)\}$
- $F^{\mathcal{M}} = \{b, c\}$.

ϕ 的实例:
- $\exists y\ R(i, y)$
- $\neg F(i)$
- $\forall x \forall y \forall z\ (R(x, y) \wedge R(x, z) \rightarrow y = z)$
- $\forall x \exists y\ R(x, y)$

- $\exists y\ R(i, y)$: there is a transition from the initial state to some state; this is true in our model, as there are transitions from the initial state a to a, b, and c.
- $\neg F(i)$: the initial state is not a final, accepting state. This is true in our model as b and c are the only final states and a is the intitial one.
- $\forall x \forall y \forall z\ (R(x, y) \wedge R(x, z) \rightarrow y = z)$: makes use of the equality predicate and states that the transition relation is *deterministic*: all transitions from any state can go to at most one state (there may be no transitions from a state as well). This is false in our model since state a has transitions to b and c.
- $\forall x \exists y\ R(x, y)$: the model is free of states that deadlock: all states have a transition to some state. This is true in our model: a can move to a, b or c; and b and c can move to c.

回到主题: 对于给定 $\mathcal{M}$ 和 ϕ, 是否满足 $\mathcal{M} \vDash \phi$?
基本方法: 类似 Propositional Logic, 枚举所有情况

1. 定义 Environment l
2. 定义 $\vDash_l$
3. 枚举 $\vDash_l$ 求解 $\vDash$

定义: Environment l

- l : var $\to A$
  - Type: from the set of variables var to A
  - A look-up table or environment for a universe A of concrete values
- $l[x \mapsto a]$
  - the look-up table
  - maps x to a and any other variable y to $l(y)$

定义: Environment l

- l : var $\rightarrow A$
  - A look-up table or environment for a universe A of concrete values

定义 $\models_l$

Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given an environment l, we define the satisfaction relation $\mathcal{M} \models_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ.

If $M \models_l \phi$ holds, we say that ϕ computes to T in the model M with respect to the environment l.

定义 $\mathcal{M} \models \phi$

$\mathcal{M} \models \phi$ holds, iff for all choices of l, $\mathcal{M} \models_l \phi$

定义: Environment l

- l : var $\to A$
  - A look-up table or environment for a universe A of concrete values

定义 $\vDash_l$

Given a model $\mathcal{M}$ for a pair $(\mathcal{F}, \mathcal{P})$ and given an environment l, we define the satisfaction relation $\mathcal{M} \vDash_l \phi$ for each logical formula ϕ over the pair $(\mathcal{F}, \mathcal{P})$ and look-up table l by structural induction on ϕ.

If $M \vDash_l \phi$ holds, we say that ϕ computes to T in the model M with respect to the environment l.

定义 $\mathcal{M} \vDash \phi$

$\mathcal{M} \vDash \phi$ holds, iff for all choices of l, $\mathcal{M} \vDash_l \phi$

P: If ϕ is of the form $P(t_1, t_2, ..., t_n)$, then we interpret the terms $t_1, t_2, ..., t_n$ in our set A by replacing all variables with their values according to l. In this way we compute concrete values $a_1, a_2, ..., a_n$ of A for each of these terms, where we interpret any function symbol $f \in \mathcal{F}$ by $f^{\mathcal{M}}$. Now $M \vDash_l P(t_1, t_2, ..., t_n)$ holds iff $(a_1, a_2, ..., a_n)$ is in the set $P^{\mathcal{M}}$.

$\forall x$: The relation $M \vDash_l \forall x\ \psi$ holds iff $M \vDash_{l[x \mapsto a]}\ \psi$ holds for all $a \in A$.

$\exists x$: $\mathcal{M} \vDash_l \exists x\ \psi$ holds iff $\mathcal{M} \vDash_l \psi$ holds for some $a \in A$.

$\neg$: The relation $\mathcal{M} \vDash_l \neg\psi$ holds iff it is not the case that $\mathcal{M} \vDash_l \psi$ holds.

$\vee$: The relation $\mathcal{M} \vDash_l \psi_1 \vee \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ or $\mathcal{M} \vDash_l \psi_2$ holds.

$\wedge$: The relation $\mathcal{M} \vDash_l \psi_1 \wedge \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_1$ and $\mathcal{M} \vDash_l \psi_2$ holds.

$\to$: The relation $\mathcal{M} \vDash_l \psi_1 \to \psi_2$ holds iff $\mathcal{M} \vDash_l \psi_2$ holds whenever $\mathcal{M} \vDash_l \psi_1$ holds.

