形式化方法导引

第 5 章 模型检测

5.2 理论

5.2.1 Fixpoint formulation | 5.2.2 BDD Algorithm

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往节内容

- Model checking
  - Modeling: Transition system
  - Specification: LTL, CTL
- Tool
  - NuSMV

本节内容:

- Basic idea of checking a model: fixpoint formulation
- Classical algorithms
  - Binary decisions diagram (BDD)
  - Bounded model checking (BMC)
  - Basic Inductive Techniques

回顾: 定义: Transition system

A transition system $\mathcal{M} = (S, \rightarrow, L)$ is

- S: a set of states
- $\rightarrow$: a transition relation: every $s \in S$ has some $s' \in S$ with $s \rightarrow s'$
- L: a label function: $L : S \rightarrow \mathcal{P}(\mathrm{Atoms})$

定义: State space

The *state space*

$$S = V_1 \times \cdots \times V_n$$

is implied by the variables $v_1, \ldots, v_n$ from finite sets $V_1, \ldots, V_n$

问题: How to check a CTL property?

问题转换: We want to check whether a CTL formula ϕ holds for all states $s \in I$ for a given set of initial states $I \subseteq S$

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基本思路:

1. *Compute the set S_ϕ* consisting of all states that satisfy ϕ
   - a state $s \in S$ satisfies ϕ if the set of all paths starting in s satisfies ϕ
2. Then the property to check is $I \subseteq S_\phi$

- $S_\bot = \emptyset$
- $S_\top = S$
- For an atomic proposition p:

$$S_p = \{s \in S \mid p(s)\}$$

- $S_{\neg\phi} = \{s \in S \mid s \notin S_\phi\}$
- $S_{\phi\vee\psi} = S_\phi \cup S_\psi$
- $S_{\phi\wedge\psi} = S_\phi \cap S_\psi$

It *remains* to describe S_ϕ for ϕ containing *CTL operator EX, EG, EU*

A state s satisfies *EX ϕ*, if there *exists* a path starting in s such that ϕ holds in the *next* state of that path. So:

$$S_{\mathrm{EX}\phi} = \{s \in S \mid \exists t \in \underline{S_\phi} : s \to t\}$$

EG and EU are harder: they deal with properties of paths *beyond a fixed finite part of the path*

思路:
Consider the first n steps for *increasing* n, *until* the corresponding set *does not change* anymore

$\mathrm{EG}\phi$ means: there *exists* a path $s_0 \to s_1 \to s_2 \cdots$ on which ϕ globally holds, that is, ϕ holds in s_i *for all i*

定义: T_n

For $n = 0, 1, 2, \ldots,$ let T_n = set of states s_0, for which there exists a path $s_0 \to s_1 \to s_2 \cdots$ on which ϕ holds *for all s_i with $i \leq n$*

Then $T_0 = S_\phi$, and for all $n = 0, 1, \ldots,$ we have

$$T_{n+1} = T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$$

算法: 求解 T_n, i.e., $S_{\mathrm{EG}\phi} = T_n$ (Fixpoint formulation)

$T_0 := S_\phi$; $n := 0$;
repeat
  $T_{n+1} := T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$; $n = n + 1$;
until $T_n = T_{n-1}$

算法: 求解 $S_{\mathrm{EG}\phi} = T_n$

$T_0 := S_\phi$; $n := 0$;
repeat
    $T_{n+1} := T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$; $n = n + 1$;
until $T_n = T_{n-1}$

The loop *terminates*, since

- the set T_n is finite
- $|T_n|$ decreases in every step

After running this algorithm we have $S_{\mathrm{EG}\phi} = T_n$

算法: 求解 $S_{\mathrm{EG}\phi} = T_n$

$T_0 := S_\phi$; $n := 0$;
repeat
  $T_{n+1} := T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$; $n = n + 1$;
until $T_n = T_{n-1}$

The loop *terminates*, since

- the set T_n is finite
- $|T_n|$ decreases in every step

After running this algorithm we have $S_{\mathrm{EG}\phi} = T_n$

After finishing this algorithm we have $T_n = T_{n-1}$, yielding $T_n = T_i$ for all $i \geq n$

So then T_n states that for every i there is a path of which the first i states satisfy ϕ

S finite $\Rightarrow$ this implies a path on which ϕ globally holds (take $i = |S|$)

So after running this algorithm we have $S_{\mathrm{EG}\phi} = T_n$

The loop terminates since all sets are finite and $|T_n|$ decreases in every step

s_0 satisfies $E[\phi\ U\ \psi]$ means: there *exists* n such that P_n holds, for

- P_n = there *exists* a path $s_0 \to s_1 \to s_2 \cdots$ on which ψ holds in s_n, and ϕ holds in s_i for all $i < n$

定义： U_n

U_n = set of states s_0 for which P_i holds for *some* $i \leq n$

Then $U_0 = S_\psi$, and for all n=0,1,..., we have

$$U_{n+1} = U_n \cup \{s \in S_\phi \mid \exists t \in U_n : s \to t\}$$

算法: 求解 $S_{\mathrm{E}[\phi \mathrm{U} \psi]} = U_n$

$U_0 := S_\psi; n := 0;$
repeat
  $U_{n+1} := U_n \cup \{s \in S_\phi \mid \exists t \in U_n : s \to t\}; n = n + 1;$
until $U_n = U_{n-1}$

