形式化方法导引

第 4 章 逻辑问题求解

4.2 理论 - (2) SMT 求解

黄文超

http://staff.ustc.edu.cn/~huangwc/fm.html

回顾: 定义: SAT 问题

SAT is the *decision* problem: given a propositional formula, is it *satisfiable*?

回顾: 定义: SMT problem

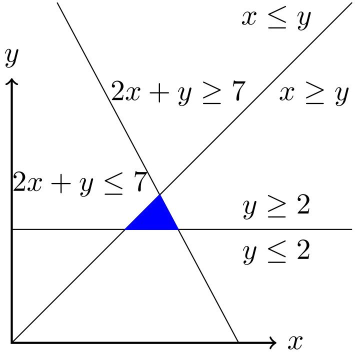
Extension of SAT, to deal with *numbers* and *inequalities*.

回顾: SMT 在软件测试中的一个重要应用: 符号执行 (*Symbolic Execution*) 问题: How to explore different program paths and for each path to

- *generate* a set of concrete *input* values exercising that path
- *check* for the presence of *various kinds* of errors

本节讲解内容: The Simplex method (单纯形法).

- dealing with *linear inequalities*



The Simplex method (单纯形法).

- dealing with *linear inequalities*

Do real numbers x, y exist such that

$$x \geq y$$
$$y \geq 2$$
$$2x + y \leq 7$$

So indeed the *blue* part describes the values x, y satisfying the requirements.

问题: How to solve for >2 variables?

分析: *No such pictures*: we want to do this for *thousands* of inequalities /variables

For SMT the underlying approach is the *simplex method* for *linear optimization* = *linear programming* (线性规划)

- Not only determines existence of solution, but finds an *optimal*
  - one for which a given linear expression has the *highest possible* value

问题定义: The Simplex Method

Among all real values $x_1, \ldots, x_n \geq 0$ find the *maxmal* value of a linear *goal function*

$$v + c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

satisfying k linear constraints

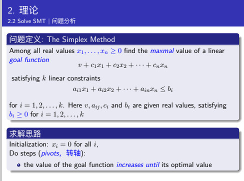
$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, 2, \ldots, k$. Here v, a_{ij}, c_i and b_i are given real values, satisfying $b_i \geq 0$ for $i = 1, 2, \ldots, k$

求解思路

Initialization: $x_i = 0$ for all i,
Do steps (*pivots*, 转轴):

- the value of the goal function *increases until* its optimal value



So $x_i \geq 0$, and choosing $x_i = 0$ for all i satisfies all constraints and yields the value v for the goal function

Probably this is not the maximal value

Approach of the simplex method:
Start from this solution with $x_i = 0$ for all i, do steps:

- the value of the goal function increases until its optimal value

Applies to more general format:

- in an inequality $\geq$, multiply both sides by -1:

$$x_1 - 2x_2 + 3x_3 \geq -5 \equiv -x_1 + 2x_2 - 3x_3 \leq 5$$

- an equality $A = B$ is replaced by two inequalities $A \leq B$ and $B \leq A$
- if one wants to minimize, multiply goal function by -1
- if a variable x runs over all reals (positive and negative), replace it by $x_1 - x_2$ for fresh variables x_1, x_2 satisfying $x_i \geq 0$ for $i = 1, 2$

Later we will see how to deal if there is no trivial start solution (in case b_i<0).

定义: Slack form 松弛型

For every linear inequality

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

a fresh variable $y_i \geq 0$ is introduced

The linear inequality is *replaced* by the equality

$$y_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n$$

Together with $y_i \geq 0$, this is equivalent to the original inequality

This format with equalities is called the *slack form*

定义: basic/non-basic variables, basic solution

Some terminology on a slack form with equations

$$y_i = b_i - a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n$$

for $i = 1, 2, \ldots, k$

- The solution $y_i = b_i$, for $i = 1, 2, \ldots, k$ and $x_j = 0$ for $j = 1, 2, \ldots, k$ is called the *basic solution*
- The variables y_i for $i = 1, 2, \ldots, k$ are called *basic variables* (基变量)
- The variables x_j for $j = 1, 2, \ldots, k$ are called *non-basic variables* (非基变量)

算法思路: The simplex algorithm consists of a *repetition* of pivots

- A *pivot chooses* a *basic* variable and a non-basic variable, *swaps* their roles, and makes a *new slack form* that is *equivalent* to the original one, but with a *higher value* for the goal function.

回顾: 问题定义: The Simplex Method

Among all real values $x_1, \ldots, x_n \geq 0$ find the *maxmal* value of a linear *goal function*

$$v + c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

satisfying k linear constraints

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, 2, \ldots, k$.