头疼的问题: 计算复杂度相比于命题逻辑的复杂度似乎更大
更头疼的问题: 先考虑可计算性?

回顾: 问题可以解么? – 问题 4

Given a set $A \subseteq S$, and $x \in S$, *whether there is a machine* that can compute whether $x \in A$.

- Define a new machine, named *Turing machine,* 图灵机.
- If yes, i.e., there is a Turing machine M for A, language A is *decidable*.
- If no, but there is a Turing machine M that can only accept s, if $s \in A$, language A is *still* *Turing-recognizable*.

定理 (*Undecidability* in First-order logic)

The decision problem of validity in predicate logic is *undecidable*: no program exists which, given any ϕ, decides whether $\models \phi$.

形式化方法导引

└定义 $\mathcal{M}$ 和 ϕ? Logics

定义 $\mathcal{M}$ 和 ϕ? Logics
2. First-order Logic | Evaluation

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定理 (*Undecidability* in First-order logic)

The decision problem of validity in predicate logic is *undecidable*: no program exists which, given any ϕ, decides whether $\vDash \phi$.

针对非确定性问题，可以参考两本书（讲述同一块内容）：Post correspondence problem (PCP)

- Page 132 in 《Logic in Computer Science
  - 这里简单介绍了例子
- Page 227 in 《Introduction to the theory of Compuation》
  - 这里介绍了证明方法

另一个问题: First-order Logic 的表达能力?

- 能表达所有问题么?

解答: 考虑一个反例——有向图 (directed graph) 的建模

- Software models, design standards, and execution models of hardware or programs often are described in terms of directed graphs.

反例:

Given a set of states $A = \{s_0, s_1, s_2, s_3\}$, let $R^{\mathcal{M}}$ be the set $\{(s_0, s_1), (s_1, s_0), (s_1, s_1), (s_1, s_2), (s_2, s_0), (s_3, s_0), (s_3, s_2)\}$. We may depict this model as *a directed graph* in a figure, where an edge (a transition) leads from a node s to a node s' iff $(s, s') \in R^{\mathcal{M}}$.

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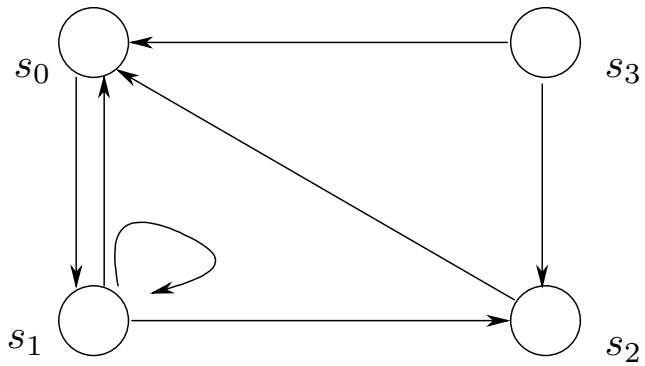
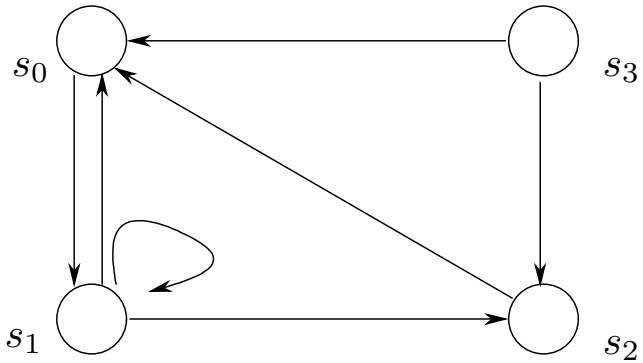


反例: How to define Reachability as ϕ

Given nodes n and n' in a directed graph, is there a finite path of transitions from n to n'?



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Given nodes n and n' in a directed graph, is there a finite path of transitions from n to n'?

反例：一种答案

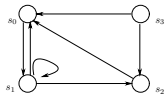
$$(u = v) \vee \exists x(R(u, x) \wedge R(x, v)) \vee \exists x_1 \exists x_2(R(u, x_1) \wedge R(x_1, x_2) \wedge R(x_2, v)) \vee \ldots$$

- This is infinite, so it's *not* a well-formed formula.
- Can we find a well-formed formula with the same meaning? *No!*

形式化方法导引

└定义 $\mathcal{M}$ 和 ϕ? Logics

详细证明见 P-137 in 《Logic in Computer Science》



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进一步问题: 既然 First-order Logic 不能表达 ϕ，那怎么表达

- 使用 Second-order Logic
- 怎么用?
  - This can be realized by applying *quantifiers* not only to variables, but also to *predicate symbols*.