Concluding,

- For an *arbitrary CTL formula* ϕ we saw how to compute the set S_ϕ, being the set of states that satisfy ϕ
  - Basics: $S_\bot$, $S_\top$, S_p, $S_{\neg\phi}$, $S_{\phi\vee\psi}$, $S_{\phi\wedge\psi}$
  - CTL Related: $S_{\mathrm{EX}\phi}$, $S_{\mathrm{EG}\phi}$, $S_{\mathrm{EU}[\phi \mathrm{U} \psi]}$
- In this computation we *only* needed the computation of the sets $T \cup U$, $T \cap U$, complements, and $\{s \in T \mid \exists t \in U : s \to t\}$, for given sets T, U

问题: In *explicit state based* model checking the complexity of this algorithm will be *at least the order of the size of the state space* S

可选方案: 2.2 Binary decisions diagram (BDD), 2.3 Bounded model checking (BMC), 2.4 k-induction

回顾:

- In *explicit state based* model checking the complexity of this algorithm will be *at least the order of the size of the state space S*
    - Key words: state spaces

Now we study BDD

- a method of *symbolic* model checking.
- adopted by NuSMV

基本思路: *Firstly*, investigate desired *requirements* for *alternative representations* for *boolean functions*

Outline of a BDD algorithm

1. Stage 1: Boolean variables
2. Stage 2: Boolean functions
3. Stage 3: Decision Tree
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6. Stage 6: ROBDD (merging and elimination)
7. Stage 7: Compute ROBDD
   - Stage 7.1: Compute $ROBDD(\phi)$
   - Stage 7.2: Compute $\diamond(T, U)$
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第 *1* 阶: **Boolean Variables**:

In NuSMV, the variable types are finite sets, in particular *boolean* or *integers with a restricted range*, like

```
VAR
a : 1..100;
```

We may *assume* we *only have* **boolean variables**

- by representing variables in *binary* notation, e.g., using *7 bits* for the range 1..100,
- *Atomic propositions*, e.g., $a < b + 5$, can be expressed in *binary* notation too

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第 2 阶: **Boolean Functions:**

Write $B = \{0, 1\}$, then for n *boolean variables* the *state space* is $S = B^n$

We want to represent and manipulate *subsets* of $S = B^n$

A *subset* $U \subseteq B^n$ can be identified by a **boolean function**

$$f_U : B^n \rightarrow B$$

defined by

$$s \in U \leftrightarrow f_U(s) = 1$$

问题: We want to *represent* and *manipulate* **boolean functions** efficiently, more precisely:

- Every boolean function has a *unique representation* (见后: 问题 2)
  (does not hold for formula representation: $p \wedge (p \wedge q)$ and $q \wedge p$ are distinct formulas representing the same boolean function)
- All operations needs for CTL model checking, including $\neg, \vee, \wedge$ should be *efficiently* computable
  (does not hold for *explicit state representation*: false corresponds to the empty set, but $\neg$ false $=$ true corresponds to the set $S = B^n$ *having* 2^n *elements, infeasible for* $n \geq 30$)
- *Many* boolean functions have an *efficient representation*
  - *Why not for all boolean functions*?

Why not for all boolean functions?

On n variables a truth table consists of 2^n lines

- Hence on n variables there are 2^{2^n} distinct boolean functions
- Indeed, there are $2^{64} \approx 20,000,000,000,000,000,000$ distinct boolean functions on six variables
- If all of these 2^{2^n} distinct boolean functions should have a distinct representation, then *on average* at least 2^n bits are needed for that, begin untractable for $n > 30$

This information theoretic argument shows that it is *unavoidable* that most of the boolean functions have *untractable* (困难的) representation

So the *best* we may hope for is that we meet *in practice* is among the *minor part of all boolean functions* that have an *efficient representation*

结论: BDD provide a *data structure* for *boolean functions* —Decision Tree

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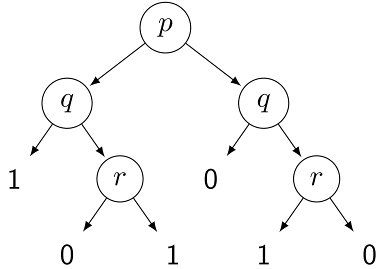
第 *3* 阶: **Decision Tree:**

定义: A decision tree is a binary tree in which

- Every *node* is labeled by a *boolean variable*
- Every *leaf* is labeled by 1 or 0, representing true or false respectively



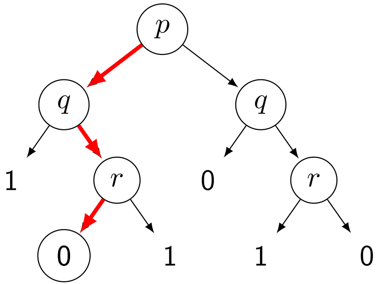
语义: If every variable has a boolean value then the corresponding function value is obtained by

- Start at the root
- For any node: go to the *left* if the corresponding variable is *true*; *otherwise*, go to the *right*
- Repeat until a *leaf* has been reached
- If the leaf is 1 then the result is true, otherwise false

Hence every node is interpreted as an if-then-else

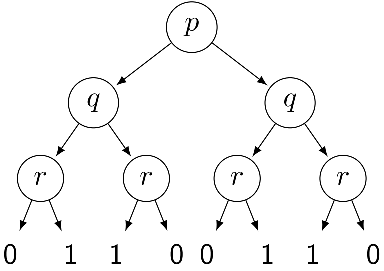
Consider our example for the values
- p: true
- q: false
- r: true

Hence, the result is 0



问题 1: Does every boolean function on finitely many boolean variables have a representation as decision tree?