例：

Maximize $z = 3 + x_1 + x_3$ satisfying the constraints:

$$\begin{aligned} -x_1 + x_2 - x_3 &\leq 2 \\ x_1 + x_3 &\leq 3 \\ 2x_1 - x_2 &\leq 4 \end{aligned}$$

例：

Maximize $z = 3 + x_1 + x_3$ satisfying the constraints:

$$\begin{aligned} -x_1 + x_2 - x_3 &\leq 2 \\ x_1 + x_3 &\leq 3 \\ 2x_1 - x_2 &\leq 4 \end{aligned}$$

步骤 0: Slack form 松弛型

$$\begin{aligned} y_1 &= 2 + x_1 - x_2 + x_3 \\ y_2 &= 3 - x_1 - x_3 \\ y_3 &= 4 - 2x_1 + x_2 \end{aligned}$$

Goal: Maximize $z = 3 + x_1 + x_3$, while keeping $\forall i.x_i \geq 0$ and $\forall i.y_i \geq 0$

步骤 0: Slack form 松弛型

$$\begin{aligned} y_1 &= 2 + x_1 - x_2 + x_3 \\ y_2 &= 3 - x_1 - x_3 \\ y_3 &= 4 - 2x_1 + x_2 \end{aligned}$$

Goal: Maximize $z = 3 + x_1 + x_3$, while keeping $\forall i.x_i \geq 0$ and $\forall i.y_i \geq 0$

步骤 1: basic solution

$$x_1 = x_2 = x_3 = 0$$

$$y_1 = 2, \; y_2 = 3, \; y_3 = 4$$

Goal: Maximize $z = 3 + x_1 + x_3$, while keeping $\forall i.x_i \geq 0$ and $\forall i.y_i \geq 0$

Observation: if we increase x_1, and x_2 and x_3 remain 0, then the goal function z will increase

分析: We want to *increase* x_1 as much as possible, keeping $x_2 = x_3 = 0$, while in the equations

$$
\begin{aligned}
y_1 &= 2 + x_1 - x_2 + x_3 \\
y_2 &= 3 - x_1 - x_3 \\
y_3 &= 4 - 2x_1 + x_2
\end{aligned}
$$

keeping $\forall i.x_i \geq 0$ and $\forall i.y_i \geq 0$

- $y_1 = 2 + x_1 \geq 0$: OK if x_1 increases
- $y_2 = 3 - x_1 \geq 0$: *only* OK if $x_1 \leq 3$
- $y_3 = 4 - 2x_1 \geq 0$: *only* OK if $x_1 \leq 2$

So $y_i \geq 0$ only holds for all i if $x_1 \leq 2$

The highest allowed value for x_1 is 2, and then y_3 will get the value 0

Highest allowed value for x_1 is 2, then y_3 will be 0

步骤 2: *pivot*: swap x_1 and y_3

Recall that

x_1 is *non-basic*: right from '$=$',$=$0 in basic solution

y_3 is *basic*: left from '$=$', possibly $\geq$ 0 in basic solution

By the *pivot*

- the *non-basic* variable x_1 will become *basic*
- the *basic* variable y_3 will become *non-basic*

问题: How to implement *pivot*? (见下页)

pivot: swap x_1 and y_3 in

$$\begin{aligned} y_1 &= 2 + x_1 - x_2 + x_3 \\ y_2 &= 3 - x_1 - x_3 \\ y_3 &= 4 - 2x_1 + x_2 \end{aligned}$$

In the equation $y_3 = 4 - 2x_1 + x_2$ move x_1 to the left and y_3 to the right:

$$x_1 = 2 + \frac{1}{2}x_2 - \frac{1}{2}y_3$$

Next replace every x_1 by $2 + \frac{1}{2}x_2 - \frac{1}{2}y_3$ in the equations for z, y_1, y_2, yielding the following *slack form*:

maximize $z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$

$$\begin{aligned} x_1 &= 2 + \frac{1}{2}x_2 - \frac{1}{2}y_3 \\ y_1 &= 4 - \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3 \\ y_2 &= 1 - \frac{1}{2}x_2 - x_3 + \frac{1}{2}y_3 \end{aligned}$$