回顾: 定义: First-order Logic in *BNF*

$$\phi ::= P(t_1, t_2, \ldots, t_n) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \mid (\forall x\ \phi) \mid (\exists x\ \phi)$$

where $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, t_i are terms over $\mathcal{F}$ and x is a variable.

回顾: 定义: First-order Logic in *BNF*

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where $P \in \mathcal{P}$ is a predicate symbol of arity $n \geq 1$, t_i are terms over $\mathcal{F}$ and x is a variable.

解决思路

For a predicate symbol P with $n \geq 1$ arguments, consider formulas of the form:

$$\exists P\ \phi$$

where ϕ is a formula of predicate logic in which P *occurs*.

反例：一种答案 First-order Logic (Not well-formed)

$$(u = v) \vee \exists x(R(u, x) \wedge R(x, v)) \vee \exists x_1 \exists x_2(R(u, x_1) \wedge R(x_1, x_2) \wedge R(x_2, v)) \vee \ldots$$

具体答案: Second-order Logic

$$\neg \exists P \forall x \forall y \forall z \ (C_1 \wedge C_2 \wedge C_3 \wedge C_4)$$

where

$$C_1 \stackrel{\text{def}}{=} P(x, x)$$

$$C_2 \stackrel{\text{def}}{=} P(x, y) \wedge P(y, z) \rightarrow P(x, z)$$

$$C_3 \stackrel{\text{def}}{=} P(u, v) \rightarrow \bot$$

$$C_4 \stackrel{\text{def}}{=} R(x, y) \rightarrow P(x, y)$$

下一个问题: 有没有 Third-order Logic, Fourth-order Logic,...?
答案: 有

- First-order logic quantifies only variables that range over individuals
- Second-order logic, in addition, also quantifies *over sets*
  - e.g., we can define $P(x, y) \stackrel{\text{def}}{=} (x, y) \in \mathbf{P}$, where $\mathbf{P}$ is a set.
- Third-order logic also quantifies over sets of sets, and so on.

Higher-order logic is the union of first-, second-, third-, ..., nth-order logic

- i.e., higher-order logic admits quantification over sets that are nested arbitrarily deeply.

作业

a. Compute the complete truth table of the formula:

1. $((p \to q) \to p) \to p$
2. $(p \wedge q) \to (p \vee q)$
3. $(p \to q) \vee (p \to \neg q)$
4. $((p \vee q) \to r) \to ((p \to r) \vee (q \to r))$

作业

1. Use the predicates

$$\begin{aligned} A(x,y): &\quad x \text{ admires } y \\ B(x,y): &\quad x \text{ attended } y \\ P(x): &\quad x \text{ is a professor} \\ S(x): &\quad x \text{ is a student} \\ L(x): &\quad x \text{ is a lecture} \end{aligned}$$

and the nullary function symbol (constant)

$$m: \quad \text{Mary}$$

to translate the following into predicate logic:

(a) Mary admires every professor.
(The answer is not $\forall x\, A(m, P(x))$.)
(b) Some professor admires Mary.
(c) Mary admires herself.
(d) No student attended every lecture.
(e) No lecture was attended by every student.
(f) No lecture was attended by any student.

作业

2. Consider the sentence $\phi \stackrel{\text{def}}{=} \forall x\, \exists y\, \exists z\, (P(x,y) \wedge P(z,y) \wedge (P(x,z) \rightarrow P(z,x)))$. Which of the following models satisfies ϕ?
   (a) The model $\mathcal{M}$ consists of the set of natural numbers with $P^{\mathcal{M}} \stackrel{\text{def}}{=} \{(m,n) \mid m < n\}$.
   (b) The model $\mathcal{M}'$ consists of the set of natural numbers with $P^{\mathcal{M}'} \stackrel{\text{def}}{=} \{(m, 2 * m) \mid m \text{ natural number}\}$.
   (c) The model $\mathcal{M}''$ consists of the set of natural numbers with $P^{\mathcal{M}''} \stackrel{\text{def}}{=} \{(m,n) \mid m < n+1\}$.