答: *Yes*: it can be defined by a *truth table*, and any truth table on n variables having 2^n lines can be represented as a decision tree with 2^n leaves



问题 2: Does every boolean function on finitely many boolean variables have a *unique representation* as decision tree?

答: *No*

The following two decision trees both represent the boolean function on p, q that yields *true* in case p *is true* and q *is false*, and false otherwise

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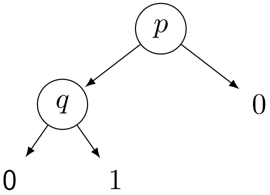
问题 2 的一种解决方法: Observe that in one case p is on top of q, while in the other case q is on top of p

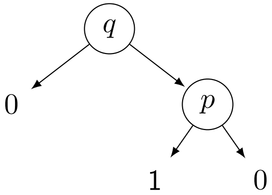
Fix an order $<$ on the boolean variables, like $p < q$

第 4 阶: **Ordered decision tree**: An *ordered decision tree* with respect to $<$ is a decision tree such that if node n is on top of node n', then

$$\text{label}(n) < \text{label}(n')$$

So the left one is ordered with respect to $p < q$, the *right one is not*



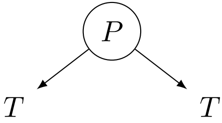


问题 3: Fixing the order $<$ on the boolean variables, does *every* boolean function have a *unique* representation as an *ordered* decision tree with respect to $<$?

No

Let T be any ordered decision tree, and let p be a variable less than the variables in T

Then T and



are two ordered decision trees representing *the same boolean function*

We are looking for a *small* representing, hence among these two we *prefer* T and exclude the latter.

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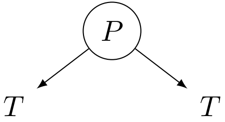
第 *5* 阶*:* **Reduced ordered decision tree:**

Replacing 

by T is called *elimination*

An ordered decision tree on which no elimination is possible is called a *reduced ordered decision tree*

定理

For a fixed order $<$ on boolean variables, every boolean function has a *unique* representation as a *reduced ordered decision tree*

证明过程: 略

定理

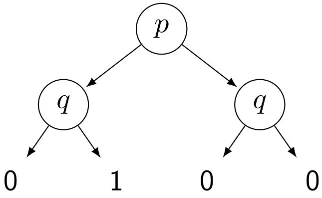
For a fixed order $<$ on boolean variables, every boolean function has a *unique* representation as a *reduced ordered decision tree*

证明过程: 略

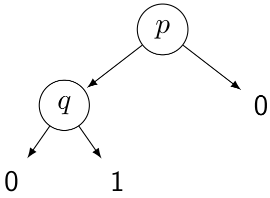
Proof sketch:

- Existence: Start by the ordered decision tree reflecting the truth table, and apply elimination anywhere in the decision tree as long as possible
- Elimination Strictly decreases the size, so cannot go on forever
- During elimination orderedness is maintained
- So at the end we have a reduced ordered decision tree representing the given boolean function

例: For the boolean function defined by the formula $p \wedge \neg q$, for the order $p < q$ the ordered decision tree reflecting the truth table is



Applying elimination on the right q yields



Now no elimination is possible any more, so this is a *reduced ordered decision tree*

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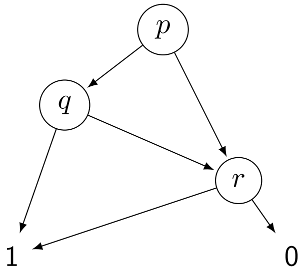
第 *6* 阶: *ROBDD*: **Reduced Ordered Binary Decisions Diagrams**

- A *particular* example of Binary Decisions Diagrams (*BDDs*)
- uniquely represent boolean functions by *merging* and *elimination*

The formula $(p \wedge q) \vee r$ describes the boolean function that yields true if both p and q are true, or r is true

With respect to the order $p < q < r$ its ROBDD is



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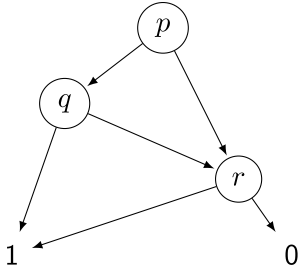


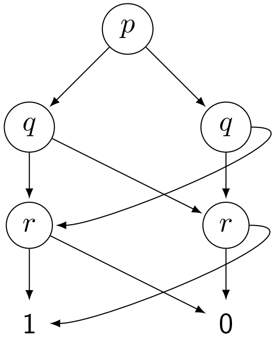
In such a ROBDD, *every node* represents a *boolean function* itself

- The ROBDD of this function is the part of the original ROBDD of which the indicated node is the *root*

All nodes of a ROBDD represent *distinct* boolean functions

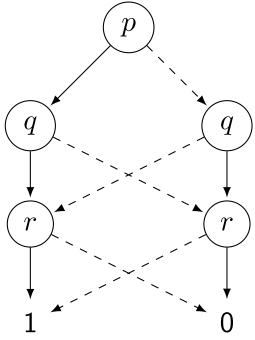
- since if two would represent the same, then they can be shared by applying *merging* and *elimination* steps, contradicting being *R*(educed)

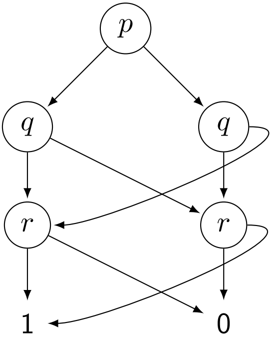
Merge and share: For $p < q < r$ the ROBDD of $p \leftrightarrow q \leftrightarrow r$ is