This is the *end* of the *first pivot*

Observe

- all equations are replaced by *equivalent* equations, so the optimization problem is *equivalent* to the original one
- Now the *basic variables* are x_1, y_1, y_2 and the *non-basic variables* are x_2, x_3, y_3
- By construction again, we have a *slack form* with a *basic solution* in which *all non-basic variables are 0*, and *all basic variables are* ≥ 0
- In this new basic solution the goal function

$$z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$$

has *value 5*, *improving* the original *value 3*

In this goal function

$$z = 5 + \frac{1}{2}x_2 + x_3 - \frac{1}{2}y_3$$

the non-basic variable x_2 has a *positive* factor $\frac{1}{2}$: increasing x_2 is the goal of the *next pivot*, while the other non-basic variables remain 0

$$\begin{aligned} x_1 &= 2 + \tfrac{1}{2}x_2 - \tfrac{1}{2}y_3 \\ y_1 &= 4 - \tfrac{1}{2}x_2 + x_3 - \tfrac{1}{2}y_3 \\ y_2 &= 1 - \tfrac{1}{2}x_2 - x_3 + \tfrac{1}{2}y_3 \end{aligned}$$

$x_1 = 2 + \frac{1}{2}x_2 \geq 0.$ OK if x_2 increases
$y_1 = 4 - \frac{1}{2}x_2 \geq 0.$ only OK if $x_2 \leq 8$
$y_2 = 1 - \frac{1}{2}x_2 \geq 0,$ only OK if $x_2 \leq 2$
So the maximal allowed value for x_2 is 2, in which case y_2 will get the value 0 $\Longrightarrow$ *pivot swapping x_2 and y_2*

步骤 3: *pivot swapping x_2 and y_2*

$$y_2 = 1 - \frac{1}{2}x_2 - x_3 + \frac{1}{2}y_3$$

yields $x_2 = 2 - 2x_3 - 2y_2 + y_3$, so in the goal function and in the other equations we replace x_2 by $2 - 2x_3 - 2y_2 + y_3$, yielding
Maximize $z = 6 - y_2$
satisfying

$$\begin{aligned} x_1 &= 3 - x_3 - y_2 \\ x_2 &= 2 - 2x_3 - 2y_2 + y_3 \\ y_1 &= 3 + 2x_3 + y_2 - y_3 \end{aligned}$$

Observation: since in

$$z = 6 - y_2$$

y_2 should be ≥ 0, the value z will always be ≤ 6

On the other hand, in the basic solution with

$$x_3 = y_2 = y_3 = 0$$

and

$$x_1 = 3, x_2 = 2, y_1 = 3$$

we obtain $z = 6$, so this basic solution yields the *maximal value* for z

算法: Simplex Method (单纯形法)

Solve a linear optimization problem in slack form maximize $z = v + \sum_{j=1}^{n} c_j x_j$ under a set of constraints of the shape

$$y_i = b_i + \sum_{j=1}^{n} a_{ij} x_j$$

with $b_i \geq 0$,for $i = 1, \ldots, k$

As long there exists j such that $c_j > 0$ do a *pivot*, that is

- find the highest value for x_j for which $b_i + a_{ij} x_j \geq 0$ for all i, and $b_i + a_{ij} x_j = 0$ for one particular i
- swap x_j and y_i and bring the result in slack form

At the end $c_j \leq 0$ for all j, from which can be concluded that the basic solution of this last slack form yields the maximal value for z

General remarks

- Optimization problems may be *unbounded*; in this mechanism this will be encountered if no equation yields an upper bound on x_j
- A pivot only requires complexity O(kn)
- If $c_j > 0$ for more than one value of j, then the procedure is non-deterministic
- Always choose smallest j with $c_j > 0$: then repetition of pivots will terminate
- Worst case: number of pivots may be *exponential*.
  - In practice the simplex method is very efficient

新问题：Until now: *only* find *optimal value* when starting by *basic solution* However, for SMT the situation is *opposite*: *no interest* in *optimal solution*, only in *existence* of *a solution*

定义：feasible

A set of constraints is called *feasible* if it admits a solution.

问题：how to apply the *simplex method* presented so far

- to determine *feasibility* of *any* given set of inequalities

问题定义：feasibility

For a set of linear inequalities

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, \ldots, k$, and $x_j \geq 0$ for $j = 1, \ldots, n$.
Note: (not all $b_i \geq 0$)

问题定义: feasibility

For a set of linear inequalities

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$$

for $i = 1, \ldots, k$, and $x_j \geq 0$ for $j = 1, \ldots, n$.
Note: (not all $b_i \geq 0$)

方法思路: introduce a *fresh variable* $z \geq 0$
Extend the set of inequalities to

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - z \leq b_i$$

for $i = 1, \ldots, k$

The extended problem

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - z \leq b_i$$

is *always* feasible, even if $b_i \leq 0$: choose z very *large*

定理

Original problem feasible $\Leftrightarrow$ maximal value of $-z$ is 0 in extended problem