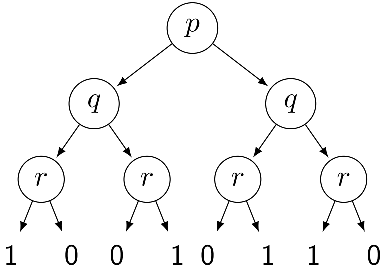
yields *true* if and only if the *number* of variables that is false, is *even*

Alternative notation to avoid curved arrows: use *solid* arrows for true-branches and *dashed* arrows for false-branches





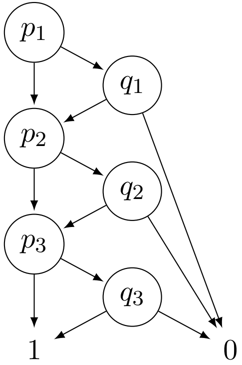
问: 为什么要 *merging?* Sharing really helps: without sharing (so the unique reduced ordered decision tree) for $p \leftrightarrow q \leftrightarrow r$ instead of, we would obtain



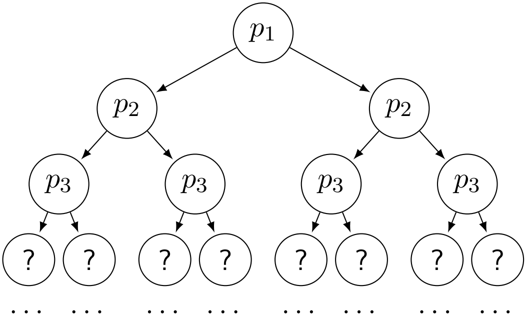
Doing the same for $p_1 \leftrightarrow p_2 \leftrightarrow \cdots \leftrightarrow p_n$ yields a ROBDD of $2n - 1$ nodes, and a reduced ordered decision tree of $2^n - 1$ nodes: an *exponential* gap

问题 *4: 如何选择 order?* The ROBDD of $(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge (p_3 \vee q_3)$ with respect to

$p_1 < q_1 < p_2 < q_2 < p_3 < q_3$ is:



w.r.t. $p_1 < p_2 < p_3 < q_1 < q_2 < q_3$ it is:



where all ? nodes represent distinct boolean functions on q_1, q_2, q_3, so cannot be shared

More general, the ROBDD of

$$(p_1 \vee q_1) \wedge (p_2 \vee q_2) \wedge \cdots \wedge (p_3 \vee q_3)$$

with respect to the order

- $p_1 < q_1 < p_2 < q_2 < \cdots < p_n < q_n$: has exactly $2n$ nodes
- $p_1 < p_2 < \cdots < p_n < q_1 < q_2 < \cdots < q_n$: has more than 2^n nodes

So *distinct orders* may result in ROBDDs of sizes with an *exponential* gap in between

Heuristic

choose the order in such a way that *variables close to each other* in the formula are also *close in the order*

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第 7 阶: How to *compute* the Reduced Ordered Binary Decision Diagram (*ROBDD*) of a given formula?

问题 *5*: The methods in the former stages should *not* be used to compute ROBDDs in *practice*

- since the *size* of this *decision tree* is always *exponential*, so *unfeasible*

解决方法: operate directly on the *formula* $\underline{\phi}$, instead of *decision tree*

Observation: Every formula is of the shape:

- false or true, or
- p for a variable p, or
- $\neg\phi$, or
- $\phi \diamond \psi$ for $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$

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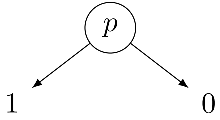
Stage 7.1: *Compute ROBDD*(ϕ). So, the ROBDD *ROBDD*(ϕ) of a formula $\underline{\phi}$ will be constructed recursively according this recursive structure of the formulas: ▸ Basic Idea

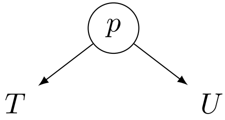
- *ROBDD*(**F**)=0, *ROBDD*(**T**)=1
- *ROBDD*($\underline{p}$)= 

- *ROBDD*($\neg\phi$)=*ROBDD*($\phi \to$ **F**)
- *ROBDD*($\phi \diamond \psi$)= *apply*(*ROBDD*(ϕ), *ROBDD*(ψ), $\diamond$)

So it remains to find an algorithm *apply* having two ROBDDs and a binary operation $\diamond \in \{\vee, \wedge, \to, \leftrightarrow\}$ as input, and having the desired ROBDD as its output

续: So it remains to find an algorithm *apply* having two ROBDDs and a binary operation $\diamond \in \{\vee, \wedge, \rightarrow, \leftrightarrow\}$ as input, and having the desired ROBDD as its output

write $p(T, U)$ for the BDD having root p for which the left branch is T and the right branch is U:



write $\diamond(T, U)$ instead of *apply*$(T, U, \diamond)$

- As the basis of the recursive we define $\diamond(T, U)$ (见下页)

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Stage 7.2: *Compute* $\diamond(T, U)$ recursively for cases:

1. if $T, U \in \{\mathbf{T}, \mathbf{F}\}$: return value according the *truth table of* $\diamond$
2. if T, U not both in $\{\mathbf{T}, \mathbf{F}\}$: let p be the smallest variable occurring in T and U.
   1. p is on top of both T and U ▸ Details

      $$\diamond(p(T_1, T_2), p(U_1, U_2)) = p(\diamond(T_1, U_1), \diamond(T_2, U_2))$$