证明:
($\Leftarrow$) If extended problem has solution with $-z = 0$, then it is solution of the original problem, so feasible
($\Rightarrow$) If original problem feasible, then extended one has solution with $-z = 0$

- Since $z \geq 0$, this is the maximal value for $-z$

定理

Original problem feasible ⇔ maximal value of $-z$ is 0 in extended problem

So now we want to *maximize $-z$* in the extended problem by the *simplex method*

例: 问题: Check feasibility

Find values $x, y \geq 0$ satisfying

$$-x - 3y \leq -12$$
$$x + y \leq 10$$
$$-x + y \leq -7$$

步骤 1:

Introduce z for maximizing $-z$:

$$-x - 3y - z \leq -12$$
$$x + y - z \leq 10$$
$$-x + y - z \leq -7$$

例: 问题: Check feasibility

Fine values $x, y \geq 0$ satisfying

$$-x - 3y \leq -12$$
$$x + y \leq 10$$
$$-x + y \leq -7$$

步骤 1:

Introduce z for maximizing $-z$:

$$-x - 3y - z \leq -12$$
$$x + y - z \leq 10$$
$$-x + y - z \leq -7$$

步骤 2: Slack form:

Maximize $-z$ satisfying

$$y_1 = -12 + x + 3y + z$$
$$y_2 = 10 - x - y + z$$
$$y_3 = -7 + x - y + z$$

问题分析: Since b_i may be negative, we may have *no* basic solution with all variables ≥ 0

解决技巧: As the first pivot swap, the non-basic variable z with the basic variable y_i for i for which b_i is the *most negative*

步骤 2:

Maximize $-z$ satisfying

$$y_1 = -12 + x + 3y + z$$
$$y_2 = 10 - x - y + z$$
$$y_3 = -7 + x - y + z$$

Indeed $x = y = z = 0$ *does not yield a basic solution*, since then $y_1 = -12$ and $y_3 = -7$ *do not satisfy* $y_i \geq 0$

Indeed now we have a *basic solution*:

$x = y = y_1 = 0$,
$z = 12, y_2 = 22, y_3 = 5$,
all ≥ 0

Most negative b_i is $b_1 = -12$, so do pivot on z and y_1

步骤 3: pivot z and y, replace every $z = 12 - x - 3y + y_1$

Maximize $-12 + x + 3y - y_1$ satisfying

$$z = 12 - x - 3y + y_1$$
$$y_2 = 22 - 2x - 4y + y_1$$

步骤 3: pivot z and y, replace every $z = 12 - x - 3y + y_1$

Maximize $-12 + x + 3y - y_1$ satisfying

$$z = 12 - x - 3y + y_1$$
$$y_2 = 22 - 2x - 4y + y_1$$
$$y_3 = 5 - 4y + y_1$$

From now on: proceed by simplex algorithm as before

Maximize $-12 + x + 3y - y_1$, so swap x with z, y_2 or y_3

步骤 4: pivot: swap x with y_2

Maximize $-1 + y - \frac{1}{2}y_1 - \frac{1}{2}y_2$ satisfying

$$x = 11 - 2y + \frac{1}{2}y_1 - \frac{1}{2}y_2$$
$$z = 1 - y + \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$y_3 = 5 - 4y + y_1$$

步骤 4: pivot: swap x with y_2

Maximize $-1 + y - \frac{1}{2}y_1 - \frac{1}{2}y_2$ satisfying

$$x = 11 - 2y + \frac{1}{2}y_1 - \frac{1}{2}y_2$$
$$z = 1 - y + \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$y_3 = 5 - 4y + y_1$$

步骤 5: pivot: swap y with z

Maximize $-z$ satisfying

$$x = 9 \cdots z \cdots y_1 \cdots y_2$$
$$y = 1 - z + \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$y_3 = 1 \cdots z \cdots y_1 \cdots y_2$$

步骤 5: pivot: swap y with z

Maximize $-z$ satisfying

$$x = 9 \cdots z \cdots y_1 \cdots y_2$$
$$y = 1 - z + \frac{1}{2}y_1 + \frac{1}{2}y_2$$
$$y_3 = 1 \cdots z \cdots y_1 \cdots y_2$$