   2. p is on top of T but does not occur in U ▸ Details

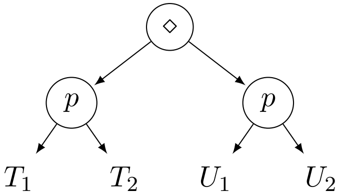
      $$\diamond(p(T_1, T_2), U) = p(\diamond(T_1, U), \diamond(T_2, U)), \text{ if } p \text{ does not occur in } U$$

   3. p is on top of U but does not occur in T ▸ Details

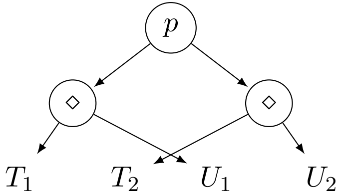
      $$\diamond(T, p(U_1, U_2)) = p(\diamond(T, U_1), \diamond(T, U_2)), \text{ if } p \text{ does not occur in } T$$

Intuitively: for two BDDs T, U computing $\diamond(T, U)$ is done by pushing $\diamond$ *downwards*, meanwhile *combining T and U*, until $\diamond$ applied to **T**/**F** has to be computed, which is replaced by its value according to the truth table

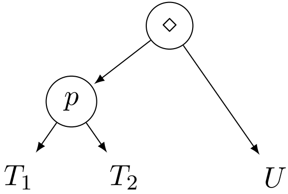
In a picture: (1) Replace



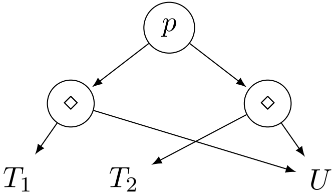
by



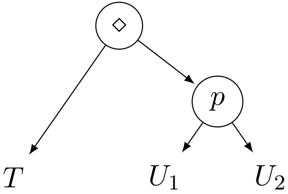
(2) If p not in U, then Replace



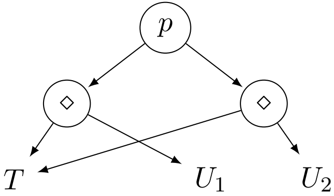
by



(3)If p not in T, then replace



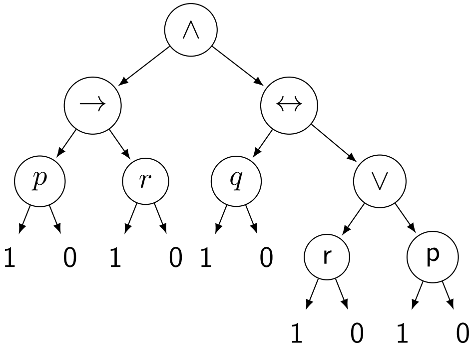
by



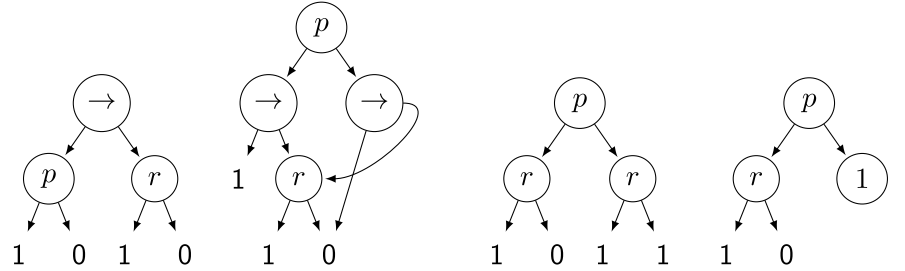
例子: We choose the formula

$$(p \to r) \wedge (q \leftrightarrow (r \vee p))$$

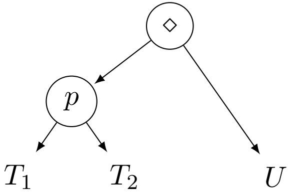
and the order $p < q < r$. in a picture:



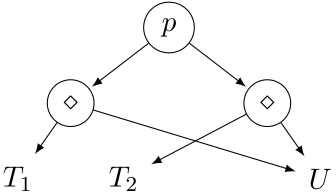
Left argument:



回顾:(2) If p not in U, then Replace

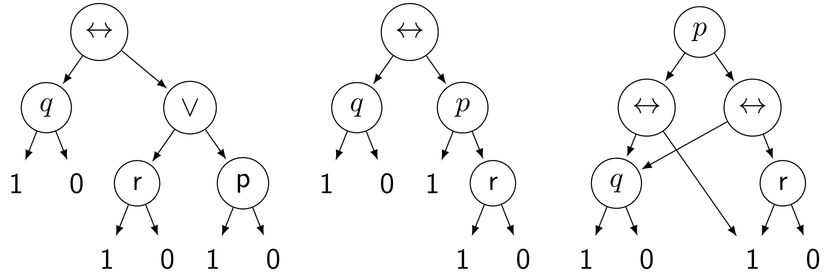


by

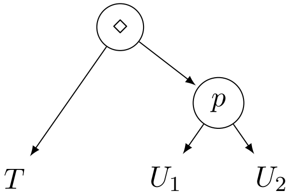


回顾:(3)If

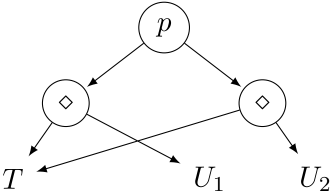
Right argument:



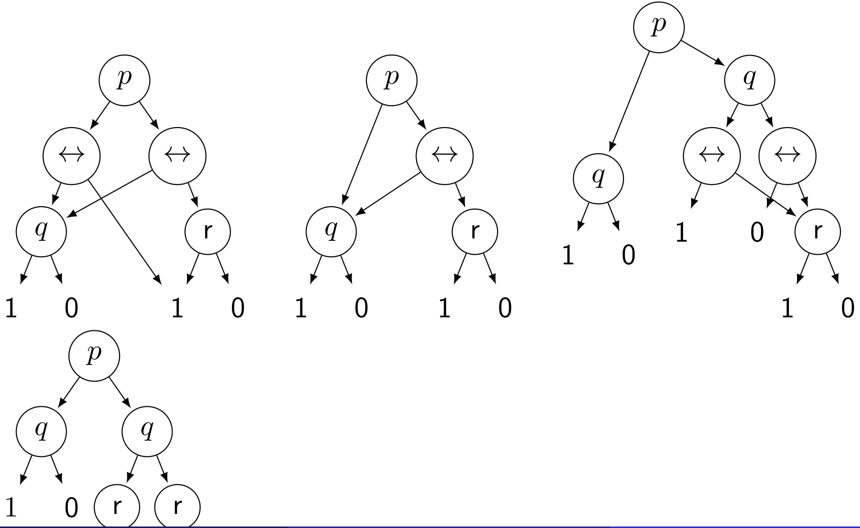
回顾:(3)If p not in T, then replace



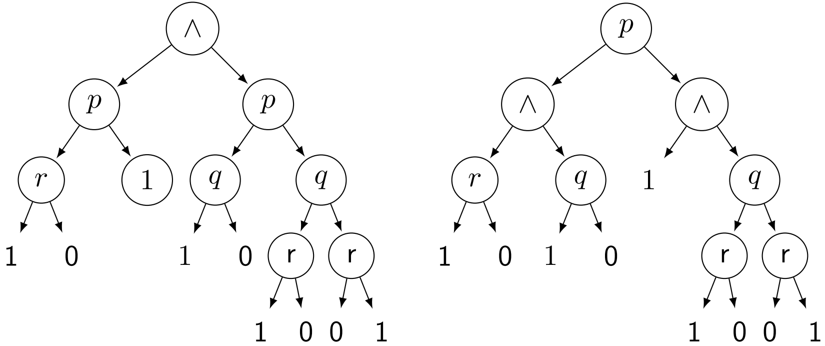
by



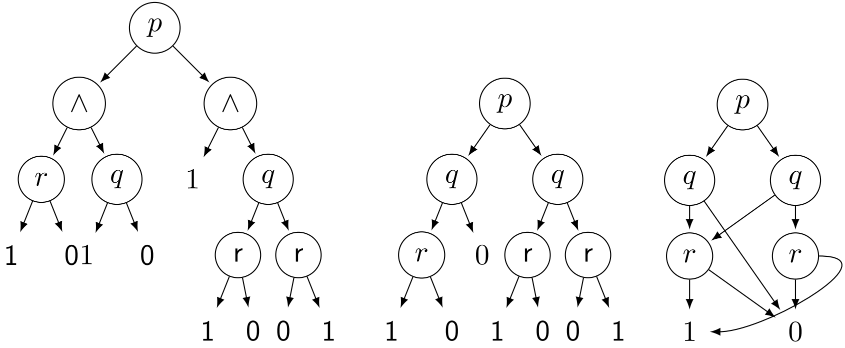
Right argument: (续上页):



Now we have computed the ROBDDs of both arguments of ∧ in the original formula, and it remains to apply ∧ on these two



回顾:(1) Replace


p
∧
∧
r
q
1
q
1
01
0
r
r
1
0
0
1
p
q
q
r
0
r
r
1
0
1
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0
1
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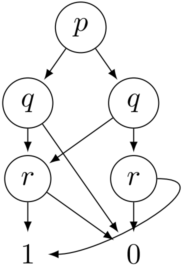
q
q
r
r
1
0



We computed the ROBDD of the formula

$$(p \to r) \wedge (q \leftrightarrow (r \vee p))$$

w.r.t the order $p < q < r$.

Doing this by hand in all detail is quite some work, but the steps are very systematic and suitable for implementation

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第 8 阶: **CTL model checking by ROBDDs**

How can we do CTL model checking by *representing large sets of states* by ROBDDs?

First *recall* the abstract algorithm for CTL model checking ▸ Basic Idea

核心思路: Iteratively update ROBDD of a *boolean function*, e.g., ROBDD(f_n), until ROBDD(f_n)=ROBDD(f_{n-1})

例: 求解 $S_{\mathrm{EG}\phi} = T_n$

回顾: 算法: 求解 $S_{\mathrm{EG}\phi} = T_n$

$T_0 := S_\phi$; $n := 0$;
repeat
    $T_{n+1} := T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$; $n = n + 1$;
until $T_n = T_{n-1}$

Here, $f_n(s) = 1 \leftrightarrow s \in T_n$

For convenience, define ROBDD$(T_n) \equiv$ ROBDD(f_n)

问题转换: How to compute ROBDD($T_n \cap \{s \in S_\phi \mid \exists t \in T_n : s \to t\}$)?

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回顾:

$$\neg \mathrm{AF}\ \phi \equiv \mathrm{EG}\ \neg\phi$$

$$\neg \mathrm{EF}\ \phi \equiv \mathrm{AG}\ \neg\phi$$

$$\neg \mathrm{AX}\ \phi \equiv \mathrm{EX}\ \neg\phi$$

$$\mathrm{AF}\ \phi \equiv \mathrm{A}[\top\ \mathrm{U}\ \phi]$$

$$\mathrm{EF}\ \phi \equiv \mathrm{E}[\top\ \mathrm{U}\ \phi]$$

Stage 8.1: *All CTL operators* can be *expressed* in (1) *boolean operators* $(\neg, \wedge, \rightarrow, \vee)$ and (2) *EX, EG, EU*

- (1) solved in stage 7 Stage 7.1 Stage 7.2
- (2) to be solved in Stage 8.2 Stage 8.2

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Stage 8.2: For *computing* EX, EG, EU, we only needed the building blocks:

- set
  - **基本思路:** Sets are described by **boolean functions**: an element is in the set if and only if the boolean function yields true $f_U : B^n \to B$
    $$s \in U \leftrightarrow f_U(s) = 1$$
    $$\mathrm{ROBDD}(U) \equiv \mathrm{ROBDD}(f_U(s)),\ \text{a.k.a.,}\ \mathrm{ROBDD}(S_\phi) \equiv \mathrm{ROBDD}(\phi)$$
- union $\cup$, intersection $\cap$
  - **基本思路:** Union and intersection correspond to $\vee$ and $\wedge$, for which we already gave an algorithm for ROBDD representations
  - $S_{\phi\vee\psi} = S_\phi \cup S_\psi,\ S_{\phi\wedge\psi} = S_\phi \cap S_\psi$
  - $\mathrm{ROBDD}(S_\phi \cup S_\psi) = \mathrm{ROBDD}(S_{\phi\vee\psi}) \equiv \mathrm{ROBDD}(\phi \vee \psi)$
  - $\mathrm{ROBDD}(S_\phi \cap S_\psi) = \mathrm{ROBDD}(S_{\phi\wedge\psi}) \equiv \mathrm{ROBDD}(\phi \wedge \psi)$
- computing
  $$\{s \in T \mid \exists t \in U : s \to t\}$$
  for a given transition relation $\to$ and given sets T, U

Stage 8.2: 小总结: 如何转换

Variables $a, b, \ldots \equiv$ *boolean variables* $a_1, a_2, \ldots, a_n$

- As all variables are from finite sets, they can be encoded *binary*
- So we assume that we *only* have *boolean variables* $a_1, \ldots, a_n$

Before

$a, b, \ldots$: 1..100;

After

$a_1, a_2, \ldots, a_n$: BOOL;

A *state* $s = (a_1, a_2, \ldots, a_n)$
The *sets* S, T and U of *states*

- boolean functions on the variables: $s \in U \leftrightarrow f_U(s) = 1$

ROBDD on S: $\mathrm{ROBDD}(S_\phi) \equiv \mathrm{ROBDD}(\phi)$
ROBDD on $\cup$ *and* $\cap$:

- $\mathrm{ROBDD}(S_\phi \cup S_\psi) = \mathrm{ROBDD}(S_{\phi \vee \psi}) \equiv \mathrm{ROBDD}(\phi \vee \psi)$
- $\mathrm{ROBDD}(S_\phi \cap S_\psi) = \mathrm{ROBDD}(S_{\phi \wedge \psi}) \equiv \mathrm{ROBDD}(\phi \wedge \psi)$

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Stage 8.3: Computing

$$V = \{s \in T \mid \exists t \in U : s \to t\}$$

for a given transition relation $\to$ and given sets T, U

准备:

- Write a'_i as shorthand for next(a_i)
- The transition relation $\to$ is given by a boolean function (P) on $a_1, \ldots, a_n, a'_1, \ldots, a'_n$, again in ROBDD representation

$$P(a_1, \ldots, a_n, a'_1, \ldots, a'_n) \Leftrightarrow P_1 \wedge P_2$$

where

- $P_1 = (a_1, \ldots, a_n) \to (a'_1, \ldots, a'_n)$
- $P_2 = (a'_1, \ldots, a'_n) \in U$

Step 1: Compute ROBDD of $P(\cdots)$, i.e., ROBDD(P)

$$P(a_1, \ldots, a_n, a'_1, \ldots, a'_n) \Leftrightarrow P_1 \wedge P_2$$

where

- $P_1 = (a_1, \ldots, a_n) \rightarrow (a'_1, \ldots, a'_n)$
  - ROBDD(P_1) was given by translation relation $\rightarrow$
- $P_2 = (a'_1, \ldots, a'_n) \in U$
  - When using the order $a_1 < \cdots < a_n < a'_1 < \cdots < a'_n$, ROBDD($P_2$) is obtained by just replacing every a_i by a'_i in ROBDD(U)
- Now, ROBDD(P)=ROBDD($P_1 \wedge P_2$)=*apply*(*ROBDD*(P_1), *ROBDD*(P_2), $\wedge$)

Step 2: Compute ROBDD(S_{P_e})

$$S_{P_e} = \{(a_1, \ldots, a_n) \mid \exists a'_1, \ldots, a'_n : P(a_1, \ldots, a_n, a'_1, \ldots, a'_n)\}$$

Observe that for every boolean variable x:

$$\exists x : \phi \equiv \underbrace{\phi[x := \mathbf{T}] \vee \phi[x := \mathbf{F}]}_{\text{computable}}$$