Surprise: we are back at our original maximization function $-z$
Since it has no positive factors anymore, the resulting basic solution

$$z = y_1 = y_2 = 0, x = 9, y = 1, y_3 = 1$$

yields the optimal value $-z = 0$
Since $-z = 0$, the original set of inequalities

$$-x - 3y \leq 12 \wedge x + y \leq 10 \wedge -x + y \leq -7$$

is satisfiable: we found the solution $x = 9, y = 1$

Concluding:

- We saw how the check on *feasibility* of a set of linear inequalities can be executed by *adding a fresh variable* z and maximize $-z$ by the original *simplex* approach

引申: *Linear programming* （线性规划问题）

- Given a set of linear inequalities on real valued variables and a linear goal function, determine whether this is feasible
- If so, find the maximal value of the goal function satisfying the inequalities

复杂度: *Worst case* the number of pivots is exponential, but *in practice* exponential blow-up is *rare*.

之后的问题: How to apply *simplex method* to SMT

Concluding:

- We saw how the check on *feasibility* of a set of linear inequalities can be executed by *adding a fresh variable* z and maximize $-z$ by the original *simplex* approach

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之后的问题: How to apply *simplex method* to SMT

求解 Linear programming 的方法:

- First check feasibility,
- if so, apply the basic Simplex method starting from the found solution

复杂度补充: a more complicated *ellipsoid algorithm* is worst case polynomial, but usually not better in practice

We saw how the *simplex method*

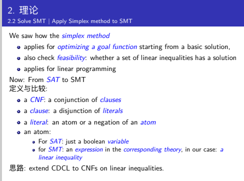
- applies for *optimizing a goal function* starting from a basic solution,
- also check *feasibility*: whether a set of linear inequalities has a solution
- applies for linear programming

Now: From *SAT* to SMT

定义与比较:

- a *CNF*: a conjunction of *clauses*
- a *clause*: a disjunction of *literals*
- a *literal*: an atom or a negation of an *atom*
- an atom:
  - For *SAT*: just a boolean *variable*
  - for *SMT*: an *expression* in the *corresponding theory*, in our case: *a linear inequality*

思路: extend CDCL to CNFs on linear inequalities.



The CDCL approach applies on CNF and consists of apply rules **Unipropagate**, **Backtrack**,...

Requirements for applying these rules consist of checking whether combinations of literals are contradictory

For instance, $M \models \neg C$ means: M extended by any literal from C is contradictory

In SAT being contradictory means a combination of p and $\neg p$ for some variable p

If literals consists of linear inequalities, being contradictory coincides with being unfeasible

So the CDCL appraoch can be extended to CNFs on linear inequalities, where for applicability of the rules the Simplex method is applied.

Consider the CNF of the following three clauses

$$(1)\ x \geq y+1 \vee z \geq y+1$$
$$(2)\ y \geq z$$
$$(3)\ z \geq x+1$$

解:

$(y \geq z)$ — **Unipropagate** on (2)

$(y \geq z)\ (x \geq y+1)$ — **Unipropagate** on (1)

since $y \geq z \wedge z \geq y+1$ is unfeasible by Simplex

Fail on (3)

since $y \geq z \wedge x \geq y+1 \wedge z \geq x+1$ is unfeasible

This proves that the given CNF is unsatisfiable

Also for more complicated examples with **Decide** and **Backtrack**

This is how current SMT solvers (Z3, CVC4, Yices, ...) deal with linear inequalities

Theories:

- *Simplex* is called very often for combination of M and literals from clauses
- Other theories: combine CDCL with efficient method for checking whether a conjunction of literals is contradictory
- *Integer Linear programming (ILP),* 整数线性规划 is harder (even NP-complete), but *effectively* supported by current SMT solvers

Also for more complicated examples with **Decide** and **Backtrack**

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Theories:

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要求所有的未知量都为整数的线性规划问题叫做整数规划（integer programming, IP）或整数线性规划（integer linear programming, ILP）问题。相对于即使在最坏情况下也能有效率地解出的线性规划问题，整数规划问题的最坏情况是不确定的，在某些实际情况中（有约束变量的那些）为 NP 困难问题。

作业

使用 Simplex Method, 求解如下问题:

1. Maximize $z = 1 + 2x_1 + 3x_2 + 6x_3$ satisfying the constraints:

$$
\begin{aligned}
x_1 + 2x_2 + 3x_3 &\leq 10 \\
x_1 - x_3 &\leq 3 \\
-x_2 + 2x_3 &\leq 5
\end{aligned}
$$

where $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$

2. Fine values $x, y \geq 0$ satisfying

$$
\begin{aligned}
x - y &\leq -3 \\
2x + y &\leq 7 \\
-x - 2y &\leq -9
\end{aligned}
$$