Applying this n times, for $x = a'_1, a'_2, \ldots, a'_n$, all '$\exists$'s are eliminated, yielding an ROBDD over $a_1, \ldots, a_n$

Step 3: Compute ROBDD of V

$$V = \{s \in T \mid \exists t \in U : s \to t\} = T \cap S_{P_e}$$

ROBDD(V)=ROBDD($T \cap S_{P_e}$)=*apply*(*ROBDD*(T), *ROBDD*(S_{P_e}), $\wedge$)

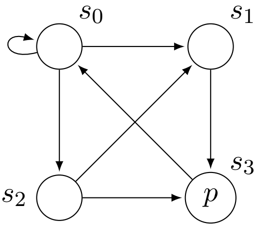
Stage 8 步骤小结:

- Stage 8.1: Express *all CTL operators* in
  1. *boolean operators* $(\neg, \wedge, \rightarrow, \vee)$
     - Compute ROBDD of boolean operators: ▸ Stage 7.1 ▸ Stage 7.2
  2. *EX, EG, EU*
- Stage 8.2: Compute ROBDD of *EX, EG, EU* ▸ Example: $S_{\mathrm{EG}\phi} = t_n$
  - solved: t_0, S_ϕ, $\cap$
  - problems left: $\mathrm{ROBDD}(V)$, where $V = \{s \in T \mid \exists t \in U : s \rightarrow t\}$
- Stage 8.3:
  - step 1: compute ROBDD(P), where $P = P_1 \wedge P_2$, $P_1 = (a_1, \ldots, a_n) \rightarrow (a'_1, \ldots, a'_n)$, $P_2 = (a'_1, \ldots, a'_n) \in U$
  - step 2: compute ROBDD(S_{P_e}), where
    $$S_{P_e} = \{(a_1, \ldots, a_n) \mid \exists a'_1, \ldots, a'_n : P(a_1, \ldots, a_n, a'_1, \ldots, a'_n)\}$$
  - step 3: compute ROBDD(V), where
    - $V = T \cap S_{P_e}$

例: ROBDD-CTL 求解

Given a transition system $\mathcal{M} = (S, \rightarrow, L)$, where $S = \{s_0, s_1, s_2, s_3\}$ $\rightarrow = \{(s_0, s_0), (s_0, s_1), (s_0, s_2), (s_1, s_3), (s_2, s_1), (s_2, s_3), (s_3, s_0)\}$, $L(s_3) = \{p\}$.

Verify: $\mathcal{M}, s_0 \vDash \mathsf{AF}\ p$

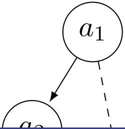


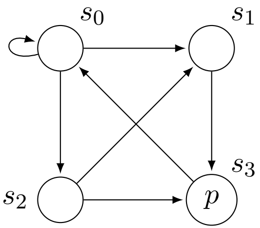
Stage 8.1: compute ROBDD(AF p)

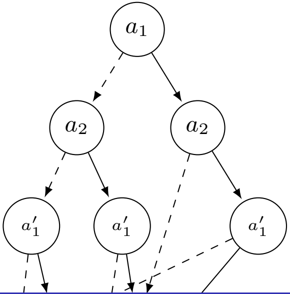
1. AF p = ¬EG¬p
2. ROBDD(AF p) = ROBDD(EG¬$p \rightarrow$ **F**)
   =apply(ROBDD(EG¬p), ROBDD(**F**), $\rightarrow$)
3. compute ROBDD(EG¬p) in *Stage 8.2*

Stage 8.2: Compute ROBDD(EG¬p) ▸ $S_{\mathrm{EG}\phi} = t_n$

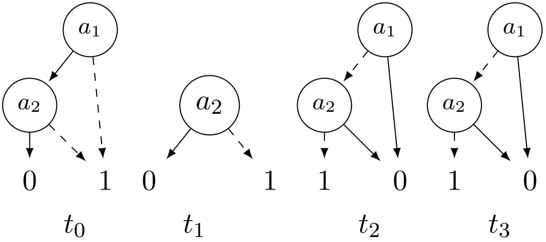
1. Define a state as a pair of variables (a_1, a_2), where $a_1, a_2 \in \{0, 1\}$
   - i.e. $s_0 = (0,0)$, $s_1 = (0,1)$, $s_2 = (1,0)$




s_0
s_1
s_3
s_2
p


a_1
a_2
a_2
a'_1
a'_1
a'_1

Similarly, compute t_2,t_3... (作业: 计算 t_2, t_3)



Observe that $t_2 = t_3$

Back to Stage 8.1: ROBDD(EG $\neg p$)=t_2

$S_{\text{EG}\neg p} = \{(0,0)\} = \{s_0\}$

$S_{\text{AF}p} = S_{\neg\text{EG}\neg p} = \{s_1, s_2, s_3\}$

So, $\mathcal{M}, s_0 \not\models$ AF p

Stage 8 小结:

- Combining this gives an algorithm to compute the ROBDD of the set states satisfying any CTL formula
- This is essentially the algorithm as it is used in tools like *NuSMV* to do *symbolic model checking*
- In contrast to *explicit state based model checking*, it can deal with very large state spaces.

作业

作业: 模仿 t_1 的求解过程，手工运算 t_2, t_3，给出运算过程.

实验大作业 (可选): 实现 ROBDD 算法, 要求:

- 可以实现至不同 stage，例如，可实现至 ROBDD, 或 ROBDD-CTL。实现的越完整，给分越高。
- 提供源代码、可执行程序、测试文件、相关文档